Mathematical Aspects from fluid-solid dynamics

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Outline

1. Introduction : the Euler and Navier-Stokes equations
2. Fluid-solid dynamics I : ideal (or almost ideal) fluids
   - D'Alembert's paradox (1752)
   - Boundary layer theory
3. Fluid-solid dynamics II : viscous fluids
   - Navier-Stokes type models
   - Weak and strong solutions
   - Drag computation and the no-collision paradox

Chronology

Good reference : O. Darrigol, *Worlds of flow*.

- Derivation of the equations goes back to the 18th (Euler) and 19th (Navier-Stokes) century.
- Result of a gradual process, with many inputs from many scientists.

Roughly, two phases in the development of mathematical hydrodynamics:

1. *Extension of Newton's laws to hydrodynamics.*

   New concepts, both in maths (partial derivatives, vector equations) and physics (notion of fluid particle, internal pressure).

   Big names: D. Bernoulli (1700-1782), J. Bernoulli (1667-1748), D'Alembert (1717-1783), Euler (1707-1783), Lagrange (1736-1813)

   <u>Peak</u>: Writing of the Euler equation in 1755.

2. *Understanding and modeling of viscosity*.

Euler equation turned ineffective for many practical problems.

Example: drag computation (see D'Alembert's paradox later).

Very early, mathematical hydrodynamics got rejected by the engineering community.

Quoting Nobel laureate Cyril Hinshelwood :

> *... an unfortunate split between the field of hydraulics, observing phenomena which could not be explained, and theoretical fluid mechanics explaining phenomena which could not be observed.*

Viscous models allowed to gather the communities.

The Navier-Stokes equation was derived several times: Navier (1821), Cauchy (1823), Saint Venant (1837), Stokes (1845).

Derivation of the equations

These equations are *macroscopic models* : one does not follow the dynamics of each individual molecule.

Approximation : Fluid considered as a *continuum.*

Marked by a space variable x, in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$.

Key notion : *Fluid particle* (at time t and point x).

It is an elementary volume $V(t, x)$, with diameter d satisfying:

- $d \ll L$, $\quad L$: typical scale of the flow.
- $d \gg l$, $\quad l$: mean free path.

Allows to define averages for each fluid particle.

Example : Fluid velocity field

$$\boxed{u(t,x) = \frac{1}{N}\sum_{i=1}^{n} V_i}$$

One can also define density $\rho(t,x)$, temperature $T(t,x)\dots$

These averages are robust, and define smooth functions of t and x.

Problem : What model for these macroscopic quantities ?

Idea: Fluid particles $\approx$ material points.

$x(t,y)$: position at time t of the fluid particle initially at y.

$$\boxed{\partial_t x(t,y) \;=\; u(t,x(t,y)).}$$

Conservation laws

Volume conservation (*incompressible* fluids) : $\det \nabla x(t, y) = 1$.

By Liouville's theorem: $\boxed{\operatorname{div} u = 0}$

Mass conservation: $\frac{d}{dt} \int_{V_t} \rho(t, x) dx \;=\; 0, \quad V_t = x(t, V_0)$.

Together with incompressibility : $\partial_t \rho + u \cdot \nabla \rho = 0$.

Homogenenous fluid: $\boxed{\rho = cst}$

Conservation of momentum:

$$\boxed{\rho(\partial_t u + u \cdot \nabla u) = \operatorname{div} \sigma + \rho f}$$

σ : stress tensor (surface forces).

f : volume forces (like gravity).

Euler equation

Leonhard Euler
(1707-1783)

Hypothesis : Stress is normal to the surface: $\sigma = -p\, I_d$, p : pressure.

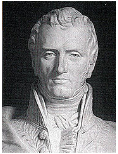
$$\boxed{\begin{cases} \rho(\partial_t u + u \cdot \nabla u) + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases}}$$

Remarks:

- $d+1$ scalar equations for $d+1$ unknowns.
- No evolution equation on the pressure (see later).

Navier-Stokes equation

Claude-Louis Navier (1785-1836)



George Stokes (1819-1903)

Takes *viscosity* into account.

Newtonian model for the stress tensor. The viscous resistance is proportional to the shearing of the fluid (symmetric part of ∇u).

$$\boxed{\begin{cases} \rho(\partial_t u + u \cdot \nabla u) + \nabla p - \mu \Delta u = 0, \\ \hfill \operatorname{div} u = 0. \end{cases}}$$

μ : coefficient of molecular viscosity.

Boundary conditions

Time: initial velocity : $\boxed{u|_{t=0} = u_0}$.

Space: depend on the choice of the fluid domain Ω.

- If Ω is a bounded domain, classical conditions are:

  for Euler : $\boxed{u \cdot n|_{\partial\Omega} = 0}$ (no penetration).

  for Navier-Stokes : $\boxed{u|_{\partial\Omega} = 0}$ (no slip).

- If $\Omega = \mathbb{R}^d$, one can impose a decay condition on u at infinity.

- Other choice (simpler for maths) : $\Omega = \mathbb{T}^d$, corresponding to a box with periodic boundary conditions.

Role of the pressure

The pressure gradient allows to preserve the incompressibility constraint through the evolution.

Case $\Omega = \mathbb{R}^3$, Euler: apply div to the equation:

$$\boxed{\Delta p = -\text{div}\,(u \cdot \nabla u)}$$

If enough decay and regularity

$$\nabla p = -\nabla \Delta^{-1} \text{div}\,(u \cdot \nabla u) = \nabla \frac{1}{4\pi |x|} \star \text{div}\,(u \cdot \nabla u)$$

and finally

$$\partial_t u + (Id - \nabla \Delta^{-1} \text{div}\,)(u \cdot \nabla u) = 0$$

One single evolution equation on u.

Other point of view:

$\mathbb{P} = Id - \nabla\Delta^{-1}\text{div}$ is the orthogonal projection in $L^2(\mathbb{R}^3)^3$ over $L^2_\sigma(\mathbb{R}^3) := \Big\{u \in L^2(\mathbb{R}^3)^3, \quad \text{div}\, u = 0\Big\}$.

Proof. Fourier.

We apply $\mathbb{P}$ to the Euler equation. We note:

- $\mathbb{P}u = u$.
- $\mathbb{P}\nabla p = 0$ because $\int_{\mathbb{R}^3} \nabla p \cdot u = -\int_{\mathbb{R}^3} p\text{div}\, u = 0$ (if enough decay and regularity).

The Euler equation leads to

$$\boxed{\partial_t u + \mathbb{P}(u \cdot \nabla u) = 0}$$

Question: can we go back to the original formulation ?

1. To recover the divergence-free condition: if $u|_{t=0}$ is divergence-free, then

$$u(t) \ = \ u|_{t=0} \ + \ \int_0^t \mathbb{P}\left(u \cdot \nabla u\right)(s) ds$$

is in the range of $\mathbb{P}$, thus divergence-free at all times.

2. To recover the pressure :

Proposition (Helmoltz decomposition)

Any field $v \in L^2(\mathbb{R}^3)^3$ has an orthogonal decomposition

$$v = \mathbb{P}v \ + \ \nabla q, \quad \text{for some scalar function } q \in L^2_{loc}(\mathbb{R}^3).$$

Proof. Two steps.

- *Step 1* : v in C_c^∞. Then, by definition of $\mathbb{P}$,

$$v = \mathbb{P}v + \nabla q, \quad q = \frac{1}{4\pi|x|} \star \operatorname{div} v.$$

- *Step 2* : v in L^2. One can find v_n in C_c^∞ such that $v_n \to v$ in L^2.

$$v_n = \mathbb{P}v_n + \nabla q_n, \quad \text{so that } \nabla q_n \to Q := v - \mathbb{P}v \text{ in } L^2$$

Question: can we write $Q = \nabla q$ for q in L^2_{loc} ? (closedness issue).

Hint: *Poincaré inequality.*

For each k, $q_n^k := q_n - \frac{1}{k^3}\int_{B(0,k)} q_n$ satisfies

$$\|q_n^k - q_m^k\|_{L^2(B(0,k))} \;\le\; C\|\nabla q_n - \nabla q_m\|_{L^2(B(0,k))}.$$

Thus, $q_n^k \to q^k$ in $L^2(B(0,k))$. Convergence of ∇q_n :

$$q^{k+1} \;:=\; q^k + C^k \quad \text{over } B(0,k).$$

Allows to define a global q:

$$q = q^1 \quad \text{over } B(0,1), q = q^2 - C^2 \quad \text{over } B(0,2), \ldots$$

Back to Euler: use Helmoltz with $v := \partial_t u + u \cdot \nabla u$.

Vorticity

Another way to get rid of the pressure is to take the curl of the equation.

$$\boxed{\omega \;=\; \nabla \times u := \begin{cases} (\partial_2 u_3 - \partial_3 u_2, \dots)^t & \text{for d=3,} \\ \partial_1 u_2 - \partial_2 u_1 & \text{for d=2.} \end{cases}}$$

Example : d=3, $u(x) = \Omega \times x$. Then, $\omega = 2\Omega$.

Vorticity equation: (still for Euler)

$$\boxed{\begin{aligned} \partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = 0 \quad & \text{for } d = 3, \\ \partial_t \omega + u \cdot \nabla \omega = 0 \quad & \text{for } d = 2. \end{aligned}}$$

Crucial remark: In 2d, the vorticity satisfies a transport equation:

$$\omega(t, x) = \omega(0, y(t, x)) \quad \text{with } y(t, \cdot) \text{ the inverse of } x(t, \cdot).$$

The L^∞ or L^p norms are conserved.

Mathematical properties of Euler and NS much better in 2d than in 3d.

Remark: *A priori*, the vorticity equation is not closed: it involves u.

Biot-Savart law: the divergence-free field u can be expressed in terms of its curl ω:

$$\boxed{\begin{array}{l} u(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(t,y)}{|x-y|^3}dy \quad \text{for } d=3, \\ \\ u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(t,y)dy \quad \text{for } d=2. \end{array}}$$

Remark: in both cases, $\nabla u = K \star \omega$, with $K \sim 1/r^d$.

More precisely, ∇u is expressed in terms of ω by a singular integral.

The theory of singular integrals yields :

$$\boxed{\|\nabla u\|_{L^p} \;\le\; C_p\,\|\omega\|_{L^p} \quad \forall\, 1 < p < \infty}$$

Mathematical issues

Euler and NS can be written in the abstract form: $\frac{du}{dt} = F(u)$,

F acting nonlinearly on functions of x. *But not an ODE !*

F contains derivatives: not $C^0(X)$ for any reasonable Banach space X.

Action of F on Sobolev spaces: for all $s \in \mathbb{N}$,

$$H^s(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d)^d, \quad \partial^\alpha u \in L^2(\mathbb{R}^d)^d \quad \forall |\alpha| \leq s \right\}.$$

Solenoidal vector fields: $\boxed{H^s_\sigma(\mathbb{R}^d) = \left\{ u \in H^s(\mathbb{R}^d)^d, \quad \mathrm{div}\, u = 0 \right\}.}$

$H^s(\mathbb{R}^d)$ *is an algebra for* $s > \frac{d}{2}$. Then, for $s > \frac{d}{2} + 1$:

- Euler: $F(u) = -\mathbb{P}(u \cdot \nabla u)$ sends H^s_σ to H^{s-1}_σ
- NS: $F(u) = -\mathbb{P}(u \cdot \nabla u) + \Delta u$ sends H^s_σ to H^{s-2}_σ.

Still, some results similar to those for ODE.

Two kinds of results:

- Cauchy-Lipschitz type : for appropriate initial data, local in time regular solutions, unique and continuous with respect to the data.
- Cauchy-Peano type : global in time weak solutions, possibly non-unique.

Cauchy-Lipschitz type

Theorem (local existence for Euler or NS)

Let $s > \frac{d}{2} + 1$, $u_0 \in H^s_\sigma(\mathbb{R}^d)$. There exists $T > 0$ and a unique

$$u \in C([0,T]; H^s_\sigma(\mathbb{R}^d)) \cap C^1([0,T]; H^{s-2}_\sigma(\mathbb{R}^d))$$

satisfying ($\nu \geq 0$) :

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) - \nu \Delta u, \quad u|_{t=0} = u_0.$$

Elements of proof.

Fixed point approach does not work.

Construction of a solution through an approximation process.

Convergence of the approximation requires stability properties of the Euler or Navier-Stokes equation.

First stability property: conservation of energy.

Multiply the equation by u:

$$\boxed{\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = 0}$$

Not enough ! Possible transfer of energy from low to high frequencies.

The quadratic interaction of oscillations can be an obstacle to the convergence of the approximation process.

Example:

$$u_n(x) = \sin(nx) \rightharpoonup 0, \quad u_n(x)^2 \not\rightharpoonup 0.$$

Need for a control of high frequencies, or higher derivatives.

Apply ∂^α to the equation, $|\alpha| = s$.

$$\partial_t \partial^\alpha u + \mathbb{P}(u \cdot \nabla \partial^\alpha u) - \nu \Delta \partial^\alpha u \;=\; \mathbb{P}\,[\partial^\alpha, u \cdot \nabla]\, u.$$

Energy estimate:

$$\boxed{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2 \;\leq\; \|\,[\partial^\alpha, u \cdot \nabla]\, u\|_{L^2}\,\|\partial^\alpha u\|_{L^2}}$$

Commutator estimate: no more than s derivatives on u. More precisely:

$$\|\,[\partial^\alpha, u \cdot \nabla]\, u\|_{L^2} \;\leq\; C\,\|\nabla u\|_{L^\infty}\|u\|_{H^s}$$

Finally, as $s > \frac{d}{2} + 1$,

$$\frac{d}{dt}\|u\|_{H^s}^2 \;\leq\; C\,\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2 \;\leq\; C'\,\|u\|_{H^s}^3.$$

$\varphi(t) = \|u(t)\|^2_{H^s}$ satisfies $\frac{d}{dt}\varphi \leq C\,\varphi^{3/2}$. Provides a control in small time.

As for ODE's: *maximal solution*

$$u \in C\left([0, T_*[, H^s_\sigma(\mathbb{R}^d)\right) \cap C^1\left([0, T_*[, H^{s-2}_\sigma(\mathbb{R}^d)\right)$$

with $T_* < +\infty \Rightarrow \lim_{t \to T_*} \|u(t)\|_{H^s} = +\infty$.

The 2d case

<u>Question</u>: $T_* = +\infty$?

Theorem (global existence of smooth solutions in 2d)

In dimension $d = 2$, one has $T_* = +\infty$.

Elements of proof.

Last energy estimate :

$$\frac{d}{dt}\|u\|_{H^s}^2 \;\leq\; C\,\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2$$

Gronwall inequality:

$$\boxed{\|u(t)\|_{H^s}^2 \;\leq\; \|u_0\|_{H^s}^2\, e^{C\int_0^t \|\nabla u(\sigma)\|_{L^\infty}\, d\sigma}}$$

Thus : $T_* < +\infty \;\Rightarrow\; \int_0^{T_*} \|\nabla u(\sigma)\|_{L^\infty} d\sigma = +\infty.$

Beale-Kato-Majda:

$$\boxed{T_* < +\infty \;\Rightarrow\; \int_0^{T_*} \|\mathrm{curl}\, u(\sigma)\|_{L^\infty} d\sigma = +\infty}$$

This criterion is valid in dimensions 2 and 3.

In dimension 2, $\omega = \operatorname{curl} u$ satisfies

$$\boxed{\partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = 0.}$$

For finite T_*, $\int_0^{T_*} \|\omega\|_{L^\infty}$ does not explode.

<u>Remark</u>: The BKM criterium is not so easy to establish: remember that

$$\|\nabla u\|_{L^p} \ \leq \ C_p \|\omega\|_{L^p}, 1 < p < +\infty$$

But false for $p = \infty$. Logarithmic loss :

$$\boxed{\|\nabla u\|_{L^\infty} \ \leq \ C \left(1 + (1 + \ln^+ \|u\|_{H^3})\|\omega\|_{L^\infty} + \|\omega\|_{L^2}\right)}$$

<u>Question</u> : What about $d = 3$?

Big open question (10^6\$) ! No one knows if the smooth Euler or Navier-Stokes solutions experience singularities in finite time.

One exception : *Navier-Stokes with small initial data*.

Theorem (global NS solutions for small data)

Let $\nu > 0$. There exists $\varepsilon > 0$ (depending on viscosity coefficient) such that

$$\|u_0\|_{H^s} \leq \varepsilon \Rightarrow T_* = +\infty$$

Cauchy-Peano type

Two problems with the previous results:

- they are local in time for $d = 3$.
- they do not allow to consider irregular flows.

Example (d=2) : *vortex patch* : $\omega|_{t=0} = 1_\Omega$, Ω open set of $\mathbb{R}^2$.

Idea: to build global in time *weak* solutions, through a compactness argument.

Problem: control of the kinetic energy is not enough. It does not prevent space oscillations.

Global weak solutions for Euler :
Only for $d = 2$, using the transport of vorticity by fluid particles.

Idea:

- An initial bound on the vorticity will be propagated by the evolution.
- Will give (at least partially) a global control of derivatives of u.
- Will prevent space oscillations, and lead to the convergence of an approximation process.

Known results:

- Yudovich, 1963 : Initial vorticity $\omega_0 \in L^1 \cap L^\infty$: existence **and uniqueness** of a global weak solution.
- DiPerna-Majda, 1987 : Initial vorticity in $\omega_0 \in L^1 \cap L^p$, for finite p : existence of a global weak solution.

- Delort, 1991 : ω_0 is in $\mathcal{M}^+(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$.

Includes the case of classical vortex sheets.

Global weak solutions for Navier-Stokes :

Theorem (Leray, 1933)

For any $u_0 \in L^2_\sigma(\mathbb{R}^d)$, existence of a distributional solution

$$u \in L^\infty(\mathbb{R}_+; L^2_\sigma(\mathbb{R}^d)), \quad \text{with} \quad \nabla u \in L^2(\mathbb{R}_+; L^2(R^d)^d)$$

satisfying the energy inequality: for a.e. t

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u\|^2_{L^2} \leq \|u_0\|^2_{L^2}.$$

Idea: the L^2 bound on ∇u provided by viscosity kills space oscillations.

Thus, the Cauchy-Peano argument, based on compactness of a sequence of approximations, works.

More precisely, a key ingredient is

Theorem (Aubin-Lions)

Let $1 \leq p, q \leq +\infty$, $T > 0$. Let $X \hookrightarrow Y \hookrightarrow Z$ Banach spaces such that the imbedding $X \hookrightarrow Y$ is compact. Let (v_n) a sequence satisfying

$$(v_n) \text{ bounded in } L^p(0, T; X),\ (\partial_t v_n) \text{ bounded in } L^q(0, T; Z)$$

Then :

- If $p < \infty$, (v_n) has a subsequence converging strongly in $L^p(0, T; Y)$.
- If $p = \infty$ and $q > 1$, (v_n) has a subsequence converging strongly in $C([0, T]; Y)$.

In the NS case, one takes : $X = H^1_\sigma, \quad Y = L^2_\sigma, \quad Z = (H^1_\sigma)'$.

One can show that a sequence of smooth approximations (u_n) satisfies :

$$\boxed{(u_n) \text{ bounded in } L^2(0,T;X),\ (\partial_t u_n) \text{ bounded in } L^{4/d}(0,T;Z)}$$

Gives compactness of (u_n) in $L^2(0,T;L^2_\sigma)$.

Gives that $u_n \cdot \nabla u_n \ = \ \mathrm{div}\,(u_n \otimes u_n)$ converges to $u \cdot \nabla u$ distributions.

<u>Remark</u> : the limit weak solutions only satisfy the energy *inequality*.
Approximations u_n do not oscillate at leading order, but ∇u_n could.

Basic example : $u_n(x) = \frac{1}{n}\sin(nx)$.

<u>Remark</u> : the uniqueness of the Leray solutions is a big open problem.

$d = 2$: one can show that the Leray solutions are unique and satisfy the energy inequality.

Who is D'Alembert ?

Jean Le Rond D'Alembert
(1717-1783)

- Illegitimate child of Mme de Tencin and army officer Destouches.
  Left on the steps of Saint Jean Le Rond church in Paris.
  Put in an orphanage, then given to a nurse.
- French mathematician and philosopher. Representative of the Age of Enlightenment (co-founder of "L'Encyclopédie" with Denis Diderot).
- Huge scientific contributions: wave equation, mechanics (notably fluid mechanics)...

Statement of the paradox

"In an ideal incompressible fluid, bodies moving at constant speed do not experience any drag, or lift."

$\Rightarrow$ Failure of the Euler equation as a model for fluid-solid interaction.

The origin of the problem is the following:

Theorem ("Incompressible potential flows generate no force on obstacles")

Let $u = u(x)$ be a smooth 3D field, defined outside a smooth bounded domain $\mathcal{O}$.

Assume that u is a divergence-free gradient field, tangent at $\partial\mathcal{O}$, uniform at infinity. Then:

1. u is a (steady) solution of the Euler equation outside $\mathcal{O}$:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}.$$

2. $F := \int_{\partial\mathcal{O}} p n d\sigma \ = \ 0.$

Proof of the theorem: Assumptions on u:

$$u = u_\infty + \nabla\eta, \quad \Delta\eta = 0, \quad \nabla\eta \xrightarrow[|x|\to\infty]{} 0, \quad \partial_n\eta|_{\partial\mathcal{O}} = -u_\infty \cdot n.$$

1. u satisfies the Euler equation, due to the algebraic identity

$$u \cdot \nabla u = -u \times \operatorname{curl} u \; + \; \frac{1}{2}\nabla|u|^2 \quad \left(p := -\frac{1}{2}|u|^2\right).$$

2. To prove that the force is zero: one uses a representation formula:

$$\eta(x) = \eta_\infty + \int_{\partial\mathcal{O}} \partial_{n_y} G(x,y)\eta(y)d\sigma(y) \; + \; \int_{\partial\mathcal{O}} u_\infty \cdot n(y) G(x,y) d\sigma(y)$$

where $G(x,y) = -\frac{1}{4\pi|x-y|}$.

Allows to prove that: $u(x) = u_\infty + O(|x|^{-3}), \quad p = p_\infty + O(|x|^{-3})$.

Back to the Euler equation:

$$\boxed{u \cdot \nabla u + \nabla p = 0}$$

the fast decay of $u - u_\infty$ and $p - p_\infty$ allows to integrate by parts "up to infinity":

$$\boxed{\int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} (u \cdot \nabla u + \nabla p) \; = \; \int_{\partial \mathcal{O}} pn \; = \; 0.}$$

How does it imply the paradox ?

Example: A plane, initially at rest.

- Initially, the air around the plane is at rest, so curl-free.

- The curl-free condition is preserved by Euler.

- When the plane reaches its cruise speed, the conditions of the theorem are fulfilled (up to a change of frame).

$\Rightarrow$ No drag, no lift !

Quoting D'Alembert:

> *It seems to me that the theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers to elucidate*

What is the flaw of the Euler model ? How to escape the paradox ?

Took 150 years to find an escape ! Still some people arguing today...

Large consensus: in domains Ω with boundaries, one should add viscosity, and consider the *Navier-Stokes equations*:

$$\boxed{\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} \;=\; 0, & \mathbf{x}\in\Omega\subset\mathbb{R}^2 \text{ ou } \mathbb{R}^3, \\ \nabla\cdot\mathbf{u} \;=\; 0, & \mathbf{x}\in\Omega. \end{cases}} \tag{NS}$$

2 possible meanings for ν:

- Dimensionalized system: $\nu = \nu_K$, *kinematic viscosity*.
- Dimensionless system: $\nu = \nu_K \;/(U\,L)$,

  U, L : typical speed and length, $1/\nu$: *Reynolds number*.

Main point: The curl-free condition is not preserved by the Navier-Stokes equation in domains with boundaries.

Allows to get out of d'Alembert's paradox...

... but still: in most experiments, ν is very small:

Example: Flows around planes: $\nu \approx 10^{-6}$.

Hence, Euler equations ($\nu = 0$) should be a good approximation !

Indeed, for smooth solutions in domains *without boundaries*, it is true !

But in domains *with boundaries*, not clear !

The problem comes from boundary conditions.

- For $\nu \neq 0$ (NS), classical *no-slip condition*:

$$\boxed{\mathbf{u}|_{\partial\Omega} \; = \; 0} \tag{D}$$

- For $\nu = 0$ (Euler), one needs to relax this condition:

$$\boxed{\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} \; = \; 0}$$

$\Rightarrow u_\nu$ concentrates near $\partial\Omega$: *boundary layer*.

Problem: Impact of this boundary layer on the asymptotics $\nu \to 0$?

This problem can be further specified:

Theorem [Kato, 1983]

Let Ω a bounded open domain. Let $\mathbf{u}_\nu$ and $\mathbf{u}_0$regular solutions of (NS)-(D) and Euler, with the same initial data. Then

$$\mathbf{u}_\nu \to \mathbf{u}_0 \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad \text{if and only if}$$

$$\nu \int_0^T \int_{d(\mathbf{x},\partial\Omega)\leq\nu} |\nabla \mathbf{u}_\nu|^2 \to 0.$$

Remarks:

- Yields a quantitative and optimal criterium for convergence.
- The convergence is related to concentration at scale ν (and not at parabolic scale $\sqrt{\nu}$).

Still, the convergence from NS to Euler is (mostly) an open question.

Prandtl's approach (1904)

Case $\Omega \subset \mathbb{R}^2$: we introduce

- curvilinear coordinates (x, y) near the boundary:

$$\mathbf{x} = \tilde{\mathbf{x}}(x) + y\,\mathbf{n}(x), \qquad \text{with } \tilde{\mathbf{x}} \in \partial\Omega, \; x \text{ arc length}, \; y \geq 0.$$

- Frénet basis: $\mathbf{u}_\nu(t, \mathbf{x}) = u_\nu(t, x, y)\,\mathbf{t}(x) + v_\nu(t, x, y)\,\mathbf{n}(x)$,

Asymptotic model for $\mathbf{u}_\nu$, based on two different expansions.

- outside the boundary layer: Euler:

$$\boxed{\mathbf{u}_\nu \approx \mathbf{u}_0 = (u_0, v_0)(t, x, y).}$$

- in the boundary layer: typical scale $\sqrt{\nu}$:

$$\boxed{\mathbf{u}_\nu \approx \left(u\left(t, x, y/\sqrt{\nu}\right), \sqrt{\nu}\, v\left(t, x, y/\sqrt{\nu}\right)\right)}$$

where $u = u(t, x, Y)$, $v = v(t, x, Y)$, $(x, Y) \in \mathbb{R} \times [0, +\infty[$.

Prandtl equation is the PDE satisfied formally by (u, v).

Formally (with y instead of Y)

$$\boxed{\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_y^2 u = 0, & x \in \mathbb{R},\ y > 0, \\ \partial_y p = 0, & x \in \mathbb{R},\ y > 0, \\ \partial_x u + \partial_y v = 0, & x \in \mathbb{R},\ y > 0. \end{cases}}$$

Boundary conditions:

- no-slip: $u(t, x, 0) = v(t, x, 0) = 0.$
- matching to the Euler flow:

$$\begin{aligned} u(t, x, +\infty) &= U(t, x) := u_0(t, x, 0), \\ p(t, x, +\infty) &= P(t, x) := p_0(t, x, 0). \end{aligned}$$

Finally, Prandtl equation reads (with $(x, y) \in \mathbb{R}^2_+$):

$$\boxed{\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = -\partial_x P, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \\ u|_{y=+\infty} = U. \end{cases}} \tag{P}$$

<u>Remarks</u>:

- No curvature term in the operators ($\neq$ 3D).
- Curvature is involved through U and P, and through the domain of definition of x. Classical choices:

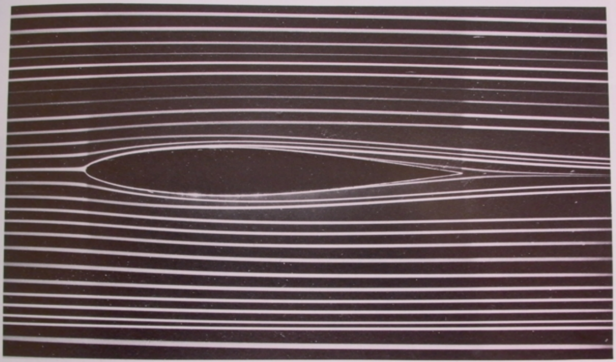
  a) $x \in \mathbb{R}, \mathbb{T}$ (local study in x, outside of a convex obstacle)

  b) $x \in (0, L)$, with an "initial" condition at $x = 0$.

<u>Questions:</u> Experimental evidence ? Mathematical justification ?

More precisely: is (P) well-posed ? Asymptotic expansion of $\mathbf{u}_\nu$?

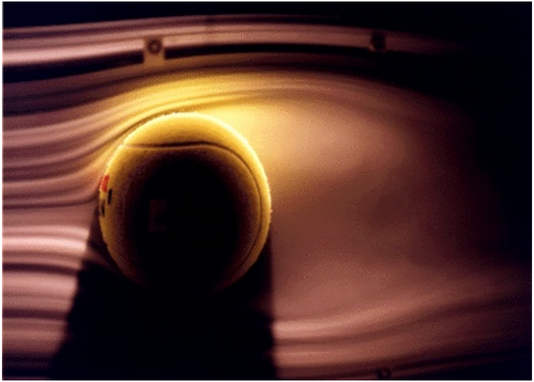
Many experimental studies (flows around obstacles)...



23. Symmetric plane flow past an airfoil. An NACA 64A015 profile is at zero incidence in a water tunnel. The Reynolds number is 7000 based on the chordlength. Streamlines are shown by colored fluid introduced upstream. The flow is evidently laminar and appears to be unseparated, though one might anticipate a small separated region near the trailing edge. *ONERA photograph, Werlé 1974*

... which exhibit many instabilities.

Example: Boundary layer separation.



Explanation for the separation: *adverse pressure gradient*.

$U > 0$, $-\partial_x P < 0$. Loss of monotonicity (in y), followed by separation.

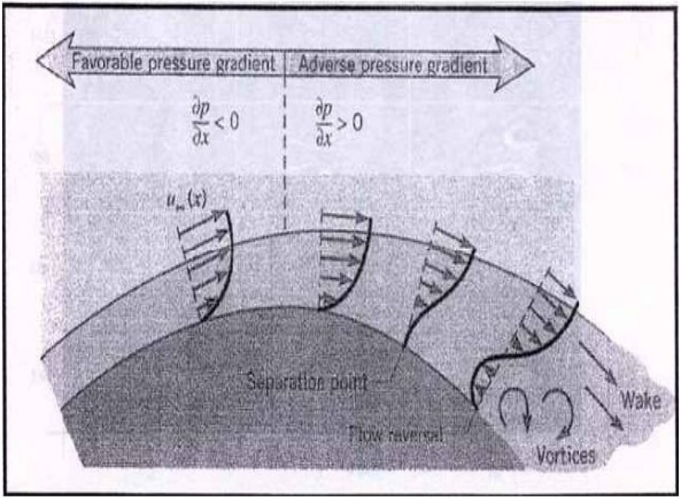


Figure 1. Décollement de la couche limite

Mathematical results

Problem 1: Cauchy theory for Prandtl ?

Problem 2: Justification of the expansion ?

For both pbs, *the choice of the functional spaces is crucial*.

Problem 1:

- $(x, Y) \in \mathbb{R} \times \mathbb{R}_+$, *analyticity in* x. *Well-posed locally in time* ([Sammartino 1998], [Cannone 2003]).
- $(x, Y) \in (0, L) \times \mathbb{R}_+$, *monotonicity in* y. *Well-posed locally in time, globally under further assumptions* ([Oleinik 1967], [Xin 2004]).

Remark: Without monotonicity, there are solutions that blow up in finite time: [E 1997].

Problem 2:

- Analytic framework: the asymptotics holds [Sammartino 1998].
- Sobolev framework: the asymptotics does not always hold in H^1 [Grenier,2000]. Relies on Rayleigh instability.

Natural question: Is Prandtl well-posed in Sobolev type spaces ?

We consider the case: $x \in \mathbb{T}$, $u^0 = 0$:

$$\boxed{\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, \quad (x,y) \in \mathbb{T} \times \mathbb{R}_+ \\ \partial_x u + \partial_y v = 0, \quad (x,y) \in \mathbb{T} \times \mathbb{R}_+ \\ (u,v)|_{y=0} = (0,0). \end{cases}} \tag{P}$$

Well- or ill-posed ?

Pb: To guess the correct answer !

No standard estimate available for the linearized system.

Example: Let $U(t,y)$ satisfying $\partial_t U - \partial_y^2 U \,=\, 0,\; U|_{y=0} = 0$.

The field $(U(t,y), 0)$ satisfies (P).

Linearized equation:

$$\boxed{\begin{cases} \partial_t u + U\partial_x u + v\partial_y U - \partial_y^2 u = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+, \\ (u,v)|_{y=0} = (0,0), & \lim\limits_{y\to+\infty} u = 0. \end{cases}} \tag{PL}$$

L^2 estimate: the annoying term is $\int v\partial_y U\, u \;\sim\; O\left(\int |\partial_x u|\,|u|\right)$.

A priori, loss of an x-derivative.

Another clue for ill-posedness: Freezing the coefficients, leads to the dispersion relation

$$\omega \;=\; k_x\; U \;+\; i\,\partial_y U\,\frac{k_x}{k_y} - ik_y^2.$$

Suggests that the equation is strongly ill-posed ... But this is misleading !

Simpler situation: no vertical diffusion, $U = U_s'(y)$:

$$\boxed{\begin{cases} \partial_t u + U_s\partial_x u + v\,U_s' = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+, \\ v|_{y=0} = 0. & \end{cases}}$$

- Frozen coefficients: bad dispersion relation.
- But an explicit computation yields

$$u(t,x,y) \;=\; u_0(x - U_s(y)t, y) \;+\; t\,U_s'(y)\int_0^y \partial_x u_0(x - U_s(z)t, z)\,dz.$$

"Weakly" well-posed (loss of a finite number of derivatives).

Back to the nonlinear setting: *The inviscid Prandtl equation is weakly well-posed* [Hong 2003].

In fact, the solution is explicit through the methods of characteristics.

Conclusion: The study without diffusion suggests well-posedness of the Prandtl equation.

But . . .

We show: (P) is strongly ill-posed.

Tricky but violent instability mechanism.

Ingredients: diffusion and critical points of the velocity field.

Does not contradict the previous existing results.

Theorems

The main theorem is on the linearization (PL) (around $U = U_s(y)$)

$$\begin{cases} \partial_t u + U_s \partial_x u + v U_s' - \partial_y^2 u = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ (u, v)|_{y=0} = (0,0), & \lim_{y \to +\infty} u = 0. \end{cases} \tag{PL}$$

Theorem (Linear ill-posedness in the Sobolev setting) (with E. Dormy)

There exists $U_s \in C_c^\infty(\mathbb{R}_+)$ such that: for all $T > 0$, one can find u_0 satisfying

1. $e^y u_0 \in H^\infty(\mathbb{T} \times \mathbb{R}_+)$
2. Equation (PL) has no distributional solution u with

$$u \in L^\infty(0, T; L^2(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y u \in L^2(0, T \times \mathbb{T} \times \mathbb{R}_+)$$

and initial data u_0.

‘The k-th Fourier mode grows like $e^{c\sqrt{k}t}$”

Pondering on this linear result, one can establish a nonlinear result (joint work with T. NGuyen)

"If the nonlinear Prandtl equation (P) *generates a flow, this flow is not Lipschitz continous from bounded sets of* $e^{-y} H^m(\mathbb{T} \times \mathbb{R}_+)$ *to* $H^1(\mathbb{T} \times \mathbb{R}_+)$*, for arbitrarily small times."*

A few hints at the proof of the linear result

1. *The non-existence of solutions for some initial data amounts to the non-continuity of the semigroup.*

   Simple consequence on the closed graph theorem.

2. Proof of non-continuity.
   - High frequency analysis of (PL) in the x variable.

     Construction of approximate high-frequency eigenvectors, in the spirit of WKB expansions.

     Existence of an unstable eigenvalue comes down to a spectral problem for a differential operator on $\mathbb{R}$.
   - Resolution of the spectral problem.
   - Consequence on the semigroup.

Key Assumption: $\boxed{U_s'(c) = 0, \quad U_s''(c) < 0.}$

One looks for solutions that read $\begin{cases} u(t,x,y) \;=\; i\, e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}\, v_\varepsilon'(y), \\ v(t,x,y) \;=\; \varepsilon^{-1}\, e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}\, v_\varepsilon(y). \end{cases}$

System:

$$\boxed{\begin{cases} (\omega(\varepsilon)+U_s)v_\varepsilon' \;-\; U_s' v_\varepsilon \;+\; i\varepsilon v_\varepsilon^{(3)} \;=\; 0, \quad y>0, \\ v_\varepsilon|_{y=0}=0, \quad v_\varepsilon'|_{y=0}=0. \end{cases}}$$

Remark: Singular perturbation problem in y.

Simpler case: $\varepsilon = 0$ (inviscid version):

$$\begin{cases} (\omega+U_s)v' \;-\; U_s' v \;=\; 0, \quad y>0, \\ v|_{y=0}=0. \end{cases}$$

One parameter family of eigenelements:

$$\boxed{\omega=\omega_a \;:=\; -U_s(a), \quad v=v_a \;:=\; H(y-a)\,(U_s-U_s(a)).}$$

Remarks:

- Wether $a = c$ or not, v_a is more or less regular at $y = a$.
- $\omega_a \in \mathbb{R}$: high frequency oscillations $e^{i\frac{\omega_a t}{\varepsilon}}$.

How are these oscillations affected by the singular perturbation $i\varepsilon v_\varepsilon^{(3)}$?

Remark: Analogy with the incompressible limit of NS in bounded domains.

- The high frequency oscillations are the acoustic waves, $e^{i\lambda_k t/\varepsilon}$, $k \in \mathbb{N}$.
- The singular perturbation is the diffusion in Navier-Stokes.

[Desjardins et al 1999]:

Diffusion generates a boundary layer.

$\longrightarrow$ singular dependence of the spectrum with respect to ε.

$O(\sqrt{\varepsilon})$ correction of each λ_k, with positive imaginary part.

Leads to a damping of the waves, with typical time $\sqrt{\varepsilon}$.

Prandtl case:

Diffusion yields a shear layer correction of each v_a near $y = a$.

For $a = c$, $O(\sqrt{\varepsilon})$ of the eigenvalue, with negative imaginary part.

Leads to exponential growth, with typical time $\sqrt{\varepsilon}$.

Ansatz for the unstable eigenvalue and eigenvector:

- Eigenvalue: correction of order $\sqrt{\varepsilon}$:

$$\boxed{\omega(\varepsilon) \approx \omega_c + \sqrt{\varepsilon}\tau}$$

- Eigenvector:

$$\boxed{v_\varepsilon(y) \approx v_c + \sqrt{\varepsilon}\,\tau\, H(y - y_c) + \sqrt{\varepsilon} V\left(\frac{y - y_c}{\varepsilon^{1/4}}\right).}$$

V corresponds to a viscous "shear layer",which corrects the non-smooth inviscid part.

Formally: $V = V(z)$, $z \in \mathbb{R}$, satisfies:

$$\begin{cases} \left(\tau + U_s''(y_c)\dfrac{z^2}{2}\right) V' \;-\; U_s''(y_c)\, z\, V \;+\; i\, V^{(3)} \;=\; 0, \quad z \neq 0, \\ [V]_{|_{z=0}} \;=\; -\tau, \quad [V']_{|_{z=0}} \;=\; 0, \quad [V'']_{|_{z=0}} = -U''(a), \\ \lim\limits_{\pm\infty} V \;=\; 0. \end{cases}$$

Remark: Too many constraints, so the parameter τ.

Hope: There is a solution (τ, V) with $\mathcal{I}m\,\tau < 0$. After changes of variables, it amounts to

(SC) : there is $\tau \in \mathbb{C}$ with $\mathcal{I}m\,\tau < 0$, and a solution W of

$$\boxed{(\tau - z^2)^2 \frac{d}{dz} W \;+ i\, \frac{d^3}{dz^3}\Big((\tau - z^2) W\Big) = 0,} \qquad \text{(ODE)}$$

such that $\lim\limits_{z \to -\infty} W = 0$, $\lim\limits_{z \to +\infty} W = 1$.

The spectral condition (SC)

Remark: (ODE) is an equation on $X = W'$:

$$\boxed{i(\tau - z^2)X'' - 6i\, z\, X' + \Big((\tau - z^2)^2 - 6i\Big) X = 0.} \qquad \text{(EDO2)}$$

Complex change of variable : amounts to positive spectrum for

$$\boxed{Au := \frac{1}{z^2+1} u'' + \frac{6z}{(z^2+1)^2} u' + \frac{6}{(z^2+1)^2} u}$$

Proposition

$A : D(A) \mapsto \mathcal{L}^2$ selfadjoint, with

$$D(A) := \Big\{ u \in \mathcal{H}^1,\ Au \in \mathcal{L}^2 \Big\},$$

$$\mathcal{L}^2 := \Big\{ u \in L^2_{loc},\ \int_{\mathbb{R}} (z^2+1)^4 |u|^2 < +\infty \Big\},$$

$$\mathcal{H}^1 := \Big\{ u \in H^1_{loc}\ \int_{\mathbb{R}} (z^2+1)^4 |u|^2 + \int_{\mathbb{R}} (z^2+1)^3 |u'|^2 < +\infty \Big\}.$$

Proposition

A has a positive eigenvalue.

Proof: One has $Au = A_1 u + A_2 u$, with

$$\boxed{A_1 u := \frac{1}{z^2+1} u'' + \frac{6z}{(z^2+1)^2} u'}$$

selfadjoint and negative in $\mathcal{L}^2$, and

$$\boxed{A_2 u := \frac{6}{(z^2+1)^2} u}$$

selfadjoint and A_1-*compact*. So $\Sigma_{ess}(A) = \Sigma_{ess}(A_1) \subset \mathbb{R}_-$.

Moreover, $(Au, u) > 0$ for $u(z) = e^{-2z^2}$.

Conclusion:

For each k, perturbation with frequency k in x that grows like $e^{\sqrt{|k|}t}$.

Numerics on (PL): the unstable eigenmode mentioned above dominates.

Maximal growth seen in numerics: $e^{\delta\sqrt{k}t}$, $\delta > 0$.

Questions:

- Can we confirm theoretically this observation on instabilities ?
- Is it still true at the nonlinear level ?

Idea: Show that (P) is locally well-posed for data whose Fourier coeffts in x decay like $e^{-\sigma\sqrt{k}}$, $\sigma > 0$.

Intermediate between the analytic framework ($\sim e^{-\sigma k}$) and the Sobolev framework ($\sim k^{-s}$).

Cauchy pb in Gevrey spaces

Definition:

Let $m \geq 1$. $G^m(\mathbb{T})$ is the set of $f = f(x)$ s.t.

$$\exists C, \tau > 0, \quad |\text{Âă}f^{(k)}(x)|\text{Âă} \;\leq\; C\,\tau^{-k}(k!)^m, \quad C, \tau > 0, \quad \forall\, k,\, x.$$

Remark:

- $m = 1$: analytic functions.
- $m > 1$: $G^m(\mathbb{T})$ contains compactly supported functions.

Proposition

$f \in G^m(\mathbb{T})$ iff $\exists C, \sigma > 0, \quad |\hat{f}(k)| \leq Ce^{-\sigma k^{1/m}}$

Question: Is Prandtl well-posed in a space of type G^2 in x?

"Theorem" (with N. Masmoudi)

The full Prandtl equation is well-posed in a space of type $G^{\frac{7}{4}}$ in x !

(exponent 7/4 instead of 2 is due to a technical restriction.)

The proof relies on the use of nonlinear energy functionals. The kinetic energy is not appropriate to boundary layer systems...

Justification of the Prandtl expansion

Consider data for which the Prandtl expansion is well defined:

- General Gevrey data or monotonic data.
- Special data, like shear flows.

  For $(u, v)|_{t=0} = (U_s(y), 0)$, Prandtl equation has the solution

$$\boxed{(u, v) = (U(t, y), 0), \quad \text{with} \quad \partial_t U - \partial_y^2 U = 0, \quad U|_{t=0} = U_s}$$

<u>Question</u>:

Does it give some asymptotic description of an exact solution $\mathbf{u}_\nu$ of (NS) ?

If

$$\begin{cases} \mathbf{u}_\nu = \mathbf{u}_0 & \text{away from the boundary,} \\ \mathbf{u}_\nu = (u(t, x, y/\sqrt{\nu}), \sqrt{\nu} v(t, x, y/\sqrt{\nu})) & \text{close to the boundary} \end{cases}$$

at initial time, is it still true over a time $T > 0$ independant of ν ?

In other words : are boundary layer solutions of (NS) stable ?
(over a time that does not shrink to zero with ν).

Analytic data : Yes (Sammartino-Caflisch 1998).

Analytic setting is the only one where convergence of Navier-Stokes to Euler in the energy space is known.

Remark: the analyticity assumption models a *clean* environment, with a fast decay of the noise. Is it realistic ?

N.B. The question is relevant to many pbs (like Landau damping).

Question: What about less regular data : Gevrey or Sobolev ?

Special case : shear flow $\boxed{\mathbf{u}_\nu^{bl} = (U(t, y/\sqrt{\nu}), 0)}$.

It is an almost exact solution of the NS equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} = 0$$

Moreover, $(U(t, Y), 0)$ is an exact solution of Prandtl.

Question: Stability of $\mathbf{u}_\nu^{bl}$?

Rescale everything : $\boxed{T = t/\sqrt{\nu},\ X = x/\sqrt{\nu},\ Y = y/\sqrt{\nu}}$.

Becomes the stability of $(U(\sqrt{\nu}T, Y), 0)$, solution of

$$\partial_T \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{X}} \mathbf{u} + \nabla_{\mathbf{X}} p - \sqrt{\nu} \Delta_{\mathbf{x}} \mathbf{u} = 0$$

Underlying inviscid problem:

Stability of $(U_s, 0)$, solution of the Euler equation

$$\partial_T \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{X}} \mathbf{u} + \nabla_{\mathbf{X}} p = 0$$

Many studies : Rayleigh, Arnold, ...

Rayleigh criterion: Instability requires that U_s *has an inflexion point.*

There are explicit examples with unstable eigenmodes, say (σ_u, V_u).

Idea: take the initial data

$$\boxed{\mathbf{u}_0 \;=\; (U_s(Y), 0) \;+\; \nu^N V_u(\mathbf{X}), \quad N \gg 1}$$

One can show that for times $T = O(|\ln \nu|)$, the solution of (NS) with viscosity $\sqrt{\nu}$, behaves like (variables $T, \mathbf{X}$)

$$\boxed{\mathbf{u} \sim \;(U_s(Y), 0) \;+\; \nu^N e^{\sigma_u T} V_u(\mathbf{X})}$$

Back to the original variables:

$$\boxed{\mathbf{u}_\nu \sim (U_s(y\sqrt{\nu}), 0) \;+\; \nu^N e^{\sigma_u \frac{t}{\sqrt{\nu}}} V_u(\mathbf{x}/\sqrt{\nu})}$$

for times $t = O(\sqrt{\nu}|\ln \nu|)$.

Strong instability : high frequency $k = \frac{1}{\sqrt{\nu}}$ in x can grow like $e^{|k|t}$.

In such cases, the stability of the Prandtl expansion is only possible in short times, for analytic data.

Question : What about shear flows without inflexion points ?

Stable for Euler...but the full equation is :

$$\boxed{\partial_T \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{X}} \mathbf{u} + \nabla_{\mathbf{X}} p - \sqrt{\nu} \Delta_{\mathbf{x}} \mathbf{u} = 0}.$$

Intuition: The additional viscosity should be stabilizing.

Wrong ! Viscosity can be destabilizing (Heisenberg, Tollmien,Schlichting, Drazin and Reid)

Possible unstable eigenvalues $\sigma_u \sim \nu^{\alpha}$.

If $\alpha < 1/2$, could lead instability of the expansion.

Need for clarification...

Solids in a Navier-Stokes flow

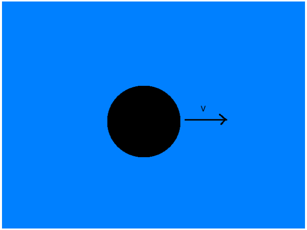
The previous lecture has shown the limitations of the Euler model as regards fluid-solid interaction.

Idea: to consider the Navier-Stokes equations...

...but it raises modeling issues as well !

Example 1: The Stokes paradox

An infinite cylinder can not move at constant speed in a Stokes flow.



Theorem (Ladyzhenskaya 1969, Heywood 1974)

Let Ω be the exterior of the unit disk, and u be a weak solution of the Stokes equation satisfying

$$u|_{\partial\Omega} = V, \quad \int_{\Omega} |\nabla u|^2 < +\infty$$

Then, $u \equiv V$ over Ω

In particular, u does not go to zero at infinity.

Proof: The field $v = u - V$ satisfies

$$\boxed{-\Delta v + \nabla p = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.}$$

Hence,

$$\int_\Omega \nabla v \cdot \nabla \varphi = 0, \quad \forall \varphi \in \mathcal{D}_\sigma(\Omega).$$

But

$$\mathcal{D}_\sigma(\Omega) \quad \text{is dense in} \quad \{v \quad \text{s.t.} \quad \int_\Omega |\nabla u|^2 < +\infty, \; v|_{\partial\Omega} = 0\}.$$

so that $\int_\Omega |\nabla v|^2 = 0$.

Remarks:

- The density result does not hold in 3d, the same for Stokes paradox.
- The Stokes approximation is not justified: the low Reynolds number limit has no meaning (no typical scale in the problem).
- As soon as the Navier-Stokes flow, or the linear Oseen flow is considered, the paradox does not hold.

Example 2: The non-collision paradox

In a NS flow, rigid bodies sink, but never hit the bottom !

This paradox will be discussed later.

Governing equations

Framework:

- One rigid solid, in a cavity full of an incompressible viscous fluid.
- Both the solid and the fluid are homogeneous.

Cavity: domain Ω of $\mathbb{R}^d$, $d = 2$ or 3:

$$\Omega \;:=\; \overline{S(t)} \cup F(t)$$

$S(t)$, $F(t)$: solid and fluid subdomains at time t.

- Navier-Stokes equations in $F(t)$:

$$\begin{cases} \rho_F \left(\partial_t u_F + u_F \cdot \nabla u_F\right) - \mu \Delta u_F \;=\; -\nabla p \;+\; \rho_F\, f, \\ \operatorname{div} u_F \;=\; 0. \end{cases} \tag{NS}$$

- Classical mechanics for the solid.
  - Rigid velocity field:

$$u_S(t,x) \,=\, \dot{x}(t) \,+\, \omega(t) \times (x - x(t))$$

  - Conservation of the linear momentum

$$m_S\, \ddot{x}(t) \,=\, \int_{\partial S(t)} \Sigma n \, d\sigma \,+\, \int_{S(t)} \rho_S\, f,$$

  - Conservation of the angular momentum

$$\frac{d}{dt}\,(J_S(t)\,\dot{\omega}(t)) \,=\, \int_{\partial S(t)} (x - x(t)) \times (\Sigma n)\, d\sigma + \int_{S(t)} (x - x(t)) \times \rho_S f$$

Notations: $x(t)$: center of mass, m_S : total mass of the solid,

Σ : stress tensor at the solid surface, J_S : inertial tensor.

$$J_S(t) = \int_{S(t)} |x - x(t)|^2 - (x - x(t) \otimes (x - x(t)))$$

<u>Remark</u>: $J_S(t) \; = \; Q(t) J_S(0) Q(t)^{-1}$, $\quad Q(t)$: orthogonal matrix.

- Continuity constraints at the fluid solid interface

$$\begin{cases} (\Sigma n)\,|_{\partial S(t)} \; = \; (2\mu D(u) n - pn)\,|_{\partial S(t)} \\ \quad u_F|_{\partial S(t)} \; = \; u_S|_{\partial S(t)} \end{cases}$$

- No slip condition at the boundary.

$$u_F|_{\partial\Omega} = 0.$$

Definitions

Many works on the well-posedness of viscous fluid-solid systems.

Key : *Global variational formulation over* Ω. Let

$$u(t,x) \;:=\; u_S(t,x) \text{ si } \; x \in S(t), \quad u_F(t,x) \;\text{ if }\; x \in F(t),$$

$$\rho(t,x) \;=\; \rho_S \mathbf{1}_{S(t)}(x) \;+\; \rho_F \mathbf{1}_{F(t)}(x), \quad \chi^S(t,x) \;=\; \chi_S \mathbf{1}_{S(t)}(x).$$

- Constraints:

$$\nabla \cdot u = 0, \quad u|_{\partial\Omega} = 0, \quad \chi^S D(u) = 0. \tag{Co}$$

- Conservation of mass: for all $T > 0$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad \partial_t \chi^S u + \operatorname{div}(\chi^S u) = 0. \tag{CM}$$

- Conservation of momentum in weak form: for all $T > 0$,

$$\int_0^T \int_\Omega \Big(\rho u \cdot \partial_t \varphi + \rho u \otimes u : D(\varphi) - \mu D(u) : D(\varphi) + \rho f \cdot \varphi \Big) dxds$$
$$+ \int_\Omega \rho_0 u_0 \cdot \varphi(0) \; = \; 0, \quad \text{(VF)}$$

for all φ in the test space

$$\mathcal{T} \; = \; \Big\{ \varphi \in \mathcal{D}([0, T) \times \Omega), \quad \nabla \cdot \varphi = 0, \quad \chi^S(t) D(\varphi) = 0, \, \forall t \Big\}$$

Remark: Close of the inhomogeneneous incompressible NS system

$$\begin{cases} \partial_t \rho + \operatorname{div} (\rho u) = 0, \\ \partial_t (\rho u) + \operatorname{div} (\rho u \otimes u) + \nabla p - \mu \Delta u = \rho f, \quad \operatorname{div} u = 0 \end{cases}$$

Main difference: The test space depends on the solution itself.

Data: $S(0) \Subset \Omega, \quad u^0 \in L^2_\sigma(\Omega), f \in L^2_{loc}\left(0, +\infty; L^2(\Omega)\right).$

Definition (weak solution)

A *weak solution* over $(0, T)$, $T > 0$, is a triple (S, F, u) such that :

- $S(t)$ is a connected open set Ω, for all $0 < t < T$, and $F(t) = \Omega \setminus \overline{S(t)}$.
- The field u, and functions ρ, χ^S as above, satisfy

  $$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \quad \rho, \chi^S \in L^\infty(0, T \times \Omega)$$

  as well as equations (Co), (VF).
- The following energy inequality holds for a.e. $t \in (0, T)$

  $$\frac{1}{2}\int_\Omega \rho(t)|u(t)|^2 + \mu \int_0^t \int_\Omega |\nabla u(s)|^2 ds \leq \frac{1}{2}\int_\Omega \rho_0 |u_0|^2 + \int_0^t \rho f(s) \cdot u(s) ds$$

Definition (strong solution)

A *strong solution* over $(0, T)$, $T > 0$, is a weak solution with additional regularity:

$$u \in L^{\infty}\left(0, T; H^1_0(\Omega)\right) \cap L^2\left(0, T;\ W^{1,p}(\Omega)\right) \text{ for all finite } p,$$
$$\partial_t u \in L^2(0, T;\ L^2(\Omega)).$$

Remark: The situation is similar to the one of Navier-Stokes. Broadly

- Weak solutions are defined globally in time, even after possible collision between the solid and the boundary of the cavity.
- They are unique up to collision in 2d.
- They are not unique after collision (lack of a bouncing law).
- Strong solutions exist locally in time, up to collision in 2d.

Existence of weak solutions

Theorem

There exists a weak solution over $(0, T)$ for all T.

Refs : [Desjardins et al, 1999], [Hoffman et al, 1999], [San Martin et al, 2002], [Feireisl, 2003].

A few ideas from the proof.

Borrows to the inhomogeneous Navier-Stokes. Approximations are constructed by *relaxing the rigidity constraint inside the solid.*

Typically:

$$\boxed{\begin{cases} \partial_t \rho^n + \mathrm{div}\,(\rho^n u^n) \;=\; 0, \quad \partial_t \chi_S^n \;+\; \mathrm{div}\,(\chi_S^n\, u^n) = 0 \\ \partial_t(\rho^n u^n) \;+\; \ldots \;-\mathrm{div}\;(\mu^n D(u^n)) \;=\; \ldots, \end{cases}}$$

with $\mu^n \;:= \mu\,(1 - \chi_S^n) \;+\; n\,\chi_S^n.$

Energy estimates yield standard bounds on ρ^n, u^n, and weak limits ρ, u.

- Strong compactness of (ρ^n):

Follow from DiPerna-Lions results on the transport equation.

$\Rightarrow$ compactness in $C([0, T]; L^p)$ for all finite p.

$\Rightarrow$ the relaxation term yields the rigid constraint of u.

- Strong compactness of (u^n) ?

No control of the time derivative of $\rho^n u^n$, due to the penalized term.

Classical in singular perturbations problems: apply the projector on the kernel of the penalized operator.

<u>Problem:</u> The penalized operator depends on n. Requires some uniformity.

Define:

$P^s_{S(\tau)}$ the projector in $H^s_\sigma(\Omega)$ on the subspace of all rigid fields over $S(\tau)$,

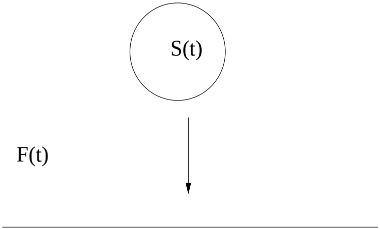
and $P^{s,*}_{S(\tau)}$ its dual operator.

- One proves, locally around each time τ, some some strong compactness for $\left(P^{s,*}_{S(\tau)}(\rho u^n)\right), \quad s < 1.$
- One shows that $P^s_{S(\tau)}(u^n)$ is "uniformly close" to u^n.

Combining both yields the strong convergence of (u^n).

Motivations

One homogeneous solid, in a viscous fluid, above a wall.



Fluid and solid at time t : $F(t), S(t)$.

<u>Aim</u> : To describe solid's dynamics near the wall.

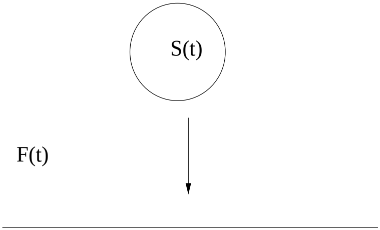
<u>Questions</u> : What is the asymptotics of the drag ? Effect of solid roughness on the drag ?

At least two reasons to wonder about the roughness effect:

Reason 1: The no-collision paradox

Remark: Fluid-solid interaction is full of paradoxes !

Example: Immersed sphere, falling above a wall under the action of gravity.



Question: Will the sphere touch the wall ?

Archimedes ($\sim$ 265 B.C.): *If $\rho_S > \rho_F$, collision.*

Relies on the hydrostatic approximation :

$$\text{Stress tensor :} \quad \Sigma := (-p_{atm} - \rho_F\, g\, z)\, I_3.$$

Force on the disk :

$$\boxed{f = -\rho_S g e_z\, |S(t)| + \int_{\partial S(t)} \Sigma n = (\rho_F - \rho_S)\; g\, |S(t)|\, e_z.}$$

Pb : The drag due to molecular pressure and viscosity is neglected.

Refined model: The one we have seen :

- Stokes or Navier-Stokes for the liquid.
- Classical laws of mechanics for the solid.
- *The stress tensor at the solid surfaces includes the newtonian tensor of the fluid.*

Surprise : *In this framework, there is no collision between the sphere and the wall !!*

Refs: Stokes : [Brenner et al, 1963], [Cooley et al, 1969]. NS : [Hillairet, 2007]

Question : What is the flaw of the model ?

Refs : [Davis et al, 1986], [Barnocky et al, 1989], [Smart et al, 1989], [Davis et al, 2003].

Most popular idea:

> *Nothing is as smooth as a sphere. The irregularity of the solid surface can change the solids' dynamics.*

$\Rightarrow$ *Need to compute the drag, notably for rough boundaries.*

Reason 2: Microfluidics

Goal: To make fluids flow through very small devices.

Example: Microchannels with diameter $\sim \mu\mathrm{m}$.

Pb: The Reynolds number is very small.

To minimize (viscous) friction at the walls is crucial.

Many theoretical and experimental works.
Refs : [Tabeling, 2004], [Bocquet, 2007 and 2012], [Vinogradova, 2009 and 2012].

Summary: At such scales, the no-slip condition usually satisfied by a viscous fluid at a wall is not always satisfied. *Some rough surfaces (*hydrophobic*) increase the slip*.

Pb:

- To maximize slip (shape optimization).
- To derive an equivalent macroscopic boundary condition (*wall law*).

Idea [Vinogradova, 2009]

- measure of the drag exerted on a solid that gets closer and closer to the rough surface.
- comparison with the asymptotics predicted by the wall laws.

$\Rightarrow$ *To obtain an approximate expression for the drag, for various models of roughness.*

Main models and results

One rough solid above a rough wall.

$S(t)$: rough sphere. P: rough plane. Fluid: $F(t)$.

We denote $h(t) := \operatorname{dist}(S(t), P)$.

Restriction: the solid translates along a vertical axis.

Remarks: For this constraint to be preserved with time:

- One needs good symmetry properties for the solid and the wall. They will be satisfied in our models.
- The mathematical model must have a good Cauchy theory (uniqueness problem).

Remark: the geometry of the domain in characterized by h:

$$S(t) = S_{h(t)} = h(t)\, e_z + S, \quad F(t) = F_{h(t)},$$

$S_h = h\, e_z + S$, F_h: domains frozen at distance h.

Equations:

- Stokes equations in the fluid: $x \in F(t), t > 0$:

$$\boxed{-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0.}$$

- Classical mechanics for the solid:

$$\boxed{\ddot{h}(t) = \int_{\partial S(t)} (2D(u)n - pn)\, d\sigma \cdot e_z}$$

n : outward normal, $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$.

Boundary conditions: will have the following general form:

- No penetration: $\boxed{u \cdot n|_P = 0, \quad \left(u - \dot{h}(t)\, e_z\right) \cdot n|_{\partial S(t)} = 0}$.

- Tangential stress

$$\boxed{\begin{cases} u \times n|_P = -2\, \beta_P\, D(u)n \times n|_P, \\ (u - \dot{h}(t)\, e_z) \times n|_{\partial S(t)} = -2\, \beta_S\, D(u)n \times n|_{\partial S(t)}. \end{cases}}$$

$\beta_S, \beta_P \geq 0$: slip lengths.

If $= 0$: no-slip (Dirichlet). If > 0: slip (Navier).

<u>Crucial remark</u>: This system turns into an ODE

$$\ddot{h}(t) \;=\; -\dot{h}(t)\, f_{h(t)}. \tag{ED}$$

with drag

$$\boxed{f_h \;=\; -\int_{\partial S_h} (2D(u_h)n - p_h n)\; d\sigma \cdot e_z}$$

where (u_h, p_h) solution of

$$\begin{cases} -\,\Delta u_h + \nabla p_h \;=\; 0, \quad \operatorname{div} u_h = 0, \\ u_h \cdot n|_P = 0, \quad (u_h -\, e_z) \cdot n|_{\partial S_h} = 0, \\ u_h \times n|_P \;=\; -2\;\beta_P\; D(u_h)n \times n|_P \\ (u_h -\, e_z) \times n|_{\partial S_h} \;=\; -2\;\beta_S\; D(u_h)n \times n|_{\partial S_h} \end{cases} \tag{S}$$

Remark: One can forget about the dynamics.

Goal: Study of f_h, h small, for various models of roughness.

Model 1: Non-smooth surface.

Cylindrical coordinates : (r, θ, z).

- $P : \{z = 0\}$
- S : ball of radius 1, perturbed near the south pole by a $C^{1,\alpha}$ "tip", $0 < \alpha < 1$. Locally, for $r < r_0$:

$$\boxed{z = 1 - \sqrt{1 - r^2} + \varepsilon r^{1+\alpha}}$$

- $\beta_P = \beta_S = 0$.

Remark: With this irregularity, $(\nabla u_h, p_h)$ is not H^1 near the boundary.

But one can show that : $(\nabla u_h, p_h) \in W^{s,\tau}$ for some s, τ with $s > 1/\tau$.
Allows to define f_h.

Model 2: Wall law of Navier type.

- $P : \{z = 0\}$.
- S : ball of radius 1.
- β_P or $\beta_S > 0$.

Model 3: Oscillations of small amplitude and wavelength.

- $P : \{z = \varepsilon\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\}$,
  with γ periodic, smooth, ≤ 0, $\gamma(0,0) = 0$.
- S : ball of radius 1.
- $\beta_P = \beta_S = 0$.

Remark: The study is limited to the case $\varepsilon \ll h$.

Remark: Limit case : $\varepsilon \to 0$, $\beta_S, \beta_P \to 0$:

One recovers the well-known case of a sphere and a plane. Cooley-O'Neil, Cox-Brenner:

$$\boxed{f_h \sim \frac{6\pi}{h}, \quad h \to 0.}$$

(which implies no-collision).

Pb: Relies on the computation of the exact solution. Heavy and restricted to simple geometries.

The study of roughness effects requires another approach ...

Proposition (Expression of the drag for model 1):

Let $\beta := \varepsilon h^{\frac{\alpha-1}{2}}$.

- In the regime $h \to 0,\ \beta \to 0$:

$$\boxed{f_h \sim \frac{6\pi}{h}(1 + c\,\beta)} \quad c = c(\alpha) \text{ explicit.}$$

- In the regime $h \to 0$, $\beta \to \infty$ (and $\varepsilon = O(1)$):
  - If $\alpha > \frac{1}{3}$,

$$\boxed{f_h \sim c\,\varepsilon^{\frac{-4}{1+\alpha}}\, h^{-\frac{3\alpha-1}{\alpha+1}}} \quad c = c(\alpha) \text{ explicit.}$$

  - If $\alpha = \frac{1}{3}$,

$$\boxed{f_h \sim c\,\varepsilon^{-3}\,|\ln h|} \quad c \text{ explicit.}$$

  - If $\alpha < \frac{1}{3}$,

$$\boxed{f_h = c\,\varepsilon^{\frac{-2}{1-\alpha}} + O(|\ln \varepsilon|)} \quad c = c(\alpha) \text{ explicit.}$$

Remarks:

- Collisions are allowed by the model for all $\alpha < 1$. Not allowed for $C^{1,1}$ boundaries.
- The more the boundary is irregular, the less the drag is.
- One recovers the classical result as $\varepsilon = 0$ (with a much simpler proof).

Proposition (Expression of the drag for model 2):

- In the regime $h \to 0$, $\beta_S, \beta_P = O(1)$, with h/β_S or h/β_P uniformly lower bounded, one has

$$\boxed{\frac{c}{h} \le f_h \le \frac{C}{h}} \quad c, C > 0.$$

- In the regime $h \to 0$, $\beta_S, \beta_P = O(1)$, with $h/\beta_S \to 0$ and $h/\beta_P \to 0$, one has

$$\boxed{f_h = 2\pi \left(\frac{1}{\beta_S} + \frac{1}{\beta_P}\right) |\ln h| + O\left(\frac{1}{\beta_S} + \frac{1}{\beta_P}\right)}$$

Remark:

- This roughness model also allows for collision, if β_P *and* $\beta_S > 0$.
- Agrees with formal calculations of Hocking (1973)

Proposition (Expression of the drag for model 3):

In the regime $\varepsilon \ll h \ll 1$:

$$\boxed{\frac{6\pi}{h + c\,\varepsilon} + O(|\ln(h+\varepsilon)|) \le f_h \le \frac{6\pi}{h} + O(|\ln h|)}$$

Remark: With homogenization techniques, one has

$$\boxed{f_h \sim \frac{6\pi}{h+\alpha\varepsilon}}$$

(if $\varepsilon/h \to 0$ fast enough.)

α explicit, associated to some boundary layer problem.

Sketch of proof

Step 1: Variational characterization of the drag

$$f_h \;=\; \min_{u \in \mathcal{A}_h} \mathcal{E}_h(u) \;=\; \mathcal{E}_h(u_h).$$

for a good energy functional $\mathcal{E}_h$ and a good admissible set $\mathcal{A}_h$.

Dirichlet case (Models 1 and 3): $\quad \mathcal{E}_h(u) \;:=\; \int_{F_h} |\nabla u|^2, \quad$ and

$$\mathcal{A}_h \;:=\; \left\{ u \in H^1_{loc}(F_h), \quad \operatorname{div} u = 0, \quad u|_P = 0, \quad u|_{\partial S_h} = e_z \right\}.$$

Navier case (Model 2):

$$\mathcal{E}_h(u) \;:=\; \int_{F_h} |\nabla u|^2 \;+\; \frac{1}{\beta_P} \int_P |u \times n|^2 \;+\; \left(\frac{1}{\beta_S} + 1 \right) \int_{\partial S_h} |(u - e_z) \times n|^2,$$

$$\mathcal{A}_h \;:=\; \left\{ u \in H^1_{loc}(F_h), \; \operatorname{div} u = 0, \; u \cdot n|_P = (u - e_z) \cdot n|_{\partial S_h} = 0 \right\}.$$

Step 2: Approximate computation of f_h, *via* some relaxed minimization problem.

Rough idea: To find $\tilde{\mathcal{E}}_h \leq \mathcal{E}_h$, and $\tilde{\mathcal{A}}_h \supset \mathcal{A}_h$, such that:

1. $\min_{u \in \tilde{\mathcal{A}}_h} \tilde{\mathcal{E}}_h(u)$ and the associate minimizer can be computed easily.
2. The minimizer $\tilde{u}_h$ belongs to $\mathcal{A}_h$.

It will follow that:

$$\tilde{\mathcal{E}}_h(\tilde{u}_h) \leq f_h \leq \mathcal{E}_h(\tilde{u}_h)$$

If the relaxed pb is close enough to the original one, it will yield a good approximation of the drag.

Remark: this rough idea requires a few adaptations: modification of the minimizer $\tilde{u}_h$ to have it belong to $\mathcal{A}_h$, ...

Remark: The difficulty lies in the choice of the good relaxed problem.

Example: Model 1 ($C^{1,\alpha}$ tip).

Idea: Simplification due to axisymmetry. The minimizer $u = u_h$ reads

$$\boxed{u \;=\; -\partial_z \phi(r,z) e_r \;+\; \frac{1}{r}\partial_r(r\phi) e_z.} \tag{R}$$

with $\phi = -\int_0^z u_r$. One restricts to fields in $\mathcal{A}_h$ of the type (R).

Boundary conditions on ϕ:

- Wall:

$$\partial_z \phi(r,0) = 0, \quad \phi(r,0) = 0, \tag{cl1}$$

- Near the south pole:

$$\partial_z \phi(r, h + \gamma_\varepsilon(r)) = 0, \quad \phi(r, h + \gamma_\varepsilon(r)) = \frac{r}{2}, \quad r < r_0 \tag{cl2}$$

  where $\gamma_\varepsilon(r) = 1 - \sqrt{1 - r^2} + \varepsilon r^{1+\alpha}$.

$$\mathcal{E}_h(u) \;=\; \int_{F_h} |\partial_z^2 \phi|^2 \;+\; \int_{F_h} |\partial_{rz}^2 \phi|^2 \;+\; \dots$$

Idea: The first term is the leading one. Only the zone near $r = 0$ matters.

Relaxed problem:

$$\tilde{\mathcal{A}}_h = \Big\{ u \in H^1_{loc}(F_h), \text{ satisfying (R)-(cl1)-(cl2) } \Big\},$$

$$\tilde{\mathcal{E}}_h(u) \;=\; \int_0^{r_0} \int_0^{\gamma_\varepsilon(r)} |\partial_z^2 \phi|^2 \, dz \, dr$$

1D minimization problems in z, parametrized by r. Minimizer:

$$\tilde{\phi}_h(r,z) = \frac{r}{2} \Phi\Big(\frac{z}{h + \gamma_\varepsilon(r)}\Big), \quad \Phi(t) = t^2(3 - 2t).$$

The minimum for the relaxed problem (lower bound for f_h) is

$$\begin{aligned}\tilde{f}_h &= 12\pi \int_0^1 \frac{r^3 dr}{(h+\gamma_\varepsilon(r))^3}\, dr \\ &= 12\pi \int_0^1 \frac{r^3 dr}{(h+\frac{r^2}{2}+\varepsilon r^{1+\alpha})^3}\, dr \;+\; \ldots \qquad\qquad = \mathcal{I}(\beta) \;+\; \ldots\end{aligned}$$

with $\beta := \varepsilon\, h^{\frac{\alpha-1}{2}}$, and

$$\boxed{\mathcal{I}(\beta) := \int_0^{+\infty} \frac{s^3 dr}{(1+\frac{s^2}{2}+\beta s^{1+\alpha})^3}.}$$

Integral with a parameter, the asymptotics of which can be computed in all regimes.

Similar drag computations are available for the other models.

Extension to Navier-Stokes (Dirichlet)

One solid $S(t)$ in a cavity Ω (bounded domains). Fluid: $F(t) := \Omega \setminus \overline{S(t)}$.

- *Navier-Stokes equations in $F(t)$*:

$$\begin{cases} \rho_F \left(\partial_t u_F + u_F \cdot \nabla u_F\right) - \Delta u_F \;=\; -\nabla p \;-\; \rho_F\, g e_z, \\ \operatorname{div} u_F \;=\; 0. \end{cases} \tag{NS}$$

- *Solid mechanics in $S(t)$*:

$$\begin{cases} u_S(t,x) \;=\; U(t) \;+\; \omega(t) \times (x - x(t)), \quad \text{with} \\ m_S\, \dot{U}(t) \;=\; \displaystyle\int_{\partial S(t)} \Sigma n\, d\sigma \;+\; \int_{S(t)} \rho_S\, g e_z, \\ J_S\, \dot{\omega}(t) \;=\; J_S \omega(t) \times \omega(t) + \displaystyle\int_{\partial S(t)} (x - x(t)) \times (\Sigma n)\, d\sigma \\ +\; \displaystyle\int_{S(t)} (x - x(t)) \times \rho_S g e_z \end{cases} \tag{MS}$$

- Conditions at the interface :

$$\begin{cases} (\Sigma n)\,|_{\partial S(t)} &= (2D(u)n - pn)\,|_{\partial S(t)} - \rho_F g n|_{\partial S(t)} \\ u_F|_{\partial S(t)} &= u_S|_{\partial S(t)} \end{cases} \tag{In}$$

- No slip conditions at the boundary of the cavity :

$$u_F|_{\partial\Omega} = 0. \tag{Pa}$$

Dynamics of the solid near $\partial\Omega$

One considers "model 1": $\partial\Omega$ is locally flat, the sphere $S(t)$ has a $C^{1,\alpha}$ tip and is in vertical translation.

Theorem

For any weak solution satisfying the assumptions of model 1, the solid touches the wall in finite time iff $\alpha < 1$.

Remark: Similar results in dimension 2. Collision in finite time iff $\alpha < 1/2$.

Idea for the proof

Choose $\varphi(t,x) = u_{h(t)}(x)$ in the variational formulation.

One has:

$$\boxed{-\mathcal{F}(h(t)) + (\rho_s - \rho_F)\, g|S(0)|\, t = R(t)}$$

where

$$\mathcal{F}(h) = \int_{h_0}^{h} f_{h'}\, dh'.$$

and $R(t)$ is a "remainder", coming from the transport in the Navier-Stokes equation.

Pb: u_h is not available.

Key: Replace u_h by $\tilde{u}_h$, minimizer of the relaxed problem.