Reliability of Coherent Systems with Dependent Component Lifetimes

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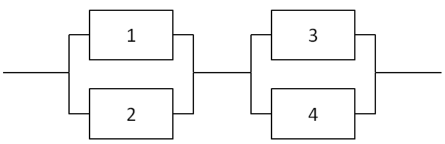
Abstract

In reliability theory, coherent systems represent a classical framework for describing the structure of technical systems. Starting from the notion of coherent systems, the connection between the lifetimes of components and the lifetime of the system itself can be obtained. If the lifetimes of components are modeled via continuous independent and identically distributed random variables, then the distribution of the system lifetime has a representation as a mixture of the distributions of the ordered component lifetimes. The weights in this mixture distribution are known as Samaniego's signature. This representation of the system lifetime distribution remains valid under weaker assumptions, for instance, the iid assumption on the components can be replaced by exchangeability. In the lecture, properties of systems with such dependent components are examined.

1 Introduction to reliability theory

In this section, we give some basic introduction to reliability theory. More detailed introductions can be found, for example, in Barlow and Proschan (1981) and Samaniego (2007).

Consider a system consisting of several components. The system can be displayed via a block diagram like



and described in a mathematical rigorous way by a structure function

$$\begin{aligned}\phi(x_1, x_2, x_3, x_4) &= (x_1 x_2 - x_1 - x_2) \cdot (x_3 x_4 - x_3 - x_4) \\ &= \min(\max(x_1, x_2), \max(x_3, x_4))\end{aligned}$$

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with the interpretation

$$x_i = \begin{cases} 1, & \text{if component } i \text{ is functioning} \\ 0, & \text{if component } i \text{ is failed} \end{cases}$$

of the component states. The value of the structure function is interpreted analogously as the system state.

Definition 1.1. *The structure function ϕ of a system with n components is a mapping $\phi : \{0,1\}^n \to \{0,1\}$. n is called order of the system.*

Usually, systems satisfy the next two intuitive properties.

Definition 1.2. *(a) The component i is called irrelevant if*

$$\phi(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = \phi(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$$

for every $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \{0,1\}^{n-1}$.

(b) A system is called monotone if its structure function ϕ is increasing, i.e.,

$$\phi(x) \leq \phi(y) \quad \text{for all } x, y \in \{0,1\}^n \text{ with } x \leq y.$$

Definition 1.3. *A system is called coherent if it is monotone and none of its components is irrelevant.*

Remark. *The structure function of a coherent system satisfies*

$$\phi(0, \dots, 0) = 0, \quad \phi(1, \dots, 1) = 1.$$

In order to obtain representations of the structure function of coherent systems, the following notions are helpful.

Definition 1.4. *Let the system be coherent with n components.*

(a) $P \subset \{1, \dots, n\}$ is called path set if the system works whenever all components i with $i \in P$ work. A path set is called minimal if it contains no proper subset which is also a path set.

(b) $C \subset \{1, \dots, n\}$ is called cut set if the system fails whenever all components i with $i \in C$ fail. A cut set is called minimal if it contains no proper subset which is also a cut set.

Lemma 1.5. *The structure function of a coherent system with minimal path sets $P_1, \dots, P_k$ and minimal cut sets $C_1, \dots, C_r$ has the representations:*

$$\begin{aligned} \phi(x_1, \dots, x_n) &= 1 - \prod_{j=1}^{k} \left(1 - \prod_{i \in P_j} x_i \right) = \max_{1 \leq j \leq k} \min_{i \in P_j} x_i \\ &= \prod_{j=1}^{r} \left(1 - \prod_{i \in C_j} (1 - x_i) \right) = \min_{1 \leq j \leq r} \max_{i \in C_j} x_i \end{aligned}$$

Important examples of coherent systems are k-out-of-n systems. Two definitions of these systems need to be distinguished.

Definition 1.6. *A system with n components is called*

- *k-out-of-n:F system if it fails whenever k or more components fail,*
- *k-out-of-n:G system if it works whenever k or more components work.*

Clearly, a k-out-of-n:F system is equal to a $n-k+1$-out-of-n:G system. A 1-out-of-n:F system (or n-out-of-n:G system) is called series system. An n-out-of-n:F system (or 1-out-of-n:G system) is called parallel system.

Remark. *The structure function of the series system is given by*

$$\phi(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i = \min_{1 \le i \le n} x_i$$

and the structure function of the parallel system is given by

$$\phi(x_1, \dots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i) = \max_{1 \le i \le n} x_i.$$

Consequently, due to Lemma 1.5 every coherent system can be interpreted as parallel (series) system consisting of series (parallel) systems. Among the latter series (parallel) systems, components may appear multiple times.

A simple approach for comparing system structures ϕ and ψ is to consider ϕ to perform better than ψ if $\phi(x) \ge \psi(x)$ for all $x \in \{0,1\}^n$. Clearly, the structure functions of the series and parallel systems are lower and upper bounds for every structure function ϕ of a coherent system, i.e.,

$$\min_{1 \le i \le n} x_i \le \phi(x_1, \dots, x_n) \le \max_{1 \le i \le n} x_i$$

for every $x_1, \dots, x_n \in \{0,1\}$. Furthermore, the following result illustrates that componentwise redundancy is better than systemwise redundancy.

Theorem 1.7. *Let ϕ be the structure function of a coherent system with n components. Then, for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \{0,1\}^n$,*

$$\phi(\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\}) \ge \max\{\phi(x), \phi(y)\}.$$

Now, random states of the components are introduced. Let $\widetilde{X}_1, \dots, \widetilde{X}_n$ be independent random variables with

$$P(\widetilde{X}_i = 1) = p_i, \quad P(\widetilde{X}_i = 0) = 1 - p_i,$$

i.e., $\widetilde{X}_i \sim b(1, p_i)$ for $p_i \in [0,1], i = 1, \dots, n$. The random variable $\widetilde{X}_i$ describes the (random) state of component i at a fixed time.

Definition 1.8. *The reliability of a coherent system with independent components is defined as*

$$h(p_1, \dots, p_n) := P(\phi(\widetilde{X}_1, \dots, \widetilde{X}_n) = 1) = E\phi(\widetilde{X}_1, \dots, \widetilde{X}_n).$$

Lemma 1.9. *The reliability satisfies*

$$\begin{aligned} h(p_1, \dots, p_n) =& p_i h(p_1, \dots, p_{i-1}, 1, p_{i+1}, \dots, p_n) \\ &+ (1 - p_i) h(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_n), \quad i = 1, \dots, n. \end{aligned}$$

Theorem 1.10. *The reliability $h(p_1, \dots, p_n)$ is strictly increasing in each $p_i, i = 1, \dots, n$, on $(0,1)^n$. Moreover, $h(0, \dots, 0) = 0$ and $h(1, \dots, 1) = 1$.*

If additionally the probability of being intact is the same for every component, then the reliability can be expressed as a polynomial.

Definition 1.11. *If $\widetilde{X}_1, \dots, \widetilde{X}_n$ are iid with $\widetilde{X}_i \sim b(1, p), p \in [0, 1]$, then*

$$h(p) := P(\phi(\widetilde{X}_1, \dots, \widetilde{X}_n) = 1) = E\phi(\widetilde{X}_1, \dots, \widetilde{X}_n)$$

is called reliability polynomial.

Corollary 1.12. *The reliability polynomial h is a strictly increasing function on $[0, 1]$ with $h(0) = 0, h(1) = 1$.*

2 Lifetime distributions of coherent systems

Let $X_1, \dots, X_n$ be non-negative random variables that describe the lifetimes of the components. Then,

$$\widetilde{X}_i := \mathbb{1}_{\{X_i > t\}} = \begin{cases} 1, & X_i > t, \\ 0, & X_i \le t, \end{cases}$$

describes the random state of component i at a fixed time $t \ge 0$. Furthermore, the lifetime of the coherent system can be defined by

$$T := \sup\{t \ge 0 : \phi(\mathbb{1}_{\{X_1 > t\}}, \dots, \mathbb{1}_{\{X_n > t\}}) = 1\}.$$

By extending the domain of the structure function ϕ in the max-min or min-max representation in Lemma 1.5 from $\{0, 1\}^n$ to $\mathbb{R}^n$, we get

$$T = \phi(X_1, \dots, X_n).$$

In this context, ϕ is called coherent life function (see Esary and Marshall 1970). Moreover,

$$P(T > t) = P(\phi(\widetilde{X}_1, \dots, \widetilde{X}_n) = 1) = E\phi(\widetilde{X}_1, \dots, \widetilde{X}_n).$$

It follows from the results in Section 1 that this reliability can be expressed in the case of iid component lifetimes with an underlying continuous distribution function F as

$$P(T > t) = h(1 - F(t)), \quad t \geq 0,$$

with the reliability polynomial h of ϕ. Alternatively, the reliability can be expressed as a particular mixture distribution (see Samaniego 1985, 2007).

Theorem 2.1. *Let $X_1, \ldots, X_n$ be iid non-negative random variables with continuous distribution function. Let $T = \phi(X_1, \ldots, X_n)$ be the lifetime of a coherent system. Then, the reliability at time $t \geq 0$ is given by*

$$P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t),$$

where $X_{1:n} \leq \cdots \leq X_{n:n}$ are the ordered lifetimes of the components and

$$s_i = P(T = X_{i:n}), \quad i = 1, \ldots, n.$$

Definition 2.2. *The vector $\boldsymbol{s} = (s_1, \ldots, s_n)$ is called signature of the system.*

The signature depends only on the structure function ϕ. In particular, it does not depend on the joint distribution of $X_1, \ldots, X_n$. The distribution of these random variables enters into the representation of the reliability only via the marginal distributions of their order statistics.

Theorem 2.1 can be proven by using the following Lemma 2.3 and Theorem 2.4 (cf. Samaniego 1985, Kochar et al. 1999, Marichal and Mathonet 2011, Marichal et al. 2011).

Lemma 2.3. *(a) Under the assumptions of Theorem 2.1,*

$$P(T = X_{i:n}) = \frac{|A_i|}{n!}$$

holds, where

$$A_i = \{\sigma \in S_n : \phi(x_1, \ldots, x_n) = x_{i:n} \text{ if } x_{\sigma(1)} < \cdots < x_{\sigma(n)}\}$$

with the coherent life function ϕ and the set S_n of permutations of the numbers $1, \ldots, n$.

(b) In general, for coherent systems with structure function ϕ,

$$\frac{|A_i|}{n!} = \phi_{n-i+1} - \phi_{n-i}$$

holds with the set A_i from (a) and

$$\phi_i = \frac{1}{\binom{n}{i}} \sum_{\{x \in \{0,1\}^n : |x| = i\}} \phi(x), \quad i = 1, \ldots, n, \quad \phi_0 := 0,$$

with $|x| := x_1 + \cdots + x_n$ for $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$.

Theorem 2.4. *Under the assumptions of Theorem 2.1, the reliability is given by*

$$P(T > t) = \sum_{i=1}^{n} \rho_i P(X_{i:n} > t)$$

with $\rho_i = \phi_{n-i+1} - \phi_{n-i}, i = 1, \dots, n$.

From the proofs of both results, it is seen that the assumptions can be weakened.

Remark. *(a) Lemma 2.3 (a) remains valid for exchangeable* $X_1, \dots, X_n$ *without ties.*

(b) Theorem 2.4 holds for arbitrary exchangeable $X_1, \dots, X_n$.

In particular, Theorem 2.1 can be extended to exchangeable random variables without ties (see also Navarro and Rychlik 2007, Navarro et al. 2008b).

Remark. *Let* $T = \phi(X_1, \dots, X_n)$ *be the lifetime of a coherent system with arbitrary random variables* $X_1, \dots, X_n$. *Then,* $\boldsymbol{s} = (s_1, \dots, s_n)$ *with* $s_i = P(T = X_{i:n})$ *is also called probability-signature and* $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ *with*

$$\rho_i = \frac{|A_i|}{n!} = \phi_{n-i+1} - \phi_{n-i}$$

is called structure-signature (cf. Navarro et al. 2010, Spizzichino and Navarro 2012). If $X_1, \dots, X_n$ *are exchangeable without ties, then probability-signature and structure-signature coincide.*

Remark. *Systems with a different structure may have the same structure-signature. For example, the coherent system with four components determined by the minimal cut sets* $\{1, 2\}, \{3, 4\}$ *has the signature vector* $\boldsymbol{\rho} = (0, 1/3, 2/3, 0)$, *but this is also the signature of the system with the minimal cut sets* $\{1, 2\}, \{1, 3\}, \{2, 3, 4\}$.

Remark. *By construction, the entries of the structure-signature* $\boldsymbol{\rho}$ *are particular rational numbers. However, the representation in Theorem 2.4 for exchangeable random variables still defines the survival function of a mixture distribution*

$$S(t) = \sum_{i=1}^{n} \rho_i P(X_{i:n} > t), \quad t \in \mathbb{R},$$

if arbitrary non-negative real ρ_i *with* $\rho_1 + \cdots + \rho_n = 1$ *are chosen. This distribution is associated with the lifetime of a so-called mixed system (cf. Samaniego 2007).*

A characterization for the representation in Theorem 2.4 is available (see Marichal et al. 2011).

Theorem 2.5. *Let $t \geq 0$ be fixed and let $T = \phi(X_1, \dots, X_n)$ denote the lifetime of a coherent system with structure function ϕ. Then the representation*

$$P(T > t) = \sum_{i=1}^{n} \rho_i P(X_{i:n} > t)$$

with the structure-signature $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ holds for all coherent systems of order n iff the random variables $\mathbb{I}_{\{X_1 > t\}}, \dots, \mathbb{I}_{\{X_n > t\}}$ are exchangeable.

Remark. *If $X_1, \dots, X_n$ are exchangeable, then $\mathbb{I}_{\{X_1 > t\}}, \dots, \mathbb{I}_{\{X_n > t\}}$ are also exchangeable for every $t \in \mathbb{R}$. However, the converse implication is not valid (see Marichal et al. 2011).*

Remark. *Given the condition of Theorem 2.5, we have*

$$P(X_{i:n} > t) = \sum_{l=n-i+1}^{n} \binom{n}{l} P(X_1 > t, \dots, X_l > t, X_{l+1} \leq t, \dots, X_n \leq t).$$

In particular, if $\mathbb{I}_{\{X_1 > t\}}, \dots, \mathbb{I}_{\{X_n > t\}}$ are iid with $P(\mathbb{I}_{\{X_1 > t\}} = 0) = F(t)$, then

$$P(X_{i:n} > t) = \sum_{l=n-i+1}^{n} \binom{n}{l} (1 - F(t))^l (F(t))^{n-l}.$$

Of course, the latter relation is valid if $X_1, \dots, X_n$ are iid with distribution function F.

Remark. *Given the condition of Theorem 2.5, it can be also shown by using the inclusion-exclusion formula that*

$$P(X_{i:n} > t) = \sum_{l=n-i+1}^{n} a_l P(X_{1:l} > t) = \sum_{k=i}^{n} b_k P(X_{k:k} > t)$$

with appropriate $a_{n-i+1}, \dots, a_n, b_i, \dots, b_n \in \mathbb{R}$. Utilizing Theorem 2.5, the reliability of the system can be expressed as generalized mixtures

$$P(T > t) = \sum_{i=1}^{n} \alpha_i P(X_{1:i} > t) = \sum_{i=1}^{n} \beta_i P(X_{i:i} > t)$$

with $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$ such that $\alpha_1 + \dots + \alpha_n = \beta_1 + \dots + \beta_n = 1$. The vectors $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ are called minimal and maximal signature, respectively (see Navarro et al. 2007). Some entries of these signatures $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be negative.

It can be shown that the lifetime distribution of a system with arbitrary (non-independent, non-identically distributed) random variables coincides with the lifetime distribution of a system with appropriate iid random variables (cf. Navarro et al. 2011).

Theorem 2.6. *Let $T = \phi(X_1, \ldots, X_n)$ be the lifetime of a coherent system and $X_1, \ldots, X_n$ be arbitrary random variables. Then, there exists a distribution function G such that iid random variables $Y_1, \ldots, Y_n \sim G$ satisfy*

$$P(T > t) = \sum_{i=1}^{n} \rho_i P(Y_{i:n} > t), \quad t \geq 0,$$

with the structure-signature $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$.

Remark. *The distribution function G in the preceding theorem may depend on ϕ. If W denotes the structure-dependence function (see Navarro et al. 2011) of the random variables $X_1, \ldots, X_n$ and $F_1, \ldots, F_n$ are their marginal distribution functions, then G can be expressed as*

$$G(t) = 1 - h^{-1}(W(1 - F_1(t), \ldots, 1 - F_n(t)))$$

with the inverse function h^{-1} of the reliability polynomial h of ϕ. The structure-dependence function W depends on the system structure and the dependence structure of the random variables $X_1, \ldots, X_n$ (via their survival copula), but it does not depend on their marginal distributions.

Remark. *Theorem 2.6 also illustrates that systems with different joint distributions of the component lifetimes may have the same system lifetime distribution. Therefore, from a statistical point of view, the joint distribution of the components is not identifiable if only system lifetime observations are available (see also Navarro et al. 2011).*

3 Stochastic orders of system lifetimes

For comparing different systems, the following well-known stochastic orders will be considered (cf. Shaked and Shanthikumar 2007).

Definition 3.1. *Let X and Y be random variables with distribution functions F and G, respectively.*

(a) X and Y are ordered according to the (usual) stochastic order (denoted by $X \leq_{st} Y$) if

$$1 - F(t) \leq 1 - G(t)$$

for all $t \in \mathbb{R}$.

(b) X and Y are ordered according to the hazard rate order (denoted by $X \leq_{hr} Y$) if

$$\frac{1 - G(t)}{1 - F(t)}$$

is increasing in t for all t such that $F(t) < 1$ or $G(t) < 1$ (with $a/0 := \infty$ for $a > 0$).

(c) *X and Y are ordered according to the likelihood ratio order (denoted by $X \leq_{lr} Y$) if*

$$\frac{g(t)}{f(t)}$$

is increasing in t for all t in the union of the supports of X and Y (with $a/0 := \infty$ for $a > 0$), where f, g are the densities or probability mass functions of F, G, respectively.

Remark. *These stochastic orderings satisfy*

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y.$$

Remark. *Consider random variables X and Y with continuous distribution functions F and G, respectively. Let the support of their distributions be given by $[0, \infty)$. Then, it can be shown that (cf. Shaked and Shanthikumar 2007)*

$$X \leq_{hr} Y \iff (X - t|X > t) \leq_{st} (Y - t|Y > t) \text{ for every } t \geq 0.$$

Moreover, on the right hand side of this equivalence the usual stochastic order can be replaced by the hazard rate order. If the distributions of X and Y are absolutely continuous, then

$$X \leq_{hr} Y \iff \frac{f(t)}{1 - F(t)} \geq \frac{g(t)}{1 - G(t)} \text{ for every } t \geq 0,$$

where f and g are density functions of F and G, respectively. Furthermore, the likelihood ratio ordering satisfies (cf. Shaked and Shanthikumar 2007)

$$X \leq_{lr} Y \iff (X|a < X < b) \leq_{st} (Y|a < Y < b) \text{ for every } 0 \leq a < b \leq \infty.$$

Moreover, on the right hand side of this equivalence the usual stochastic order can be replaced by the likelihood ratio order.

In the following, we will consider coherent systems with given exchangeable component lifetimes $X_1, \ldots, X_n$ which have a joint distribution with the support $[0, \infty)^n$. Using the mixture representation from Section 2, it can be shown that a particular stochastic order on the signature vectors of two systems (if these signatures are interpreted as probability distributions) is transferred to the corresponding system lifetimes if the order statistics of the component lifetimes already satisfy this order (see Navarro et al. 2005, cf. also Kochar et al. 1999, Samaniego 2007). For the case of the usual stochastic order, note that the order statistics are always stochastically ordered.

Theorem 3.2. *Let $T_1 = \phi_1(X_1, \ldots, X_n)$ and $T_2 = \phi_2(X_1, \ldots, X_n)$ be the lifetimes of two coherent systems with structure-signatures $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, respectively, based on the exchangeable component lifetimes $X_1, \ldots, X_n$.*

(a) *If* $\boldsymbol{\rho}_1 \leq_{st} \boldsymbol{\rho}_2$, *then* $T_1 \leq_{st} T_2$.

(b) *If* $\boldsymbol{\rho}_1 \leq_{hr} \boldsymbol{\rho}_2$ *and* $X_{1:n} \leq_{hr} \cdots \leq_{hr} X_{n:n}$, *then* $T_1 \leq_{hr} T_2$.

(c) *If* $\boldsymbol{\rho}_1 \leq_{lr} \boldsymbol{\rho}_2$ *and* $X_{1:n} \leq_{lr} \cdots \leq_{lr} X_{n:n}$, *then* $T_1 \leq_{lr} T_2$.

Remark. *If* $X_1, \ldots, X_n$ *are iid with (absolutely continuous) distribution function* F, *then* $X_{i:n} \leq_{hr} (\leq_{lr}) X_{i+1:n}$ *for* $i = 1, \ldots, n-1$.

Moreover, the distribution of the residual lifetime of the system also has a mixture representation (cf. Navarro et al. 2008a). However, the weights in this mixture depend on both, the system signature and the joint distribution of the component lifetimes.

Theorem 3.3. *Let* $T = \phi(X_1, \ldots, X_n)$ *be the lifetime of a coherent system with structure-signature* $\boldsymbol{\rho}$ *based on the exchangeable component lifetimes* $X_1, \ldots, X_n$. *Then, the distribution of the residual lifetime of the system is given by*

$$P(T - t > x | T > t) = \sum_{i=1}^{n} \rho_i(t) P(X_{i:n} - t > x | X_{i:n} > t), \quad x, t \geq 0,$$

where the entries of $\boldsymbol{\rho}(t) = (\rho_1(t), \ldots, \rho_n(t))$ *are given by*

$$\rho_i(t) = \frac{\rho_i P(X_{i:n} > t)}{\sum_{j=1}^{n} \rho_j P(X_{j:n} > t)}, \quad i = 1, \ldots, n.$$

If $X_1, \ldots, X_n$ *have no ties, then*

$$\rho_i(t) = P(T = X_{i:n} | T > t), \quad i = 1, \ldots, n.$$

Using the preceding representation, conditions for stochastic orders of the residual lifetime distributions of systems can be obtained.

Theorem 3.4. *Let* $T_1 = \phi_1(X_1, \ldots, X_n)$ *and* $T_2 = \phi_2(X_1, \ldots, X_n)$ *be the lifetimes of two coherent systems with structure-signatures* $\boldsymbol{\rho}_1$ *and* $\boldsymbol{\rho}_2$, *respectively, based on the exchangeable component lifetimes* $X_1, \ldots, X_n$. *For fixed* $t > 0$, *let* $\boldsymbol{\rho}_1(t)$ *and* $\boldsymbol{\rho}_2(t)$ *be the corresponding vectors from Theorem 3.3.*

(a) *If* $\boldsymbol{\rho}_1(t) \leq_{st} \boldsymbol{\rho}_2(t)$ *and* $X_{1:n} \leq_{hr} \cdots \leq_{hr} X_{n:n}$, *then*

$$(T_1 - t | T_1 > t) \leq_{st} (T_2 - t | T_2 > t).$$

(b) *If* $\boldsymbol{\rho}_1(t) \leq_{hr} \boldsymbol{\rho}_2(t)$ *and* $X_{1:n} \leq_{hr} \cdots \leq_{hr} X_{n:n}$, *then*

$$(T_1 - t | T_1 > t) \leq_{hr} (T_2 - t | T_2 > t).$$

(b) If $\boldsymbol{\rho}_1(t) \leq_{lr} \boldsymbol{\rho}_2(t)$ and $X_{1:n} \leq_{lr} \cdots \leq_{lr} X_{n:n}$, then

$$(T_1 - t | T_1 > t) \leq_{lr} (T_2 - t | T_2 > t).$$

Remark. *If $X_1, \ldots, X_n$ are iid with (absolutely continuous) distribution function, then only the assumption on the stochastic order of the weights $\boldsymbol{\rho}_1(t)$ and $\boldsymbol{\rho}_2(t)$ needs to be imposed in order to obtain the corresponding order on the residual lifetimes of the systems (cf. Navarro et al. 2008a).*

4 Systems based on sequential order statistics

In this section, systems with a particular joint distribution of the component lifetimes are considered. The dependence structure of the components originates from sequential order statistics.

The model of sequential order statistics has been introduced to describe the lifetimes of k-out-of-n systems which have the following additional property: After a failure of a component, the lifetime distribution of remaining components is allowed to change. In this way, an influence on the performance of surviving components can be modeled. The increasingly ordered lifetimes of the components in such systems are described by sequential order statistics $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n}$ based on distribution functions $F_1, \ldots, F_n$. The distribution function F_i is used to model the lifetime of each surviving component before the i-th failure occurs. The exact definition of sequential order statistics and further information about this model can be found in Kamps (1995) and Cramer and Kamps (2001).

Now, a dependence model for exchangeable component lifetimes can be derived from sequential order statistics. In the following, let sequential order statistics $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n}$ be based on continuous distribution functions $F_1, F_2, \ldots, F_n$ with support $[0, \infty)$. There exist exchangeable random variables $X^*_1, X^*_2, \ldots, X^*_n$ such that their order statistics are equal to $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n}$. Moreover, the joint distribution of these exchangeable random variables $X^*_1, X^*_2, \ldots, X^*_n$ is uniquely determined. The random variables $X^*_1, X^*_2, \ldots, X^*_n$ describe component lifetimes in a system where failures may affect the lifetimes of surviving components (cf. Burkschat 2009, Navarro and Burkschat 2011). These so-called failure-dependent component lifetimes have no ties. Moreover, coherent systems with lifetime given by

$$T = \phi(X^*_1, X^*_2, \ldots, X^*_n)$$

are then called systems based on sequential order statistics.

Using the results from Sections 2 and 3, differently structured systems with failure-dependent components can be compared in several stochastic orders. The following theorem summarizes conditions for the corresponding stochastic orders in the case of sequential order statistics (see Cramer et al.

2003, Hu and Zhuang 2005, Zhuang and Hu 2007, Navarro and Burkschat 2011, Torrado et al. 2012).

Theorem 4.1. *Let* $X^*_{1:n}, X^*_{2:n}, \dots, X^*_{n:n}$ *be sequential order statistics based on absolutely continuous* $F_1, F_2, \dots, F_n$ *and let* $h_i = \frac{f_i}{1-F_i}$ *denote the hazard rate of* F_i, $i = 1, \dots, n$.

(a) If h_k/h_{k+1} *is an increasing function for every* $k = 1, 2, \dots, n-1$*, then*

$$X^*_{1:n} \le_{hr} X^*_{2:n} \le_{hr} \cdots \le_{hr} X^*_{n:n}.$$

(b) If $h_k/h_{k+1} \equiv c_k$*, i.e.* h_k/h_{k+1} *is constant, for every* $k = 1, 2, \dots, n-1$*, then*

$$X^*_{1:n} \le_{lr} X^*_{2:n} \le_{lr} \cdots \le_{lr} X^*_{n:n}.$$

(c) If $F_1 \le_{lr} F_2 \le_{lr} \cdots \le_{lr} F_n$ *and*

$$h_{i+2} + h_i \ge 2h_{i+1} \quad \textit{for } i = 1, \dots, n-2 \quad (\textit{if } n \ge 3),$$

then $X^*_{1:n} \le_{lr} X^*_{2:n} \le_{lr} \cdots \le_{lr} X^*_{n:n}$.

In Section 3, a mixture representation of the system residual lifetime distribution at time t is given. The weights in this mixture depend on the signature and the joint distribution of the component lifetimes (and the time t). In systems based on sequential order statistics, a mixture representation of the residual lifetime distribution with weights depending solely on the signature entries is obtained if it is known that the time t coincides with a failure time (see Navarro and Burkschat 2011, Burkschat and Navarro 2013).

Theorem 4.2. *Let* T *be the lifetime of a coherent system with ordered component lifetimes* $X^*_{1:n}, X^*_{2:n}, \dots, X^*_{n:n}$ *based on* $F_1, F_2, \dots, F_n$ *and the signature vector* $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$*. Then the distribution of the residual lifetime* $(T-t|X^*_{i:n} = t < T)$ *coincides with the lifetime distribution of a mixed system with* $n-i$ *components and the signature vector*

$$\boldsymbol{s}_{|i} = (s_{i+1}/S_i, s_{i+2}/S_i, \dots, s_n/S_i) \quad \textit{with } S_i = s_{i+1} + s_{i+2} + \ldots + s_n.$$

The ordered component lifetimes in this mixed system are given by sequential order statistics $Y^*_{1:n-i}, Y^*_{2:n-i}, \dots, Y^*_{n-i:n-i}$ *based on*

$$G_k(\cdot) = \frac{F_{i+k}(\cdot + t) - F_{i+k}(t)}{1 - F_{i+k}(t)}, \quad k = 1, 2, \dots, n-i.$$

Corollary 4.3. *In the situation of Theorem 4.2, let* $1 \le i < j \le n$ *and* $\boldsymbol{s} = (0, \dots, 0, s_j, \dots, s_n)$*. Then the distribution of* $(T - t|X^*_{i:n} = t)$ *coincides with the lifetime distribution of a mixed system with* $n-i$ *components and the signature vector (with* $n-i$ *entries)*

$$\boldsymbol{s}_{|i} = (0, \dots, 0, s_j, \dots, s_n).$$

The ordered component lifetimes in this mixed system are the same as in Theorem 4.2.

For studying the limiting behavior of the weights in the mixture representation of the residual lifetime distribution in Theorem 3.3, it is assumed that for the hazard rates h_i of the distribution functions F_i the limits

$$c_{i,j} = \lim_{t\to\infty} \frac{(n-i+1)h_i(t)}{(n-j+1)h_j(t)}, \quad i,j \in \{1,\ldots,n\},$$

exist and are finite. Moreover, let

$$k_m = \max\{1 \le j \le m \,|\, 1 \le c_{i,j} < \infty \text{ for every } i = 1,\ldots,m\}$$

and define $c_m = c_{k_m,m}$ for $m = 1,\ldots,n$. Then the limits of the weights can be given as follows (see Burkschat and Navarro 2014).

Theorem 4.4. *Let* $\mathbf{s} = (s_1,\ldots,s_j,0,\ldots,0)$ *with* $s_j > 0$. *Then*

$$\lim_{t\to\infty} P(T = X^*_{i:n} | T > t) = 0$$

for $1 \le i \le k_j - 1$ *and*

$$\lim_{t\to\infty} P(T = X^*_{i:n} | T > t) = s_i \Big/ \left(\sum_{d=k_j}^{i-1} s_d \prod_{m=d+1}^{i} (1-c_m) + s_i + \sum_{d=i+1}^{j} s_d \prod_{m=i+1}^{d} \frac{1}{1-c_m} \right)$$

for $k_j \le i \le r$.

Remark. *In the case of iid component lifetimes* $X_1,\ldots,X_n$ *with a continuous distribution function, the limit is given by (cf. Navarro et al. 2008a)*

$$\lim_{t\to\infty} P(T = X_{i:n} | T > t) = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

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