Lecture 1

Spaces and groups

Abstract: In the first part of this course we will discuss a useful topology on sets of transformations, the so-called *compact-open* topology. This topology will play a crucial role in the remaining part of the lectures.

We assume that the students understand the material of basic courses in algebra, general topology, calculus and linear algebra. Concepts such as group, field, homomorphism, isomorphism, kernel, range, topological space, metric space, Banach space, Hilbert space, compact space, product space, Tychonoff product topology, continuous function, etc., should be familiar.

Our primary interest is in *separable metrizable spaces.* We therefore call a separable metrizable space simply a *space* from now on. In the first lecture we will discuss some basic properties of spaces, topological groups and zero-dimensional spaces.

Lecture 2

The compact-open topology

Abstract: In this lecture we will define and study a useful topology on sets of transformations, the so-called *compact-open* topology, which will play a crucial role in the remaining part of the course. Continuity of the composition operator, continuity of the evaluation function, products, special metrics for groups of transformations, limits of continuous functions and function spaces will be dealt with in detail.

Lecture 3

Banach spaces

Abstract: Some important examples of Banach spaces are presented. For example, the spaces $C^*(X)$, functionals on them, and the weak* topology. The fundamental theorem that every Banach space is isometrically isomorphic to a closed linear subspace of $C(\mathbb{I})$ will be proved (recall that all spaces are separable). Here $\mathbb{I}$ denotes $[0, 1]$. Hence $C(\mathbb{I})$ is a 'universal' space for all Banach spaces in the same way as the Hilbert cube Q is 'universal' for spaces. We will see later that there also is a 'universal' topological group, i.e., a topological group containing an isomorphic copy of any topological group.

Lecture 4

Groups of homeomorphisms

Abstract: In this lecture we focuss our attention to function spaces of a special type endowed with the compact-open topology, namely, the groups of homeomorphisms $\mathscr{H}(X)$. The question when $\mathscr{H}(X)$ is a topological group is an important one. It will be shown that if the locally compact space X is compact or has the property that every point of it has a neighborhood that is a continuum, then $\mathscr{H}(X)$ is a topological group. If X is $\mathbb{N} \times \mathsf{C}$, then $\mathscr{H}(X)$ is not a topological group. Here C denote the familiar Cantor middle-third set. Hence for locally compact spaces in general, the compact-open topology does not 'work' for $\mathscr{H}(X)$. But, fortunately, for such spaces X, there is a strongly related and useful topology on $\mathscr{H}(X)$ that makes $\mathscr{H}(X)$ a (Polish) topological group. We will discuss that topology. It is shown that for compact X, the topological group is Polish, that $\mathscr{H}(\mathsf{C})$ is not locally compact. We also discuss briefly left and/or right invariant metrics.

Lecture 5

The Inductive Convergence Criterion and Applications

Abstract: In this lecture we will discuss among other things an important application of the completeness of $\mathscr{H}(X)$, namely, the so-called 'Inductive Convergence Criterion' which is a very useful tool in homeomorphism theory. Countable Dense Homogeneity (abbreviated: CDH) is introduced and it is shown that every connected CDH-space is homogeneous, and that every strongly locally homogeneous (abbreviated: SLH) locally compact space is CDH. Hence all the euclidean spaces $\mathbb{R}^n$ for $n \geq 1$ are CDH.

Lecture 6

Examples

Abstract: In this lecture we will present some interesting examples of homeomorphism groups. We show that $\mathscr{H}(\mathsf{C})$ is homeomorphic to $\mathbb{P}$, the space of irrational numbers. In addition, we show that $\mathscr{H}(\mathbb{I})$ is homeomorphic to $\{0,1\} \times \mathbb{R}^\infty$. The question naturally arises what can be said about the homeomorphism groups $\mathscr{H}(\mathbb{I}^n)$ for $n \geq 2$. In fact, it is natural to think of the following closed subgroup of $\mathscr{H}(\mathbb{I}^n)$:

$$\mathscr{H}_\partial(\mathbb{I}^n) = \{h \in \mathscr{H}(\mathbb{I}^n) : h \restriction \partial\mathbb{I}^n = 1_{\partial\mathbb{I}^n}\};$$

here $\partial\mathbb{I}^n$ is the union of the endfaces of $\mathbb{I}^n$. It is known that $\mathscr{H}_\partial(\mathbb{I}^2)$ is homeomorphic to $\mathbb{R}^\infty$; the proof is difficult. It is also known that $\mathscr{H}(Q) \approx \mathbb{R}^\infty$; the proof of this is also difficult. There are compact spaces X for which $\mathscr{H}(X)$ consists of the identity homeomorphism only. Such a space is called *rigid*. An example of a compact rigid space is presented.

Lecture 7

The compact-open topology is natural

Abstract: Let G be a group. There are generally many admissible topologies on G. The discrete topology is obviously admissible. MARKOV asked whether every infinite group admits a non-discrete admissible topology. Here topology means T_0 topology to make life interesting. Note that T_0 is in general a very weak separation axiom. But not so for topological groups for it is known that a T_0 topological group is completely regular. It is not difficult to see that for Abelian groups, Markov's question has a positive answer. SHELAH constructed under the Continuum Hypothesis the first example of a group of cardinality ω_1 of which the only admissible topology is the discrete topology. This suggested the question whether every countably infinite group admits a non-discrete admissible topology. This interesting question was answered by OL′SHANSKIĬ in the negative.

In this lecture we will show that the compact-open topology is natural by proving that $\mathscr{H}(\mathbb{I})$ admits a unique minimum admissible T_0 topology (one contained in all admissible T_0 topologies), which is the compact-open topology.

Lecture 8

Actions of topological groups

Abstract: We first discuss some basic material on actions of topological groups on spaces. Eventually we will apply these results to get conclusions on various relations between strong forms of homogeneity on spaces. We show that actions on second category spaces are d-open ($\equiv$ 'almost open').

Lecture 9

The Effros Theorem

Abstract: Now that we know that actions are sometime 'almost open', it is natural to ask when they are 'open'. Let G be a group acting on a space X. The action is *micro-transitive* if for every $x \in X$ and every neighborhood U of e in G the set Ux is a neighborhood of x in X. We will show that if an analytic group acts transitively on a second category space, then it acts micro-transitively. This is the so-called *Effros Theorem*, or the *open mapping principle*. This implies that if a Polish group acts transitively on a second category space X, then X is Polish. Other applications of the Effors Theorem that will be proved are:

(1) Let G and H be topological groups with G analytic and H Polish. If $\varphi\colon G \to H$ is a continuous surjective homomorphism then φ is open.
(2) Every C^n Polish group is LC^n $(0 \leq n < \infty)$.
(3) Let G be a path-connected Polish group which is not locally compact. Then G contains a copy of the Hilbert cube Q and hence is strongly infinite-dimensional.
(4) Let G and H be topological groups with G Polish. If $\varphi\colon G \to H$ is a homomorphism such that its graph

$$G(\varphi) = \big\{\big(g, \varphi(g)\big) : g \in G\big\} \subseteq G \times H$$

is analytic, then φ is continuous.

Lecture 10

Coset spaces

Let G be a topological group with closed subgroup H. If $x, y \in G$ and $xH \cap yH \neq \emptyset$ then $xH = yH$. Hence the collection of all *left cosets* $G/H = \{xH : x \in G\}$ is a partition of G in closed sets. Let $\pi\colon G \to G/H$ be defined by $\pi(x) = xH$. We endow G/H by the quotient topology. In other words, if $A \subseteq G$ then $\{xH : x \in A\}$ is open in G/H if and only if $\bigcup\{xH : x \in A\} = AH$ is open in G.

A space X is a *coset space* provided that there is a closed subgroup H of a topological group G such that X and G/H are homeomorphic. In this lecture we will consider the following basic question: which spaces are coset spaces of topological groups? We will show, as another application of the Effros Theorem, that every locally compact homogeneous space is a coset space. Moreover, every homogeneous and strongly locally homogeneous space is a coset space. As we will show, not all homogeneous spaces are coset spaces. For each i let

$$W_i = \prod_{j \neq i} [-1 + 2^{-i}, 1 - 2^{-i}]_j \times \{1\}_i \subseteq Q.$$

Then W_i is a 'shrunken' endface in the i-th coordinate direction. It was shown by Anderson, Curtis and van Mill that $Y = Q \setminus \bigcup_{i=1}^{\infty} W_i$ is homogeneous. It can be shown that Y is a coset space. As an application of the Sierpiński Theorem, we will show that the σ-compact connected and locally connected space $W = \bigcup_{i=1}^{\infty} W_i$ is also homogeneous, but is not a coset space.

Lecture 11

Descriptive complexity of transitive group actions

Abstract: We will show that for X homogeneous and SLH, if X is absolutely Borel (analytic) then there are an absolutely Borel (analytic) group G and a closed subgroup H of G such that $G/H \approx X$. Let Γ denote the class of spaces X having the following property: for every completion $\tilde{X}$ of X the 'remainder' $\tilde{X} \setminus X$ is either countable or contains a Cantor set. Let X be a space which admits a transitive action by an analytic group G. We show that if for every countable subset D of X there is an element $g \in G$ such that $D \cap gD = \emptyset$, then $X \in \Gamma$. This implies that it is consistent that there are uncountable homogeneous SLH-analytic spaces X with the following property: for every analytic group G admitting a transitive action on X there is a countable set $D \subseteq X$ such that $D \cap gD \neq \emptyset$ for every $g \in G$.

Lecture 12

Ungar's Theorem and applications

Abstract: Let X be a homogeneous compact space. We show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $\varrho(x, y) < \delta$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ while moreover f moves no point more than ε. This is Ungar's Theorem which is a very powerful tool for continuum theorists in their study of homogeneous continua.

A space X is called *uniquely homogeneous* provided that for all $x, y \in X$ there is a *unique* homeomorphism moving x to y. So a uniquely homogeneous space is a space which is 'barely' homogeneous. We show that no compact space containing more than 2 points is uniquely homogeneous. It is unknown whether there is a Polish uniquely homogeneous space. But there is an example of a Baire uniquely homogeneous space that admits the structure of a Boolean topological group.

A space X is *n-homogeneous* if for all subsets $F, G \subseteq X$ of size n there is a homeomorphism $f\colon X \to X$ such that $f(F) = G$. We will prove that if X is n-homogeneous for some $n > 1$, then X is $(n-1)$-homogeneous.

The following is a natural strengthening of n-homogeneity. A space X is *strongly n-homogeneous* if for all subsets $F, G \subseteq X$ of size n, every bijection $f\colon F \to G$ can be extended to a homeomorphism $\bar{f}\colon X \to X$. It is clear that every strongly n-homogeneous space is n-homogenous and strongly $(n-1)$-homogeneous if $n > 1$. We show that if X is an n-homogeneous compact space such that no set of at most $n-1$ points separates X. Then X is strongly n-homogeneous. As a consequence, if X is an n-homogeneous continuum, then X is strongly n-homogeneous or X is the circle.

Lecture 13

More applications of Ungar's Theorem

Abstract: We show that every 2-homogeneous continuum is locally connected. We also show that every locally compact homogeneous space is the product of a connected space and a zero-dimensional space.

Lecture 14

Equivariant compactifications

Abstract: It is natural to ask whether every action can be 'compactified' in some natural way. If so, then actions on general spaces can be studied with the same ease as actions on compact spaces. It turns out that under mild and natural conditions the answer to this question is in the affirmative. But there are also actions for which a 'compactification' does not exist.

Let G act on X. A compactification γX of X is called *equivariant* if G acts on γX while moreover for every $g \in G$ the homeomorphism of γX determined by g restricts to the homeomorphism on X determined by g.

Let G act on X. It is well-known that every homeomorphism of X can be extended to a homeomorphism of the Čech-Stone compactification βX of X. The action $G \times X \to X$ can therefore be 'extended' to an 'action' $G \times \beta X \to \beta X$. However, the extended action need not be continuous and βX need not be metrizable. It turns out that certain conditions have to be imposed on the interplay between G and X in order to be able to conclude anything of interest.

Let G act on X. We say that a function $f \in C^*(X)$ is *right-uniformly continuous* provided that for every $\varepsilon > 0$ there is a neighborhood V of e in G such that for all $g \in V$ and $x \in X$ we have

$$|f(x) - f(gx)| < \varepsilon.$$

Let $C^*_{r,G}(X) = \{f \in C^*(X) : f \text{ is right-uniformly continuous}\}$.

Suppose that G acts on X. We show that the following statements are equivalent.

(1) X has an equivariant compactification γX.

(2) If $x \in X$ and A is a closed subset of X not containing x then there exists $f \in C^*_{r,G}(X)$ such that $f(x) \notin \overline{f[A]}$.

Lecture 15

Abstract: We will prove that *locally compact* groups acting on spaces can always be equivariantly compactified. This seems to be the best result possible since we will show that it cannot be generalized to Polish groups. We also show that if the action is transitive, and X is of the second category, then it can be equivariantly compactified.