An elementary harmonic analysis of arithmetic functions

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For functions $f : \mathbb{N} \to \mathbb{C}$ that carry arithmetic information, analytic number theory often provides asymptotic formulae for the mean over an arithmetic progression. If f is suitably normalized, this will take the shape

$$\sum_{\substack{n \le x \\ n \equiv a \bmod q}} f(n) = \eta(q,a)x + o(x). \tag{1}$$

A typical instance is the Siegel-Walfisz theorem, in which one takes $f(p) = \log p$ when p is prime, and $f(n) = 0$ for composite n. Then (1) holds with $\eta(q,a) = 1/\varphi(q)$ when $(a,q) = 1$, and $\eta(q,a) = 0$ otherwise.

Alternatively, one may consider functions $f : \mathbb{N} \to \mathbb{C}$ in the style of functional analysis. The set of such functions for which

$$\limsup \frac{1}{N} \sum_{n \le N} |f(n)|^2 = \|f\|^2, \quad \text{say,}$$

is finite, forms a Banach space $\mathcal{L}$. This contains all periodic functions as a subspace, and its completion gives rise to a natural notion of limit-periodic function.

The course begins with an unpublished result of the lecturer, to the effect that whenever $f \in \mathcal{L}$ also obeys the asymptotic formulae (1), then f is limit periodic if and only if

$$\|f\|^2 = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big| \sum_{b=1}^{q} \eta(q,b) e^{2\pi i ab/q} \Big|^2.$$

Note that the "local" data $\eta(q,a)$ and the "global" norm $\|f\|^2$ suffice to decide whether a function is limit periodic. The proof prominently features the Hardy-Littlewood circle method, and the latter will appear in a new light. In some cases, we shall be able to solve binary additive problems and k-tuple problems via the circle method, much in contrast with widely tolerated tattle-tales. As the course moves on, we shall look at a suitable subspace of $\mathcal{L}$ as a convolution algebra, and gain more insight into the structure of the leading term that typically arises in k-tuple problems.

In the second week, we turn to the variance

$$V(x,Q) = \sum_{q \le Q} \sum_{a=1}^{q} \Big| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} f(n) - \eta(q,a)x \Big|^2. \tag{2}$$

Again, when f is a weighted indicator of the primes, an asymptotic formula for (2), valid when Q is rather close to x, is classical territory, known as the the Hooley-Montgomery theorem. We shall establish a similar formula for a large class of functions f. This will yield another purely arithmetical characterisation of limit-periodic functions. In short, the leading term in the proposed asymptotics vanishes if and only if f is limit-periodic. If time permits, will will also discuss suitable generalisations of the Bombieri-Vinogradov theorem.

There are other applications of the ideas developed in the course, including a new proof of Elliott's classification of limit periodic *multiplicative* functions in $\mathcal{L}$. Depending on the pre-education of the audience in this field, a brief sketch of what this is about is an optional final lecture.

Suggestions for praparatory reading:

Jörg Brüdern. *Binary additive problems and the circle method, multiplicative sequences and convergent sieves.* In: Analytic number theory, Essays in honour of Klaus Roth. W.W.L. Chen et al, eds. Cambridge University Press, Cambridge 2009 (pp. 91–132).

Peter Elliott. Probabilistic number theory. I. Mean-value theorems. Grundlehren der Mathematischen Wissenschaften 239. Springer-Verlag, New York-Berlin, 1979

Peter Elliott. Probabilistic number theory. II. Central limit theorems. Grundlehren der Mathematischen Wissenschaften 240. Springer-Verlag, Berlin-New York, 1980.