ERGODIC THEORY NOTES
TORUN, OCTOBER 2014.

JON AARONSON'S `LECTURE NOTES`

Lecture # 1 8/10/2014.

Introduction

Let $(X, \mathcal{B}, m)$ be a standard σ-finite measure space[1] A `null preserving transformation` (`NPT`) of X is only defined modulo nullsets, and is a map $T : X_0 \to X_0$ (where $X_0 \subset X$ has full measure), which is measurable and has the *null preserving property* that for $A \in \mathcal{B},\ m(T^{-1}A) = 0$ implies that $m(A) = 0$.

A *non-singular transformation* (`NST`) is a `NPT` $(X, \mathcal{B}, m, T)$ with the stronger property that for $A \in \mathcal{B},\ m(T^{-1}A) = 0$ iff $m(A) = 0$.

A *measure preserving transformation* (`MPT`) is a `NST` $(X, \mathcal{B}, m, T)$ with the additional property that $m(T^{-1}A) = m(A)\ \forall\ A \in \mathcal{B}$.

We'll call a nonsingular transformation `NS`-*invertible* if the associated map is invertible with a nonsingular inverse.

Let

$$\texttt{NST}(X, \mathcal{B}, m) := \{\texttt{nonsingular invertible transformations of } X\}$$

$$\texttt{MPT}(X, \mathcal{B}, m) := \{\texttt{invertible measure preserving transformations of } X\}$$

$$\texttt{PPT}(X, \mathcal{B}, m) := \texttt{MPT}(X, \mathcal{B}, m) \ \texttt{ in case } m(X) = 1.$$

The are all groups under composition (see the exercise below).

Equivalent invariant measures. If T is a non-singular transformation of a σ-finite measure space $(X, \mathcal{B}, m)$, and p is another measure on $(X, \mathcal{B})$ *equivalent* to m (denoted $p \sim m$ and meaning that p and m have the same nullsets), then T is a non-singular transformation of $(X, \mathcal{B}, p)$.

Thus, a non-singular transformation of a σ-finite measure space is actually a non-singular transformation of a probability space.

[1] `i.e. an uncountable Polish spec equipped with Borel sets and a non-atomic,` σ`-finite measure.`

The first question about a `NST` $(X,\mathcal{B},p,T)$ is whether it was obtained from a measure preserving transformation in this way, or, slightly more generally:

$\exists$? a σ-finite absolutely continuous invariant measure (a.c.i.m., i.e. $m \ll p$, with $m\circ T^{-1} = m$).

Radon Nikodym Derivatives

Let $(X,\mathcal{B},m,T)$ be an invertible `NST` of the probability space $(X,\mathcal{B},m)$. The measures m & $m\circ T$ are equivalent (i.e. $m\circ T \ll m$ & $m \ll m\circ T$), written $m\circ T \sim m$. By the Radon Nikodym theorem, $\exists\,!\ T' \in L^1$, $T' > 0$ a.e., so that

$$m(TA) = \int_A T' dm \ \forall\ A \in \mathcal{B}.$$

The function T' is called the *RN derivative* of T. The measurable map $f : A \to A'$ is called

- *null preserving* (`NP`) if for $C \in \mathcal{B}' \cap A'$, $m'(C) = 0 \ \Rightarrow\ m(f^{-1}C) = 0$;
- *nonsingular* (`NS`) if for $C \in \mathcal{B}' \cap A'$, $m(f^{-1}C) = 0$ iff $m'(C) = 0$; and
- *measure preserving* (`MP`) if $m(f^{-1}C) = m'(C)$ for $C \in \mathcal{B}' \cap A'$.

Exercise 1: `Chain rule for RN derivatives.`

Let $(X,\mathcal{B},m)$ be a probability space and let $S,\ T \in$ `NST` $(X,\mathcal{B},m)$.

(i) Show that $T\circ S \in$ `NST` $(X,\mathcal{B},m)$ and

$$(T\circ S)' = T'\circ S\cdot S'.$$

(ii) Let $(X,\mathcal{B},m)$ be the unit interval equipped with Borel sets and Lebesgue measure, and suppose that $T : X \to X$ is nondecreasing and C^1, then

- $T : X \to X$ is a homeomorphism iff $[T' = 0]^o = \varnothing$;
- $T^{-1} : X \to X$ is non-singular iff $m([T' = 0]) = 0$; &
- $\exists$ a C^1 homeomorphism $T : X \to X$ with $T^{-1} : X \to X$ singular.

`Transfer Operator.`

Let $(X,\mathcal{B},m,T)$ be a null-preserving transformation, then $\|f\circ T\|_\infty \le \|f\|_\infty\ \forall\ \ f \in L^\infty(m)$ and $T : L^\infty(m) \to L^\infty(m)$ where $Tf := f\circ T$.

There is an operator known as the *transfer operator* $\widehat{T} : L^\infty(m) \to L^\infty(m)$ so that $\widehat{T}^* = T$ i.e.:

$$\int_X \widehat{T}f\cdot g dm = \int_X f\cdot Tg dm \ \forall\ f \in L^1(m),\ g \in L^\infty(m).$$

This is given by $\widehat{T}f := \frac{d\nu_f \circ T^{-1}}{dm}$ where $\nu_f(A) := \int_X f dm$ (!).

Exercise 2.

Let $(X, \mathcal{B}, m, T)$ be a nonsingular transformation.

(i) Show that if T is invertible, then $\widehat{T}f = T^{-1\prime} f \circ T^{-1}$.

(ii) Show that $\exists$ an absolutely continuous invariant probability for T iff $\exists\ h \in L^1_+$ satisfying $\widehat{T}h = h$.

Examples

`Rotations of the circle.` Let X be the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1)$, $\mathcal{B}$ be its Borel sets, and m be Lebesgue measure. The *rotation* (or translation) of the circle by $x \in X$ is the transformation $r_x : X \to X$ defined by $r_x(y) = x + y \mod 1$.

Evidently $m \circ r_x = m$ for every $x \in X$ and each r_x is an invertible measure preserving transformation of $(X, \mathcal{B}, m)$.

`The adding machine.` Let $\Omega = \{0,1\}^{\mathbb{N}}$, and $\mathcal{F}$ be the σ-algebra generated by cylinders. Define the *adding machine* $\tau : \Omega \to \Omega$ by $\tau(\overline{1}) := (\overline{0})$ where $(\overline{a})_k = a\ \forall\ k \geq 1$; and

$$\tau(1, ..., 1, 0, \omega_{\ell+1}, \omega_{\ell+2}, ...) = (0, ..., 0, 1, \omega_{\ell+1}, \omega_{\ell+2}, ...)$$

for $\omega \in \Omega \setminus \{(\overline{1})\}$ where $\ell(\omega) := \min\{n \geq 1 :\ \omega_n = 0\}$.

The reason for the name "adding machine" is that

$$\sum_{k=1}^{\infty} 2^{k-1} (\tau^n \overline{0})_k = n \quad \forall\ n \geq 1.$$

We'll consider the adding machine with respect to various probabilities on Ω.

¶ For $p \in (0,1)$, define a probability μ_p on Ω by

$$\mu_p([\epsilon_1, ..., \epsilon_n]) = \prod_{k=1}^{n} p(\epsilon_k)$$

where $p(0) = 1 - p$ and $p(1) = p$.

1.3 Proposition

τ is an invertible, nonsingular transformation of $(\Omega, \mathcal{F}, \mu_p)$ with

$$\frac{d\mu_p \circ \tau}{d\mu_p} = \left(\frac{1-p}{p}\right)^{\ell - 2}.$$

Proof

We show that $\mu_p \circ \tau \sim \mu_p$ and calculate $\frac{d\mu_p \circ \tau}{d\mu_p}$. We show that for any set $A \in \mathcal{F}$,

$$\mu_p(\tau A) = \int_A \left(\frac{1-p}{p}\right)^{\ell-2} d\mu_p.$$

Consider first a cylinder set $A \subset [\ell = k] \ \ (k \geq 1)$

$$A = [\underbrace{1, \dots, 1}_{k-1 \text{ times}}, 0, a_1, \dots, a_n],$$

then

$$\tau A = [\underbrace{0, \dots, 0}_{k-1 \text{ times}}, 1, a_1, \dots, a_n],$$

and

(✈)
$$\begin{aligned}\mu_p(\tau A) &= \mu_p([\underbrace{0, \dots, 0}_{k-1 \text{ times}}, 1])\mu_p([a_1, \dots, a_n]) \\ &= \left(\frac{1-p}{p}\right)^{k-2} \mu_p(A) \\ &= \int_A \left(\frac{1-p}{p}\right)^{\ell-2} d\mu_p.\end{aligned}$$

Let

$$\mathcal{C} := \{A \in \mathcal{F} : \ (✈) \text{ holds}\}.$$

As above, $\mathcal{C} \supset \ \{$`cylinders`$\}$.

Since a any finite union of cylinders is also a finite union of disjoint cylinders, $\mathcal{C} \subset \mathcal{A}$, the algebra of finite unions of cylinders.

By σ-additivity of μ_p, $\mathcal{C}$ is a monotone class, and by the monotone class theorem, $\mathcal{C} \supseteq \sigma(\mathcal{A}) = \mathcal{B}$. ☑

Note that $\mu_{\frac{1}{2}} \circ \tau = \mu_{\frac{1}{2}}$.

Rank one constructions.

This method constructs a $T \in$ `MPT` $(X, \mathcal{B}, m)$ where $X = (0, S_T)$ is an interval, m is Lebesgue measure and where T is an invertible *piecewise translation* that is there are intervals $\{I_n : n \geq 1\}$ and numbers $a_n \in \mathbb{R} \ \ (n \geq 1)$ so that mod m:

$$X = \dot{\bigcup_{n=1}^{\infty}} I_n = \dot{\bigcup_{n=1}^{\infty}} (a_n + I_n) \quad \& \quad T(x) = x + a_n \ \text{ for } x \in I_n.$$

The `rank one transformation` $(X, \mathcal{B}, m, T)$ is an invertible piecewise translation of an interval $J_T = (0, S_T)$ where $S_T \in (0, \infty]$ which is defined as the "`limit of a refining sequence of Rokhlin towers`".

• A *Rokhlin tower* is a finite sequence of disjoint intervals $\tau = (I_1, I_2, \dots, I_n)$ of equal lengths; considered equipped with the translations $I_j \to I_{j+1}$ $(1 \le j \le n-1)$. It is thus a piecewise translation

$$T_\tau : \operatorname{Dom} T_\tau = \biguplus_{j=1}^{n-1} I_j \to \biguplus_{j=2}^{n} I_j$$

being defined everyhere on $\biguplus_{j=1}^{n} I_j$ except the last interval I_n.

• We'll say that the Rokhlin tower $\theta = (J_1, \dots, J_\ell)$ *refines* the Rokhlin tower $\tau = (I_1, I_2, \dots, I_n)$ (written $\theta \succ \tau$) if

$$\biguplus_{j=1}^{n} I_j \subset \biguplus_{k=1}^{\ell} J_k \ \&\ I_j = \biguplus_{1 \le k \le \ell,\ J_k \subset I_j} J_k.$$

This entails (!) $\biguplus_{j=1}^{n-1} I_j \subset \biguplus_{k=1}^{\ell-1} J_k$, whence $T_\theta|_{\biguplus_{j=1}^{n-1} I_j} \equiv T_\tau$.

Definition.

Let $c_n \in \mathbb{N},\ c_n \ge 2\ \ (n \ge 1)$ and let $S_{n,k} \ge 0,\ \ (n \ge 1,\ 1 \le k \le c_n)$. The *rank one transformation* with *construction data*

$$\{(c_n; S_{n,1}, \dots, S_{n,c_n}) :\ n \ge 1\}$$

is an invertible piecewise translation of the interval $J_T = (0, S_T)$ where

$$S_T := 1 + \sum_{n \ge 1} \frac{1}{c_1 \cdots c_n} \sum_{k=1}^{c_n} S_{n,k} \le \infty.$$

To obtain T, we define a refining sequence $(\tau_n)_{n \ge 1}$ of Rokhlin towers where $\tau_1 = [0,1]$ and τ_{n+1} is constructed from τ_n by

• cutting τ_n into c_n columns of equal width,

• putting $S_{n,k}$ spacer intervals (of the same width) above the k^{th} column $(1 \le k \le c_n)$;

• and stacking.

Evidently $\tau_{n+1} \succ \tau_n$. Let X be the increasing union of the intervals in the towers τ_n.

The sum of the lengths of the last intervals of the towers is $\sum_{n=1}^{\infty} \frac{1}{c_1 \dots c_n} < \infty$ and so for a.e. $x \in X$, $\exists\ n \le 1$ so that $x \in \operatorname{Dom} T_{\tau_k}\ \forall\ k \ge n$ and $T(x) := T_{\tau_k}(x)\ \forall\ k \ge n$.

The length of X is 1 plus the total length of all the spacer intervals added in the construction i.e. S_T.

Exercise 3. Show that the adding machine $(\Omega, \mathcal{F}, \mu, \tau)$ where $\mu = \mu_{\frac{1}{2}} := \prod(\frac{1}{2}, \frac{1}{2})$ is `isomorphic` to $(X, \mathcal{B}, m, T)$, the rank one transformation with construction data $\{(c_n; S_{n,1}, \dots, S_{n,c_n}) :\ \ n \ge 1\}$ with

$c_n = 2$ & $s_{n,1} = s_{n,2} = 0\ \forall\ n \ge 1$; i.e. show that there are measurable sets $X_0 \in \mathcal{B}$, $\Omega_0 \in \mathcal{F}$ of full measure so that $TX_0 = X_0$ & $\tau\Omega_0 = \Omega_0$ and $\pi : X_0 \to \Omega_0$ invertible, measure preserving so that $\pi \circ T = \tau \circ \pi$.

Kakutani skyscrapers.

Suppose that $(\Omega, \mathcal{F}, \mu, S)$ is a `NST` of the σ-finite measure space $((\Omega, \mathcal{F}, \mu)$ and that $\varphi : \Omega \to \mathbb{N}$ is measurable. The *Kakutani skyscraper* over S with *height function* φ is the transformation T of the σ-finite measure space $(X, \mathcal{B}, m)$ defined as follows.

$$X = \{(x,n) :\ x \in \Omega,\ 1 \le n \le \varphi(x)\},$$

$$\mathcal{B} = \sigma\{A \times \{n\} : n \in \mathbb{N},\ A \in \mathcal{F} \cap [\varphi \ge n]\},\ m(A \times \{n\}) = \mu(A),$$

and

$$T(x,n) = \begin{cases} (Sx, \varphi(x)) \text{ if } n = \varphi(x), \\ (x, n+1) \text{ if } 1 \le n \le \varphi(x) - 1. \end{cases}$$

Evidently T is a `NST` with

$$m(X) = \int_\Omega \varphi d\mu.$$

Moreover, if S is a `MPT`, then so is T.

- $\bigcup_{n \ge 1} T^{-n}(\Omega \times \{1\}) = X$;
- For $x \in \Omega$, let $\varphi_N(x) := \sum_{k=0}^{N-1} \varphi(S^k x)$, then $T^{\varphi_N(x)}(x,1) = (S^N x, 1)$ and

$$\{n \ge 1 :\ T^n(x,1) \in \Omega \times \{1\}\} \quad = \quad \{\varphi_N(x) :\ N \ge 1\}.$$

`Bernoulli shift.`

The (two sided) *Bernoulli shift* is defined by $X = \mathbb{R}^{\mathbb{Z}}$, $\mathcal{B}(X)$ the σ-algebra generated by *cylinder sets* of form

$$[A_1, \dots, A_n]_k := \{\underline{x} \in X : x_{j+k} \in A_j,\ 1 \le j \le n\}$$

where $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. The shift $S : X \to X$ is defined by $(Sx)_n = x_{n+1}$.

Let $p : \mathcal{B}(\mathbb{R}) \to [0,1]$ be a probability, and define $\widehat{\mu}_p : \{\texttt{cylinders}\} \to [0,1]$ by

$$\widehat{\mu}_p([A_1, ..., A_n]_k) = \prod_{k=1}^n p(A_k) \quad (A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})).$$

By Kolmogorov's existence theorem (see below) $\exists$ a probability measure $\mu_p : \mathcal{B}(X) \to [0,1]$ so that $\mu_p|_{\{\texttt{cylinders}\}} \equiv \widehat{\mu}_p$.

Evidently (!), the two sided Bernoulli shift is measure preserving.

2.1 Kolmogorov's existence theorem

Let Y be a Polish space, and suppose that for $k, \ell \in \mathbb{Z}$, $k \le \ell$ $P_{k,\ell} \in \mathcal{P}(Y^{\ell-k+1})$ are such that

$$P_{k,\ell+1}(A_k \times \cdots \times A_\ell \times Y) = P_{k-1,\ell}(Y \times A_k \times \cdots \times A_\ell) = P_n(A_k \times \cdots \times A_\ell)$$

then there is a probability measure $P \in \mathcal{P}(Y^{\mathbb{Z}})$ satisfying

$$P([A_1, \cdots, A_n]_k) = P_{k+1,n}(A_1 \times \cdots \times A_n).$$

`Vague sketch of proof`

- WLOG Y is uncountable ($\because$ any countable Polish space is measurably embeddable in an uncountable Polish space);
- WLOG $Y = \Omega := \{0,1\}^{\mathbb{N}}$ (by Kuratowski's isomorphism theorem).
- Now let $\mathcal{A}$ be the collection of cylinder subsets of Ω and set

$$\mathfrak{A} := \{[A_1, ..., A_n]_k : \ A_1, ..., A_n \in \mathcal{A}\}.$$

All sets in $\mathfrak{A}$ are both open and compact wrt the compact product topology on $\Omega^{\mathbb{Z}}$.

- Define $\mu : \mathfrak{A} \to [0,1]$ by

$$\mu([A_1, ..., A_n]_k) := P_{k+1,k+n}(A_1 \times ... \times A_n),$$

then $\mu : \mathfrak{A} \to [0,1]$ is additive and hence (!) countably subadditive.

- The reqired probability exists by Caratheodory's theorem.☑

Lecture # 2 9/10/2014.

`Interval maps.`

Let $I \subseteq \mathbb{R}$ be an interval, let m be Lebesgue measure on I, and α be a collection of disjoint open subintervals of I such that

$$m(I \setminus U_\alpha) = 0 \text{ where } U_\alpha = \bigcup_{a\in\alpha} a.$$

For $r \geq 1$, a C^r *interval map* with *basic partition* α is a map $T : I \to I$ such that

$$\text{for each } a \in \alpha,\ T|_a \text{ extends to a } C^r \text{ diffeomorphism } T : \overline{a} \to T(\overline{a}).$$

The C^r interval map is called *piecewise onto* if $T(a) = I \ \forall\ a \in \alpha$.

Transfer operator of an interval map.

Let $T : I \to I$ be a C^r interval map with basic partition α. For $a \in \alpha$, let $v_a : I \to a$ be the inverse of $T : a \to I$ (a C^r diffeomorphism). It follows from an integration variable-change argument that with respect to m:

$$\widehat{T}f = \sum_{a\in\alpha} 1_{T(a)} v_a' f \circ v_{\underline{a}}$$

Note that here $v_a' := \frac{dm\circ v_a}{dm} = |\frac{dv_a}{dx}|$.

Exercise 4.

(i) Show that for a C^1 interval map (I, T, α):

$$\widehat{T}f(x) = \sum_{y\in I,\ Ty=x} \frac{f(y)}{|T'(y)|}.$$

(ii) Show that if (I, T, α) is a piecewise onto, piecewise linear interval map (i.e. $T : a \to Ta$ is linear $\forall\ a \in \alpha$) with $\#\alpha \geq 2$, then $m \circ T^{-1} = m$ and that

$$m(\bigcap_{k=0}^{N} T^{-k} a_k) = \prod_{k=0}^{N} m(a_k) \ \ \forall\ N \geq 1,\ a_0, a_1, \dots, a_N \in \alpha.$$

Boole transformations & inner functions.

A *Boole transformation* is a map $T : \mathbb{R} \to \mathbb{R}$ of form

$$T(x) = \alpha x + \beta + \sum_{k=1}^{N} \frac{p_k}{t_k - x}$$
$$\text{where } \alpha \geq 0,\ p_1, \dots, p_N > 0\ \&\ \beta,\ t_1, \dots, t_N \in \mathbb{R}.$$

A Boole transformation T is an *inner function* of the upper half plane $\mathbb{R}^{2+} := \{\omega \in \mathbb{C} : \ \operatorname{Im}\omega > 0\}$ i.e. an analytic endomorphism of $\mathbb{R}^{2+}$ which preserves $\mathbb{R}$.

The general form of an inner function T of $\mathbb{R}^{2+}$ is given by:

(⛟) $$T(\omega) = \alpha\omega + \beta + \int_{\mathbb{R}} \frac{1 + t\omega}{t - \omega}\, d\mu(t)$$

where $\alpha \geq 0,\ \beta \in \mathbb{R}$ and μ is a finite, Lebesgue-singular, measure on $\mathbb{R}$.

If $\omega \in \mathbb{R}^{2+}$ the upper half plane, and $\omega = a + ib,\ a, b \in \mathbb{R},\ b > 0$ then

$$\mathtt{Im}\, \frac{1}{x - \omega} = \frac{b}{(x-a)^2 + b^2} = \pi\varphi_\omega(x)$$

where φ_ω is the well known *Cauchy density*.

These are the densities of the `Poisson` or `harmonic` measures on $\mathbb{R}^{2+}$:

If $\phi : \mathbb{R}^{2+} \to \mathbb{C}$ is bounded, analytic on $\mathbb{R}^{2+}$ and then for a.e. $t \in \mathbb{R}$, $\exists\ \lim_{y \to 0+} \phi(t + iy) =: \phi^*(t)$ and

(⚖) $$\phi(\omega) = \int_{\mathbb{R}} \phi^*(t) dP_\omega(t) \quad (\omega \in \mathbb{R}^{2+})$$

where $dP_\omega(t) = \varphi_\omega(t)dt$.

2.2 Boole's Formula *Let T be an inner function, then $(\mathbb{R}, \mathcal{B}, m, T)$ is non-singular and*

(📖) $$\widehat{T}\varphi_\omega = \varphi_{T(\omega)}\ \forall\ \omega \in \mathbb{R}^{2+}.$$

Proof (G.Letac) It suffices to show that $P_\omega \circ T^{-1} = P_{T(\omega)}$.

The Fourier transform of P_ω is given by

$$\widehat{P_\omega}(t) := \int_{\mathbb{R}} e^{itx} dP_\omega(x) = e^{it\omega} \quad (t \geq 0).$$

For $t > 0$, $\phi_t(\omega) = e^{it\omega}$ is a bounded analytic functions on $\mathbb{R}^{2+}$ with $\phi_t^*(x) = e^{itx}$ on $\mathbb{R}$. By (⚖),

$$\widehat{P_\omega \circ T^{-1}}(t) = \int_{\mathbb{R}} e^{itT(x)} dP_\omega(x) = e^{itT(\omega)} = \widehat{P_{T(\omega)}}(t),$$

whence (📖). ☑

Remark.

As a consequence of (📖), we see that the inner function T has an absolutely continuous invariant probability (`acip`) if $\exists\ \omega \in \mathbb{R}^{2+}$ with $T(\omega) = \omega$ (in which case P_ω is T-invariant). We'll see later that this is the only way T can have an acip.

2.3 Corollary *If T is an inner function with $\alpha > 0$ in (⛟), then $m \circ T^{-1} = \frac{1}{\alpha} \cdot m$.*

`Vague sketch of proof that` $\widehat{T}\mathbb{1} = \frac{1}{\alpha}\mathbb{1}$

- $\pi b \varphi_{ib} \xrightarrow[b\to\infty]{} 1$ unifomly on bounded subsets of $\mathbb{R}$;
- if $T(ib) = u(b) + iv(b)$, then $\frac{v(b)}{b} \xrightarrow[b\to\infty]{} \alpha$ & $\frac{u(b)}{b} \xrightarrow[b\to\infty]{} 0$.
- $$\widehat{T}\mathbb{1} \xleftarrow[b\to\infty]{} \pi b \widehat{T}\varphi_{ib} = \pi b \varphi_{T(ib)} \xrightarrow[b\to\infty]{} \frac{1}{\alpha}\mathbb{1}. \quad ☑$$

Exercise 5: Boole & Glaisher transformations.
For $\alpha,\ \beta > 0$ define $T = T_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ by $T(x) := \alpha x - \frac{\beta}{x}$.

(a) Show that if $\alpha + \beta = 1$, then $\widehat{T}\varphi_i = \varphi_i$ and T has an absolutely continuous, invariant probability (a.c.i.p.).

Consider the Glaisher transformations $T : \mathbb{R} \to \mathbb{R}$ of form

$$T_{a,b}x := ax + b\tan x \quad (a, b \geq 0,\ a + b > 0).$$

(b) Give conditions on a, b so that $T_{a,b}$ has an absolutely continuous invariant probability.

(c) Show that $T_{1,b}$ preserves Lebesgue measure.

(d) Show that $T_{0,1}x = \tan x$ preserves the measure $d\mu_0(x) := \frac{dx}{x^2}$.

`Hint:` $S := \pi \circ T_{0,1} \circ \pi^{-1}$ preserves Lebesgue measure where $\pi(x) := \frac{-1}{x}$.

Recurrence and conservativity

A set $W \in \mathcal{B},\ m(W) > 0$ is called *wandering* (for the `NPT` $(X, \mathcal{B}, m, T)$) if the sets $\{T^{-n}W\}_{n=0}^{\infty}$ are disjoint. and the `NPT` T is called *conservative* if $\mathcal{W}(T) = \varnothing$ (i.e. there are no wandering sets).

Remarks.

¶1 A conservative `NPT` $(X, \mathcal{B}, m, T)$ is non-singular. Else $\exists\ A \in \mathcal{B},\ m(A) > 0$ with $m(T^{-1}A) = 0$, whence $m(T^{-n}A) = 0\ \forall\ n \geq 1$. It follows that $W := A \setminus \bigcup_{n=1}^{\infty} T^{-n}A$ is a wandering set satisfying $m(W) = m(A)$.

¶2 Similarly, a `NPT` $(X, \mathcal{B}, m, T)$ is conservative iff (!) it is *incompressible* in the sense that $A \in \mathcal{B}$ and $T^{-1}A \subset A$ imply $A = T^{-1}A \bmod m$.

¶3 If $(X, \mathcal{B}, m, T)$ is a Kakutani skyscraper over the `NST` $(\Omega, \mathcal{F}, \mu, S)$, then T is conservative iff S is conservative.

Proof of $\Leftarrow$ If T is not conservative, then $\exists\ A \in \mathcal{F}_+,\ A \times \{1\} \in \mathcal{W}(T)$ whence $A \in \mathcal{W}(S)$. ☑

Proof of $\Rightarrow$ Let $W \in \mathcal{W}(S)$, then (!) $W \times \{1\} \in \mathcal{W}(T)$. ☑

Halmos recurrence theroem

Let $(X, \mathcal{B}, m, T)$ *be a* `NPT`. *TFAE:*

(i) T *is conservative;*

(ii) $A \overset{m}{\subset} \bigcup_{n=1}^{\infty} T^{-n}A \quad \forall\ A \in \mathcal{B}_+$;

(iii) $\sum_{n=1}^{\infty} 1_A \circ T^n = \infty$ *a.e. on* $A \quad \forall\ A \in \mathcal{B}_+$.

Proof of (i) ⇒ (iii)

Suppose that $A \in \mathcal{B},\ m(A) > 0$. The set $W := A \setminus \bigcup_{n=1}^{\infty} T^{-n}A$ is wandering if of positive measure, whence $m(W) = 0$ and $A \subseteq \bigcup_{n=1}^{\infty} T^{-n}A \mod m$. By null preservation, $T^{-N}A \subseteq \bigcup_{n=N+1}^{\infty} T^{-n}A \mod m\ \forall\ N \geq 1$, whence, mod m:

$$A \subseteq \bigcup_{n=1}^{\infty} T^{-n}A \subseteq \cdots \subseteq \bigcup_{n=N+1}^{\infty} T^{-n}A \subseteq \cdots \subseteq \bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty} T^{-n}A = \left[\sum_{n=1}^{\infty} 1_A \circ T^n = \infty\right].$$

☑

Conditions for conservativity.

2.4 Maharam Recurrence theorem

Let $(X, \mathcal{B}, m, T)$ *be* MPT.

If $\exists\ A \in \mathcal{B},\ m(A) < \infty$ *such that* $X = \bigcup_{n=1}^{\infty} T^{-n}A \mod m$*, then* T *is conservative.*

Proof We have that $\sum_{n=1}^{\infty} 1_A \circ T^n = \infty$ a.e. If $W \in \mathcal{W},\ m(W) > 0$, then $\forall\ n \geq 1$,

$$\begin{aligned} m(A) &\geq \int_{T^{-n}A}\left(\sum_{k=1}^{n} 1_W \circ T^k\right)dm = \sum_{k=1}^{n} m(T^{-k}W \cap T^{-n}A) \\ &= \sum_{j=0}^{n-1} m(W \cap T^{-j}A) = \int_W\left(\sum_{j=0}^{n-1} 1_A \circ T^j\right)dm \to \infty. \end{aligned}$$

Contradiction. ☑

For example, any PPT is conservative. This statement is known as `Poincaré's recurrence theorem`.

A MPT of a σ-finite, infinite measure space need not be conservative. For example $x \mapsto x + 1$ is a measure preserving transformation of $\mathbb{R}$ equipped with Borel sets, and Lebesgue measure, which is totally dissipative.

Example.

The original Boole transformation $T : \mathbb{R} \to \mathbb{R}$ given by

$$T(x) = x - \frac{1}{x}$$

is conservative.

Proof By corollary 2.3, $m\circ T^{-1} = m$. By inspection, $\bigcup_{n=0}^{\infty} T^{-n}[-1,1] = \mathbb{R}$. ☑

Exercise 6. Let

$$T(x) = x + \sum_{k=1}^{N} \frac{p_k}{t_k - x} \text{ where } p_1,\dots,p_N > 0 \ \&\ t_1,\dots,t_N \in \mathbb{R}.$$

Show that $\bigcup_{n=1}^{\infty} T^{-n}(u,v) = \mathbb{R} \mod m$ where $u := \min T^{-1}\{0\}$ & $v := \max T^{-1}\{0\}$; and hence that T is conservative.
`Hint` WLOG, $N \ge 2$, $u < 0 < v$ & $T(0) = 0$.

Exercise 7: Skyscaper conservativity.
Let $(X,\mathcal{B},m,T)$ be a Kakutani skyscaper over the `NST` $\Omega,\mathcal{F},\mu,S)$. Show that T is conservative iff S is conservative.

Exercise 8: Stronger recurrence properties.
Let $(X,\mathcal{B},m,T)$ be a conservative `NST`.

(i) Show that if (Y,d) is a separable, metric space and $h : X \to Y$ is measurable, then

$$\varliminf_{n\to\infty} d(h, h\circ T^n) = 0 \ \text{ a.e..}$$

(ii) What about when (Y,d) is an arbitrary metric space (not necessarily separable) and $h : X \to Y$ is measurable?

Induced transformation.
This is the "reverse" of the skyscraper construction.

Suppose $(X,\mathcal{B},m,T)$ is a `NST` and let $A \in \mathcal{B}_+$ be such that m–a.e. point of A returns to A under iterations of T (e.g. if $(X,\mathcal{B},m,T)$ is conservative). The *return time* function to A, defined for $x \in A$ by $\varphi_A(x) := \min\{n \ge 1 : T^n x \in A\}$ is finite m–a.e. on A.

The *induced transformation* on A is defined by $T_A x = T^{\varphi_A(x)} x$.

The first key observation is that $(A,\mathcal{B}\cap A,T_A,m_A)$ is a `NST` and, if T is a `MPT`, then so is T_A. These follows from

$$T_A^{-1} B = \bigcup_{n=1}^{\infty} [\varphi = n] \cap T^{-n} B.$$

It follows that $\varphi_A \circ T_A$ is defined a.e. on A and an induction now shows that all powers $\{T_A^k\}_{k\in\mathbb{N}}$ are defined a.e. on A, and satisfy

$$T_A^k x = T^{(\varphi_A)_k(x)} x \text{ where } (\varphi_A)_1 = \varphi_A,\ (\varphi_A)_k = \sum_{j=0}^{k-1} \varphi_A \circ T_A^j.$$

Exercise 9: Inducing inverse to skyscraping.

Let $(X, \mathcal{B}, m, T)$ be an invertible, conservative `NST` and suppose that $A \in \mathcal{B},\ m(A) > 0$ satisfies $\bigcup_{n=1}^{\infty} T^{-n}A = X \mod m$.

Show that

(i) $(X, \mathcal{B}, m, T)$ is isomorphic to the the Kakutani skyscraper over $(A, \mathcal{B} \cap A, m_A, T_A)$ with height function φ_A.

(ii) T is conservative $\implies$ T_A is conservative.

Both constructions can be generalized to the nonsingular case.

Hopf decomposition

Let $(X, \mathcal{B}, m, T)$ be a `NPT`. The collection $\mathcal{W}(T)$ of wandering sets is a hereditary collection (any measurable subset of a member is also a member), and T-sub-invariant (W wandering or null $\implies$ $T^{-1}W$ wandering or null).

By `exhaustion`, $\exists$ a countable union of wandering sets $\mathfrak{D}(T) \in \mathcal{B}$ with the property that any wandering set $W \in \mathcal{B}$ is contained in $\mathfrak{D}(T) \mod m$ (i.e. $m(W \smallsetminus \mathfrak{D}(T)) = 0$). This `measurable union` $\mathfrak{D}(T)$ of $\mathcal{W}(T)$ is unique mod m and $T^{-1}\mathfrak{D} \subseteq \mathfrak{D} \mod m$. It is called the *dissipative part* of the nonsingular transformation T.

Evidently T is conservative on $\mathfrak{C}(T) := X \smallsetminus \mathfrak{D}(T)$, the *conservative part* of T.

The partition $\{\mathfrak{C}(T),\ \mathfrak{D}(T)\}$ is called the *Hopf decomposition* of T.

The nonsingular transformation T is called (totally) *dissipative* if $\mathfrak{D}(T) = X \mod m$.

2.7 Proposition. *Any inner function T with $\alpha > 1$ in (🚂) is dissipative.*

Proof By corollary 2.3,

$$\sum_{n=1}^{\infty} m(T^{-n}A) < \infty \quad \forall\ A \in \mathcal{B},\ 0 < m(A) < \infty$$

and is dissipative. ☑

Exercise 10:

In this exercise, you show that if $(X, \mathcal{B}, m, T)$ is an invertible `NST`, then $\exists$ a wandering set $W \in \mathcal{B}$ such that

$$\mathfrak{D} = \biguplus_{n \in \mathbb{Z}} T^n W.$$

`Hints` For $A \in \mathcal{B}$ set $A^T := \bigcup_{n\in\mathbb{Z}} T^n A$.

WLOG, $m(X) = 1$.

- Define $\epsilon_1 := \sup\{m(W) : W \in \mathcal{W}\}$;
- choose $W \in \mathcal{W}$ with $m(W_1) \ge \frac{\epsilon_1}{2}$;
- define $\epsilon_2 := \sup\{m(W) : W \in \mathcal{W}, W \cap W_1^T = \varnothing\}$;
- choose $W_2 \in \mathcal{W}, W \cap W_1^T = \varnothing$ with $m(W_2) \ge \frac{\epsilon_2}{2}$.

Continue this process to obtain $\{W_n : n \in \mathbb{N}\} \subset \mathcal{W}$ & $\{\epsilon_n : n \in \mathbb{N}\} \subset \mathbb{R}_+$ so that

- $W_k \cap W_\ell^T = \varnothing \ \forall \ k > \ell$;
- $2m(W_n) \ge \epsilon_n := \sup\{m(W) : W \in \mathcal{W}, W \cap W_k^T = \varnothing \ \forall \ 1 \le k \le n-1\}$.

Show that $W := \dot\bigcup_{n\ge1} W_n$ is as required.

Exercise 11: `Hopf decomposition not` T`-invariant.`

Let $(X, \mathcal{B}, m, T) = ([0,2], \mathcal{B}([0,2]), \texttt{Leb})$ where $T : [0,2) \to [0,2)$ is defined by

$$T(x) := \begin{cases} 2x & x \in [0,1), \\ 1 + (2(x-1) \mod 1) & x \in [1,2). \end{cases}$$

Show that T is non-singular, $\mathfrak{D}(T) = [0,1), \mathfrak{C}(T) = [1,2)$ and that

$$T^{-1}\mathfrak{D}(T) = [0, \tfrac{1}{2}) \ \& \ m(T^{-1}\mathfrak{D}(T) \Delta \mathfrak{D}(T)) = \tfrac{1}{2}.$$

Conservativity and transfer operators

2.10 Hopf's recurrence theorem

If $T : X \to X$ is nonsingular then

(i) $$\mathfrak{C}(T) \supset [\sum_{n=1}^{\infty} \widehat{T}^k f = \infty] \quad mod \ \ m \ \forall \ f \in L^1(m)_+; \quad \&$$

(ii) $$\mathfrak{C}(T) = [\sum_{n=1}^{\infty} \widehat{T}^k f = \infty] \quad mod \ \ m \ \forall \ f \in L^1(m), f > 0.$$

Proof (i) Fix $f \in L^1(m)_+$ and $W \in \mathcal{W}_T$, then

$$\infty > \int_X f dm \ge \int_X f \left(\sum_{n\ge0} 1_W \circ T^n\right) dm = \int_W \left(\sum_{n\ge0} \widehat{T}^n f\right) dm.$$

This shows that $\mathfrak{D}(T) \subset [\sum_{n=1}^{\infty} \widehat{T}^k f < \infty]$. ☑

(ii) Assume otherwise and fix $f \in L^1(m), f > 0, \ A \in \mathcal{B}_+, \ A \subset \mathfrak{C}(T)$ s.t. $\sum_{n=1}^{\infty} \widehat{T}^k f < \infty$ on A.

WLOG $f(x) \ge c > 0 \ \forall \ x \in A$, and the series converges uniformly on A whence $\int_A (\sum_{n=1}^{\infty} \widehat{T}^k f) dm < \infty$.

On the other hand, by Halmos' recurrence theorem $\sum_{n\ge 0} 1_A \circ T^n = \infty$ a.e. on A.

Thus

$$\infty > \int_A \left(\sum_{n=0}^{\infty} \widehat{T}^k f \right) dm = \int_X f \left(\sum_{n\ge 0} 1_A \circ T^n \right) dm$$
$$\ge \int_A f \left(\sum_{n\ge 0} 1_A \circ T^n \right) dm \ge c \int_A \left(\sum_{n\ge 0} 1_A \circ T^n \right) dm = \infty \quad \boxtimes \quad ☑$$

2.11 Corollary.

If $Tx = x + \beta + \int_{\mathbb{R}} \frac{d\nu(t)}{t-x}$ where ν is a finite, Lebesgue-singular, measure on $\mathbb{R}$ with compact support, then T is conservative if $\beta = 0$ and dissipative if $\beta \ne 0$.

Proof By Hopf's recurrence theorem, it suffices to show that $\sum_{n\ge 0}^{\infty} \widehat{T}^n \varphi_\omega$ diverges a.e. for some $\omega \in \mathbb{R}^{2+}$ when $\beta = 0$; and converges a.e. for some $\omega \in \mathbb{R}^{2+}$ when $\beta \ne 0$.

By Boole's formula

$$\widehat{T}^n \varphi_\omega(x) = \varphi_{T^n\omega}(x) = \frac{1}{\pi} \cdot \frac{v_n}{(x-u_n)^2 + v_n^2} \quad \text{where } T^n\omega = u_n + iv_n.$$

Elementary estimations show that

- when $\beta \ne 0$. $\exists\ B = B(\omega) \in \mathbb{R}_+$ & $C = C(\omega) \in \mathbb{R}$ so that

$$\text{(I)} \qquad v_n \uparrow B \quad \& \quad u_n = \beta n - \frac{\nu}{\beta} \log n + C + O\Big(\frac{\log n}{n}\Big) \ \ \texttt{as} \ \ n \to \infty;$$

and

- when $\beta = 0$,

$$\text{(II)} \qquad \sup_{n\ge 1} |u_n| < \infty \quad \& \quad v_n \sim \sqrt{2\nu n} \ \ \texttt{as} \ \ n \to \infty \ \ \texttt{where} \ \ \nu := \sum_{k=1}^{n} p_k$$

It follows that T is

- conservative when $\beta = 0$ ($\because\ \widehat{T}^n \varphi_\omega \propto \frac{1}{\sqrt{n}}$ uniformly on bounded subsets of $\mathbb{R}$);
- and totally dissipative when $\beta \ne 0$ ($\because\ \widehat{T}^n \varphi_\omega \ll \frac{1}{n^{\frac{3}{2}}}$ on $\mathbb{R}$). ☑

Exercise 11: Hopf recurrence theorem for `MPT`s.

Suppose that T is a `MPT` of the σ-finite measure space $(X, \mathcal{B}, m)$. Show that

$$\left[\sum_{n=1}^{\infty} f \circ T^n = \infty \right] = \mathfrak{C}(T) \quad \text{mod } m \ \ \forall\ f \in L^1(m), f > 0.$$

Lecture # 3 15/10/2014 10-12.

ERGODICITY

A transformation T of the measure space $(X,\mathcal{B},m)$ is called *ergodic* if

$$A\in\mathcal{B},\ T^{-1}A=A \mod m \ \Rightarrow\ m(A)=0,\ \text{or } m(A^c)=0.$$

In general, let

$$\mathfrak{I}(T):=\{A\in\mathcal{B},\ T^{-1}A=A\}.$$

Remarks.

`It is not hard to see that:`

- $\mathfrak{I}(T)$ is a σ-algebra (and that T is ergodic iff $\mathfrak{I}\overset{m}{=}\{\varnothing,X\}$);
- an invertible ergodic nonsingular transformation of a non-atomic measure space is necessarily conservative;
- a nonsingular transformation $(X,\mathcal{B},m,T)$ is conservative and ergodic iff

$$\sum_{n=1}^{\infty}1_A\circ T^n=\infty \text{ a.e. } \ \forall A\in\mathcal{B}_+.$$

Exercise 13.

(i) Suppose that $(X,\mathcal{B},m,T)$ is a Kakutani skyscraper over the ergodic `NST` $(\Omega,\mathcal{F},\mu,S)$, then T is ergodic.

(ii) Suppose that $(X,\mathcal{B},m,T)$ is a conservative, `NST` and that $A\in\mathcal{B}$, $\bigcup_{n=1}^{\infty}T^{-n}A\overset{m}{=}X$, then T is ergodic $\iff$ T_A is ergodic.

Exercise 14.

Let $(X,\mathcal{B},m,T)$ be a conservative, ergodic nonsingular transformation and let (Z,d), a separable metric space. Show that if $f:X\to Z$ is a measurable map, then for a.e. $x\in X$,

$$\overline{\{f(T^nx):n\in\mathbb{N}\}}=\operatorname{spt} m\circ f^{-1}.$$

SOME ERGODIC TRANSFORMATIONS

`Rotations of the circle.` Let X be the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}=[0,1)$, $\mathcal{B}$ be its Borel sets, and m be Lebesgue measure. The *rotation* (or translation) of the circle by $x\in X$ is the transformation $r_x:X\to X$ defined by $r_x(y)=x+y \mod 1$.

Evidently $m\circ r_x=m$ for every $x\in X$ and each r_x is an invertible measure preserving transformation of $(X,\mathcal{B},m)$.

3.2 Proposition

If α is irrational, then r_α is ergodic.

Proof

We use harmonic analysis. Suppose that $f : X \to \mathbb{R}$ is bounded and measurable, and that $f \circ r_\alpha = f$, then

$$\begin{aligned}\widehat{f}(n) &= \int_{[0,1)} f(y) e^{-2\pi i n y} dy \\ &= \int_{[0,1)} f(\alpha + y) e^{-2\pi i n y} dy = \lambda^n \widehat{f}(n) \text{ where } \lambda := e^{2\pi i \alpha}.\end{aligned}$$

It follows that

$$\lambda^n = 1 \text{ whenever } \widehat{f}(n) \neq 0,$$

whence, since $\lambda^n \neq 1\ \forall\ n \neq 0$, $\widehat{f}(n) = 0$ whenever $n \neq 0$ and f is constant. ☑

Ergodicity of rank one constructions.

3.3 Proposition

Let $(X, \mathcal{B}, m, T)$ *be a rank one* `MPT` *as above, then* T *is ergodic.*

Proof Let

$$R_n = \biguplus_{I \in \mathfrak{r}_n} I \uparrow X$$

be the refining sequence of Rokhlin towers defining T; where each

$$\mathfrak{r}_n = \{T^j I_n :\ 0 \le j \le k_n - 1\}$$

is a partition of R_n into intervals with equal lengths $m(I_n) \xrightarrow[n\to\infty]{} 0$.

We claim first that it suffices to show that

For $\epsilon > 0$ & $A \in \mathcal{B}_+,\ \exists\ N = N_{\epsilon,A}$ so that

☕ $$\forall\ n > N\ \exists\ I \in \mathfrak{r}_n \texttt{ s.t. } m(A|I) > 1 - \epsilon.$$

Proof of sufficiency of ☕

Suppose that $A \in \mathcal{B}_+,\ TA = A$. We'll show assuming ☕ that $\forall\ N \ge 1$ large enough,

$$m(A \cap R_N) > (1 - \epsilon) m(R_N)\ \ \forall\ \ \epsilon > 0$$

whence $A \supset R_N \uparrow X \mod m$.

To see this, choose (by ☕) $n \ge N$ & $J \in \mathfrak{r}_n$ satisfying $m(A|J) > 1 - \epsilon$. Then for each $K = T^{i_K} J \in \mathfrak{r}_n$, we have using T-invariance of m & A:

$$m(A|K) = \frac{m(A \cap T^{i_K} J)}{m(T^{i_K} J)} = m(A|J) > 1 - \epsilon$$

whence

$$m(A \cap R_N) = \sum_{K \in \mathfrak{r}_n,\ K \subset R_N} m(A|K) m(K) > (1 - \epsilon) m(R_N).\ ☑$$

Proof of ☕

Suppose that $A \in \mathcal{B}_+$ and fix $N \geq 1$ so that $B := A \cap R_N \in \mathcal{B}_+$. For $n \geq N$, let

$$\mathfrak{s}_n := \{I \in \mathfrak{r}_n : \ I \subset R_N\}.$$

Fix $0 < \epsilon < 1$ and for $n \geq N$ let

$$\mathcal{Z}_n := \{I \in \mathfrak{s}_n : \ m(B|I) > 1 - \epsilon\} \ \& \ \mathcal{Y}_n := \mathfrak{s}_n \setminus \mathcal{Z}_n.$$

We show that $\forall \ n$ large enough, $\mathcal{Z}_n \neq \varnothing$.

Since $\sigma(\bigcup_{n \geq N} \mathfrak{s}_n) = \mathcal{B}(R_N)$, $\exists \ n \geq N$ & C_n, a union of sets in $\mathfrak{s}_n$ so that $m(B \Delta C_n) < \frac{\epsilon^2 m(B)}{9}$. It follows that

$$\begin{aligned} m(C_n) - \frac{\epsilon^2 m(B)}{9} &< m(B \cap C_n) \\ &= \sum_{I \in \mathfrak{s}_n, \ I \subset C_n} m(B|I)m(I) \\ &= \sum_{I \in \mathcal{Z}_n, \ I \subset C_n} m(B|I)m(I) + \sum_{I \in \mathcal{Y}_n, \ I \subset C_n} m(B|I)m(I) \\ &\leq \sum_{I \in \mathcal{Z}_n, \ I \subset C_n} m(I) + (1-\epsilon) \sum_{I \in \mathcal{Y}_n, \ I \subset C_n} m(I) \\ &= m(\bigcup \mathcal{Z}_n) + (1-\epsilon) m(C_n) \end{aligned}$$

whence

$$\begin{aligned} m(\bigcup \mathcal{Z}_n) &\geq m(C_n) - \frac{\epsilon^2 m(B)}{9} - (1-\epsilon) m(C_n) \\ &= \epsilon m(C_n) - \frac{\epsilon^2 m(B)}{9} \\ &> \epsilon m(B) - \frac{\epsilon^3 m(B)}{9} - \frac{\epsilon^2 m(B)}{9} \\ &> \frac{7 \epsilon m(B)}{9} > 0. \quad ☑ \end{aligned}$$

Ergodicity via stronger properties

Sometimes it's easier to prove more than ergodicity.

`One-sided Bernoulli shifts.`

Let $X = \mathbb{R}^{\mathbb{N}}$ and let $\mathcal{B}(X)$ be the σ-algebra generated by *cylinder* sets of form $[A_1, \dots, A_n] := \{\underline{x} \in X : x_j \in A_j, \ 1 \leq j \leq n\}$, where $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$ (the Borel subsets of $\mathbb{R}$), and let the *shift* $S : X \to X$ be defined by

$$(Sx)_n = x_{n+1}.$$

For $p : \mathcal{B}(\mathbb{R}) \to [0,1]$ a probability, let $\mu_p : \mathcal{B}(X) \to [0,1]$ be the probability[2] satisfying

$$\mu_p([A_1, ..., A_n]) = \prod_{k=1}^{n} p(A_k) \quad (A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})).$$

Evidently $S^{-1}[A_1, ..., A_n] = [\mathbb{R}, A_1, ..., A_n]$ whence $\mu_p \circ S^{-1} = \mu_p$.

The *one-sided Bernoulli shift* with marginal distribution p is the probability preserving transformation S of $(X, \mathcal{B}, \mu_p)$.

`Tail, exactness.` Let T be a nonsingular transformation of $(X, \mathcal{B}, m)$. The *tail* σ-algebra of T is

$$\mathfrak{T}(T) := \bigcap_{n=1}^{\infty} T^{-n}\mathcal{B}.$$

The transformation T is called *exact* if $\mathfrak{T}(T) = \{\varnothing, X\}$ mod m.

Evidently $\mathfrak{I}(T) \subset \mathfrak{T}(T)$ mod m and so exact transformations are ergodic.

3.4 Kolmogorov's zero-one law

Any one-sided Bernoulli shift is exact.

Proof

Suppose that $B \in \mathcal{B}$ is a finite union of cylinders. If the length of the longest cylinder in the union is n, then

$$\mu_p(B \cap S^{-n}C) = \mu_p(B)\mu_p(C) \quad \forall\ C \in \mathcal{B}.$$

Now suppose $A \in \mathfrak{T}$. Since, for each $n \in \mathbb{N}$,

$$A = S^{-n}A_n \text{ where } A_n \in \mathcal{B},\ \mu_p(A_n) = \mu_p(A),$$

we have that

$$\mu_p(B \cap A) = \mu_p(B)\mu_p(A)$$

for $B \in \mathcal{B}$ a finite union of cylinders, and hence (by approximation) $\forall\ B \in \mathcal{B}$. This implies that

$$0 = \mu_p(A \cap A^c) = \mu_p(A)(1 - \mu_p(A))$$

demonstrating that $\mathfrak{T}$ is trivial mod μ_p. ☑

Note that no invertible nonsingular transformation can be exact (except the identity no a 1-pt. space). Hence an irrational rotation of $\mathbb{T}$ is ergodic, but not exact.

---

[2] `Existence guaranteed by Kolmogorov's existence theorem as on p.5.`

`Two sided Bernoulli shift.`

Recall that the *two sided* Bernoulli shift is defined with $X = \mathbb{R}^{\mathbb{Z}}$, $\mathcal{B}(X)$ the σ-algebra generated by cylinder sets of form

$$[A_1, \dots, A_n]_k := \{\underline{x} \in X : x_{j+k} \in A_j,\ 1 \le j \le n\}$$

where $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. The shift $S : X \to X$ is defined as before by $(Sx)_n = x_{n+1}$, and the S-invariant probability $\mu_p : \mathcal{B}(X) \to [0,1]$ is defined (for $p : \mathcal{B}(\mathbb{R}) \to [0,1]$ a probability) by

$$\mu_p([A_1, ..., A_n]_k) = \prod_{k=1}^{n} p(A_k) \quad (A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})).$$

The two sided Bernoulli shift is an invertible measure preserving transformation (and hence cannot be exact).

3.5 Proposition.

A two sided Bernoulli shift is `mixing` in the sense that

$$\mu_p(A \cap T^{-n}B) \to \mu_p(A)\mu_p(B) \text{ as } n \to \infty \quad \forall\ A, B \in \mathcal{B}(X),$$

and hence ergodic.

Proof True in the combinatorial sense for A, B finite unions of cylinders, and hence (by approximation) $\forall\ A,\ B \in \mathcal{B}$. ☑

Exercise 15.

Show that an exact probability preserving transformation (X, T, μ) is mixing.

`Hint` Show first that if $f \in L^2,\ n_k \to \infty$ and $f \circ T^{n_k} \to g \in L^2$ weakly in L^2, then g is tail measurable.

`Nonsingular Adding Machine.`

Let $\Omega = \{0,1\}^{\mathbb{N}}$, and $\mathcal{B}$ be the σ-algebra generated by cylinders. We consider again the *adding machine* $\tau : \Omega \to \Omega$ defined by

$$\tau(1, ..., 1, 0, \epsilon_{n+1}, \epsilon_{n+2}, ...) = (0, ..., 0, 1, \epsilon_{n+1}, \epsilon_{n+2}, ...).$$

The adding machine has

the odometer property.

⏲ $\{((\tau^k x)_1, ..., (\tau^k x)_n) : 0 \le k \le 2^n - 1\} = \{0,1\}^n\ \forall\ x \in \Omega,\ n \ge 1.$

The next lemma illustrates how the odometer "parametrizes" the tail of the one-sided shift $S : \Omega \to \Omega$.

3.6 Lemma

For $x \in \widetilde{\mathbb{Z}} := \{\tau^n(\overline{0}) :\ n \in \mathbb{Z}\}$,

$$\{y \in \Omega :\ \exists\ n \geq 0,\ S^n(y) = S^n(x)\} = \{\tau^n(x) :\ n \in \mathbb{Z}\}.$$

Proof Note that $\widetilde{\mathbb{Z}} = \{x \in \Omega :\ \exists\ \lim_{n\to\infty} x_n\}$. Thus for $x \notin \widetilde{\mathbb{Z}}$, both $\ell(x) := \min\{n \geq 1 :\ x_n = 0\}$ and $(x) := \min\{n \geq 1 :\ x_n = q\}$ are finite, whence

$$\exists\ n \geq 1 \text{ s.t. } S^n x = S^n \tau(x) = S^n \tau^{-1}(x).$$

Since $\tau\widetilde{\mathbb{Z}} = \widetilde{\mathbb{Z}}$,

$$\{y \in \Omega :\ \exists\ n \geq 0,\ S^n(y) = S^n(x)\}\ \supset\ \{\tau^n(x) :\ n \in \mathbb{Z}\}.$$

For the other inclusion, suppose $S^n x = S^n y = z$, then using the odometer property,

$$(\underbrace{0, \ldots, 0}_{n \text{ times}}, z) = \tau^{-\nu_n(x)}(x) = \tau^{-\nu_n(y)}(y)$$

where $\nu_n(\omega) := \sum_{k=1}^n 2^{k-1}\omega_n$. Thus

$$y = \tau^{\nu_n(y) - \nu_n(x)}(x). \quad ☑$$

For $p \in (0,1)$, set $\mu_p = \prod(1-p, p) \in \mathcal{P}(\Omega)$ and recall that

$$\frac{d\mu_p \circ \tau}{d\,\mu_p} = \left(\frac{1-p}{p}\right)^{\phi}$$

where $\phi(x) := \min\{n \geq 1 :\ x_n = 0\} - 2 =: \ell(x) - 2$.

3.7 Proposition

τ is an invertible, conservative, ergodic nonsingular transformation of $(\Omega, \mathcal{B}, \mu_p)$.

Proof It is not hard to show, using lemma 3.6, that $\mathfrak{I}(\tau) = \mathfrak{T}(S) \mod \mu_p$ and the ergodicity of $(\Omega, \mathcal{B}, \mu_p, \tau)$ follows from the exactness of $(\Omega, \mathcal{B}, \mu_p, S)$. As above, conservativity is automatic in this case. ☑

3.8 Rigidity proposition *For $0 < p < 1$, $(\Omega, \mathcal{B}, \mu_p)$ is* `rigid` *in the sense that if $f : \Omega \to \mathbb{R}$ is measurable, then $\forall\ \epsilon > 0$,*

$$\mu_p([|f \circ \tau^{2^n} - f| \geq \epsilon]) \to 0 \textit{ as } n \to \infty.$$

Proof Firstly, note that if $f : \Omega \to \mathbb{R}$ and f is defined by $f(x) = g(x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$, then $f \circ \tau^{2^k} \equiv f$ for every $k \geq n$. To enable approximation, we show that $\exists\ \Delta > 0$ & $M > 1$ so that

$$(⚽) \qquad \mu_p(\tau^{-2^n} A) \leq M \mu_p(A)^{\Delta} \quad \forall\ A \in \mathcal{B}.$$

Proof of (⚽)

As before,

$\frac{d\mu_p\circ\tau^{-1}}{d\mu_p} = (\frac{p}{1-p})^\psi$ where $\psi(x) := \min\{n\in\mathbb{N} : x_n = 1\} - 2$;

Using the odometer property:

$$(☕)\qquad \sum_{j=0}^{2^n-1}\psi(\tau^{-k}x) = \sum_{\epsilon\in\{0,1\}^n\setminus\{\underline{1}\}}\psi(\epsilon) + n + \psi(S^n x)$$
$$= \sum_{k=1}^{n}(k-2)2^{n-k} + n + \psi(S^n x)$$
$$= \psi(S^n x).$$

By (☕)

$$\frac{d\mu_p\circ\tau^{-2^n}}{d\mu_p} = \prod_{k=0}^{2^n-1}\left(\frac{d\mu_p\circ\tau^{-1}}{d\mu_p}\right)\circ\tau^{-k}$$
$$= \prod_{k=0}^{2^n-1}\left(\frac{p}{1-p}\right)^{\psi\circ\tau^{-k}}$$
$$= \left(\frac{p}{1-p}\right)^{\psi\circ S^n}.$$

Fix (!) $q > 1$ be such that $\frac{p^q}{(1-p)^{q-1}} < 1$, then

$$M^q := \|(\tfrac{p}{1-p})^\psi\|^q_{L^q(\mu_p)} \propto \sum_{n\ge1}(\tfrac{p^q}{(1-p)^{q-1}})^n < \infty$$

and for $A\in\mathcal{B}$,

$$\mu_p(\tau^{-2^n}A) = \int_A\left(\tfrac{p}{1-p}\right)^{\psi\circ S^n}d\mu_p \le \left\|\left(\tfrac{p}{1-p}\right)^\psi\right\|_q\mu_p(A)^{\frac{q-1}{q}} = M\mu_p(A)^{\frac{q-1}{q}}$$

by Hölder's inequality. ☑(⚽)

Now, suppose that $F : \Omega\to\mathbb{R}$ is measurable, and let $\epsilon > 0$ be given. There exist $n\in\mathbb{N}$, and $f : \Omega\to\mathbb{R}$ and f defined by $f(x) = g(x_1,\dots,x_n)$ for some $g : \{0,1\}^n\to\mathbb{R}$ such that $\mu_p([|F-f|\ge\epsilon/2]) < \epsilon$. For $k\ge n$, we have $f\circ\tau^{2^k}\equiv f$, whence

$$\mu_p([|F\circ\tau^{2^k} - F|\ge\epsilon]) \le \mu_p([|F\circ\tau^{2^k} - f\circ\tau^{2^k}|\ge\epsilon/2]) + \mu_p([|F-f|\ge\epsilon/2])$$
$$\le \epsilon + M\epsilon^{\frac{1}{q'}},$$

establishing that indeed

$$F\circ\tau^{2^n}\xrightarrow{\mu_p}F. \quad ☑$$

Lecture # 4 15/10/2014 18-20.

Ergodic Maharam extension for the non-singular adding machine.

Define $\tau_\phi : \Omega \times \mathbb{Z} \to \Omega \times \mathbb{Z}$ by

$$\tau_\phi(x,z) := (\tau x, z + \phi(x)).$$

For $0 < p < 1$ define the measure $m_p : \mathcal{B}(\Omega \times \mathbb{Z}) \to [0,\infty]$ by

$$m_p(A \times \{z\}) := \mu_p(A)(\tfrac{p}{1-p})^z.$$

This kind of transformation is aka a `Maharam extension`.

3.9 Theorem *For each $0 < p < 1$, $(\Omega \times \mathbb{Z}, \mathcal{B}(\Omega \times \mathbb{Z}), m_p, \tau_\phi)$ is a conservative, ergodic measure preserving transformation.*

Proof that $m_p \circ \tau_\phi = m_p$

Any $A \in \mathcal{B}(\Omega \times \mathbb{Z})$ has a measurable decomposition $A = \biguplus_{z,\ell \in \mathbb{Z}} A_{z,\ell} \times \{z\}$ where $\phi = \ell$ on $A_{z,\ell}$. Thus:

$$\begin{aligned} m_p(\tau_\phi A) &= \sum_{z,\ell \in \mathbb{Z}} m_p(\tau_\phi(A_{z,\ell} \times \{z\})) = \sum_{z,\ell \in \mathbb{Z}} m_p(\tau A_{z,\ell} \times \{z + \ell\}) \\ &= \sum_{z,\ell \in \mathbb{Z}} \mu_p(\tau A_{z,\ell})(\tfrac{p}{1-p})^{z+\ell} = \sum_{z,\ell \in \mathbb{Z}} \mu_p(A_{z,\ell})(\tfrac{p}{1-p})^z \\ &= \sum_{z,\ell \in \mathbb{Z}} m_p((A_{z,\ell} \times \{z\})) = m_p(A). \quad ☑ \end{aligned}$$

Proof of ergodicity of τ_ϕ Suppose that $F : \Omega \times \mathbb{Z} \to \mathbb{R}$ is bounded, measurable and τ_ϕ-invariant. We'll show first that $F(x,z) = F(x, z-1)$ m_p-a.e..

A similar calculation to (☕) shows that

$$(\underline{☕}) \qquad \phi_{2^n}(x) = \phi(S^n x).$$

Iterating τ_ϕ, we have that

$$F(x,z) = F \circ \tau^{2^n}(x,z) = F(\tau^{2^n} x, z + \phi_{2^n}(x)) = F(\tau^{2^n} x, z + \phi(S^n(x))).$$

By the rigidity proposition, $\exists\ n_k \to \infty$ and $\Omega_0 \in \mathcal{B}(\Omega)$, $\mu_p(\Omega_0) = 1$ such that

$$F(\tau^{2^{n_k}} x, z) \underset{k\to\infty}{\longrightarrow} F(x,z) \quad \forall\ x \in \Omega_0,\ z \in \mathbb{Z}.$$

The events

$$A_n = [\phi \circ S^n = -1] = \{x \in \Omega : x_{n+1} = 0\}$$

are independent under μ_p, and $\mu_p(A_n) = 1 - p$.

By the Borel-Cantelli lemma, $\exists\ \Omega_1 \in \mathcal{B}(\Omega),\ \Omega_1 \subset \Omega_0,\ \mu_p(\Omega_1) = 1$ such that $\forall\ x \in \Omega_1,\ \exists\ k_\ell = k_\ell(x) \to \infty$ with

$$\phi(S^{n_{k_\ell}}x) = -1\ \forall\ \ell \geq 1,$$

whence

$$F(x,z) = F(\tau^{2^{n_{k_\ell}}}x, z + \phi(S^{n_{k_\ell}}(x)) = F(\tau^{2^{n_{k_\ell}}}x, z-1) \xrightarrow[\ell\to\infty]{} F(x,z-1).$$

Thus $\exists\ f : \Omega \to \mathbb{R}$, measurable, such that $F(x,z) = f(x)$ μ_p-a.e. $\forall\ z \in \mathbb{Z}$. Since F is τ_ϕ-invariant, f is τ-invariant and μ_p-a.e. constant by ergodicity of $(\Omega, \mathcal{B}, \mu_p, \tau)$. ☑

3.10 Corollary

The nonsingular adding machine $(\Omega, \mathcal{B}, \mu_p, \tau)$ has no σ-finite, absolutely continuous, invariant measure.

Proof Suppose otherwise, that $m \ll \mu_p$ is a σ-finite, τ-invariant measure and let $dm = hd\mu_p$ where $h \geq 0$ is measurable, then(!) $h > 0$ μ_p-a.e. ($\because m \sim \mu_p$) and

$$h = \widehat{\tau^{-1}}h = \tau' h \circ \tau \implies \tau' = \frac{h}{h \circ \tau}.$$

Since $\tau' = (\frac{1-p}{p})^\phi$ we have that $\phi = k - k \circ \tau$ where $k : \Omega \to \mathbb{R}$ satisfies $h = (\frac{1-p}{p})^k$.

Define $F : \Omega \times \mathbb{Z} \to \mathbb{R}$ by $F(x,z) = z + k(x)$, then

$$F(\tau_\phi(x,z)) = F(\tau x, z + \phi(x)) = z + \phi(x) + k(\tau x) = z + k(x) = F(x,z).$$

By ergodicity, F is constant, but it isn't ($\because$ $F(x,z+1) = F(x,z) + 1$). ☑

Exercise 16: Dissipative exact `MPT`s.

Let $\Omega = \{0,1\}^{\mathbb{N}}$ let $S : \Omega \to \Omega$ be the shift, let $\tau : \Omega \to \Omega$ be the adding machine and let $\mu_p = \prod(1-p,p) \in \mathcal{P}(\Omega)$, $(0 < p < 1)$. Define $f,\ \phi : \Omega \to \mathbb{Z}$ by

$$f(x) := x_1 \quad \& \quad \phi(x) := \ell(x) - 2), \quad \ell(x) := \min\{n \geq 1 :\ x_n = 0\}$$

and $S_f,\ \tau_\phi$ by

$$S(x,z) = (\sigma(x), z + x_1),\ T(x,z) := (\tau(x), z + \ell(x) - 2).$$

Show that

(i) $(\Omega \times \mathbb{Z}, \mathcal{B}(\Omega \times \mathbb{Z}), \mu_p \times \#, S_f)$ is a totally dissipative `MPT`;

(ii) $\mathfrak{T}(S_f) = \mathfrak{I}(\tau_\phi)$.

(iii) $(\Omega \times \mathbb{Z}, \mathcal{B}(\Omega \times \mathbb{Z}), \mu_p \times \#, S_f)$ is exact.

Ratio ergodic theorem

Suppose that $(X,\mathcal{B},m,T)$ is a conservative, nonsingular transformation.

4.6 Hurewicz's Ergodic Theorem

$$\frac{\sum_{k=1}^n \widehat{T}^k f(x)}{\sum_{k=1}^n \widehat{T}^k p(x)} \xrightarrow[n\to\infty]{} E_{m_p}\left(\frac{f}{p}|\mathfrak{I}\right)(x) \ \textit{for a.e. } x\in X,\ \forall f,p\in L^1(m),\ \ p>0,$$

where $dm_p = pdm$, *and* $\mathfrak{I}$ *is the* σ*-algebra of* T*-invariant sets in* $\mathcal{B}$.

Conditional expectations.

Here, given a probability space $(\Omega,\mathcal{F},P)$, and a sub-σ-algebra $\mathcal{C}\subset\mathcal{F}$, the *conditional expectation wrt* $\mathcal{C}$ is a linear operator $f\mapsto E_P(f|\mathcal{C}),\ \ L^1(\Omega,\mathcal{F},P)\to L^1(\Omega,\mathcal{C},P)$ satisfying

$$\int_C E_P(f|\mathcal{C})dP = \int_C fdP \ \ \forall\ C\in\mathcal{C}.$$

Such operators are unique by their defining equations,. They exist $L^2(\Omega,\mathcal{F},P)\to L^2(\Omega,\mathcal{C},P)$ as orthogonal projections and extend to L^1 by approximation.

Proof of Hurewicz's theorem

Set, for $f,p\in L^1(m),\ p>0$, $\widehat{S}_0 f=0$, and $n\in\mathbb{N}$,

$$\widehat{S}_n f := \sum_{k=0}^{n-1}\widehat{T}^k f,\quad R_n(f,p) := \frac{\widehat{S}_n f}{\widehat{S}_n p}.$$

Let

$$\mathcal{H}_p := \{f = hp + g - \widehat{T}g\in L^1(m) : h\circ T = h\in L^\infty(m), g\in L^1(m)\}.$$

We claim that for $f = hp+g-\widehat{T}g\in\mathcal{H}_p$,

$$R_n(f,p) = h + \frac{g-\widehat{T}^n g}{\widehat{S}_n p}.$$

We show that $R_n(hp,p) = h$ where $h\circ T = h\in L^\infty(m)$. For $g\in L^\infty(m),\ n\in\mathbb{N}$,

$$\int_X \widehat{T}^n(hp)\cdot gdm = \int_X phg\circ T^n dm = \int_X ph\circ T^n g\circ T^n dm = \int_X h\widehat{T}^n p\cdot gdm$$

for every whence $\widehat{T}^n f = h\widehat{T}^n p$, and $R_n(f,p) = h$. The convergence

$$R_n(f,p)\xrightarrow[n\to\infty]{} h,\ \text{a.e.}\ \forall\ f = hp+g-\widehat{T}g\in\mathcal{H}_p$$

follows immediately from the

4.7 Chacon-Ornstein Lemma

$$\frac{\widehat{T}^n g}{\widehat{S}_n p} \xrightarrow[n\to\infty]{} 0, \ \ a.e. \ \ \forall g \in L^1(m).$$

Proof Choose $\epsilon > 0$, and let $\eta_n = 1_{[\widehat{T}^n g > \epsilon \widehat{S}_n p]}$. We must show that $\sum_{n=1}^\infty \eta_n < \infty$ a.e. $\forall\ \epsilon > 0$.

We have

$$\epsilon p + \widehat{T}^{n+1} g - \epsilon \widehat{S}_{n+1} p = \widehat{T}(\widehat{T}^n g - \epsilon \widehat{S}_n p),$$

whence

$$\epsilon p + \widehat{T}^{n+1} g - \epsilon \widehat{S}_{n+1} p \le \widehat{T}(\widehat{T}^n g - \epsilon \widehat{S}_n p)_+,$$

where g_+ denotes $g \vee 0$, $f \vee g = \max\{f, g\}$.

Multiplying both sides of the inequality by η_{n+1}:

$$\begin{aligned}\eta_{n+1}\epsilon p + \eta_{n+1}(\widehat{T}^{n+1} g - \epsilon \widehat{S}_{n+1} p) &= \eta_{n+1}\epsilon p + (\widehat{T}^{n+1} g - \epsilon \widehat{S}_{n+1} p)_+ \\ &\le \eta_{n+1}\widehat{T}(\widehat{T}^n g - \epsilon \widehat{S}_n p)_+ \\ &\le \widehat{T}(\widehat{T}^n g - \epsilon \widehat{S}_n p)_+.\end{aligned}$$

Equivalently,

$$\eta_{n+1}\epsilon p \le \widehat{T} J_n - J_{n+1}$$

where $J_n := (\widehat{T}^n g - \epsilon \widehat{S}_n p)_+$.

Integrating, we get

$$\epsilon \int_X p\eta_{n+1} dm \le \int_X (J_n - J_{n+1}) dm$$

and, summing over n, we get

$$\epsilon \int_X p \sum_{n=2}^N \eta_n dm \le \int_X J_1 dm < \infty.$$

This shows that indeed

$$\sum_{n=1}^\infty \eta_n < \infty \text{ a.e.}$$

and thereby proves the lemma. ☑

We next establish that

☺ $$\overline{\mathcal{H}_p} = L^1(m).$$

To see this, we show that

$$k \in L^\infty(m), \int_X k f dm = 0 \ \forall f \in \mathcal{H}_p \quad \Rightarrow \quad k = 0 \ \text{ a.e.}$$

To see this, let

$$k \in L^\infty(m) \ \ni \ \int_X k f dm = 0 \ \forall f \in \mathcal{H}_p,$$

then, in particular

$$\int_X gk\circ T dm = \int_X \widehat{T}g\cdot k dm = \int_X gk dm \ \forall g\in L^1(m),$$

whence $k\circ T = k$ a.e., and $kp\in\mathcal{H}_p$.

Hence,

$$\int_X k^2 p dm = 0 \ \Rightarrow k = 0 \text{ a.e.}$$

☺ now follows from the Hahn-Banach theorem. ☑

Proof of Hurewicz's theorem `ctd.`

`Identification of the limit.`

We now identify the limit of $R_n(f,p)$ $f\in\mathcal{H}_p$. Define $\Phi_p : L^1(m)\to L^1(m_p)$ by

$$\Phi_p(f) := E_{m_p}(\tfrac{f}{p}\|\mathfrak{I}),$$

then

$$\|\Phi_p(f)\|_{L^1(m_p)} \le \|f\|_1 \ \forall f\in L^1(m).$$

We claim that

(✠) $$R_n(f,p)\xrightarrow[n\to\infty]{} \Phi_p(f) \ \ \forall \ f\in\mathcal{H}_p.$$

For this, it suffices that

$$\Phi_p(hp+g-\widehat{T}g) = h \ \ \forall \ f = hp+g-\widehat{T}g\in\mathcal{H}_p.$$

Indeed, if $k\circ T = k\in L^\infty(m)$, then

$$\begin{aligned}\int_X k\tfrac{f}{p}dm_p &= \int_X kf dm\\ &= \int_X k(hp+g-\widehat{T}g)dm\\ &= \int_X khp dm + \int_X k(g-\widehat{T}g)dm\\ &= \int_X kh dm_p. \ \boxtimes\end{aligned}$$

We extend (✠) to all $f\in L^1(m)$, by an approximation argument which uses the

5.1 Maximal inequality

For $f,p\in L^1$, such that $p>0$ a.e., and $t\in\mathbb{R}_+$,

$$m_p([\sup_{n\in\mathbb{N}} R_n(f,p) > t]) \le \frac{\|f\|_1}{t},$$

where $dm_p = pdm$.

Proof of theorem 4.6 given the maximal inequality

Let $f\in L^1(m)$. Fix $\epsilon>0$.

By ☺, we can write $f = g + k$, where $g \in \mathcal{H}_p$ and $\|k\|_1 < \epsilon^2$. I
t follows that

$$\overline{\lim_{n\to\infty}} |R_n(f,p) - \Phi_p(f)| \le \sup_{n\in\mathbb{N}} |R_n(k,p)| + |\Phi_p(k)|,$$

whence, by the maximal inequality, and by Tchebychev's inequality,

$$m_p([\overline{\lim_{n\to\infty}} |R_n(f,p) - \Phi_p(f)| > 2\epsilon]) \le m_p([\sup_{n\ge 1} |R_n(k,p)| > \epsilon]) + m_p([|\Phi_p(k)| > \epsilon])$$
$$\le \frac{2\|k\|_1}{\epsilon} \le 2\epsilon.$$

This last inequality holds for arbitrary $\epsilon > 0$, whence

$$\overline{\lim_{n\to\infty}} |R_n(f,p) - \Phi_p(f)| = 0 \quad \text{a.e.},$$

and the ergodic theorem is almost established, it remaining only to prove the maximal inequality.

5.2 Hopf's Maximal ergodic theorem

$$\int_{[M_n f>0]} f dm \ge 0, \ \ \forall f \in L^1(m), n \in \mathbb{N},$$

where

$$M_n f = \left(\bigvee_{k=1}^{n} \widehat{S}_k f\right)_+ = \left(\bigvee_{k=0}^{n} \widehat{S}_k f\right).$$

Proof Note first that if $M_n f(x) > 0$, then

$$M_n f(x) \le M_{n+1} f(x) = \bigvee_{k=1}^{n+1} \widehat{S}_k f(x)$$
$$= f(x) + \bigvee_{k=0}^{n} \widehat{S}_k \widehat{T} f(x) = f(x) + M_n \widehat{T} f(x).$$

Also (!) $M_n \widehat{T} f \le \widehat{T} M_n f$, whence

$$M_n f > 0 \ \Rightarrow \ f \ge M_n f - \widehat{T} M_n f,$$

and

$$\int_{[M_n f>0]} f dm \ge \int_{[M_n f>0]} (M_n f - \widehat{T} M_n f) dm.$$

Since $\widehat{T} M_n f \ge 0$ a.e., and $M_n f = 0$ on $[M_n f > 0]^c$, we get

$$\int_{[M_n f>0]} f dm \ge \int_{[M_n f>0]} M_n f dm - \int_{[M_n f>0]} \widehat{T} M_n f dm$$
$$\ge \int_X M_n f dm - \int_X \widehat{T} M_n f dm$$
$$= 0,$$

whence the theorem. ☑

Proof of the maximal inequality Suppose f, p, t are as in the maximal inequality, then

$$M_n(f - tp) > 0 \;\Leftrightarrow\; \max_{1\le k\le n} R_k(f,p) > t.$$

Thus, using Hopf's maximal ergodic theorem, we obtain

$$\int_{[M_n(f-tp)>0]} (f - tp)dm \ge 0,$$

whence

$$tm_p([\max_{1\le k\le n} R_k(f,p) > t]) \le \int_{[\max_{1\le k\le n} R_k(f,p)>t]} f dm$$
$$\le \|f\|_1.$$

The maximal inequality follows from this as $n \to \infty$. ☑

Hurewicz's ergodic theorem is now established.

Hurewicz's theorem for a conservative, ergodic nonsingular transformation T, states that

$$\frac{\sum_{k=0}^{n-1} \widehat{T}^k f(x)}{\sum_{k=0}^{n-1} \widehat{T}^k g(x)} \to \frac{\int_X f dm}{\int_X g dm} \text{ for a.e. } x \in X$$

whenever $f, g \in L^1(m),\ \int_X g dm \ne 0$.

Exercise 17: von Neuann's ergodic theorem.

Let $\mathcal{H}$ be a Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator.

Show that

(i) $\mathcal{H}_0 := \{f \in \mathcal{H} :\ Uf = f\}$ is a closed, invariant subspace of $\mathcal{H}$ and that

(ii) $$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f - Pf\right\| \xrightarrow[n\to\infty]{} 0 \quad \forall\ f \in \mathcal{H}$$

where $P : \mathcal{H} \to \mathcal{H}_0$ is orthogonal projection.

Exercise 18: Hopf's ergodic theorem.

Suppose that $(X, \mathcal{B}, m, T)$ is a conservative measure preserving transformation.

(i) Prove that

$$\frac{\sum_{k=1}^n f(T^k x)}{\sum_{k=1}^n p(T^k x)} \xrightarrow[n\to\infty]{} E_{m_p}(f|\mathfrak{I})(x) \text{for a.e. } x \in X,\ \forall f, p \in L^1(m),\quad p > 0.$$

`Hint` Hopf's ergodic theorem is a special case of Hurewicz's theorem in case T is invertible. It can be proved analogously for T non-invertible.

(ii) Now suppose that T is a conservative, ergodic, measure preserving transformation of the σ-finite, infinite measure space $(X, \mathcal{B}, m)$. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} f(T^k x) \underset{n\to\infty}{\longrightarrow} 0 \text{ for a.e. } x \in X,\ \forall f \in L^1(m).$$

Lecture # 5 16/10/2014 12-14.

Ergodicity via the ratio ergodic theorem

Boole transformations.

Let $(X,\mathcal{B},m)$ be $\mathbb{R}$ equipped with Borel sets and Lebesgue measure, and consider Boole's transformations:

$$(\text{✐})\qquad Tx = x+\beta+\sum_{k=1}^{N}\frac{p_k}{t_k-x}$$

where $N\ge 1,\ \ p_1,\dots,p_N>0$ and $\beta,\ t_1,...,t_N\in\mathbb{R}$.

By corollary 2.3, for T as in (✐), $(X,\mathcal{B},m,T)$ is a measure preserving transformation. By proposition 2.11, T is conservative iff $\beta=0$.

5.3 Proposition

(i) *If $\beta=0$, then T is conservative, ergodic.*

(ii) *If $\beta\neq 0$, then $\exists\ F:\mathbb{R}^{2+}\to\mathbb{R}^{2+}$ analytic, so that $F\circ T=F+\beta$. In particular, T is not ergodic.*

Proof sketch

For $\omega\in\mathbb{R}^{2+}$, write $T^n(\omega):=u_n+iv_n$, then

$$v_{n+1}=v_n+v_n\sum_{k=1}^{N}\frac{p_k}{(t_k-u_n)^2+v_n^2}$$
$$u_{n+1}=u_n+\beta+\sum_{k=1}^{N}\frac{p_k(t_k-u_n)}{(t_k-u_n)^2+v_n^2}.$$

As before, elementary calculations show that

- when $\beta\neq 0$. $\exists\ B=B(\omega)\in\mathbb{R}_+$ & $C=C(\omega)\in\mathbb{R}$ so that

$$\text{(I)}\qquad v_n\uparrow B\ \ \&\ \ \ u_n=\beta n-\frac{\nu}{\beta}\log n+C+O(\frac{\log n}{n})\ \ \texttt{as}\ \ n\to\infty;$$

and

- when $\beta=0$,

$$\text{(II)}\qquad \sup_{n\ge1}|u_n|<\infty\ \ \&\ \ \ v_n\sim\sqrt{2\nu n})\ \ \texttt{as}\ \ n\to\infty\ \ \texttt{where}\ \ \nu:=\sum_{k=1}^{n}p_k$$

`Proof of (i)`

Set $p:=\varphi_i$, then $\forall\ x\in\mathbb{R},\ \omega\in\mathbb{R}^{2+}$,

$$\widehat{S}_n\varphi_\omega(x):=\sum_{k=0}^{n-1}\widehat{T}^k\varphi_\omega(x)\sim\sum_{k=0}^{n-1}\frac{1}{\pi v_k}\sim a(n):=\frac{1}{\pi}\sqrt{\frac{2n}{\nu}}.$$

By Hurewicz's theorem, for $f \in L^1(m)$ and a.e. $x \in X$,

$$\frac{\widehat{S}_n f(x)}{a(n)} \sim \frac{\widehat{S}_n f(x)}{\widehat{S}_n p(x)} \xrightarrow[n\to\infty]{} E_{m_p}(f|\mathfrak{I}).$$

On the other hand, for $f = g * \varphi_{ib}$ $(g \in L^1(m))$,

$$f(x) := \int_{\mathbb{R}} g(t)\varphi_{ib}(x-t)dt = \int_{\mathbb{R}} g(t)\varphi_{t+ib}(x)dt$$

whence

$$\widehat{T}^n f = \int_{\mathbb{R}} g(t)\varphi_{T^n(t+ib)}(x)dt$$

and by (I)

$$\frac{\widehat{S}_n f(x)}{a(n)} = \int_{\mathbb{R}} g(t)\frac{\widehat{S}_n \varphi_{t+ib}}{a(n)}dt \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} g dm = \int_{\mathbb{R}} f dm$$

whence $E_{m_p}(f|\mathfrak{I})$ is constant. Since such f are dense in $L^1(m)$, T is ergodic. ☑(i)

`Proof of (ii)` By (II),

$$T^n(\omega) - n\beta + \frac{\nu}{\beta}\log n \xrightarrow[n\to\infty]{} C(\omega) + iB(\omega) =: F(\omega) \in \mathbb{R}^{2+}.$$

It follows that $F : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ is analytic. Moreover

$$\begin{aligned} F(T\omega) &\xleftarrow[n\to\infty]{} T^{n+1}(\omega) - n\beta + \frac{\nu}{\beta}\log n \\ &= (T^{n+1}(\omega) - (n+1)\beta + \frac{\nu}{\beta}\log(n+1)) + \beta + O(\frac{1}{n}) \\ &\xrightarrow[n\to\infty]{} F(\omega) + \beta. \quad ☑\text{(ii)} \end{aligned}$$

Aperiodicity and Rokhlin towers

Periodicity. Let $(X, \mathcal{B}, m)$ be a standard probability space and let $T \in \mathtt{NST}(X, \mathcal{B}, m)$.

For each $p \geq 1$ consider the set of *p-periodic points*

$$\mathtt{Per}_p(T) := \{x \in X : T^p x = x, \ \ T^j x \neq x \ \forall \ 1 \leq j < p\}.$$

Exercise 19. Show that for $p \in \mathbb{N}$:

(i) $\mathtt{Per}_p(T) \in \mathcal{B}$;

(ii) there is a set $A \in \mathcal{B}$ so that $\{T^j A : 0 \leq j \leq p-1\}$ are disjoint and

$$\mathtt{Per}_p(T) \overset{m}{=} \biguplus_{j=0}^{p-1} T^j A.$$

`Hints for (ii)` Using the polish structure of X, show that $\forall\ A \in \mathcal{B}_+,\ \exists\ B \in \mathcal{B}_+,\ B \subset A$ so that $\{T^j B:\ 0 \le j \le p-1\}$ are disjoint. Then perform an exhaustion argument.

Aperiodicity.

The non-singular transformation $(X, \mathcal{B}, m, T)$ is called *aperiodic* if $m(\mathtt{Per}_n(T)) = 0\ \forall\ n \ge 1$.

Sweepout sets. Let $(X, \mathcal{B}, m, T)$ be a `NST`. A set $A \in \mathcal{B}$ is called a *sweepout set* if $\bigcup_{n=1}^{\infty} T^{-n} A \overset{m}{=} X$.

The next exercise shows that an aperiodic, conservative `NST` has sweepout sets of arbitrarily small measure.

Note that this is immediate for a conservative, ergodic `NST` $(X, \mathcal{B}, m, T)$, for then for any $A \in \mathcal{B}_+$, $\bigcup_{n=1}^{\infty} T^{-n} A$ has positive measure and is T-invariant mod m...

Exercise 20. Let $(X, \mathcal{B}, m, T)$ be an aperiodic, conservative `NST`. Show that $\forall\ \epsilon > 0\ \exists\ E \in \mathcal{B},\ m(E) < \epsilon$ s.t. $\widetilde{E} := \bigcup_{n \ge 1} T^{-k} E = X \mod m$.

`Directions:` [3]

Fix $N > \frac{1}{\epsilon}$ and let

$$\mathcal{Z}_N := \{A \in \mathcal{B}_+:\ \{T^{-j} A:\ 0 \le j < N\} \text{ disjoint}\}.$$

¶1 Show that $\forall\ J \in \mathfrak{B}_+,\ \exists\ A \in \mathcal{Z}_N$ so that $m(A \cap J) > 0$.

`Hints` (i) Assume WLOG that $T^n x \ne x\ \forall\ x \in X,\ n \ge 0$. Fix a polish metric d on X and find (!) $C \subset J$ compact so that $m(C) > 0$ and $T^j : C \to X$ is continuous for $0 \le j \le N$.

(ii) Find $x \in C$ so that $m(C \cap B(x, \epsilon)) > 0\ \forall\ \epsilon > 0$ where $B(x, \epsilon)$ is the d-ball of radius ϵ around x and then find (!) $\eta > 0$ so that $\{T^j(C \cap B(x, \eta)):\ 0 \le j \le p-1\}$ are disjoint.

¶2 Obtain using exhaustion: sets $A_1, A_2, \cdots \in \mathcal{Z}_N$ and numbers $\epsilon_n \ge 0$ so that

$$\widetilde{A_{n+1}} \cap \widetilde{A_k} = \varnothing\ \forall\ 1 \le k \le n;$$

$$2m(\widetilde{A_{n+1}}) \ge \epsilon_{n+1} := \sup\{m(A):\ A \in \mathcal{Z}_N,\ \widetilde{A_{n+1}} \cap \widetilde{A_k} = \varnothing\ \forall\ 1 \le k \le n\}$$

and show that for some $0 \le J < N$, $T^{-J} \bigcup_{k=1}^{\infty} A_k$ is as required.

6.2 Rokhlin's tower theorem *Let T be a conservative, aperiodic nonsingular transformation of the Polish, probability space $(X, \mathcal{B}, m)$. For $N \ge 1$, and $\eta > 0$, $\exists\ E \in \mathcal{B}$ such that $\{T^{-j} E\}_{j=0}^{N-1}$ are disjoint, and $m(X \setminus \bigcup_{j=0}^{N-1} T^{-j} E) < \eta$.*

[3] Here, I'm breaking up the proof into "easy stages".

Proof

By non-singularity $\exists\ \delta > 0$ so that

$$m(A) < \delta \implies m(\bigcup_{k=0}^{N-1} T^{-k}A) < \eta.$$

Using this and exercise 20, we can choose choose $A \in \mathcal{B}$ such that $\widetilde{A} = X$ and $m(\bigcup_{k=0}^{N-1} T^{-k}A) < \eta$.

Set $A_0 := A,\ A_n := T^{-n}A \setminus \bigcup_{j=0}^{n-1} T^{-j}A,\ (n \ge 1)$, then $\{A_n :\ n \ge 0\}$ are disjoint and $\biguplus_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} T^{-n}A = X$.

Set $E := \biguplus_{p=1}^{\infty} A_{pN}$, then for $0 \le k \le N-1$:

$$T^k E \subset \biguplus_{p=1}^{\infty} A_{pN-k}$$

whence $\{T^j E\}_{j=0}^{N-1}$ are disjoint.

We claim that $\{T^{-j}E\}_{j=0}^{N-1}$ are disjoint. To see this, fix $1 \le k \le N-1$, then $E \subset T^{-k}T^k E$ whence

$$T^{-k}E \cap E \subset T^{-k}E \cap T^{-k}T^k E = T^{-k}(E \cap T^k E) = \varnothing.$$

On the other hand, for $0 \le k \le N-1$,

$$T^{-k}E \supset \biguplus_{p=1}^{\infty} A_{pN+k},$$

whence $\bigcup_{k=0}^{N-1} T^{-k}E \supset \biguplus_{n=N}^{\infty} A_n$, and

$$m(X \setminus \bigcup_{j=0}^{N-1} T^{-j}E) \le m(\biguplus_{n=0}^{N-1} A_n) = m(\bigcup_{k=0}^{N-1} T^{-k}A) < \epsilon.$$

Skew Products

Let $(X, \mathcal{B}, m, T)$ be a `NST` and let G be a locally compact, polish, abelian topological group.

Given a measurable function $\phi : X \to G$, define the *skew product* transformation $T_\phi : X \times G \to X \times G$ by $T_\phi(x, g) := (Tx, \phi(x) + g)$.

1.1 Proposition (Hopf decomposition of skew products)

Suppose that T is ergodic and either a `MPT`*, or an invertible* `NST`*. Let $\varphi : X \to G$ be measurable, then T_φ is either conservative, or totally dissipative.*

Proof By the assumption, T_ϕ is also either a `MPT`, or an invertible `NST`. In either case, $\mathfrak{D}(T_\varphi)$ is T_ϕ-invariant. We'll show that it's invariant under an ergodic action of a larger semigroup.

Let $\Gamma \subset G$ be a countable dense subgroup of G. The action of Γ on G by translation is ergodic with respect to Haar measure on G. It follows that the $\mathbb{N} \times \Gamma$ action S on $(X \times G, \mathcal{B}(X \times G), m \times m_G)$ given by $S_{(n,a)}(x,y) := (T^n x, y + a + \phi_n(x))$ is ergodic.

Let $a \in G$, then since $S_{0,a}$ is invertible and $S_{0,a} \circ T_\varphi = T_\varphi \circ S_{0,a}$ we have that $W \in \mathcal{W}(T_\varphi)$ iff $S_{0,a} W \in \mathcal{W}(T_\varphi)$, whence $S_{0,a}\mathfrak{D}(T_\varphi) = \mathfrak{D}(T_\varphi)$. Since $T_\varphi^{-1}\mathfrak{D}(T_\varphi) = \mathfrak{D}(T_\varphi)$, it follows that $\mathfrak{D}(T_\varphi)$ is S-invariant, whence the proposition by ergodicity of S. ☑

1.2 Proposition *Let $(X, \mathcal{B}, m, T)$ be a* `PPT`*, then T_ϕ is conservative iff*

$$\liminf_{n\to\infty} \|\phi_n(x)\| = 0 \textit{ for a.e. } x \in X.$$

Proof

Assume first that T_ϕ is conservative and let $\epsilon > 0$. By Halmos' recurrence theorem

$$\sum_{n=1}^{\infty} 1_{X \times B_G(0,\epsilon/2)} \circ T_\phi^n = \infty \text{ a.e. on } X \times B_G(0, \epsilon/2).$$

So for a.e. $x \in X,\ y \in B_G(0, \epsilon/2)$,

$$\sum_{n=1}^{\infty} 1_{B_G(0,\epsilon/2)}(y + \phi_n(x)) = \infty,$$

whence for a.e. $x \in X,\ \liminf_{n\to\infty} \|\phi_n(x)\| \le \epsilon$.

Now assume that

$$\liminf_{n\to\infty} \|\phi_n(x)\| = 0 \text{ for a.e. } x \in X.$$

Fix $f : G \to \mathbb{R}_+$ be continuous, positive and integrable and let $0 < \epsilon < \kappa_G$. For $y \in G$, let $\delta(y, \epsilon) := \inf_{B_G(y,\epsilon)} f$. By compactness of $B_G(y, \epsilon)$, $\delta(y, \epsilon) > 0$.

We have that $\forall\ y \in G$, for a.e. $(x, z) \in X \times B_G(y, \frac{\epsilon}{2})$,

$$\sum_{n=1}^{\infty} (1 \otimes f) \circ T_\phi^n(x, z) = \sum_{n=1}^{\infty} f(z + \phi_n(x)) \ge \delta(y, \epsilon) \sum_{n=1}^{\infty} 1_{B_G(0,\frac{\epsilon}{2})}(\phi_n(x)) = \infty$$

and T_ϕ is conservative. □

1.3 Proposition *If $\phi = \Psi - \Psi \circ T$ with $\Psi : X \to G$ measurable, then T_ϕ is conservative.*

Proof Evidently T_0 is conservative, and if ϕ is a coboundary, then T_ϕ is isomorphic to T_0. □

Persistencies and Essential values

Let $(X, \mathcal{B}, m)$ be a standard probability space, and let $T : X \to X$ be an ergodic, NST. Suppose that $\phi : X \to G$ is measurable. The collection of *persistencies*of ϕ is

$$\Pi(\phi) = \{a \in G : \forall\ A \in \mathcal{B}_+,\ \epsilon > 0,\ \exists\ n \ge 1,\ m(A \cap T^{-n} A \cap [\|\phi_n - a\| < \epsilon]) > 0\}.$$

For T invertible, the collection of *essential values* of ϕ is

$$E(\phi) = \{a \in G : \forall\ A \in \mathcal{B}_+,\ \epsilon > 0,\ \exists\ n \in \mathbb{Z},\ m(A \cap T^{-n} A \cap [\|\phi_n - a\| < \epsilon]) > 0\}.$$

2.1 Proposition [?Schm1]
Either $\Pi(\phi) = \varnothing$, or $\Pi(\phi)$ is a closed subgroup of G.

Proof
To see that $\Pi(\phi)$ is closed let $a \in \overline{\Pi(\phi)}$ and let $\epsilon > 0,\ A \in \mathcal{B}_+$.
$\exists\ a' \in \Pi(\phi)$ such that $\|a - a'\| < \epsilon/2$, and $\exists\ n \ge 1$ such that $m(A \cap T^{-n} A \cap [\|\varphi_n - a'\| < \epsilon/2]) > 0$.
It follows that
$m(A \cap T^{-n} A \cap [\|\varphi_n - a\| < \epsilon]) \ge m(A \cap T^{-n} A \cap [\|\varphi_n - a'\| < \epsilon/2]) > 0$.
Thus, $a \in \Pi(\phi)$ and $\Pi(\phi)$ is closed.

To show that $\Pi(\phi)$ is a group, we show that $a, b \in \Pi(\phi) \implies a - b \in \Pi(\phi)$.

Let $a, b \in \Pi(\phi),\ \epsilon > 0,\ A \in \mathcal{B}_+$ and let $n \ge 1$ be such that $m(A \cap T^{-n} A \cap [\|\phi_n - a\| < \epsilon/2]) > 0$.

By Rokhlin's lemma, $\exists\ B \in \mathcal{B}_+,\ B \subset A \cap T^{-n} A \cap [\|\phi_n - a\| < \epsilon/2]$ such that $B \cap T^{-k} B = \varnothing$ for $1 \le k \le n$.

Since $b \in \Pi(\phi)$, $\exists\ N \ge 1$ such that $m(B \cap T^{-N} B \cap [\|\phi_N - b\| < \epsilon/2]) > 0$. The construction of B implies that $N > n$ whence

$$\begin{aligned}
&B \cap T^{-N} B \cap [\|\phi_N - b\| < \epsilon/2] \\
&= B \cap T^{-N} B \cap [\|\phi_n - a\| < \epsilon/2] \cap [\|\phi_N - b\| < \epsilon/2] \\
&\subset B \cap T^{-N} B \cap [\|\phi_{N-n} \circ T^n - (b - a)\| < \epsilon],
\end{aligned}$$

$$\begin{aligned}
0 < &m(B \cap T^{-N} B \cap [\|\phi_{N-n} \circ T^n - (b - a)\| < \epsilon]) \\
&\le m(A \cap T^{-n} A \cap T^{-N} A \cap [\|\phi_{N-n} \circ T^n - (b - a)\| < \epsilon]) \\
&\le m(T^{-n}(A \cap T^{-(N-n)} A \cap [\|\phi_{N-n} - (b - a)\| < \epsilon]))
\end{aligned}$$

whence $m(A \cap T^{-(N-n)} A \cap [\|\phi_{N-n} - (b - a)\| < \epsilon]) > 0$ and $b - a \in \Pi(\phi)$.
$\square$

Lecture # 6 17/10/2014 12-13.

2.2 Theorem [K.Schmidt]

Let $(X,\mathcal{B},m,T)$ be a conservative NST*, and let $\phi : X \to G$, then T_ϕ is conservative $\iff$ $0 \in \Pi(\phi)$.*

Proof of $\Rightarrow$

Suppose first that T_ϕ is conservative and let $A \in \mathcal{B}_+$, $\epsilon > 0$. $\exists\ n \ge 1$ such that $m \times m_G(A \times B_G(0,\epsilon/2) \cap T_\phi^{-n} A \times B_G(0,\epsilon/2)) > 0$. Since $A \times B_G(0,\epsilon/2) \cap T_\phi^{-n} A \times B_G(0,\epsilon/2) \subset (A \cap T^{-n} A \cap [\|\phi_n\| < \epsilon]) \times B_G(0,\epsilon/2)$, we have $m(A \cap T^{-n} A \cap [\|\phi_n\| < \epsilon]) > 0$ and $0 \in \Pi(\phi)$. ☑

Proof of $\Leftarrow$

In case G is countable, every $B \in \mathcal{B}(X \times G)_+$ contains a set Conversely, suppose that T_ϕ is not conservative. Let $A \in \mathcal{B}$. Consider the sections

$$A_x := \{y \in G : \ (x,y) \in A\} \quad (x \in X).$$

A calculation shows that

$$(T_\phi^{-n} A)_x = A_{T^n x} - \phi_n(x).$$

By Fubini's theorem, $A_x \in \mathcal{B}(G)\ \forall\ x\ \&\ x \mapsto m_G(A_x)$ is measurable. Let

$$X_A := \{x \in X : \ m(A_x) > 0\},$$

then $m(X_A) > 0$. Now let $W \in \mathcal{W}(T_\phi)$. We claim that

¶ there is a measurable subset $V \subset W$ with

$$0 < m(V_x) < \infty \ \text{ for a.e. } x \in X_W.$$

Proof of ¶

Define $R : X \to [0,\infty)$ by

$$R(x) := \inf \{r > 0 : \ m(W_x \cap B(0,r)) > \min \{\tfrac{m(W_x)}{2}, 1\},$$

then

$$V_0 := \{(x,y) : \ y \in W_x \cap B(0,R(x))\}$$

is Lebesgue measurable and $m \times m_G(V_0) > 0$. It follows that $\exists\ V \in \mathcal{B}(X \times G)$, $V \subset V_0$ with $m \times m_G(V_0 \setminus V) = 0$.

It follows that for a.e. $x \in X_W$, $V_x = (V_0)_x$ whence

$$0 < m(V_x) < \infty \ \text{ for a.e. } x \in X_W. \ ☑¶$$

Let

$$\overline{\mathcal{F}} := \{f \in L^1(m_G) : \ \exists\ A \in \mathcal{B},\ f = 1_A \text{ a.e.}\},$$

then $\overline{\mathcal{F}}$ is a polish space with the metric

$$\rho([A],[B]) := \|1_A - 1_B\|_1 = m_G(A \Delta B)$$

for $A,B\in\mathcal{B},\ 0<m(A),m(B)<\infty$ where $[C]:=\{B\in\mathcal{B}(G):\ \mu(B\Delta C)=0\}$.

By Fubini's theorem, $x\mapsto[V_x]$ is a Borel map $X\to\overline{\mathcal{F}}$.

By Lusin's theorem, $\exists$ a compact set $C\in\mathcal{B}_+,\ C\subset X_W$ so that $x\mapsto V_x$ is continuous on C.

Also, for $A\in\mathcal{F}_+,\ t\mapsto m_G(A\cap(t+A))$ is continuous $G\to[0,\infty)$.

By compactness, $m_G(V_x)\le\Delta>0\ \forall\ x\in C$.

By continuity, $\exists\ \epsilon>0$ & a compact set $D\in\mathcal{B}_+,\ D\subset C$ so that

(☢) $$m_G(V_x\cap(V_y+t))\ge\epsilon\quad\forall\ x,y\in D,\ \|t\|<\epsilon.$$

Set $U=V\cap(D\times G)$ then

$$U_x=\begin{cases} V_x & x\in D,\\ \varnothing & \times\notin D.\end{cases}$$

It follows from Fubini that $m\times m_G(U)>0$ whence $U\in\mathcal{W}(T)$.

Thus, we have, for $n\ge1$

$$U\cap T_\phi^{-n}U\overset{m}{\subset}(D\cap T^{-n}D)\times G$$

and for a.e. $x\in D\cap T^{-n}D$, we have

$$\begin{aligned}\varnothing&=(U\cap T_\phi^{-n}U)_x=U_x\cap(U_{T^nx}-\phi_n(x))\\ &=U_x\cap(U_{T^nx}-\phi_n(x))=V_x\cap(V_{T^nx}-\phi_n(x)).\end{aligned}$$

By (☢),

$$U\cap T^{-n}U\subset[\|\phi_n\|\ge\epsilon]\quad\forall\ n\ge1$$

and $0\notin\Pi(\phi)$. ☑

2.3 Proposition

Suppose that $\phi,\varphi:X\to G$ are cohomologous, then $\Pi(\phi)=\Pi(\varphi)$.

Proof

By symmetry, it is sufficient to show that $\Pi(\phi)\subseteq\Pi(\varphi)$.

Suppose that $\varphi=\phi+h\circ T-h$ where $h:X\to G$ is measurable.

Let $a\in\Pi(\phi)$ and let $A\in\mathcal{B}_+,\ \epsilon>0$.

Since X is a standard space, by Lusin's theorem $\exists\ B\subset A,\ B\in\mathcal{B}_+$ such that $\|h(x)-h(y)\|<\frac{\epsilon}{2}\ \forall\ x,y\in B$.

Since $a\in\Pi(\phi),\ \exists\ n\ge1$ such that $m(B\cap T^{-n}B\cap[\|\phi_n-a\|<\frac{\epsilon}{2}])>0$.

By construction of B, if $x\in B\cap T^{-n}B$, then $\|\varphi_n(x)-\phi_n(x)\|=\|h(T^nx)-h(x)\|<\frac{\epsilon}{2}$ whence

$$m(B\cap T^{-n}B\cap[\|\varphi_n-a\|<\epsilon])\ge m(B\cap T^{-n}B\cap[\|\phi_n-a\|<\frac{\epsilon}{2}])>0,$$

and $a \in \Pi(\varphi)$. □

Periods. Define the collection of *periods* for T_ϕ-invariant functions:

$$\text{Per}\,(\phi) = \{a \in G : Q_a A = A \mod m \ \forall\ A \in \mathfrak{I}(T_\phi)\}$$

where $Q_a(x,y) = (x, y+a)$.

2.4 Theorem [K.Schmidt]

(i) *Suppose that T_ϕ is conservative, then*

$$\Pi(\phi) = \text{Per}\,(\phi).$$

(ii) *Suppose that T is invertible, then*

$$E(\phi) = \text{Per}\,(\phi).$$

Remark. (i) fails for some non-invertible T with T_ϕ dissipative

Proof of (i)

¶1 $\text{Per}\,(\phi) \subset \Pi(\phi)$

Suppose $0 \neq a \notin \Pi(\phi)$, then $\exists\ 0 < \epsilon < d(0,a)$, and $A \in \mathcal{B}_+$ such that $m(A \cap T^{-n}A \cap [\|\phi_n - a\| < 2\epsilon]) = 0\ \forall\ n \geq 1$.

For $z \in G$ & $\epsilon > 0$, set

$$B_z = \bigcup_{n \in \mathbb{N}} T_\phi^{-n}\Bigg(A \times B_G(z,\epsilon)\Bigg).$$

We have that $T_\phi^{-1} B_z \subset B_z$, whence by conservativity $T_\phi^{-1} B_z \overset{m}{=} B_z$. Moreover $1_{B_0} \circ Q_a = 1_{B_a}$.

To see that $a \notin \text{Per}\,(\phi)$, it suffices to prove that

$$m(B_0 \cap B_a) = 0.$$

This holds because $\forall\ n \in \mathbb{N}$,

$$\begin{aligned}(A \times B_G(0,\epsilon) \cap T_\phi^{-n}(A \times B_G(a,\epsilon))) \ \cup\ (A \times B_G(a,\epsilon) \cap T_\phi^{-n}(A \times B_G(0,\epsilon)))\\ \subset\ A \cap T^{-n}A \cap [\|\phi_n - a\| < 2\epsilon] \times G. \quad ☑¶1\end{aligned}$$

¶2 $\Pi(\phi) \subset \text{Per}\,(\phi)$

Now assume that $a \notin \text{Per}\,(\phi)$, then $\exists\ A, B \in \mathfrak{I}(T_\phi)_+$ disjoint such that $B = Q_a A$. Set for $x \in X$,

$$A_x = \{y \in G : (x,y) \in A\}$$

Note that

$$A_{Tx} = \{y \in G : (Tx, y) = T_\phi(x, y - \phi(x)) \in A\} = A_x + \phi(x),$$

whence $m_G(A_x) = m_G(A_{Tx})$, and by ergodicity, $m_G(A_x) = m \times m_G(A) > 0$ for m-a.e. $x \in X$.

Next, as in the proof of $\Leftarrow$ in theorem 2.2:

- $\exists\ \theta \in \mathcal{B}(A)$ such that $0 < m_G(\theta_x) < \infty$ a.e.;
- $\exists\ \epsilon > 0$ and $D \in \mathcal{B}(X)_+$ such that

$$m_G(\theta_x \cap (\theta_y + t)) \ge \epsilon\ \forall\ x, y \in D,\ \|t\| < \epsilon.$$

Lastly, we show that $a \notin \mathit{\Pi}(\phi)$. This will follow from

$$D \cap T^{-n} D \cap [\|\phi_n(x) - a\| < \epsilon] = \varnothing\ \forall\ n \ge 1.$$

Indeed, supposing that $x, T^n x \in D$, we note that

$$\Big(a + \theta_{T^n x}\Big) \cap \Big(\theta_x + \phi_n(x)\Big) \subset B_{T^n x} \cap A_{T^n x} = \varnothing,$$

whence,

$$m_G(\theta_x \cap (\theta_{T^n x} + a - \phi_n(x))) = m_G((a + \theta_{T^n x}) \cap (\theta_x + \phi_n(x))) \le m_G(B_{T^n x} \cap A_{T^n x}) = 0$$

and

$$\|\phi_n(x) - a\| \ge \epsilon.$$

$\square$

Exercise 21: Essential values.

Let $(X, \mathcal{B}, m, T)$ be an invertible `NST` and let $\phi : X \to \mathbb{G}$ be measurable ($\mathbb{G}$ a `LCAP` group). Show that

(i) $E(\phi) = \mathit{\Pi}(\phi) \cup \{0\}$; (ii) $E(\phi) = \text{Per}(\phi)$.

Exercise 22: Dissipative exact example.

This is a counterexample to theorem 2.4 for dissipative, non-invertible skew products..

Let $(X, \mathcal{B}, m, S)$ be an `EPPT` and let $f : X \to \mathbb{Z}$ be such that S_f is an ergodic, totally dissipative `MPT` (as in e.g. exercise 16).

Show that

(i) $\mathit{\Pi}(f, S) = \varnothing$;

(ii) `Per` $(f, S) = \mathbb{Z}$.

End of minicourse