Lecture # 1 8/10/2014.

INTRODUCTION

Let \((X, \mathcal{B}, m)\) be a standard \(\sigma\)-finite measure space\(^1\) A null preserving transformation (NPT) of \(X\) is only defined modulo nullsets, and is a map \(T : X_0 \to X_0\) (where \(X_0 \subset X\) has full measure), which is measurable and has the null preserving property that for \(A \in \mathcal{B}\), \(m(T^{-1}A) = 0\) implies that \(m(A) = 0\).

A non-singular transformation (NST) is a \(\text{NPT} (X, \mathcal{B}, m, T)\) with the stronger property that for \(A \in \mathcal{B}\), \(m(T^{-1}A) = 0\) iff \(m(A) = 0\).

A measure preserving transformation (MPT) is a \(\text{NST} (X, \mathcal{B}, m, T)\) with the additional property that \(m(T^{-1}A) = m(A) \forall A \in \mathcal{B}\).

We'll call a nonsingular transformation \(\text{NS-invertible}\) if the associated map is invertible with a nonsingular inverse.

Let
\[
\text{NST}(X, \mathcal{B}, m) := \{\text{nonsingular invertible transformations of } X\}
\]
\[
\text{MPT}(X, \mathcal{B}, m) := \{\text{invertible measure preserving transformations of } X\}
\]
\[
\text{PPT}(X, \mathcal{B}, m) := \text{MPT}(X, \mathcal{B}, m) \text{ in case } m(X) = 1.
\]
The are all groups under composition (see the exercise below).

Equivalent invariant measures. If \(T\) is a non-singular transformation of a \(\sigma\)-finite measure space \((X, \mathcal{B}, m)\), and \(p\) is another measure on \((X, \mathcal{B})\) equivalent to \(m\) (denoted \(p \sim m\) and meaning that \(p\) and \(m\) have the same nullsets), then \(T\) is a non-singular transformation of \((X, \mathcal{B}, p)\).

Thus, a non-singular transformation of a \(\sigma\)-finite measure space is actually a non-singular transformation of a probability space.

\(^1\)i.e. an uncountable Polish spec equipped with Borel sets and a non-atomic, \(\sigma\)-finite measure.

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The first question about a NST \((X,\mathcal{B},p,T)\) is whether it was obtained from a measure preserving transformation in this way, or, slightly more generally:

\[ \exists \; \sigma \text{-finite absolutely continuous invariant measure (a.c.i.m., i.e. } m \ll p, \text{ with } m \circ T^{-1} = m. \]

**RADON NIKODYM DERIVATIVES**

Let \((X,\mathcal{B},m,T)\) be an invertible NST of the probability space \((X,\mathcal{B},m)\). The measures \(m \& m \circ T\) are equivalent (i.e. \(m \circ T \ll m \& m \ll m \circ T\)), written \(m \circ T \sim m\). By the Radon Nikodym theorem, \(\exists ! T' \in L^1, T' > 0\) a.e., so that

\[ m(TA) = \int_A T' dm \; \forall \; A \in \mathcal{B}. \]

The function \(T'\) is called the RN derivative of \(T\). The measurable map \(f : A \to A'\) is called
- null preserving (NP) if for \(C \in \mathcal{B}' \cap A'\), \(m'(C) = 0 \Rightarrow m(f^{-1}C) = 0\);
- nonsingular (NS) if for \(C \in \mathcal{B}' \cap A'\), \(m(f^{-1}C) = 0\) iff \(m'(C) = 0\); and
- measure preserving (MP) if \(m(f^{-1}C) = m'(C)\) for \(C \in \mathcal{B}' \cap A'\).

**Exercise 1:** Chain rule for RN derivatives.

Let \((X,\mathcal{B},m)\) be a probability space and let \(S, T \in \text{NST}(X,\mathcal{B},m)\).

(i) Show that \(T \circ S \in \text{NST}(X,\mathcal{B},m)\) and \((T \circ S)' = T' \circ S \cdot S'\).

(ii) Let \((X,\mathcal{B},m)\) be the unit interval equipped with Borel sets and Lebesgue measure, and suppose that \(T : X \to X\) is nondecreasing and \(C^1\), then

- \(T : X \to X\) is a homeomorphism iff \([T' = 0]^o = \emptyset\);
- \(T^{-1} : X \to X\) is non-singular iff \(m([T' = 0]) = 0\); &
- \(\exists\) a \(C^1\) homeomorphism \(T : X \to X\) with \(T^{-1} : X \to X\) singular.

**Transfer Operator.**

Let \((X,\mathcal{B},m,T)\) be a null-preserving transformation, then \(\|f \circ T\|_\infty \leq \|f\|_\infty \; \forall \; f \in L^\infty(m)\) and \(T : L^\infty(m) \to L^\infty(m)\) where \(Tf := f \circ T\).

There is an operator known as the transfer operator \(\hat{T} : L^\infty(m) \to L^\infty(m)\) so that \(\hat{T}^* = T\) i.e.:

\[ \int_X \hat{T} f \cdot g dm = \int_X f \cdot Tg dm \; \forall \; f \in L^1(m), \; g \in L^\infty(m). \]
This is given by \( \hat{T}f := \frac{d\nu_f \circ T^{-1}}{dm} \) where \( \nu_f(A) := \int_X f \, dm \) (!).

**Exercise 2.** Let \((X, \mathcal{B}, m, T)\) be a nonsingular transformation.

(i) Show that if \(T\) is invertible, then \( \hat{T}f = T^{-1}f \circ T^{-1} \).

(ii) Show that \( \exists \) an absolutely continuous invariant probability for \(T\) iff \( \exists h \in L^1_+ \) satisfying \( \hat{T}h = h \).

**Examples**

**Rotations of the circle.** Let \(X\) be the circle \( \mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)\), \(\mathcal{B}\) be its Borel sets, and \(m\) be Lebesgue measure. The rotation (or translation) of the circle by \(x \in X\) is the transformation \(r_x : X \to X\) defined by \(r_x(y) = x + y \mod 1\).

Evidently \(m \circ r_x = m\) for every \(x \in X\) and each \(r_x\) is an invertible measure preserving transformation of \((X, \mathcal{B}, m)\).

**The adding machine.** Let \(\Omega = \{0, 1\}^\mathbb{N}\), and \(\mathcal{F}\) be the \(\sigma\)-algebra generated by cylinders. Define the adding machine \(\tau : \Omega \to \Omega\) by \(\tau(\overline{0}) := (0)\) where \(\tau(\overline{a})_k = a \forall k \geq 1\); and

\[
\tau(1, \ldots, 1, 0, \omega_{\ell+1}, \omega_{\ell+2}, \ldots) = (0, \ldots, 0, 1, \omega_{\ell+1}, \omega_{\ell+2}, \ldots)
\]

for \(\omega \in \Omega \setminus \{(\overline{1})\}\) where \(\ell(\omega) := \min\{n \geq 1 : \omega_n = 0\}\).

The reason for the name "adding machine" is that

\[
\sum_{k=1}^{\infty} 2^{k-1}(\tau^n \overline{0})_k = n \quad \forall \ n \geq 1.
\]

We'll consider the adding machine with respect to various probabilities on \(\Omega\).

For \(p \in (0, 1)\), define a probability \(\mu_p\) on \(\Omega\) by

\[
\mu_p([\epsilon_1, \ldots, \epsilon_n]) = \prod_{k=1}^{n} p(\epsilon_k)
\]

where \(p(0) = 1 - p\) and \(p(1) = p\).

**1.3 Proposition**

\(\tau\) is an invertible, nonsingular transformation of \((\Omega, \mathcal{F}, \mu_p)\) with

\[
\frac{d\mu_p \circ \tau}{d\mu_p} = \left(\frac{1 - p}{p}\right)^{\ell-2}.
\]

**Proof**
We show that $\mu_p \circ \tau \sim \mu_p$ and calculate $\frac{d\mu_p \circ \tau}{d\mu_p}$. We show that for any set $A \in \mathcal{F}$,

$$\mu_p(\tau A) = \int_A \left( \frac{1-p}{p} \right)^{\ell-2} d\mu_p.$$ 

Consider first a cylinder set $A \subset [\ell = k]$ ($k \geq 1$)

$$A = [1, \ldots, 1, 0, a_1, \ldots, a_n],$$

then

$$\tau A = [0, \ldots, 0, 1, a_1, \ldots, a_n],$$

and

$$(\mathcal{C}) \quad \mu_p(\tau A) = \mu_p([0, \ldots, 0, 1]) \mu_p([a_1, \ldots, a_n])$$

$$= \left( \frac{1-p}{p} \right)^{k-2} \mu_p(A)$$

$$= \int_A \left( \frac{1-p}{p} \right)^{\ell-2} d\mu_p.$$ 

Let

$$\mathcal{C} := \{ A \in \mathcal{F} : (\mathcal{C}) \text{ holds} \}.$$ 

As above, $\mathcal{C} \supset \{ \text{cylinders} \}$.

Since a any finite union of cylinders is also a finite union of disjoint cylinders, $\mathcal{C} \subset \mathcal{A}$, the algebra of finite unions of cylinders.

By $\sigma$-additivity of $\mu_p$, $\mathcal{C}$ is a monotone class, and by the monotone class theorem, $\mathcal{C} \supset \sigma(\mathcal{A}) = \mathcal{B}$. 

Note that $\mu_{\frac{1}{2}} \circ \tau = \mu_{\frac{1}{2}}$.

**Rank one constructions.**

This method constructs a $T \in \mathbb{MPT}(X, B, m)$ where $X = (0, S_T)$ is an interval, $m$ is Lebesgue measure and where $T$ is an invertible *piecewise translation* that is there are intervals $\{I_n : n \geq 1\}$ and numbers $a_n \in \mathbb{R}$ ($n \geq 1$) so that mod $m$:

$$X = \bigsqcup_{n=1}^{\infty} I_n = \bigsqcup_{n=1}^{\infty} (a_n + I_n) \quad \& \quad T(x) = x + a_n \quad \text{for } x \in I_n.$$ 

The rank one transformation $(X, B, m, T)$ is an invertible piecewise translation of an interval $J_T = (0, S_T)$ where $S_T \in (0, \infty]$ which is defined as the “limit of a refining sequence of Rokhlin towers”. 

• A Rokhlin tower is a finite sequence of disjoint intervals \( \tau = (I_1, I_2, \ldots, I_n) \) of equal lengths; considered equipped with the translations \( I_j \to I_{j+1} \) \((1 \leq j \leq n - 1)\). It is thus a piecewise translation

\[
T_\tau : \text{Dom } T_\tau = \bigcup_{j=1}^{n-1} I_j \to \bigcup_{j=2}^n I_j
\]

being defined everywhere on \( \bigcup_{j=1}^n I_j \) except the last interval \( I_n \).

• We’ll say that the Rokhlin tower \( \theta = (J_1, \ldots, J_\ell) \) refines the Rokhlin tower \( \tau = (I_1, I_2, \ldots, I_n) \) (written \( \theta > \tau \)) if

\[
\bigcup_{j=1}^n I_j \subset \bigcup_{k=1}^\ell J_k \& I_j = \bigcup_{1 \leq k \leq \ell, J_k \subset I_j} J_k.
\]

This entails (!) \( \bigcup_{j=1}^{n-1} I_j \subset \bigcup_{k=1}^{\ell-1} J_k \), whence \( T_\theta|_{\bigcup_{j=1}^{n-1} I_j} \equiv T_\tau \).

**Definition.**

Let \( c_n \in \mathbb{N}, c_n \geq 2 \ (n \geq 1) \) and let \( S_{n,k} \geq 0, \ (n \geq 1, 1 \leq k \leq c_n) \). The rank one transformation with construction data

\[
\{(c_n; S_{n,1}, \ldots, S_{n,c_n}) : n \geq 1\}
\]

is an invertible piecewise translation of the interval \( J_T = (0, S_T) \) where

\[
S_T := 1 + \sum_{n \geq 1} \frac{1}{c_1 \cdots c_n} \sum_{k=1}^{c_n} S_{n,k} \leq \infty.
\]

To obtain \( T \), we define a refining sequence \( (\tau_n)_{n \geq 1} \) of Rokhlin towers where \( \tau_1 = [0,1] \) and \( \tau_{n+1} \) is constructed from \( \tau_n \) by

- cutting \( \tau_n \) into \( c_n \) columns of equal width,
- putting \( S_{n,k} \) spacer intervals (of the same width) above the \( k \)th column \((1 \leq k \leq c_n)\);
- and stacking.

Evidently \( \tau_{n+1} > \tau_n \). Let \( X \) be the increasing union of the intervals in the towers \( \tau_n \).

The sum of the lengths of the last intervals of the towers is \( \sum_{n=1}^{\infty} \frac{1}{c_1 \cdots c_n} < \infty \) and so for a.e. \( x \in X \), \( \exists n \leq 1 \) so that \( x \in \text{Dom } T_{\tau_k} \forall k \geq n \) and \( T(x) := T_{\tau_k}(x) \forall k \geq n \).

The length of \( X \) is 1 plus the total length of all the spacer intervals added in the construction i.e. \( S_T \).

**Exercise 3.** Show that the adding machine \((\Omega, \mathcal{F}, \mu, \tau)\) where \( \mu = \mu_{\frac{1}{2}} \) is isomorphic to \((X, \mathcal{B}, m, T)\), the rank one transformation with construction data \( \{(c_n; S_{n,1}, \ldots, S_{n,c_n}) : n \geq 1\} \) with
\[ c_n = 2 \; \& \; s_{n,1} = s_{n,2} = 0 \; \forall \; n \geq 1; \; \text{i.e. show that there are measurable sets } X_0 \in \mathcal{B}, \; \Omega_0 \in \mathcal{F} \text{ of full measure so that } TX_0 = X_0 \; \& \; \tau \Omega_0 = \Omega_0 \] and \( \pi : X_0 \to \Omega_0 \) invertible, measure preserving so that \( \pi \circ T = \tau \circ \pi \).

**Kakutani skyscrapers.**

Suppose that \( (\Omega, \mathcal{F}, \mu, S) \) is a NST of the \( \sigma \)-finite measure space \( (\Omega, \mathcal{F}, \mu) \) and that \( \varphi : \Omega \to \mathbb{N} \) is measurable. The Kakutani skyscraper over \( S \) with height function \( \varphi \) is the transformation \( T \) of the \( \sigma \)-finite measure space \( (X, \mathcal{B}, m) \) defined as follows.

\[
X = \{ (x, n) : x \in \Omega, \; 1 \leq n \leq \varphi(x) \},
\[
\mathcal{B} = \sigma \{ A \times \{ n \} : n \in \mathbb{N}, \; A \in \mathcal{F} \cap [\varphi \geq n] \}, \; m(A \times \{ n \}) = \mu(A),
\]
and

\[
T(x, n) = \begin{cases} (Sx, \varphi(x)) & \text{if } n = \varphi(x), \\ (x, n + 1) & \text{if } 1 \leq n \leq \varphi(x) - 1. \end{cases}
\]

Evidently \( T \) is a NST with

\[
m(X) = \int_{\Omega} \varphi d\mu.
\]

Moreover, if \( S \) is a MPT, then so is \( T \).

- \( \bigcup_{n \geq 1} T^{-n}(\Omega \times \{ 1 \}) = X \);
- For \( x \in \Omega \), let \( \varphi_N(x) := \sum_{k=0}^{N-1} \varphi(S^k x) \), then \( T^{\varphi_N(x)}(x, 1) = (S^N x, 1) \) and

\[
\{ n \geq 1 : T^n(x, 1) \in \Omega \times \{ 1 \} \} = \{ \varphi_N(x) : N \geq 1 \}.
\]

**Bernoulli shift.**

The (two sided) Bernoulli shift is defined by \( X = \mathbb{R}^\mathbb{Z}, \) \( \mathcal{B}(X) \) the \( \sigma \)-algebra generated by cylinder sets of form

\[
[A_1, \ldots, A_n] := \{ x \in A_j \in \mathcal{B}(\mathbb{R}), \; 1 \leq j \leq n \}
\]
where \( A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}) \). The shift \( S : X \to X \) is defined by \( (Sx)_n = x_{n+1} \).

Let \( p : \mathcal{B}(\mathbb{R}) \to [0, 1] \) be a probability, and define \( \widehat{\mu}_p : \{ \text{cylinders} \} \to [0, 1] \) by

\[
\widehat{\mu}_p([A_1, \ldots, A_n]) = \prod_{k=1}^{n} p(A_k) \; (A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})).
\]

By Kolmogorov’s existence theorem (see below) \( \exists \) a probability measure \( \mu_p : \mathcal{B}(X) \to [0, 1] \) so that \( \mu_p|_{\{ \text{cylinders} \}} = \widehat{\mu}_p \).

Evidently (!), the two sided Bernoulli shift is measure preserving.
2.1 Kolmogorov’s existence theorem

Let $Y$ be a Polish space, and suppose that for $k, \ell \in \mathbb{Z}$, $k \leq \ell$, $P_{k,\ell} \in \mathcal{P}(Y^{\ell-k+1})$ are such that

$$P_{k,\ell+1}(A_k \times \cdots \times A_\ell \times Y) = P_{k-1,\ell}(Y \times A_k \times \cdots \times A_\ell) = P_n(A_k \times \cdots \times A_\ell)$$

then there is a probability measure $P \in \mathcal{P}(Y^\mathbb{Z})$ satisfying

$$P([A_1, \ldots, A_n]_k) = P_{k+1,n}(A_1 \times \cdots \times A_n).$$

Vague sketch of proof

- WLOG $Y$ is uncountable (; any countable Polish space is measurable embeddable in an uncountable Polish space);
- WLOG $Y = \Omega := \{0, 1\}^\mathbb{N}$ (by Kuratowski’s isomorphism theorem).
- Now let $\mathcal{A}$ be the collection of cylinder subsets of $\Omega$ and set
  $$\mathfrak{A} := \{[A_1, \ldots, A_n]_k : A_1, \ldots, A_n \in \mathcal{A}\}.$$  

All sets in $\mathfrak{A}$ are both open and compact wrt the compact product topology on $\Omega^\mathbb{Z}$.
- Define $\mu : \mathfrak{A} \to [0, 1]$ by
  $$\mu([A_1, \ldots, A_n]_k) := P_{k+1,k+n}(A_1 \times \cdots \times A_n),$$

then $\mu : \mathfrak{A} \to [0, 1]$ is additive and hence (!) countably subadditive.
- The required probability exists by Caratheodory’s theorem.\[}
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Interval maps.
Let \( I \subseteq \mathbb{R} \) be an interval, let \( m \) be Lebesgue measure on \( I \), and \( \alpha \) be a collection of disjoint open subintervals of \( I \) such that
\[
m(I \setminus \bigcup_{a \in \alpha} a) = 0
\]
where \( U_a = \bigcup_{a \in \alpha} a \).

For \( r \geq 1 \), a \( C^r \) interval map with basic partition \( \alpha \) is a map \( T : I \to I \) such that for each \( a \in \alpha \), \( T|_a \) extends to a \( C^r \) diffeomorphism \( T : a \to T(a) \).

The \( C^r \) interval map is called piecewise onto if \( T(a) = I \) \( \forall \ a \in \alpha \).

Transfer operator of an interval map.
Let \( T : I \to I \) be a \( C^r \) interval map with basic partition \( \alpha \). For \( a \in \alpha \), let \( v_a : I \to a \) be the inverse of \( T : a \to I \) (a \( C^r \) diffeomorphism). It follows from an integration variable-change argument that with respect to \( m \):
\[
\hat{T}f = \sum_{a \in \alpha} 1_{T(a)} v'_a f \circ v_a
\]
Note that here \( v'_a := \frac{d \text{vol}_a}{dm} = |\frac{dv_a}{dx}| \).

Exercise 4.
(i) Show that for a \( C^1 \) interval map \( (I, T, \alpha) \):
\[
\hat{T}f(x) = \sum_{y \in I, T(y) = x} \frac{f(y)}{|T'(y)|}.
\]

(ii) Show that if \( (I, T, \alpha) \) is a piecewise onto, piecewise linear interval map (i.e. \( T : a \to Ta \) is linear \( \forall \ a \in \alpha \) with \#\( \alpha \geq 2 \), then \( m \circ T^{-1} = m \) and that
\[
m(\bigcap_{k=0}^{N} T^{-k} a_k) = \prod_{k=0}^{N} m(a_k) \quad \forall \ N \geq 1, \ a_0, a_1, \ldots, a_N \in \alpha.
\]

Boole transformations & inner functions.
A Boole transformation is a map \( T : \mathbb{R} \to \mathbb{R} \) of form
\[
T(x) = \alpha x + \beta + \sum_{k=1}^{N} \frac{p_k}{t_k - x}
\]
where \( \alpha \geq 0, \ p_1, \ldots, p_N > 0 \) & \( \beta, \ t_1, \ldots, t_N \in \mathbb{R} \).

A Boole transformation \( T \) is an inner function of the upper half plane \( \mathbb{R}^{2^+} := \{ \omega \in \mathbb{C} : \ \text{Im} \omega > 0 \} \) i.e. an analytic endomorphism of \( \mathbb{R}^{2^+} \) which preserves \( \mathbb{R} \).
The general form of an inner function $T$ of $\mathbb{R}^2^+$ is given by:

\begin{equation}
T(\omega) = \alpha \omega + \beta + \int_{\mathbb{R}} \frac{1 + t\omega}{t - \omega} d\mu(t)
\end{equation}

where $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $\mu$ is a finite, Lebesgue-singular, measure on $\mathbb{R}$.

If $\omega \in \mathbb{R}^2^+$ the upper half plane, and $\omega = a + ib$, $a, b \in \mathbb{R}$, $b > 0$ then

\[
\Im \frac{1}{x - \omega} = \frac{b}{(x - a)^2 + b^2} = \pi \phi_\omega(x)
\]

where $\phi_\omega$ is the well known Cauchy density.

These are the densities of the Poisson or harmonic measures on $\mathbb{R}^2^+$.

If $\phi : \mathbb{R}^2^+ \to \mathbb{C}$ is bounded, analytic on $\mathbb{R}^2^+$ and then for a.e. $t \in \mathbb{R}$, $\exists \lim_{y \to 0^+} \phi(t + iy) =: \phi^*(t)$ and

\begin{equation}
\phi(\omega) = \int_{\mathbb{R}} \phi^*(t) dP_\omega(t) \quad (\omega \in \mathbb{R}^2^+)
\end{equation}

where $dP_\omega(t) = \varphi_\omega(t) dt$.

2.2 Boole’s Formula Let $T$ be an inner function, then $(\mathbb{R}, \mathcal{B}, m, T)$ is non-singular and

\begin{equation}
\mathcal{T}\varphi_\omega = \varphi_{T(\omega)} \forall \omega \in \mathbb{R}^2^+.
\end{equation}

Proof (G.Letac) It suffices to show that $P_\omega \circ T^{-1} = P_{T(\omega)}$.

The Fourier transform of $P_\omega$ is given by

\[
\mathcal{F}_{\omega}(t) := \int_{\mathbb{R}} e^{itx} dP_\omega(x) = e^{it\omega} \quad (t \geq 0).
\]

For $t > 0$, $\phi_t(\omega) = e^{it\omega}$ is a bounded analytic functions on $\mathbb{R}^2^+$ with $\phi_t^*(x) = e^{itx}$ on $\mathbb{R}$. By (10),

\[
P_\omega \circ T^{-1}(t) = \int_{\mathbb{R}} e^{itT(x)} dP_\omega(x) = e^{itT(\omega)} = \mathcal{F}_{T(\omega)}(t),
\]

whence (11). $\Box$

Remark.

As a consequence of (11), we see that the inner function $T$ has an absolutely continuous invariant probability (acip) if $\exists \omega \in \mathbb{R}^2^+$ with $T(\omega) = \omega$ (in which case $P_\omega$ is $T$-invariant). We’ll see later that this is the only way $T$ can have an acip.

2.3 Corollary If $T$ is an inner function with $\alpha > 0$ in (10), then $m \circ T^{-1} = \frac{1}{\alpha} \cdot m$.
Exercise 5: Boole & Glaisher transformations.

For \( \alpha, \beta > 0 \) define \( T = T_{\alpha, \beta} : \mathbb{R} \to \mathbb{R} \) by \( T(x) := \alpha x - \frac{\beta}{x} \).

(a) Show that if \( \alpha + \beta = 1 \), then \( \pi b \varphi_i = \varphi_i \) and \( T \) has an absolutely continuous, invariant probability (a.c.i.p.).

Consider the Glaisher transformations \( T : \mathbb{R} \to \mathbb{R} \) of form

\[
T_a,b x := ax + b \tan x \quad (a, b \geq 0, \ a + b > 0).
\]

(b) Give conditions on \( a, b \) so that \( T_{a,b} \) has an absolutely continuous invariant probability.

(c) Show that \( T_{1,b} \) preserves Lebesgue measure.

(d) Show that \( T_{0,1} \) preserves the measure \( d \mu_0(x) := \frac{dx}{x^2} \).

Hint: \( S := \pi \circ T_{0,1} \circ \pi^{-1} \) preserves Lebesgue measure where \( \pi(x) := \frac{1}{x} \).

**Recurrence and conservativity**

A set \( W \in \mathcal{B} \), \( m(W) > 0 \) is called wandering (for the NPT \( (X, \mathcal{B}, m, T) \)) if the sets \( \{T^{-n}W\}_{n=0}^{\infty} \) are disjoint. and the NPT \( T \) is called conservative if \( \mathcal{W}(T) = \emptyset \) (i.e. there are no wandering sets).

**Remarks.**

1. A conservative NPT \( (X, \mathcal{B}, m, T) \) is non-singular. Else \( \exists A \in \mathcal{B}, \ m(A) > 0 \) with \( m(T^{-1}A) = 0 \), whence \( m(T^{-n}A) = 0 \ \forall \ n \geq 1 \). It follows that \( W := A \setminus \bigcup_{n=1}^{\infty} T^{-n}A \) is a wandering set satisfying \( m(W) = m(A) \).

2. Similarly, a NPT \( (X, \mathcal{B}, m, T) \) is conservative iff (!) it is incompressible in the sense that \( A \in \mathcal{B} \) and \( T^{-1}A \subset A \) imply \( A = T^{-1}A \mod m \).

3. If \( (X, \mathcal{B}, m, T) \) is a Kakutani skyscraper over the NST \( (\Omega, \mathcal{F}, \mu, S) \), then \( T \) is conservative iff \( S \) is conservative.

**Proof of \( \Leftarrow \)** If \( T \) is not conservative, then \( \exists A \in \mathcal{F}, \ A \times \{1\} \in \mathcal{W}(T) \) whence \( A \in \mathcal{W}(S) \).

**Proof of \( \Rightarrow \)** Let \( W \in \mathcal{W}(S) \), then (!) \( W \times \{1\} \in \mathcal{W}(T) \).

**Halmos recurrence theorem**

Let \( (X, \mathcal{B}, m, T) \) be a NPT. TFAE:

(i) \( T \) is conservative;
(ii) $A^m \subset \bigcup_{n=1}^\infty T^{-n}A \quad \forall \ A \in \mathcal{B}_+$;

(iii) $\sum_{n=1}^\infty 1_A \circ T^n = \infty \text{ a.e. on } A \quad \forall \ A \in \mathcal{B}_+$.

Proof of (i) $\Rightarrow$ (iii)

Suppose that $A \in \mathcal{B}$, $m(A) > 0$. The set $W := A \setminus \bigcup_{n=1}^\infty T^{-n}A$ is wandering if of positive measure, whence $m(W) = 0$ and $A \subset \bigcup_{n=1}^\infty T^{-n}A \mod m$. By null preservation, $T^{-N}A \subset \bigcup_{n=N+1}^\infty T^{-n}A$ mod $m \forall N \geq 1$, whence, mod $m$:

$$A \in \bigcup_{n=1}^\infty T^{-n}A \subset \cdots \subset \bigcap_{n=N+1}^{\infty} T^{-n}A \subset \cdots \subset \bigcap_{j=n}^{\infty} \bigcup_{n=j}^{\infty} T^{-n}A = \left[ \sum_{n=1}^\infty 1_A \circ T^n = \infty \right].$$

$\varnothing$

Conditions for conservativity.

2.4 Maharam Recurrence theorem

Let $(X, \mathcal{B}, m, T)$ be MPT.

If $\exists \ A \in \mathcal{B}$, $m(A) < \infty$ such that $X = \bigcup_{n=1}^\infty T^{-n}A \mod m$, then $T$ is conservative.

Proof We have that $\sum_{n=1}^\infty 1_A \circ T^n = \infty \text{ a.e.}$ If $W \in \mathcal{W}$, $m(W) > 0$, then $\forall \ n \geq 1$,

$$m(A) \geq \int_{T^{-n}A} \left( \sum_{k=1}^n 1_W \circ T^k \right) dm = \sum_{k=1}^n m(T^{-k}W \cap T^{-n}A)$$

$$= \sum_{j=0}^{n-1} m(W \cap T^{-j}A) = \int_W \left( \sum_{j=0}^{n-1} 1_A \circ T^j \right) dm \to \infty.$$

Contradiction. $\varnothing$

For example, any PPT is conservative. This statement is known as Poincaré’s recurrence theorem.

A MPT of a $\sigma$-finite, infinite measure space need not be conservative. For example $x \mapsto x + 1$ is a measure preserving transformation of $\mathbb{R}$ equipped with Borel sets, and Lebesgue measure, which is totally dissipative.

Example.

The original Boole transformation $T : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$T(x) = x - \frac{1}{x}$$

is conservative.
Proof. By corollary 2.3, \( m \circ T^{-1} = m \). By inspection, \( \bigcup_{n=0}^{\infty} T^{-n}[-1,1] = \mathbb{R} \). ∎

Exercise 6. Let
\[
T(x) = x + \sum_{k=1}^{N} \frac{p_k}{t_k - x}
\]
where \( p_1, \ldots, p_N > 0 \) & \( t_1, \ldots, t_N \in \mathbb{R} \).

Show that \( \bigcup_{n=1}^{\infty} T^{-n}(u,v) = \mathbb{R} \mod m \) where \( u := \min T^{-1}(0) \) & \( v := \max T^{-1}(0) \); and hence that \( T \) is conservative.

Hint. WLOG, \( N \geq 2, u < 0 < v \) & \( T(0) = 0 \).

Exercise 7: Skyscraper conservativity.

Let \((X, \mathcal{B}, m, T)\) be a Kakutani skyscraper over the NST \( \Omega, F, \mu, S \).

Show that \( T \) is conservative iff \( S \) is conservative.

Exercise 8: Stronger recurrence properties.

Let \((X, \mathcal{B}, m, T)\) be a conservative NST.

(i) Show that if \((Y, d)\) is a separable, metric space and \( h : X \to Y \) is measurable, then
\[
\lim_{n \to \infty} d(h, h \circ T^n) = 0 \text{ a.e.}
\]

(ii) What about when \((Y, d)\) is an arbitrary metric space (not necessarily separable) and \( h : X \to Y \) is measurable?

Induced transformation.

This is the “reverse” of the skyscraper construction.

Suppose \((X, \mathcal{B}, m, T)\) is a NST and let \( A \in \mathcal{B} \) be such that \( m \)-a.e. point of \( A \) returns to \( A \) under iterations of \( T \) (e.g. if \((X, \mathcal{B}, m, T)\) is conservative). The return time function to \( A \), defined for \( x \in A \) by
\[
\varphi_A(x) := \min\{n \geq 1 : T^n x \in A\}
\]
is finite \( m \)-a.e. on \( A \).

The induced transformation on \( A \) is defined by \( T_A x = T^{\varphi_A(x)} x \).

The first key observation is that \((A, \mathcal{B} \cap A, T_A, m_A)\) is a NST and, if \( T \) is a MPT, then so is \( T_A \). These follows from
\[
T_A^{-1} B = \bigcup_{n=1}^{\infty} [\varphi = n] \cap T^{-n} B.
\]

It follows that \( \varphi_A \circ T_A \) is defined a.e. on \( A \) and an induction now shows that all powers \( \{T_A^k\}_{k \in \mathbb{N}} \) are defined a.e. on \( A \), and satisfy
\[
T_A^k x = T^{(\varphi_A)_k(x)} x \text{ where } (\varphi_A)_1 = \varphi_A, \ (\varphi_A)_k = \sum_{j=0}^{k-1} \varphi_A \circ T_A^j.
\]
Exercise 9: Inducing inverse to skyscraping.

Let \((X, \mathcal{B}, m, T)\) be an invertible, conservative NST and suppose that \(A \in \mathcal{B}, \ m(A) > 0\) satisfies \(\bigcup_{n=1}^{\infty} T^{-n}A = X \mod m\).

Show that

(i) \((X, \mathcal{B}, m, T)\) is isomorphic to the the Kakutani skyscraper over \((A, \mathcal{B} \cap A, m_A, T_A)\) with height function \(\varphi_A\).

(ii) \(T\) is conservative \(\implies T_A\) is conservative.

Both constructions can be generalized to the nonsingular case.

**HOPF DECOMPOSITION**

Let \((X, \mathcal{B}, m, T)\) be a NPT. The collection \(\mathcal{W}(T)\) of wandering sets is a hereditary collection (any measurable subset of a member is also a member), and \(T\)-sub-invariant (\(W\) wandering or null \(\implies T^{-1}W\) wandering or null).

By exhaustion, \(\exists\) a countable union of wandering sets \(\mathcal{D}(T) \in \mathcal{B}\) with the property that any wandering set \(W \in \mathcal{B}\) is contained in \(\mathcal{D}(T) \mod m\) (i.e. \(m(W \setminus \mathcal{D}(T)) = 0\)). This measurable union \(\mathcal{D}(T)\) of \(\mathcal{W}(T)\) is unique mod \(m\) and \(T^{-1}\mathcal{D} \subseteq \mathcal{D} \mod m\). It is called the dissipative part of the nonsingular transformation \(T\).

Evidently \(T\) is conservative on \(\mathcal{C}(T) := X \setminus \mathcal{D}(T)\), the conservative part of \(T\).

The partition \(\{\mathcal{C}(T), \mathcal{D}(T)\}\) is called the Hopf decomposition of \(T\).

The nonsingular transformation \(T\) is called (totally) dissipative if \(\mathcal{D}(T) = X \mod m\).

2.7 Proposition. Any inner function \(T\) with \(\alpha > 1\) in \((\mathcal{B})\) is dissipative.

Proof By corollary 2.3,

\[
\sum_{n=1}^{\infty} m(T^{-n}A) < \infty \quad \forall \ A \in \mathcal{B}, \ 0 < m(A) < \infty
\]

and is dissipative. \(\forall\)

Exercise 10:

In this exercise, you show that if \((X, \mathcal{B}, m, T)\) is an invertible NST, then \(\exists\) a wandering set \(W \in \mathcal{B}\) such that

\[
\mathcal{D} = \bigcup_{n \in \mathbb{Z}} T^nW.
\]
HINTS
For $A \in \mathcal{B}$ set $A^T := \bigcup_{n \in \mathbb{Z}} T^n A$.

WLOG, $m(X) = 1$.

- Define $\epsilon_1 := \sup \{m(W) : W \in \mathcal{W}\}$;
- choose $W \in \mathcal{W}$ with $m(W_1) \geq \frac{\epsilon_1}{2}$;
- define $\epsilon_2 := \sup \{m(W) : W \in \mathcal{W}, W \cap W_1^T = \emptyset\}$;
- choose $W_2 \in \mathcal{W}$, $W \cap W_1^T = \emptyset$ with $m(W_2) \geq \frac{\epsilon_2}{2}$.

Continue this process to obtain $\{W_n : n \in \mathbb{N}\} \subset \mathcal{W}$ and $\{\epsilon_n : n \in \mathbb{N}\} \subset [0, \infty)$ so that

- $W_k \cap W_1^T = \emptyset \quad \forall \ k > \ell$;
- $2m(W_n) \geq \epsilon_n := \sup \{m(W) : W \in \mathcal{W}, W \cap W_1^k = \emptyset \quad \forall \ 1 \leq k \leq n-1\}$.

Show that $W := \bigcup_{n \geq 1} W_n$ is as required.

**Exercise 11:** Hopf decomposition not $T$-invariant.

Let $(X, \mathcal{B}, m, T) = ([0, 2], \mathcal{B}([0, 2]), \text{Leb})$ where $T : [0, 2) \to [0, 2)$ is defined by

$$T(x) := \begin{cases} 
2x & x \in [0, 1), \\
1 + (2(x - 1) \mod 1) & x \in [1, 2).
\end{cases}$$

Show that $T$ is non-singular, $\mathcal{D}(T) = [0, 1), \mathcal{C}(T) = [1, 2)$ and that

$$T^{-1}\mathcal{D}(T) = [0, \frac{1}{2}) \quad \text{and} \quad m(T^{-1}\mathcal{D}(T) \Delta \mathcal{D}(T)) = \frac{1}{2}.$$

**Conservativity and Transfer Operators**

### 2.10 Hopf’s recurrence theorem

If $T : X \to X$ is nonsingular then

(i) $\mathcal{C}(T) \supset [\sum_{n=1}^{\infty} \tilde{T}^k f = \infty] \quad \text{mod} \ m \ \forall f \in L^1(m)_{+}; \quad \&$

(ii) $\mathcal{C}(T) = [\sum_{n=1}^{\infty} \tilde{T}^k f = \infty] \quad \text{mod} \ m \ \forall f \in L^1(m), f > 0.$

**Proof** (i) Fix $f \in L^1(m)_{+}$ and $W \in \mathcal{W}_T$, then

$$\infty > \int_X f dm \geq \int_X f \left( \sum_{n=0}^{\infty} 1_W \circ T^n \right) dm = \int_W \left( \sum_{n=0}^{\infty} \tilde{T}^n f \right) dm.$$

This shows that $\mathcal{D}(T) \subset [\sum_{n=1}^{\infty} \tilde{T}^k f < \infty]$.

(ii) Assume otherwise and fix $f \in L^1(m), f > 0$, $A \in \mathcal{B}_{+}$, $A \in \mathcal{C}(T)$ s.t. $\sum_{n=1}^{\infty} \tilde{T}^k f < \infty$ on $A$.

WLOG $f(x) \geq c > 0 \ \forall \ x \in A$, and the series converges uniformly on $A$ whence $\int_A (\sum_{n=1}^{\infty} \tilde{T}^k f) dm < \infty.$
On the other hand, by Halmos’ recurrence theorem \( \sum_{n \geq 0} 1_A \circ T^n = \infty \) a.e. on \( A \).

Thus
\[
\infty > \int_A \left( \sum_{n=0}^{\infty} \hat{T}^n f \right) dm = \int_X f \left( \sum_{n \geq 0} 1_A \circ T^n \right) dm \\
\geq \int_A f \left( \sum_{n \geq 0} 1_A \circ T^n \right) dm \geq c \int_A \left( \sum_{n \geq 0} 1_A \circ T^n \right) dm = \infty \quad \blacksquare \quad \blacksquare
\]

2.11 Corollary.
If \( T x = x + \beta + \int_{\mathbb{R}} \frac{d\nu(t)}{1-x} \) where \( \nu \) is a finite, Lebesgue-singular, measure on \( \mathbb{R} \) with compact support, then \( T \) is conservative if \( \beta = 0 \) and dissipative if \( \beta \neq 0 \).

Proof By Hopf’s recurrence theorem, it suffices to show that \( \sum_{n \geq 0} \hat{T}^n \varphi_\omega \) diverges a.e. for some \( \omega \in \mathbb{R}^2 \) when \( \beta = 0 \); and converges a.e. for some \( \omega \in \mathbb{R}^2 \) when \( \beta \neq 0 \).

By Boole’s formula
\[
\hat{T}^n \varphi_\omega(x) = \varphi_{T^n \omega}(x) = \frac{1}{\pi} \cdot \frac{v_n}{(x-u_n)^2 + v_n^2}
\]
where \( T^n \omega = u_n + iv_n \).

Elementary estimations show that
- when \( \beta \neq 0 \), \( \exists B = B(\omega) \in \mathbb{R}_+ \& C = C(\omega) \in \mathbb{R} \) so that
\[
(I) \quad v_n \uparrow B \& u_n = \beta n - \frac{\nu}{\beta} \log n + C + O\left(\frac{\log n}{n}\right) \quad \text{as} \quad n \to \infty;
\]
and
- when \( \beta = 0 \),

\[
(II) \quad \sup_{n \geq 1} |u_n| < \infty \quad \& \quad v_n \sim \sqrt{2\nu n} \quad \text{as} \quad n \to \infty \quad \text{where} \quad \nu := \sum_{k=1}^{n} p_k
\]

It follows that \( T \) is
- conservative when \( \beta = 0 \) \( \therefore \hat{T}^n \varphi_\omega \ll \frac{1}{\sqrt{n}} \) uniformly on bounded subsets of \( \mathbb{R} \);
- and totally dissipative when \( \beta \neq 0 \) \( \therefore \hat{T}^n \varphi_\omega \ll \frac{1}{n^{3/2}} \) on \( \mathbb{R} \). \( \blacksquare \)

Exercise 11: Hopf recurrence theorem for MPTs.
Suppose that \( T \) is a MPT of the \( \sigma \)-finite measure space \( (X, \mathcal{B}, m) \). Show that
\[
\left[ \sum_{n=1}^{\infty} f \circ T^n = \infty \right] = \mathcal{C}(T) \mod m \quad \forall f \in L^1(m), f > 0.
\]
Ergodicity

A transformation $T$ of the measure space $(X, \mathcal{B}, m)$ is called **ergodic** if

$$A \in \mathcal{B}, \quad T^{-1}A = A \mod m \Rightarrow m(A) = 0, \text{ or } m(A^c) = 0.$$ 

In general, let

$$\mathcal{I}(T) := \{A \in \mathcal{B}, \ T^{-1}A = A\}.$$ 

Remarks.

It is not hard to see that:

- $\mathcal{I}(T)$ is a $\sigma$-algebra (and that $T$ is ergodic iff $\mathcal{I} = \{\emptyset, X\}$);
- an invertible ergodic nonsingular transformation of a non-atomic measure space is necessarily conservative;
- a nonsingular transformation $(X, \mathcal{B}, m, T)$ is conservative and ergodic iff

$$\sum_{n=1}^{\infty} 1_A \circ T^n = \infty \ a.e. \ \forall A \in \mathcal{B}_+.$$ 

Exercise 13.

(i) Suppose that $(X, \mathcal{B}, m, T)$ is a Kakutani skyscraper over the ergodic NST $(\Omega, \mathcal{F}, \mu, S)$, then $T$ is ergodic.

(ii) Suppose that $(X, \mathcal{B}, m, T)$ is a conservative, NST and that $A \in \mathcal{B}$, $\bigcup_{n=1}^{\infty} T^{-n}A = X$, then $T$ is ergodic $\iff T_A$ is ergodic.

Exercise 14.

Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic nonsingular transformation and let $(Z, d)$, a separable metric space. Show that if $f : X \to Z$ is a measurable map, then for a.e. $x \in X$,

$$\{f(T^n x) : n \in \mathbb{N}\} = \text{spt } m \circ f^{-1}.$$ 

Some Ergodic Transformations

Rotations of the circle. Let $X$ be the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$, $\mathcal{B}$ be its Borel sets, and $m$ be Lebesgue measure. The rotation (or translation) of the circle by $x \in X$ is the transformation $r_x : X \to X$ defined by $r_x(y) = x + y \mod 1$.

Evidently $m \circ r_x = m$ for every $x \in X$ and each $r_x$ is an invertible measure preserving transformation of $(X, \mathcal{B}, m)$.

3.2 Proposition

*If $\alpha$ is irrational, then $r_\alpha$ is ergodic.*
Proof
We use harmonic analysis. Suppose that \( f : X \to \mathbb{R} \) is bounded and measurable, and that \( f \circ r_\alpha = f \), then
\[
\hat{f}(n) = \int_{[0,1)} f(y) e^{-2\pi iny} dy = \int_{[0,1)} f(\alpha + y) e^{-2\pi iny} dy = \lambda^n \hat{f}(n) \quad \text{where} \quad \lambda := e^{2\pi i \alpha}.
\]
It follows that \( \lambda^n = 1 \) whenever \( \hat{f}(n) \neq 0 \), whence, since \( \lambda^n \neq 1 \) \( \forall \ n \neq 0 \), \( \hat{f}(n) = 0 \) whenever \( n \neq 0 \) and \( f \) is constant.

Ergodicity of rank one constructions.

3.3 Proposition
Let \((X, \mathcal{B}, m, T)\) be a rank one MPT as above, then \( T \) is ergodic.

Proof
Let
\[
R_n = \bigcup_{I \in \mathcal{R}_n} I \uparrow X
\]
be the refining sequence of Rokhlin towers defining \( T \); where each
\[
\mathcal{R}_n = \{T^j I_n : 0 \leq j \leq k_n - 1\}
\]
is a partition of \( R_n \) into intervals with equal lengths \( m(I_n) \xrightarrow{n \to \infty} 0 \).

We claim first that it suffices to show that

For \( \epsilon > 0 \) \& \( A \in \mathcal{B}_+ \), \( \exists \ N = N_{\epsilon, A} \) so that
\[
\forall \ n > N \ \exists \ I \in \mathcal{R}_n \text{ s.t. } m(A|I) > 1 - \epsilon.
\]

Proof of sufficiency of \( \clubsuit \)
Suppose that \( A \in \mathcal{B}_+ \), \( TA = A \). We’ll show assuming \( \clubsuit \) that \( \forall \ N \geq 1 \) large enough,
\[
m(A \cap R_N) > (1 - \epsilon)m(R_N) \quad \forall \ \epsilon > 0
\]
whence \( A \supset R_N \uparrow X \mod m \).

To see this, choose (by \( \clubsuit \)) \( n \geq N \) \& \( J \in \mathcal{R}_n \) satisfying \( m(A|J) > 1 - \epsilon \). Then for each \( K = T^{i\kappa} J \in \mathcal{R}_n \), we have using \( T \)-invariance of \( m \) \& \( A \):
\[
m(A|K) = \frac{m(A \cap T^{i\kappa} J)}{m(T^{i\kappa} J)} = m(A|J) > 1 - \epsilon
\]
whence
\[
m(A \cap R_N) = \sum_{K \in \mathcal{R}_n, K \subseteq R_N} m(A|K) m(K) > (1 - \epsilon)m(R_N). \quad \square
\]
**Proof of**

Suppose that \( A \in \mathcal{B}_+ \) and fix \( N \geq 1 \) so that \( B := A \cap R_N \in \mathcal{B}_+ \). For \( n \geq N \), let

\[
\mathfrak{s}_n := \{ I \in \mathfrak{r}_n : I \subset R_N \}.
\]

Fix \( 0 < \epsilon < 1 \) and for \( n \geq N \) let

\[
\mathcal{Z}_n := \{ I \in \mathfrak{s}_n : m(B \mid I) > 1 - \epsilon \} \quad \& \quad \mathcal{Y}_n := \mathfrak{s}_n \setminus \mathcal{Z}_n.
\]

We show that \( \forall n \) large enough, \( Z_n \neq \emptyset \).

Since \( \sigma(\cup_{n\geq N} \mathfrak{s}_n) = \mathcal{B}(R_N) \), \( \exists n \geq N \) & \( C_n \), a union of sets in \( \mathfrak{s}_n \) so that \( m(B \Delta C_n) < \frac{\epsilon^2 m(B)}{9} \). It follows that

\[
m(C_n) - \frac{\epsilon^2 m(B)}{9} < m(B \cap C_n)
\]

whence

\[
m(\bigcup \mathcal{Z}_n) \geq m(C_n) - \frac{\epsilon^2 m(B)}{9} - (1 - \epsilon)m(C_n)
\]

\[
= \epsilon m(C_n) - \frac{\epsilon^2 m(B)}{9}
\]

\[
> \epsilon m(B) - \frac{\epsilon^3 m(B)}{9} - \frac{\epsilon^2 m(B)}{9}
\]

\[
> \frac{7\epsilon m(B)}{9} > 0. \quad \square
\]

**Ergodicity via stronger properties**

Sometimes it’s easier to prove more than ergodicity.

**One-sided Bernoulli shifts.**

Let \( X = \mathbb{R}^N \) and let \( \mathcal{B}(X) \) be the \( \sigma \)-algebra generated by cylinder sets of form \( [A_1, \ldots, A_n] := \{ x \in X : x_j \in A_j, \ 1 \leq j \leq n \} \), where \( A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}) \) (the Borel subsets of \( \mathbb{R} \)), and let the shift \( S : X \to X \) be defined by

\[
(Sx)_n = x_{n+1}.
\]
For $p : \mathcal{B}(\mathbb{R}) \to [0, 1]$ a probability, let $\mu_p : \mathcal{B}(X) \to [0, 1]$ be the probability$^2$ satisfying

$$\mu_p([A_1, \ldots, A_n]) = \prod_{k=1}^{n} p(A_k) \quad (A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})).$$

Evidently $S^{-1}[A_1, \ldots, A_n] = [\mathbb{R}, A_1, \ldots, A_n]$ whence $\mu_p \circ S^{-1} = \mu_p$.

The one-sided Bernoulli shift with marginal distribution $p$ is the probability preserving transformation $S$ of $(X, \mathcal{B}, \mu_p)$.

**Tail, exactness.** Let $T$ be a nonsingular transformation of $(X, \mathcal{B}, m)$. The *tail* $\sigma$-algebra of $T$ is

$$\mathcal{T}(T) := \bigcap_{n=1}^{\infty} T^{-n}\mathcal{B}.$$ 

The transformation $T$ is called *exact* if $\mathcal{T}(T) = \{\emptyset, X\} \bmod m$.

Evidently $\mathcal{I}(T) \subset \mathcal{T}(T) \bmod m$ and so exact transformations are ergodic.

**3.4 Kolmogorov’s zero-one law**

*Any one-sided Bernoulli shift is exact.*

**Proof**

Suppose that $B \in \mathcal{B}$ is a finite union of cylinders. If the length of the longest cylinder in the union is $n$, then

$$\mu_p(B \cap S^{-n}C) = \mu_p(B)\mu_p(C) \quad \forall \ C \in \mathcal{B}.$$ 

Now suppose $A \in \mathcal{T}$. Since, for each $n \in \mathbb{N}$,

$$A = S^{-n}A_n \text{ where } A_n \in \mathcal{B}, \mu_p(A_n) = \mu_p(A),$$

we have that

$$\mu_p(B \cap A) = \mu_p(B)\mu_p(A)$$

for $B \in \mathcal{B}$ a finite union of cylinders, and hence (by approximation) $\forall \ B \in \mathcal{B}$. This implies that

$$0 = \mu_p(A \cap A^c) = \mu_p(A)(1 - \mu_p(A))$$

demonstrating that $\mathcal{T}$ is trivial mod $\mu_p$. \[QED\]

Note that no invertible nonsingular transformation can be exact (except the identity on a 1-pt. space). Hence an irrational rotation of $\mathbb{T}$ is ergodic, but not exact.

---

$^2$Existence guaranteed by Kolmogorov’s existence theorem as on p.5.
Two sided Bernoulli shift.
Recall that the two-sided Bernoulli shift is defined with $X = \mathbb{R}^\mathbb{Z}$, $\mathcal{B}(X)$ the $\sigma$-algebra generated by cylinder sets of form

$$[A_1, \ldots, A_n]_k := \{ z \in \mathbb{R}^\mathbb{Z} : x_{j+k} \in A_j, \ 1 \leq j \leq n \}$$

where $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$. The shift $S : X \to X$ is defined as before by $(Sx)_n = x_{n+1}$, and the $S$-invariant probability $\mu_p : \mathcal{B}(X) \to [0, 1]$ is defined (for $p : \mathcal{B}(\mathbb{R}) \to [0, 1]$ a probability) by

$$\mu_p([A_1, \ldots, A_n]_k) = \prod_{k=1}^n p(A_k) \ (A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})).$$

The two sided Bernoulli shift is an invertible measure preserving transformation (and hence cannot be exact).

3.5 Proposition.
A two sided Bernoulli shift is mixing in the sense that

$$\mu_p(A \cap T^{-n}B) \to \mu_p(A) \mu_p(B) \text{ as } n \to \infty \ \forall \ A, B \in \mathcal{B}(X),$$

and hence ergodic.

Proof True in the combinatorial sense for $A, B$ finite unions of cylinders, and hence (by approximation) $\forall \ A, B \in \mathcal{B}$. $\Box$

Exercise 15.
Show that an exact probability preserving transformation $(X, T, \mu)$ is mixing.

Hint Show first that if $f \in L^2$, $n_k \to \infty$ and $f \circ T^{n_k} \to g \in L^2$ weakly in $L^2$, then $g$ is tail measurable.

Nonsingular Adding Machine.
Let $\Omega = \{0, 1\}^{\mathbb{N}}$, and $\mathcal{B}$ be the $\sigma$-algebra generated by cylinders. We consider again the adding machine $\tau : \Omega \to \Omega$ defined by

$$\tau(1, \ldots, 1, 0, \epsilon_{n+1}, \epsilon_{n+2}, \ldots) = (0, \ldots, 0, 1, \epsilon_{n+1}, \epsilon_{n+2}, \ldots).$$

The adding machine has

the odometer property.

$$\{(\tau^k x)_1, \ldots, (\tau^k x)_n) : 0 \leq k \leq 2^n - 1 \} = \{0, 1\}^n \ \forall \ x \in \Omega, \ n \geq 1.$$

The next lemma illustrates how the odometer “parametrizes” the tail of the one-sided shift $S : \Omega \to \Omega$. 

3.6 Lemma

For \( x \in \bar{\mathbb{Z}} := \{ \tau^n(0) : n \in \mathbb{Z} \} \),
\[
\{ y \in \Omega : \exists \ n \geq 0, \ S^n(y) = S^n(x) \} = \{ \tau^n(x) : n \in \mathbb{Z} \}.
\]

**Proof** Note that \( \bar{\mathbb{Z}} = \{ x \in \Omega : \exists \ \lim_{n \to \infty} x_n \} \). Thus for \( x \notin \bar{\mathbb{Z}} \), both \( \ell(x) := \min\{n \geq 1 : x_n = 0\} \) and \( (x) := \min\{n \geq 1 : x_n = q\} \) are finite, whence
\[
\exists \ n \geq 1 \text{ s.t. } S^n(x) = S^n(x) = S^n(x).
\]

Since \( \tau \bar{\mathbb{Z}} = \bar{\mathbb{Z}} \),
\[
\{ y \in \Omega : \exists \ n \geq 0, \ S^n(y) = S^n(x) \} \supset \{ \tau^n(x) : n \in \mathbb{Z} \}.
\]

For the other inclusion, suppose \( S^n(x) = S^n(y) = z \), then using the odometer property,
\[
(0, \ldots, 0, z) = \tau^{-\nu_n(x)}(x) = \tau^{-\nu_n(y)}(y)
\]
n times
where \( \nu_n(\omega) := \sum_{k=1}^n 2^{k-1} \omega_n \). Thus
\[
y = \tau^{\nu_n(y) - \nu_n(x)}(x).
\]

For \( p \in (0, 1) \), set \( \mu_p = \prod (1 - p, p) \in \mathcal{P}(\Omega) \) and recall that
\[
\frac{d\mu_p \circ \tau}{d\mu_p} = \left( \frac{1 - p}{p} \right)^{\phi}
\]
where \( \phi(x) := \min\{n \geq 1 : x_n = 0\} - 2 =: \ell(x) - 2 \).

3.7 Proposition

\( \tau \) is an invertible, conservative, ergodic nonsingular transformation of \( (\Omega, \mathcal{B}, \mu_p) \).

**Proof** It is not hard to show, using lemma 3.6, that \( I(\tau) = I(S) \mod \mu_p \) and the ergodicity of \( (\Omega, \mathcal{B}, \mu_p, \tau) \) follows from the exactness of \( (\Omega, \mathcal{B}, \mu_p, S) \). As above, conservativity is automatic in this case. \( \square \)

3.8 Rigidity proposition For \( 0 < p < 1 \), \( (\Omega, \mathcal{B}, \mu_p) \) is rigid in the sense that if \( f : \Omega \to \mathbb{R} \) is measurable, then \( \forall \ \epsilon > 0, \)
\[
\mu_p([|f \circ \tau^{2^n} - f| \geq \epsilon]) \to 0 \text{ as } n \to \infty.
\]

**Proof** Firstly, note that if \( f : \Omega \to \mathbb{R} \) and \( f \) is defined by \( f(x) = g(x_1, \ldots, x_n) \) for some \( n \in \mathbb{N} \), then \( f \circ \tau^{2^k} = f \) for every \( k \geq n \). To enable approximation, we show that \( \exists \ \Delta > 0 \text{ & } M > 1 \) so that
\[
\mu_p(\tau^{-2^n}A) \leq M \mu_p(A)^{\Delta} \quad \forall \ A \in \mathcal{B}.
\]
Proof of (ii)

As before, 

\[ \frac{d\mu_p\circ \tau^{-1}}{d\mu_p} = \left( \frac{p}{1-p} \right)^\psi \] where \( \psi(x) := \min\{n \in \mathbb{N} : x_n = 1\} - 2; \)

Using the odometer property:

\[ \sum_{j=0}^{2^n-1} \psi(\tau^{-k}x) = \sum_{\epsilon \in \{0,1\}^n \setminus \{1\}} \psi(\epsilon) + n + \psi(S^n x) = \sum_{k=1}^{n} (k-2)2^{n-k} + n + \psi(S^n x) = \psi(S^n x). \]

By (vi)

\[ \frac{d\mu_p \circ \tau^{-2^n}}{d\mu_p} = \prod_{k=0}^{2^n-1} \left( \frac{d\mu_p \circ \tau^{-1}}{d\mu_p} \right) \circ \tau^{-k} = \prod_{k=0}^{2^n-1} \left( \frac{p}{1-p} \right)^{\psi \circ \tau^{-k}} = \left( \frac{p}{1-p} \right)^{\psi \circ S^n}. \]

Fix (!) \( q > 1 \) be such that \( \frac{p^q}{(1-p)^{q-1}} < 1 \), then

\[ M_q := \left\| (\frac{p}{1-p})^\psi \right\|_{L^q(\mu_p)} ≈ \sum_{n \geq 1} \left( \frac{p^q}{(1-p)^{q-1}} \right)^n < \infty \]

and for \( A \in \mathcal{B} \),

\[ \mu_p(\tau^{-2^n} A) = \int_A \left( \frac{p}{1-p} \right)^{\psi \circ S^n} d\mu_p \leq \left\| \left( \frac{p}{1-p} \right)^\psi \right\|_q \mu_p(A) \frac{2q-1}{q} = M \mu_p(A) \frac{2q-1}{q} \]

by Hölder’s inequality. (ii)

Now, suppose that \( F : \Omega \to \mathbb{R} \) is measurable, and let \( \epsilon > 0 \) be given. There exist \( n \in \mathbb{N} \), and \( f : \Omega \to \mathbb{R} \) and \( f \) defined by \( f(x) = g(x_1, \ldots, x_n) \) for some \( g : \{0,1\}^n \to \mathbb{R} \) such that \( \mu_p(\{ |F - f| \geq \epsilon/2 \}) < \epsilon \). For \( k \geq n \), we have \( f \circ \tau^{2^k} \equiv f \), whence

\[ \mu_p(\{ |F \circ \tau^{2^k} - F| \geq \epsilon \}) \leq \mu_p(\{ |F \circ \tau^{2^k} - f \circ \tau^{2^k}| \geq \epsilon/2 \}) + \mu_p(\{ |F - f| \geq \epsilon/2 \}) \leq \epsilon + M \epsilon^{\frac{1}{q}}, \]

establishing that indeed

\[ F \circ \tau^{2^n} \xrightarrow{\mu_p} F. \]
Ergodic Maharam extension for the non-singular adding machine.

Define $\tau_\phi : \Omega \times \mathbb{Z} \to \Omega \times \mathbb{Z}$ by
$$
\tau_\phi(x, z) := (\tau x, z + \phi(x)).
$$

For $0 < p < 1$ define the measure $m_p : \mathcal{B}(\Omega \times \mathbb{Z}) \to [0, \infty]$ by
$$
m_p(A \times \{z\}) := \mu_p(A) \left( \frac{p}{1-p} \right)^z.
$$

This kind of transformation is aka a Maharam extension.

3.9 Theorem  For each $0 < p < 1$, $(\Omega \times \mathbb{Z}, \mathcal{B}(\Omega \times \mathbb{Z}), m_p, \tau_\phi)$ is a conservative, ergodic measure preserving transformation.

Proof that $m_p \circ \tau_\phi = m_p$

Any $A \in \mathcal{B}(\Omega \times \mathbb{Z})$ has a measurable decomposition $A = \bigcup_{z, \ell \in \mathbb{Z}} A_{z, \ell} \times \{z\}$ where $\phi = \ell$ on $A_{z, \ell}$. Thus:
$$
m_p(\tau_\phi A) = \sum_{z, \ell \in \mathbb{Z}} m_p(\tau_\phi(A_{z, \ell} \times \{z\})) = \sum_{z, \ell \in \mathbb{Z}} m_p(\tau A_{z, \ell} \times \{z + \ell\})
$$
$$
= \sum_{z, \ell \in \mathbb{Z}} \mu_p(\tau A_{z, \ell})(\frac{p}{1-p})^{z+\ell} = \sum_{z, \ell \in \mathbb{Z}} \mu_p(A_{z, \ell})(\frac{p}{1-p})^z
$$
$$
= \sum_{z, \ell \in \mathbb{Z}} m_p((A_{z, \ell} \times \{z\})) = m_p(A). \quad \square
$$

Proof of ergodicity of $\tau_\phi$  Suppose that $F : \Omega \times \mathbb{Z} \to \mathbb{R}$ is bounded, measurable and $\tau_\phi$-invariant. We’ll show first that $F(x, z) = F(x, z-1)$ $m_p$-a.e..

A similar calculation to (II) shows that

(II)  $\phi_{2^n}(x) = \phi(S^n x)$.

Iterating $\tau_\phi$, we have that

$$
F(x, z) = F \circ \tau_{2^n}(x, z) = F(\tau_{2^n}x, z + \phi_{2^n}(x)) = F(\tau_{2^n}x, z + \phi(S^n(x)) = F(x, z).
$$

By the rigidity proposition, $\exists n_k \to \infty$ and $\Omega_0 \in \mathcal{B}(\Omega)$, $\mu_p(\Omega_0) = 1$ such that

$$
F(\tau_{2^{n_k}}x, z) \xrightarrow{k \to \infty} F(x, z) \quad \forall x \in \Omega_0, \ z \in \mathbb{Z}.
$$

The events

$$
A_n = [\phi \circ S^n = -1] = \{x \in \Omega : x_{n+1} = 0\}
$$

are independent under $\mu_p$, and $\mu_p(A_n) = 1 - p$. 

By the Borel-Cantelli lemma, \( \exists \Omega_1 \in \mathcal{B}(\Omega), \ \Omega_1 \subset \Omega_0, \ \mu_p(\Omega_1) = 1 \) such that \( \forall \ x \in \Omega_1, \ \exists \ k_\ell = k_\ell(x) \to \infty \) with 
\[
\phi(S^{n k_\ell}x) = -1 \ \forall \ \ell \geq 1,
\]
whence
\[
F(x, z) = F(\tau^{2 n k_\ell}x, z + \phi(S^{n k_\ell}x)) = F(\tau^{2 n k_\ell}x, z - 1) \to F(x, z - 1).
\]
Thus \( \exists f : \Omega \to \mathbb{R}, \) measurable, such that \( F(x, z) = f(x) \mu_p\text{-a.e.} \ \forall \ z \in \mathbb{Z}. \) Since \( F \) is \( \tau_\phi\)-invariant, \( f \) is \( \tau\)-invariant and \( \mu_p\text{-a.e.} \) constant by ergodicity of \( (\Omega, \mathcal{B}, \mu_p, \tau). \)

3.10 Corollary

The nonsingular adding machine \( (\Omega, \mathcal{B}, \mu_p, \tau) \) has no \( \sigma\)-finite, absolutely continuous, invariant measure.

Proof Suppose otherwise, that \( m \ll \mu_p \) is a \( \sigma\)-finite, \( \tau\)-invariant measure and let \( dm = h d\mu_p \) where \( h \geq 0 \) is measurable, then(!) \( h > 0 \) \( \mu_p\text{-a.e.} \) (\( : \ \mu = \mu_p \)) and
\[
\tau' = (1 - p)^\phi \text{ we have that } \phi = k - k \circ \tau \text{ where } k : \Omega \to \mathbb{R} \text{ satisfies } h = (1 - p)^k.
\]
Define \( F : \Omega \times \mathbb{Z} \to \mathbb{R} \) by \( F(x, z) = z + k(x), \) then
\[
F(\tau_\phi(x, z)) = F(\tau x, z + \phi(x)) = z + \phi(x) + k(\tau x) = z + k(x) = F(x, z).
\]
By ergodicity, \( F \) is constant, but it isn’t (\( : \ \) \( F(x, z + 1) = F(x, z) + 1). \)

Exercise 16: Dissipative exact MPTs.

Let \( \Omega = \{0, 1\}^\mathbb{N} \) let \( S : \Omega \to \Omega \) be the shift, let \( \tau : \Omega \to \Omega \) be the adding machine and let \( \mu_p = \prod (1 - p, p) \in \mathcal{P}(\Omega), \ \ (0 < p < 1). \) Define \( f, \ \phi : \Omega \to \mathbb{Z} \) by
\[
f(x) := x_1 \ \& \ \phi(x) := \ell(x) - 2), \ \ell(x) := \min \{n \geq 1 : x_n = 0\}
\]
and \( S_f, \ \tau_\phi \) by
\[
S(x, z) = (\sigma(x), z + x_1), \ \tau(x, z) := (\tau(x), z + \ell(x) - 2).
\]
Show that
(i) \( (\Omega \times \mathbb{Z}, \mathcal{B}(\Omega \times \mathbb{Z}), \mu_p \times \#, S_f) \) is a totally dissipative MPT;
(ii) \( \mathfrak{F}(S_f) = \mathfrak{F}(\tau_\phi). \)
(iii) \( (\Omega \times \mathbb{Z}, \mathcal{B}(\Omega \times \mathbb{Z}), \mu_p \times \#, S_f) \) is exact.
Ratio ergodic theorem

Suppose that \((X, \mathcal{B}, m, T)\) is a conservative, nonsingular transformation.

4.6 Hurewicz’s Ergodic Theorem

\[
\sum_{k=1}^{n} \mathcal{T}^k f(x) \quad \text{for a.e. } x \in X, \quad \forall f, p \in L^1(m), \quad p > 0,
\]

where \(dm_p = pdm\), and \(\mathcal{I}\) is the \(\sigma\)-algebra of \(T\)-invariant sets in \(\mathcal{B}\).

Conditional expectations.

Here, given a probability space \((\Omega, \mathcal{F}, P)\), and a sub-\(\sigma\)-algebra \(\mathcal{C} \subset \mathcal{F}\),

the **conditional expectation** wrt \(\mathcal{C}\) is a linear operator \(f \mapsto E_P(f|\mathcal{C})\), \(L^1(\Omega, \mathcal{F}, P) \rightarrow L^1(\Omega, \mathcal{C}, P)\) satisfying

\[
\int_C E_P(f|\mathcal{C}) dP = \int_C f dP \quad \forall C \in \mathcal{C}.
\]

Such operators are unique by their defining equations,. They exist \(L^2(\Omega, \mathcal{F}, P) \rightarrow L^2(\Omega, \mathcal{C}, P)\) as orthogonal projections and extend to \(L^1\) by approximation.

Proof of Hurewicz’s theorem

Set, for \(f, p \in L^1(m), \quad p > 0\), \(S_0 f = 0\), and \(n \in \mathbb{N}\),

\[
S_n f := \sum_{k=0}^{n-1} \mathcal{T}^k f, \quad R_n(f, p) := \frac{S_n f}{S_n p}.
\]

Let

\[
\mathcal{H}_p := \{ f = hp + g - \mathcal{T}g \in L^1(m) : h \circ T = h \in L^\infty(m), g \in L^1(m) \}.
\]

We claim that for \(f = hp + g - \mathcal{T}g \in \mathcal{H}_p\),

\[
R_n(f, p) = h + \frac{g - \mathcal{T}^n g}{S_n p}.
\]

We show that \(R_n(hp, p) = h\) where \(h \circ T = h \in L^\infty(m)\). For \(g \in L^\infty(m), \quad n \in \mathbb{N}\),

\[
\int_X \mathcal{T}^n (hp) \cdot g dm = \int_X ph g \circ T^n dm = \int_X ph \circ T^n g \circ T^n dm = \int_X h \mathcal{T}^n p \cdot g dm
\]

for every whence \(\mathcal{T}^n f = h \mathcal{T}^n p\), and \(R_n(f, p) = h\). The convergence

\[
R_n(f, p) \longrightarrow h, \quad \text{a.e. } \forall f = hp + g - \mathcal{T}g \in \mathcal{H}_p
\]

follows immediately from the
4.7 Chacon-Ornstein Lemma

\[ \frac{\hat{T}^n g}{\hat{S}_n p} \to 0, \text{ a.e. } \forall g \in L^1(m). \]

**Proof**  Choose \( \epsilon > 0 \), and let \( \eta_n = 1_{[\hat{T}^n g \geq \epsilon \hat{S}_n p]} \). We must show that \( \sum_{n=1}^{\infty} \eta_n < \infty \) a.e. \( \forall \epsilon > 0 \).

We have

\[ \epsilon p + \hat{T}^{n+1} g - \epsilon \hat{S}_{n+1} p = \hat{T}(\hat{T}^n g - \epsilon \hat{S}_n p), \]

whence

\[ \epsilon p + \hat{T}^{n+1} g - \epsilon \hat{S}_{n+1} p \leq \hat{T}(\hat{T}^n g - \epsilon \hat{S}_n p)_+, \]

where \( g_+ \) denotes \( g \vee 0, f \vee g = \max\{f, g\} \).

Multiplying both sides of the inequality by \( \eta_{n+1} \):

\[ \eta_{n+1} \epsilon p + \eta_{n+1}(\hat{T}^{n+1} g - \epsilon \hat{S}_{n+1} p) = \eta_{n+1} \epsilon p + (\hat{T}^{n+1} g - \epsilon \hat{S}_{n+1} p)_+ \]

\[ \leq \eta_{n+1} \hat{T}(\hat{T}^n g - \epsilon \hat{S}_n p)_+ \]

\[ \leq \hat{T}(\hat{T}^n g - \epsilon \hat{S}_n p)_+. \]

Equivalently,

\[ \eta_{n+1} \epsilon p \leq \hat{T}J_n - J_{n+1} \]

where \( J_n := (\hat{T}^n g - \epsilon \hat{S}_n p)_+ \).

Integrating, we get

\[ \epsilon \int_X p \eta_{n+1} dm \leq \int_X (J_n - J_{n+1}) dm \]

and, summing over \( n \), we get

\[ \epsilon \int_X p \sum_{n=2}^{N} \eta_n dm \leq \int_X J_1 dm < \infty. \]

This shows that indeed

\[ \sum_{n=1}^{\infty} \eta_n < \infty \text{ a.e.} \]

and thereby proves the lemma. \( \Box \)

We next establish that

\[ \overline{H}_p = L^1(m). \]

To see this, we show that

\[ k \in L^\infty(m), \int_X kf \, dm = 0 \forall f \in \mathcal{H}_p \Rightarrow k = 0 \text{ a.e.} \]

To see this, let

\[ k \in L^\infty(m) \ni \int_X kf \, dm = 0 \forall f \in \mathcal{H}_p, \]
then, in particular
\[ \int_X g k \circ T \, dm = \int_X \widehat{T} g \cdot k \, dm = \int_X g k dm \quad \forall g \in L^1(m), \]
whence \( k \circ T = k \) a.e., and \( kp \in \mathcal{H}_p \).

Hence,
\[ \int_X k^2 p dm = 0 \Rightarrow k = 0 \text{ a.e.} \]
\( \square \)
now follows from the Hahn-Banach theorem. \( \checkmark \)

**Proof of Hurewicz’s theorem** ctd.

Identification of the limit.

We now identify the limit of \( R_n(f, p) \) \( f \in \mathcal{H}_p \). Define \( \Phi_p : L^1(m) \to L^1(m_p) \) by
\[ \Phi_p(f) := E_{m_p}(\frac{f}{p} \, \mathcal{H}), \]
then
\[ \| \Phi_p(f) \|_{L^1(m_p)} \leq \| f \|_1 \quad \forall f \in L^1(m). \]

We claim that
\[ (\star) \quad R_n(f, p) \xrightarrow{n \to \infty} \Phi_p(f) \quad \forall f \in \mathcal{H}_p. \]

For this, it suffices that
\[ \Phi_p(hp + g - \widehat{T} g) = h \quad \forall f = hp + g - \widehat{T} g \in \mathcal{H}_p. \]

Indeed, if \( k \circ T = k \in L^\infty(m) \), then
\[
\begin{align*}
\int_X k \frac{f}{p} dm_p &= \int_X kf dm \\
&= \int_X k(hp + g - \widehat{T} g) dm \\
&= \int_X kh dm + \int_X k(g - \widehat{T} g) dm \\
&= \int_X khdm_p.
\end{align*}
\]

We extend \((\star)\) to all \( f \in L^1(m) \), by an approximation argument which uses the

**5.1 Maximal inequality**

For \( f, p \in L^1, \) such that \( p > 0 \) a.e., and \( t \in \mathbb{R}_+ \),
\[ m_p([\sup_{n \in \mathbb{N}} R_n(f, p) > t]) \leq \frac{\| f \|_1}{t}, \]
where \( dm_p = pdm. \)

**Proof of theorem 4.6 given the maximal inequality**

Let \( f \in L^1(m) \). Fix \( \epsilon > 0. \)
By ⊗, we can write \( f = g + k \), where \( g \in \mathcal{H}_p \) and \( \| k \|_1 < \epsilon^2 \). It follows that
\[
\lim_{n \to \infty} |R_n(f, p) - \Phi_p(f)| \leq \sup_{n \in \mathbb{N}} |R_n(k, p)| + |\Phi_p(k)|,
\]
whence, by the maximal inequality, and by Tchebychev’s inequality,
\[
m_p\left(\lim_{n \to \infty} |R_n(f, p) - \Phi_p(f)| > 2\epsilon\right) \leq m_p\left(\sup_{n \geq 1} |R_n(k, p)| > \epsilon\right) + m_p\left(\| \Phi_p(k) \| > \epsilon\right)
\leq \frac{2\| k \|_1}{\epsilon} \leq 2\epsilon.
\]
This last inequality holds for arbitrary \( \epsilon > 0 \), whence
\[
\lim_{n \to \infty} |R_n(f, p) - \Phi_p(f)| = 0 \quad \text{a.e.},
\]
and the ergodic theorem is almost established, it remaining only to prove the maximal inequality.

5.2 Hopf’s Maximal ergodic theorem
\[
\int_{[M_n f > 0]} f \, dm \geq 0, \quad \forall f \in L^1(m), \ n \in \mathbb{N},
\]
where
\[
M_n f = \left( \bigvee_{k=1}^n \hat{S}_k f \right)_+ = \left( \bigvee_{k=0}^n \hat{S}_k f \right).
\]

Proof Note first that if \( M_n f (x) > 0 \), then
\[
M_n f (x) \leq M_{n+1} f (x) = \bigvee_{k=1}^{n+1} \hat{S}_k f (x)
= f (x) + \bigvee_{k=0}^n \hat{S}_k \tilde{T} f (x) = f (x) + M_n \tilde{T} f (x).
\]
Also (!) \( M_n \tilde{T} f \leq \tilde{T} M_n f \), whence
\[
M_n f > 0 \Rightarrow f \geq M_n f - \tilde{T} M_n f,
\]
and
\[
\int_{[M_n f > 0]} f \, dm \geq \int_{[M_n f > 0]} (M_n f - \tilde{T} M_n f) \, dm.
\]
Since \( \tilde{T} M_n f \geq 0 \) a.e., and \( M_n f = 0 \) on \( [M_n f > 0]^c \), we get
\[
\int_{[M_n f > 0]} f \, dm \geq \int_{[M_n f > 0]} M_n f \, dm - \int_{[M_n f > 0]} \tilde{T} M_n f \, dm
\geq \int_X M_n f \, dm - \int_X \tilde{T} M_n f \, dm
= 0,
\]
whence the theorem. \( \Box \)
Proof of the maximal inequality  Suppose $f, p, t$ are as in the maximal inequality, then

$$M_n(f - tp) > 0 \iff \max_{1 \leq k \leq n} R_k(f, p) > t.$$  

Thus, using Hopf’s maximal ergodic theorem, we obtain

$$\int_{[M_n(f - tp) > 0]} (f - tp) \, dm \geq 0,$$

whence

$$tm_p(\left\{ \max_{1 \leq k \leq n} R_k(f, p) > t \right\}) \leq \int_{[\max_{1 \leq k \leq n} R_k(f, p) > t]} f \, dm \leq \|f\|_1.$$

The maximal inequality follows from this as $n \to \infty$. \qedsymbol

Hurewicz’s ergodic theorem is now established.

Hurewicz’s theorem for a conservative, ergodic nonsingular transformation $T$, states that

$$\frac{\sum_{k=0}^{n-1} \hat{T}^k f(x)}{\sum_{k=0}^{n-1} \hat{T}^k g(x)} \to \frac{\int_X f \, dm}{\int_X g \, dm} \text{ for a.e. } x \in X$$

whenever $f, g \in L^1(m)$, $\int_X g \, dm \neq 0$.

Exercise 17: von Neuann’s ergodic theorem.

Let $\mathcal{H}$ be a Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator.

Show that

(i) $\mathcal{H}_0 := \{ f \in \mathcal{H} : Uf = f \}$ is a closed, invariant subspace of $\mathcal{H}$ and that

(ii) 

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k f - Pf \right\| \to 0 \quad \forall \, f \in \mathcal{H}$$

where $P : \mathcal{H} \to \mathcal{H}_0$ is orthogonal projection.

Exercise 18: Hopf’s ergodic theorem.

Suppose that $(X, \mathcal{B}, m, T)$ is a conservative measure preserving transformation.

(i) Prove that

$$\frac{\sum_{k=1}^{n} f(T^k x)}{\sum_{k=1}^{n} p(T^k x)} \to E_{m_p}(f|\mathcal{F})(x) \text{ for a.e. } x \in X, \forall f, p \in L^1(m), \ p > 0.$$
**Hint** Hopf’s ergodic theorem is a special case of Hurewicz’s theorem in case $T$ is invertible. It can be proved analogously for $T$ non-invertible.

(ii) Now suppose that $T$ is a conservative, ergodic, measure preserving transformation of the $\sigma$-finite, infinite measure space $(X, \mathcal{B}, m)$. Prove that

$$\frac{1}{n} \sum_{k=1}^{n} f(T^k x) \underset{n \to \infty}{\longrightarrow} 0 \text{ for a.e. } x \in X, \forall f \in L^1(m).$$
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ERGODICITY VIA THE RATIO ERGODIC THEOREM

Boole transformations.

Let \((X, \mathcal{B}, m)\) be \(\mathbb{R}\) equipped with Borel sets and Lebesgue measure, and consider Boole’s transformations:

\[
T_x = x + \beta + \sum_{k=1}^{N} \frac{p_k}{t_k - x}
\]

where \(N \geq 1, \ p_1, \ldots, p_N > 0\) and \(\beta, \ t_1, \ldots, t_N \in \mathbb{R}\).

By corollary 2.3, for \(T\) as in \((\mathcal{F})\), \((X, \mathcal{B}, m, T)\) is a measure preserving transformation. By proposition 2.11, \(T\) is conservative iff \(\beta = 0\).

5.3 Proposition

(i) If \(\beta = 0\), then \(T\) is conservative, ergodic.
(ii) If \(\beta \neq 0\), then \(\exists F : \mathbb{R}^2^+ \rightarrow \mathbb{R}^2^+\) analytic, so that \(F \circ T = F + \beta\). In particular, \(T\) is not ergodic.

Proof sketch

For \(\omega \in \mathbb{R}^2^+\), write \(T^n(\omega) := u_n + iv_n\), then

\[
v_{n+1} = v_n + v_n \sum_{k=1}^{N} \frac{p_k}{(t_k - u_n)^2 + v_n^2}
\]

\[
u_{n+1} = u_n + \beta + \sum_{k=1}^{N} \frac{p_k(t_k - u_n)}{(t_k - u_n)^2 + v_n^2}.
\]

As before, elementary calculations show that

- when \(\beta \neq 0\), \(\exists B = B(\omega) \in \mathbb{R}_+ \& C = C(\omega) \in \mathbb{R}\) so that

\[
(I) \quad v_n \uparrow B \quad \& \quad u_n = \beta n - \frac{\nu}{\beta} \log n + C + O\left(\frac{\log n}{n}\right) \quad \text{as} \quad n \to \infty;
\]

and

- when \(\beta = 0\),

\[
(II) \quad \sup_{n \geq 1} |u_n| < \infty \quad \& \quad v_n \sim \sqrt{2n \nu} \quad \text{as} \quad n \to \infty \quad \text{where} \quad \nu := \sum_{k=1}^{n} p_k
\]

Proof of (i)

Set \(p := \varphi_i\), then \(\forall x \in \mathbb{R}, \ \omega \in \mathbb{R}^2^+\),

\[
\overline{S}_n \varphi \omega(x) := \sum_{k=0}^{n-1} \overline{T}^k \varphi \omega(x) \sim \sum_{k=0}^{n-1} \frac{1}{\pi v_k} \sim a(n) := \frac{1}{\pi} \sqrt{\frac{2n}{\nu}}.
\]
By Hurewicz’s theorem, for $f \in L^1(m)$ and a.e. $x \in X$,

$$\frac{\widehat{S}_n f(x)}{a(n)} \sim \frac{\widehat{S}_n p(x)}{\widehat{S}_n p} \rightarrow E_{m_p}(f|\mathcal{J}).$$

On the other hand, for $f = g \ast \varphi_{ib}$ ($g \in L^1(m)$),

$$f(x) := \int_{\mathbb{R}} g(t) \varphi_{ib}(x-t) \, dt = \int_{\mathbb{R}} g(t) \varphi_{t+ib}(x) \, dt$$

whence

$$\widehat{T}^n f = \int_{\mathbb{R}} g(t) \varphi_{T^n(t+ib)}(x) \, dt$$

and by (I)

$$\frac{\widehat{S}_n f(x)}{a(n)} = \int_{\mathbb{R}} g(t) \frac{\widehat{S}_n \varphi_{t+ib}}{a(n)} \, dt \rightarrow \int_{\mathbb{R}} g \, dm = \int_{\mathbb{R}} f \, dm$$

whence $E_{m_p}(f|\mathcal{J})$ is constant. Since such $f$ are dense in $L^1(m)$, $T$ is ergodic. $\Box$(i)

Proof of (ii) By (II),

$$T^n(\omega) - n\beta + \frac{\nu}{\beta} \log n \rightarrow C(\omega) + iB(\omega) =: F(\omega) \in \mathbb{R}^{2+}.$$ 

It follows that $F : \mathbb{R}^{2+} \rightarrow \mathbb{R}^{2+}$ is analytic. Moreover

$$F(T\omega) \longleftrightarrow \frac{T^{n+1}(\omega) - n\beta + \frac{\nu}{\beta} \log n}{n \rightarrow \infty} = (T^{n+1}(\omega) - (n + 1)\beta + \frac{\nu}{\beta} \log(n + 1)) + \beta + O\left(\frac{1}{n}\right)$$

$$\rightarrow F(\omega) + \beta. \quad \Box(\text{ii})$$

APERIODICITY AND ROKHLIN TOWERS

Perodicity. Let $(X, \mathcal{B}, m)$ be a standard probability space and let $T \in \text{NST}(X, \mathcal{B}, m)$.

For each $p \geq 1$ consider the set of $p$-periodic points

$$\text{Per}_p(T) := \{x \in X : T^p x = x, \ T^j x \neq x \ \forall \ 1 \leq j < p\}.$$ 

Exercise 19. Show that for $p \in \mathbb{N}$:

(i) $\text{Per}_p(T) \in \mathcal{B}$;

(ii) there is a set $A \in \mathcal{B}$ so that $\{T^j A : 0 \leq j \leq p - 1\}$ are disjoint and

$$\text{Per}_p(T) \sim \bigcup_{j=0}^{p-1} T^j A.$$
Hints for (ii) Using the polish structure of $X$, show that $\forall \ A \in \mathcal{B}_+, \ \exists \ B \in \mathcal{B}_+, \ B \in A$ so that $\{T^jB : 0 \leq j \leq p-1\}$ are disjoint. Then perform an exhaustion argument.

**Aperiodicity.**

The non-singular transformation $(X, \mathcal{B}, m, T)$ is called **aperiodic** if $m(\text{Per}_n(T)) = 0 \ \forall \ n \geq 1$.

**Sweepout sets.** Let $(X, \mathcal{B}, m, T)$ be a NST. A set $A \in \mathcal{B}$ is called a **sweepout set** if $\bigcup_{n=1}^{\infty} T^{-n}A \supseteq X$.

The next exercise shows that an aperiodic, conservative NST has sweepout sets of arbitrarily small measure.

Note that this is immediate for a conservative, ergodic NST $(X, \mathcal{B}, m, T)$, for then for any $A \in \mathcal{B}_+$, $\bigcup_{n=1}^{\infty} T^{-n}A$ has positive measure and is $T$-invariant mod $m$...

**Exercise 20.** Let $(X, \mathcal{B}, m, T)$ be an aperiodic, conservative NST. Show that $\forall \ \epsilon > 0 \ \exists \ E \in \mathcal{B}$, $m(E) < \epsilon$ s.t. $\tilde{E} := \bigcup_{n=1}^{\infty} T^{-k}E \equiv X \mod m$.

**Directions:**

1. Fix $N > \frac{1}{\epsilon}$ and let $Z_N := \{A \in \mathcal{B}_+ : \{T^{-j}A : 0 \leq j < N\} \text{ disjoint}\}$.

2. Show that $\forall \ J \in \mathcal{B}_+, \ \exists \ A \in Z_N \text{ so that } m(A \cap J) > 0$.

**Hints**

(i) Assume WLOG that $T^n x \neq x \ \forall \ x \in X, \ n \geq 0$. Fix a polish metric $d$ on $X$ and find (!) $C \subset J$ compact so that $m(C) > 0$ and $T^j : C \rightarrow X$ is continuous for $0 \leq j \leq N$.

(ii) Find $x \in C$ so that $m(C \cap B(x, \epsilon)) > 0 \ \forall \ \epsilon > 0$ where $B(x, \epsilon)$ is the $d$-ball of radius $\epsilon$ around $x$ and then find (!) $\eta > 0$ so that $\{T^j(C \cap B(x, \eta)) : 0 \leq j \leq p-1\}$ are disjoint.

3. Obtain using exhaustion: sets $A_1, A_2, \cdots \in Z_N$ and numbers $\epsilon_n \geq 0$ so that $\overline{A_{n+1}} \cap \overline{A_k} = \emptyset \ \forall \ 1 \leq k \leq n$;

$2m(\overline{A_{n+1}}) \geq \epsilon_{n+1} := \sup \{m(A) : \ A \in Z_N, \ \overline{A_{n+1}} \cap \overline{A_k} = \emptyset \ \forall \ 1 \leq k \leq n\}$

and show that for some $0 \leq J < N$, $T^{-J} \bigcup_{k=1}^{\infty} A_k$ is as required.

**6.2 Rokhlin’s tower theorem**

Let $T$ be a conservative, aperiodic nonsingular transformation of the Polish, probability space $(X, \mathcal{B}, m)$. For $N \geq 1$, and $\eta > 0$, $\exists \ E \in \mathcal{B}$ such that $\{T^{-j}E\}_{j=0}^{N-1}$ are disjoint, and $m(X \setminus \bigcup_{j=0}^{N-1} T^{-j}E) < \eta$.

\[\text{Here, I’m breaking up the proof into “easy stages”}\.]
Proof
By non-singularity \( \exists \delta > 0 \) so that
\[
m(A) < \delta \implies m(\bigcup_{k=0}^{N-1} T^{-k}A) < \eta.
\]
Using this and exercise 20, we can choose \( A \in \mathcal{B} \) such that \( A = X \) and \( m(\bigcup_{k=0}^{N-1} T^{-k}A) < \eta \).
Set \( A_0 := A, A_n := T^{-n}A \setminus \bigcup_{j=0}^{n-1} T^{-j}A, (n \geq 1) \), then \( \{A_n : n \geq 0\} \) are disjoint and \( \bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} T^{-n}A = X \).
Set \( E := \bigcup_{p=1}^{\infty} A_{pN} \), then for \( 0 \leq k \leq N-1 \):
\[
T^kE \subset \bigcup_{p=1}^{\infty} A_{pN-k}
\]
whence \( \{T^jE\}_{j=0}^{N-1} \) are disjoint.
We claim that \( \{T^{-j}E\}_{j=0}^{N-1} \) are disjoint. To see this, fix \( 1 \leq k \leq N-1 \), then \( E \subset T^{-k}T^kE \) whence
\[
T^{-k}E \cap E \subset T^{-k}E \cap T^{-k}T^kE = T^{-k}(E \cap T^kE) = \emptyset.
\]
On the other hand, for \( 0 \leq k \leq N-1 \),
\[
T^{-k}E \supset \bigcup_{p=1}^{\infty} A_{pN+k},
\]
whence \( \bigcup_{k=0}^{N-1} T^{-k}E \supset \bigcup_{n=N}^{\infty} A_n \), and
\[
m(X \setminus \bigcup_{j=0}^{N-1} T^{-j}E) \leq m(\bigcup_{n=0}^{N-1} A_n) = m(\bigcup_{k=0}^{N-1} T^{-k}A) < \epsilon.
\]

Skew Products

Let \((X, \mathcal{B}, m, T)\) be a NST and let \( G \) be a locally compact, polish, abelian topological group.
Given a measurable function \( \phi : X \to G \), define the skew product transformation \( T_\phi : X \times G \to X \times G \) by \( T_\phi(x, g) := (Tx, \phi(x) + g) \).

1.1 Proposition (Hopf decomposition of skew products)
Suppose that \( T \) is ergodic and either a MPT, or an invertible NST. Let \( \varphi : X \to G \) be measurable, then \( T_\varphi \) is either conservative, or totally dissipative.

Proof By the assumption, \( T_\varphi \) is also either a MPT, or an invertible NST. In either case, \( \mathcal{D}(T_\varphi) \) is \( T_\varphi \)-invariant. We’ll show that it’s invariant under an ergodic action of a larger semigroup.
Let $\Gamma \subset G$ be a countable dense subgroup of $G$. The action of $\Gamma$ on $G$ by translation is ergodic with respect to Haar measure on $G$. It follows that the $\mathbb{N} \times \Gamma$ action $S$ on $(X \times G, \mathcal{B}(X \times G), m \times m_G)$ given by $S_{(n,a)}(x,y) := (T^nx, y + a + \phi_n(x))$ is ergodic.

Let $a \in G$, then since $S_{0,a}$ is invertible and $S_{0,a} \circ T^\varphi = T^\varphi \circ S_{0,a}$ we have that $W \in \mathcal{W}(T^\varphi)$ iff $S_{0,a}W \in \mathcal{W}(T^\varphi)$, whence $S_{0,a}D(T^\varphi) = D(T^\varphi)$. Since $T^{-1}D(T^\varphi) = D(T^\varphi)$, it follows that $D(T^\varphi)$ is $S$-invariant, whence the proposition by ergodicity of $S$. \qed

1.2 Proposition Let $(X, \mathcal{B}, m, T)$ be a PPT, then $T^\phi$ is conservative iff

$$\liminf_{n \to \infty} \|\phi_n(x)\| = 0 \text{ for a.e. } x \in X.$$  

Proof Assume first that $T^\phi$ is conservative and let $\epsilon > 0$. By Halmos’ recurrence theorem

$$\sum_{n=1}^\infty 1_{X \times B_G(0,\epsilon/2)} \circ T^\phi_n = \infty \text{ a.e. on } X \times B_G(0,\epsilon/2).$$

So for a.e. $x \in X$, $y \in B_G(0,\epsilon/2),$

$$\sum_{n=1}^\infty 1_{B_G(0,\epsilon/2)}(y + \phi_n(x)) = \infty,$$

whence for a.e. $x \in X$, $\liminf_{n \to \infty} \|\phi_n(x)\| \leq \epsilon$.

Now assume that

$$\liminf_{n \to \infty} \|\phi_n(x)\| = 0 \text{ for a.e. } x \in X.$$  

Fix $f : G \to \mathbb{R}_+$ be continuous, positive and integrable and let $0 < \epsilon < \kappa_G$. For $y \in G$, let $\delta(y,\epsilon) := \inf_{B_G(y,\epsilon)} f$. By compactness of $B_G(y,\epsilon)$, $\delta(y,\epsilon) > 0$.

We have that $\forall$ $y \in G$, for a.e. $(x,z) \in X \times B_G(y,\epsilon/2),$

$$\sum_{n=1}^\infty (1 \otimes f) \circ T^\phi_n(x,z) = \sum_{n=1}^\infty f(z + \phi_n(x)) \geq \delta(y,\epsilon) \sum_{n=1}^\infty 1_{B_G(0,\epsilon/2)}(\phi_n(x)) = \infty$$

and $T^\phi$ is conservative. \qed

1.3 Proposition If $\phi = \Psi - \Psi \circ T$ with $\Psi : X \to G$ measurable, then $T^\phi$ is conservative.

Proof Evidently $T_0$ is conservative, and if $\phi$ is a coboundary, then $T^\phi$ is isomorphic to $T_0$. \qed
Persistencies and Essential Values

Let \((X, \mathcal{B}, m)\) be a standard probability space, and let \(T : X \to X\) be an ergodic, finitely supported measure-preserving transformation. Suppose that \(\phi : X \to G\) is measurable. The collection of persistencies of \(\phi\) is

\[
\Pi(\phi) = \{ a \in G : \forall A \in \mathcal{B}_+, \epsilon > 0, \exists n \geq 1, m(A \cap T^{-n}A \cap \{\|\phi_n - a\| < \epsilon\}) > 0\}.
\]

For \(T\) invertible, the collection of essential values of \(\phi\) is

\[
E(\phi) = \{ a \in G : \forall A \in \mathcal{B}_+, \epsilon > 0, \exists n \in \mathbb{Z}, m(A \cap T^{-n}A \cap \{\|\phi_n - a\| < \epsilon\}) > 0\}.
\]

2.1 Proposition [Schm1]

Either \(\Pi(\phi) = \emptyset\), or \(\Pi(\phi)\) is a closed subgroup of \(G\).

Proof

To see that \(\Pi(\phi)\) is closed let \(a \in \overline{\Pi(\phi)}\) and let \(\epsilon > 0\), \(A \in \mathcal{B}_+\).

\(\exists a' \in \Pi(\phi)\) such that \(\|a - a'\| < \epsilon/2\), and \(\exists n \geq 1\) such that

\[
m(A \cap T^{-n}A \cap \{\|\phi_n - a'\| < \epsilon/2\}) > 0.
\]

It follows that

\[
m(A \cap T^{-n}A \cap \{\|\phi_n - a\| < \epsilon\}) \geq m(A \cap T^{-n}A \cap \{\|\phi_n - a'\| < \epsilon/2\}) > 0.
\]

Thus, \(a \in \Pi(\phi)\) and \(\Pi(\phi)\) is closed.

To show that \(\Pi(\phi)\) is a group, we show that \(a, b \in \Pi(\phi) \implies a - b \in \Pi(\phi)\).

Let \(a, b \in \Pi(\phi), \epsilon > 0\), \(A \in \mathcal{B}_+\) and let \(n \geq 1\) be such that \(m(A \cap T^{-n}A \cap \{\|\phi_n - a\| < \epsilon/2\}) > 0\).

By Rokhlin’s lemma, \(\exists B \in \mathcal{B}_+, B \subset A \cap T^{-n}A \cap \{\|\phi_n - a\| < \epsilon/2\}\) such that \(B \cap T^{-k}B = \emptyset\) for \(1 \leq k \leq n\).

Since \(b \in \Pi(\phi)\), \(\exists N \geq 1\) such that \(m(B \cap T^{-N}B \cap \{\|\phi_N - b\| < \epsilon/2\}) > 0\).

The construction of \(B\) implies that \(N > n\) whence

\[
B \cap T^{-N}B \cap \{\|\phi_N - b\| < \epsilon/2\} = B \cap T^{-N}B \cap \{\|\phi_n - a\| < \epsilon/2\} \cap \{\|\phi_N - b\| < \epsilon/2\} \subset B \cap T^{-N}B \cap \{\|\phi_{N-n} \circ T^n - (b-a)\| < \epsilon\},
\]

\[
0 < m(B \cap T^{-N}B \cap \{\|\phi_{N-n} \circ T^n - (b-a)\| < \epsilon\}) \leq m(A \cap T^{-n}A \cap T^{-N}A \cap \{\|\phi_{N-n} \circ T^n - (b-a)\| < \epsilon\}) \leq m(T^{-n}(A \cap T^{-N-n}A \cap \{\|\phi_{N-n} - (b-a)\| < \epsilon\}))\]

whence \(m(A \cap T^{-(N-n)}A \cap \{\|\phi_{N-n} - (b-a)\| < \epsilon\}) > 0\) and \(b - a \in \Pi(\phi)\). \(\Box\)
2.2 Theorem [K. Schmidt]

Let \((X, \mathcal{B}, m, T)\) be a conservative NST, and let \(\phi: X \to G\), then \(T_\phi\) is conservative \(\iff 0 \in \Pi(\phi)\).

**Proof of \(\Rightarrow\)**
Suppose first that \(T_\phi\) is conservative and let \(A \in \mathcal{B}_+\), \(\epsilon > 0\). \(\exists n \geq 1\) such that \(m \times m_G(A \times B_G(0, \epsilon/2) \cap T_\phi^{-n} A \times B_G(0, \epsilon/2)) > 0\). Since \(A \times B_G(0, \epsilon/2) \cap T_\phi^{-n} A \times B_G(0, \epsilon/2) \subset (A \cap T^{-n} A \cap [\|\phi_n\| < \epsilon]) \times B_G(0, \epsilon/2)\), we have \(m(A \cap T^{-n} A \cap [\|\phi_n\| < \epsilon]) > 0\) and \(0 \in \Pi(\phi)\). \(\Box\)

**Proof of \(\Leftarrow\)**
In case \(G\) is countable, every \(B \in \mathcal{B}(X \times G)_x\) contains a set Conversely, suppose that \(T_\phi\) is not conservative. Let \(A \in \mathcal{B}\). Consider the sections
\[
A_x := \{y \in G: (x, y) \in A\} \quad (x \in X).
\]
A calculation shows that
\[
(T_\phi^{-n} A)_x = A_{T^n x} - \phi_n(x).
\]
By Fubini’s theorem, \(A_x \in \mathcal{B}(G) \quad \forall x \in X\) and \(x \mapsto m_G(A_x)\) is measurable. Let
\[
X_A := \{x \in X: m(A_x) > 0\},
\]
then \(m(X_A) > 0\). Now let \(W \in \mathcal{W}(T_\phi)\). We claim that
\(\exists W'\) there is a measurable subset \(V \subset W\) with
\[
0 < m(V_x) < \infty \quad \text{for a.e. } x \in X_W.
\]

**Proof of \(\exists W'\)**
Define \(R: X \to [0, \infty)\) by
\[
R(x) := \inf \{r > 0: m(W_x \cap B(0, r)) > m(W_x) - 1\},
\]
then
\[
V_0 := \{(x, y): y \in W_x \cap B(0, R(x))\}
\]
is Lebesgue measurable and \(m \times m_G(V_0) > 0\). It follows that \(\exists V \in \mathcal{B}(X \times G), \ V \subset V_0\) with \(m \times m_G(V_0 \setminus V) = 0\).

It follows that for a.e. \(x \in X_W\), \(V_x = (V_0)_x\) whence
\[
0 < m(V_x) < \infty \quad \text{for a.e. } x \in X_W. \ \Box\]

Let
\[
\mathcal{F} := \{f \in L^1(m_G): \exists A \in \mathcal{B}, f = 1_A \ \text{a.e.}\},
\]
then \(\mathcal{F}\) is a polish space with the metric
\[
\rho([A], [B]) := \|1_A - 1_B\|_1 = m_G(A \Delta B)
\]
for $A, B \in \mathcal{B}$, $0 < m(A), m(B) < \infty$ where $[C] := \{ B \in \mathcal{B}(G) : \mu(B \Delta C) = 0 \}$.

By Fubini’s theorem, $x \mapsto [V_x]$ is a Borel map $X \to \bar{F}$.

By Lusin’s theorem, there exists a compact set $C \in \mathcal{B}_+$, $C \subset X_{W}$ so that $x \mapsto V_x$ is continuous on $C$.

Also, for $A \in \mathcal{F}_+$, $t \mapsto m_G(A \cap (t + A))$ is continuous $G \to [0, \infty)$.

By compactness, $m_G(V_x) \leq \Delta > 0$ $\forall x \in C$.

By continuity, there exists $\epsilon > 0$ such that $x \mapsto V_x$ is continuous on $C$.

Thus, we have, for $n \geq 1$

$$U \cap T^{-n}U \subseteq (D \cap T^{-n}D) \times G$$

and for a.e. $x \in D \cap T^{-n}D$, we have

$$\emptyset = (U \cap T^{-n}U)_x = U_x \cap (U_{T^n x} - \phi_n(x))$$

$$= U_x \cap (U_{T^n x} - \phi_n(x)) = V_x \cap (V_{T^n x} - \phi_n(x)).$$

By (\textbullet\textbullet),

$$U \cap T^{-n}U \subseteq [\|\phi_n\| \geq \epsilon] \quad \forall \ n \geq 1$$

and $0 \notin \Pi(\phi)$. \(\Box\)

2.3 Proposition

Suppose that $\phi, \varphi : X \to G$ are cohomologous, then $\Pi(\phi) = \Pi(\varphi)$.

Proof

By symmetry, it is sufficient to show that $\Pi(\phi) \subseteq \Pi(\varphi)$.

Suppose that $\varphi = \phi + h \circ T - h$ where $h : X \to G$ is measurable.

Let $a \in \Pi(\phi)$ and let $A \in \mathcal{B}_+$, $\epsilon > 0$.

Since $X$ is a standard space, by Lusin’s theorem there exists $B \subset A$, $B \in \mathcal{B}_+$ such that $\|h(x) - h(y)\| < \frac{\epsilon}{3}$ $\forall x, y \in B$.

Since $a \in \Pi(\phi)$, there exists $n \geq 1$ such that $m(B \cap T^{-n}B \cap [\|\phi_n - a\| < \frac{\epsilon}{3}]) > 0$.

By construction of $B$, if $x \in B \cap T^{-n}B$, then $\|\varphi_n(x) - \phi_n(x)\| = \|h(T^n x) - h(x)\| < \frac{\epsilon}{3}$ whence

$$m(B \cap T^{-n}B \cap [\|\varphi_n - a\| < \epsilon]) \geq m(B \cap T^{-n}B \cap [\|\phi_n - a\| < \frac{\epsilon}{2}]) > 0,$$
and \( a \in \Pi(\varphi) \).

**Periods.** Define the collection of periods for \( T_\varphi \)-invariant functions:

\[
\text{Per}(\varphi) = \{ a \in G : Q_a A = A \mod m \ \forall \ A \in \mathcal{I}(T_\varphi) \}
\]

where \( Q_a(x, y) = (x, y + a) \).

2.4 Theorem [K.Schmidt]

(i) Suppose that \( T_\varphi \) is conservative, then

\[
\Pi(\varphi) = \text{Per}(\varphi).
\]

(ii) Suppose that \( T \) is invertible, then

\[
E(\varphi) = \text{Per}(\varphi).
\]

**Remark.** (i) fails for some non-invertible \( T \) with \( T_\varphi \) dissipative

**Proof of (i)**

\[\text{1} \quad \text{Per}(\varphi) \subset \Pi(\varphi)\]

Suppose \( 0 \neq a \notin \Pi(\varphi) \), then \( \exists 0 < \epsilon < d(0, a) \), and \( A \in \mathcal{B}_+ \) such that

\[
m(A \cap T^{-n} A \cap [\|\phi_n - a\| < 2\epsilon]) = 0 \ \forall \ n \geq 1.
\]

For \( z \in G \) & \( \epsilon > 0 \), set

\[
B_z = \bigcup_{n \in \mathbb{N}} T_{\phi}^{-n} \left( A \times B_G(z, \epsilon) \right).
\]

We have that \( T_{\phi}^{-1} B_z \subset B_z \), whence by conservativity \( T_{\phi}^{-1} B_z \overset{m}{=} B_z \). Moreover \( 1_{B_0} \circ Q_a = 1_{B_a} \).

To see that \( a \notin \text{Per}(\varphi) \), it suffices to prove that

\[
m(B_0 \cap B_a) = 0.
\]

This holds because \( \forall \ n \in \mathbb{N} \),

\[
(A \times B_G(0, \epsilon) \cap T_{\phi}^{-n}(A \times B_G(a, \epsilon))) \cup (A \times B_G(a, \epsilon) \cap T_{\phi}^{-n}(A \times B_G(0, \epsilon)))
\]

\[
\subset A \cap T^{-n} A \cap [\|\phi_n - a\| < 2\epsilon] \times G.
\]

\[\text{2} \quad \Pi(\varphi) \subset \text{Per}(\varphi)\]

Now assume that \( a \notin \text{Per}(\varphi) \), then \( \exists A, B \in \mathcal{I}(T_\varphi)_+ \), disjoint such that \( B = Q_a A \). Set for \( x \in X \),

\[
A_x = \{ y \in G : (x, y) \in A \}\]
Note that
\[ A_{Tx} = \{ y \in G : (Tx, y) = T\phi(x, y - \phi(x)) \in A \} = A_x + \phi(x), \]
whence \( m_G(A_x) = m_G(A_{Tx}) \), and by ergodicity, \( m_G(A_x) = m \times m_G(A) > 0 \) for \( m \)-a.e. \( x \in X \).

Next, as in the proof of \( \Leftarrow \) in theorem 2.2:

• \( \exists \theta \in B(A) \) such that \( 0 < m_G(\theta_x) < \infty \) a.e.;
• \( \exists \epsilon > 0 \) and \( D \in B(X) \) such that \( m_G(\theta_x \cap (\theta_y + t)) \geq \epsilon \forall x, y \in D, \|t\| < \epsilon. \)

Lastly, we show that \( a \notin \Pi(\phi) \). This will follow from
\[ D \cap T^{-n}D \cap [\|\phi_n(x) - a\| < \epsilon] = \emptyset \forall n \geq 1. \]

Indeed, supposing that \( x, T^n x \in D \), we note that
\[ \left( a + \theta_{T^n x} \right) \cap \left( \theta_x + \phi_n(x) \right) \subset B_{T^n x} \cap A_{T^n x} = \emptyset, \]
whence,
\[ m_G(\theta_x \cap (\theta_{T^n x} + a - \phi_n(x))) \leq m_G(B_{T^n x} \cap A_{T^n x}) = 0 \]
and
\[ \|\phi_n(x) - a\| \geq \epsilon. \]

\[ \Box \]

Exercise 21: Essential values.
Let \((X, \mathcal{B}, m, T)\) be an invertible NST and let \( \phi : X \to G \) be measurable (\( G \) a LCA group). Show that
(i) \( E(\phi) = \Pi(\phi) \cup \{0\} \); (ii) \( E(\phi) = \text{Per}(\phi) \).

Exercise 22: Dissipative exact example.
This is a counterexample to theorem 2.4 for dissipative, non-invertible skew products..
Let \((X, \mathcal{B}, m, S)\) be an EPPT and let \( f : X \to \mathbb{Z} \) be such that \( Sf \) is an ergodic, totally dissipative MPT (as in e.g. exercise 16).
Show that
(i) \( \Pi(f, S) = \emptyset \);
(ii) \( \text{Per}(f, S) = \mathbb{Z} \).

\[ \text{End of minicourse} \]