An elementary harmonic analysis of arithmetic functions

Jörg Brüdern

For functions $f : \mathbb{N} \to \mathbb{C}$ that carry arithmetic information, analytic number theory often provides asymptotic formulae for the mean over an arithmetic progression. If $f$ is suitably normalized, this will take the shape

$$\sum_{n \leq x} f(n) = \eta(q,a)x + o(x).$$

(1)

A typical instance is the Siegel-Walfisz theorem, in which one takes $f(p) = \log p$ when $p$ is prime, and $f(n) = 0$ for composite $n$. Then (1) holds with $\eta(q,a) = 1/\varphi(q)$ when $(a,q) = 1$, and $\eta(q,a) = 0$ otherwise.

Alternatively, one may consider functions $f : \mathbb{N} \to \mathbb{C}$ in the style of functional analysis. The set of such functions for which

$$\limsup \frac{1}{N} \sum_{n \leq N} |f(n)|^2 = \|f\|^2,$$

say, is finite, forms a Banach space $\mathcal{L}$. This contains all periodic functions as a subspace, and its completion gives rise to a natural notion of limit-periodic function.

The course begins with an unpublished result of the lecturer, to the effect that whenever $f \in \mathcal{L}$ also obeys the asymptotic formulae (1), then $f$ is limit periodic if and only if

$$\|f\|^2 = \sum_{q=1}^{\infty} \sum_{a=1}^{q} \left| \sum_{b=1}^{q} \eta(q,b)e^{2\pi ib/q} \right|^2.$$

Note that the “local” data $\eta(q,a)$ and the “global” norm $\|f\|^2$ suffice to decide whether a function is limit periodic. The proof prominently features the Hardy-Littlewood circle method, and the latter will appear in a new light. In some cases, we shall be able to solve binary additive problems and $k$-tuple problems via the circle method, much in contrast with widely tolerated tattle-tales. As the course moves on, we shall look at a suitable subspace of $\mathcal{L}$ as a convolution algebra, and gain more insight into the structure of the leading term that typically arises in $k$-tuple problems.

In the second week, we turn to the variance

$$V(x,Q) = \sum_{q \leq Q} \sum_{a=1}^{q} \left| \sum_{n \leq x \mod q} f(n) - \eta(q,a)x \right|^2.$$

(2)

Again, when $f$ is a weighted indicator of the primes, an asymptotic formula for (2), valid when $Q$ is rather close to $x$, is classical territory, known as the the Hooley-Montgomery theorem. We shall establish a similar formula for a large class of functions $f$. This will yield another purely arithmetical characterisation of limit-periodic functions. In short, the leading term in the proposed asymptotics vanishes if and only if $f$ is limit-periodic. If time permits, will will also discuss suitable generalisations of the Bombieri-Vinogradov theorem.

There are other applications of the ideas developed in the course, including a new proof of Elliott’s classification of limit periodic multiplicative functions in $\mathcal{L}$. Depending on the pre-education of the audience in this field, a brief sketch of what this is about is an optional final lecture.
Suggestions for preparatory reading:

