Exercises for 1st lecture.

Assume that X is an uncountable polish metric space.

1) Show that $|X| = \mathfrak{C}$.
2) Prove that there exists compact subset $C \subseteq X$ which is a Cantor set that is $C \cong \{0,1\}^{\mathbb{N}}$.
3) Prove that $[0,1]^{\mathbb{N}}$ is isomorphic to a Borel subset of $\{0,1\}^{\mathbb{N}}$.
4) Show that any polish subset of X is G_δ.
5) Show that X is homeomorphic to Y, a polish subset of $[0,1]^{\mathbb{N}}$.
6) Using the measurable version of the proof of Cantor-Bernstein's theorem, deduce from above exercises Kuratowski's Isomorphism Theorem saying that if X is an uncountable Polish space then X is Borel isomorphic with $[0,1]$.

Let $(X, \mathcal{B}, m)$ be a probability space and $S, T \in NST(X, \mathcal{B}, m)$.

7) Show that if $f \in L^1(m)$, then $\int_A f d(m \circ T) = \int_A T' \dot{f} dm$.
8) Show that $S \circ T \in NST(X, \mathcal{B}, m)$ and $(T \circ S)' = T'(S) \cdot S'$.
9) Assume that $T, T^{-1} \in NST(X, \mathcal{B}, m)$. Knowing T', derive the formula for $(T^{-1})'$.

Let $T \in NST(X, \mathcal{B}, m)$.

10) Show that if T is invertible then $\hat{T}f = (T^{-1})' f \circ T^{-1}$.
11) Show that the following are equivalent:
   a) there exists absolutely coninuous to m, T-invariant probability;
   b) There exists non-negative $h \in L^1(m)$, such that $\int_X hdm > 0$ and $\hat{T}h = h$.

12) Prove that μ_p is invariant probability for the adding machine iff $p = 2$.
13) Describe the inverse to adding machine.
14) Prove that for almost every $x \in (X, \mathcal{B}, m)$ in the construction of rank one system, $T(x)$ is not defined only on finitely many steps.
15) Prove that the transformation obtained in the construction of rank one system is well defined almost everywhere. 16) Show that Kakutani Skycsrapper is measure preserving iff the base transformation is measure preserving.
17) Prove that Bernoulli shift is a probability preserving transformation.

Let $\Omega = \{0,1\}^{\mathbb{N}}$ and let $\mathcal{C}$ be the set of all cylinders in Ω. Let $X := \Omega^{\mathbb{Z}}$ with product topology nad let

$$\mathfrak{A} := \{[A_1, \ldots, A_n]_k; \quad \text{where} \quad A_1, \ldots, A_n \in \mathcal{C} \quad \text{and} \quad k \in \mathbb{N}\}.$$

18) Prove that all elements of $\mathfrak{A}$ are compact and closed and that $\mathfrak{a}$ is a ring of sets.
19) Let $\mu([A_1, \ldots, A_n]_k := P_{k+1,n}(A_1 \times \ldots \times A_n)$. Prove that μ is additive,

$$\mu^*(E) := inf\{\sum_{n=1}^{\infty} \mu(A_n); \quad A_n \in \mathfrak{A}, \quad E \subseteq \bigcup_{n=1}^{\infty} A_n\}$$

is an outer measure and for all $A \in \mathfrak{A}$ we have $\mu^*(A) = \mu(A)$.
20) Using above exercises, prove Kolmogorov Existence Theorem.

Exercises for 2nd lecture

1) When is a piecewise onto interval map an isomorphism?
2) Show that for C^1 interval map (I, T, α) we have

$$\hat{T}f(x) = \sum_{y \in I, Ty = x} \frac{f(y)}{|T'(y)|}.$$

3) Prove that if (I,T,α) is piecewise onto, piecewise linear interval map then $m\circ T^{-1}=m$ and

$$m(\bigcap k=0^N T^{-k}a_k)=\prod_{k=0}^{N} m(a_k),$$

for every $N>0$ and every choice of $a_1,\ldots,a_n\in\alpha$.
4) Find the transfer operator for Gauss map.
5) Find an invariant measure for Gauss map.
6) Show that in the proof of Boole's formula it is indeed enough to show that $P_\omega\circ T^{-1}=P_{T\omega}$.
7) Let T be inner function with $\alpha>0$. Prove that for $b\in\mathbb{R}$ if $T(ib)=u(b)+iv(b)$, then $\frac{v(b)}{b}\to\alpha$ and $\frac{u(b)}{b}\to 0$ as $b\to\infty$.
8) Using above exercise and Boole's formula, show that $\hat{T}\mathbf{1}=\frac{1}{\alpha}\mathbf{1}$.

Let $(X,\mathcal{B},m,T)$ be a null-preserving transformation.
9) Prove that for all sets $A\in\mathcal{B}$ if $A\setminus\bigcup_{n=1}^{\infty}T^{-n}A$ has positive measure, then it is a wandering set.
10) Prove Poincare Reccurence Theorem without using Maharam Reccurence Theorem.
11) Let $T:\mathbb{R}\to\mathbb{R}$ such that $T(x)=x-\frac{1}{x}$. Prove that for every $x\in\mathbb{R}$ there exists $n(x)\in\mathbb{N}$ such that $|T^{n(x)}x|<1$.
12) Use previous exercise to prove that T is conservative.
13) Generalise previous result to arbitrary Boole's transformation.
14) Suppose that $(X,\mathcal{B},m,T)$ is conservative transformation and (Y,d) is a separable metric space and let $h:X\to Y$ be a measurable function. Then

$$\liminf_{n\to\infty} d(h(T^n x),h(x))=0$$

for a.e. $x\in X$.
15) Prove that Kakutani skyscrapper over a conservative transformation is conservative.
16) Prove that 1st return map of measure preserving transformation is induced measure preserving.
17) Let $(X,\mathcal{B},m,T)$ be conservative measure preserving transformation and $m(\Omega)>0$. Show that:
a) first return map to Ω is conservative;
b) if $(X,\mathcal{B},m,T)$ is invertible then it is isomorphic to the Kakutani skyscrapper over T_Ω with height function being first return time function.
c) if Ω is a sweeping set then conservativity of first return map implies conservativity of $(X,\mathcal{B},m,T)$.
18) Show that if T is invertible, then $D(T)$ is an invariant set.
19) Prove that the inner transformation $T\omega=\omega+\beta+\int_{\mathbb{R}}\frac{1}{t-\omega}d\nu(t)$, with $\nu\perp Leb$ and ν having bounded support, is conservative iff $\beta=0$.

Exercises for the 3rd lecture

1) Prove that an invertible nonsingular transformation of a non-atomic measure space is conservative. Is it true for measures with atoms?
2) Let $(X,\mathcal{B},m,T)$ be nonsingular. Show that it is ergodic and conservative iff $\sum_{n=1}^{\infty}\mathbf{1}_A\circ T^n=\infty$ a.e. for $A\in\mathcal{B}_+$.
3) Prove that Kakutani skyscrapper over nonsingular ergodic transformation is ergodic.

4) Suppose that $(X,\mathcal{B},m,T)$ is nonsingular conservative transformation. Prove that for $A\in\mathcal{B}$ such that $\bigcup_{n=1}^{\infty}T^{-n}A=X$ a.e. then T is ergodic iff T_A is ergodic.
5) Show that $(X,\mathcal{B},m,T)$ is ergodic iff for every $f\in L^1(m)$, $f\circ T=f$ implies that f is constant a.e.
6) Show that if $(X,\mathcal{B},m,T)$ is ergodic, then it does not imply that $(X\times X,\mathcal{B}\otimes\mathcal{B},m\otimes m,T\times T)$ is ergodic.
7) Using the example of rotation show that the ergodicity T does not imply the ergodicity of T^2.
8) We say that MPT $(X,\mathcal{B},m,T)$ is weakly mixing if for $f\in L^2(m)$, $f\circ T=\lambda f$ implies f constant. Show that a rotation on a circle is never weakly mixing.
9) Show that if for a sequence $\{f_n\}_{n\in\mathbb{N}}\subset L^2(X,\mathcal{B},\nu)$ and σ-algebra $\mathcal{C}\subseteq\mathcal{B}$ such that f_n is $\mathcal{C}$-measurable and $f_n\to f$ weakly in L^2, then f is $\mathcal{C}$-measurable.
10) Show that for PPT $(X,\mathcal{B},m,T)$ if $f\circ T^{n_k}\to g$ weakly in L^2 then g is tail measurable.
11) Prove that the previous exercise implies that exactness of probability preserving transformation implies mixing.
12) For every mixing PPT construct Kakutani skyscrapper which is not mixing.
13) Prove that for mixing PPTs $(X,\mathcal{B},m,T)$ and $(Y,\mathcal{C},\hat{m},S)$, the product transformation $(X\times Y,\mathcal{B}\otimes\mathcal{C},m\otimes\hat{m},T\times S)$ is also mixing. 14) Prove that the σ-algebra of invariant sets of the odometer is equal to the tail σ-algebra of the one-sided Bernoulli shift.

Exercises for the 4th lecture

1) For $x\in\{0,1\}^{\mathbb{N}}$ show that $\phi_{2^n}(x)=\phi(S^n(x))$, where $\phi:=min\{n\geq 1 x_n=0\}-2$ and S is a right shift.
2) Show that for $p=\frac{1}{2}$, the Rigidity Proposition implies that odometer is not mixing.

Let $(\Omega,\mathcal{B},\mu_p,\tau)$ be an adding machine and let S be shift on Ω. Let $f(x):=x_1$ and $\phi(x):=l(x)-2$, where $l(x):=:=min\{n\geq 1\; x_n=0\}$. Let also $S_f(x,z):=(S(x),z+f(x))$ and $\tau_\phi:=(\tau(x),z+l(x)-2)$.
3) Show that $(\Omega\times\mathbb{Z},\mathcal{B}\otimes 2^{\mathbb{Z}},\mu_p\otimes\#,S_f)$ is totally dissipative.
4) Show that $\mathcal{T}(S_f)=\mathcal{I}(\tau_\phi)$.
5) Deduce that $(\Omega\times\mathbb{Z},\mathcal{B}\otimes 2^{\mathbb{Z}},\mu_p\otimes\#,S_f)$ is exact.

6) Using Hurewicz's theorem prove that a PPT transformation $(X,\mathcal{B},m,T)$ is ergodic iff for any sets $A,B\in\mathcal{B}$ we have $\frac{1}{n}\sum_{k=0}^{n-1}\mu(T^kA\cap B)\to\mu(A)\mu(B)$.

7) Let U be an unitary operator on a Hilbert space $\mathcal{H}$ and let $\mathcal{H}_0^U$ be the subspace of invariant elements of $\mathcal{H}$. Let U^* be the adjoint operator. Show that $\mathcal{H}_0^U=\mathcal{H}_0^{U^*}$.
8) Taking the notation from previous exercise, show that for elements of the form $F=Ug-g$, where $g\in\mathcal{H}$, we have $\left\|\frac{1}{n}\sum_{k=0}^{n-1}U^kF\right\|_{\mathcal{H}}\to 0$.
9) Using previous exercises prove the von Neumann's Ergodic Theorem, that is

$$\lim_{n\to\infty}\left\|frac1n\sum_{k=0}^{n-1}U^kh-P_Uh\right\|_{\mathcal{H}}=0,$$

for any $h\in\mathcal{H}$, where P_U is an orthogonal projection on $\mathcal{H}_0^U$.
10) Suppose that T is a conservative, ergodic, measure preserving transformation

of the σ-finite measure space $(X, \mathcal{B}, \nu)$. Prove that for every $f \in L^1(m)$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k) \to 0 \quad \text{for a.e.} \quad x \in X.$$

Exercises for the 5th and 6th lecture

1) Show that if $(X, \mathcal{B}, m, T)$ and there exists $F \in L^1(m)$ such that $F \circ T = F + \beta$ for some $\beta \in \mathbb{R}$, then T is not ergodic.
2) Show that set of p-periodic points is measurable.
3) Let $(X, \mathcal{B}, m)$ be a standard probability space. Prove that every set of positive measure $A \in \mathcal{B}_+$ contains set $B \in \mathcal{B}_+$ of positive measure such that sets $T^j B$ are disjoint for $j = 0, \dots, p-1$.
4) Prove that there exists set $A \in B$ such that $\{T^j A\}$ are disjoint for $j = 0, \dots, p-1$ and $\mathtt{Per}_p(T) = \bigcup_{j=0}^{p-1} T^j A$, m-almost everywhere.
5) Prove that if NST $(X, \mathcal{B}, m, T)$ is conservative and ergodic then every set of positive measure is a sweepout set.
6) Suppose that $(X, \mathcal{B}, m, T)$ is aperiodic and conservative NST. Show that for each set of positive measure $J \in \mathcal{B}_+$ and for every $N \in \mathbb{N}$ we can find a subset $A \subset J$ of positive measure, such that $T^{-j}A$ are disjoint for $j = 1, \dots, N-1$.
7) Using the previous exercise prove that for $(X, \mathcal{B}, m, T)$ there exists arbitrarily small sweepout set.
8) Show that Roklin's tower theorem implies the statement from previous exercise.
9) Suppose that a PPT $(X, \mathcal{B}, m, T)$ is ergodic and $f \in L^1(m)$ such that $\int_X f dm \neq 0$. Show that the skew product T_f is totally dissipative.
10) Let $\phi = \xi - \xi \circ T$ be a coboundary with $\xi : X \to G$ measurable. Prove that T_ϕ is isomorphic to T_0.
11) Let T be a PPT and ϕ be measurable. Show that T_ϕ is ergodic iff $E(\phi) = G$.
12) Show that for $(X, \mathcal{B}, m, T)$ invertible and nonsingular and for $\phi : X \to G$, where G is locally compact polish abelian group, we have $E(\phi) = \Pi(\phi) \cup \{0\}$ and $E(\phi) = \mathtt{Per}(\phi)$.
13) Suppose that $(X, \mathcal{B}, m, S)$ is an ergodic PPT and $f : X \to \mathbb{Z}$ be such that S_f is an ergodic, totally dissipative MPT. Prove that $\Pi(f, S) = \emptyset$ and $\mathtt{Per}(f, S) = \mathbb{Z}$.