Advance material for the lectures "Topics in the (non-)amenability of Banach algebras"

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Acknowledgments

The spirit of these lectures, if not the precise content or the approach to amenability, owes much to Volker Runde's book [33]. A less obvious influence on the approach to cohomology by way of explicit averaging is the book of Allan Sinclair and Roger Smith [34]; although the lectures will not focus on cohomology, learning how those calculations worked has shaped what follows. Most of all, I would like to thank Michael White for many conversations about homological aspects of Banach algebra theory, both during my PhD studies and afterwards. His insights and explanations have been a strong influence on how I view the subject. (Of course, I take full responsibility for any errors or lack of mathematical taste in what follows.)

Some words on the scope and purpose of this document

> "In my experience, Miss Cripslock tends to write down *exactly* what one says," Vetinari observed. "It's a terrible thing when journalists do that. It spoils the fun. One feels instinctively that it's cheating, somehow."
>
> (From *Going Postal* by Terry Pratchett.)

My aim with these lectures is twofold: to give the audience an introduction to the basic concepts and techniques; but also to showcase some results in the area which I think deserve to be known to non-specialists. **This document deliberately contains more than I will present in lectures**, so that choices can be made depending on the audience's interests and background. It also includes some commentary that I have included here in case I forget to mention it in lectures, or run out of time. However, it omits most of the proofs (which I plan to go through in lectures).

In the early sections, a lot more has been written than will actually be delivered in spoken presentation, just to make sure that the audience has precise statements and definitions to refer to. Similarly, there are remarks in later sections that are included for sake of completeness, but which may be skipped during the lectures. On the other hand, during the lectures I may supplement some of the arguments with extra lemmas or clarification. The actual coverage depends on the level of detail that the audience find helpful or interesting, and on how much time we are left with.

In most of the sections, I have included a subsection called "Possible further topics". I hope to discuss some of them during the lectures; but if not, then they provide suggestions and pointers for those who are interested in learning more.

Suggestion. Section 0 is long, and mainly exists so that the audience has something to refer back to while we go through the later sections. It is not intended that we start at the beginning of Section 0 and crunch our way through all the details. **Instead, I propose to start with Section 1**, and refer back to Section 0 where necessary. Nevertheless, if you have time to quickly look through Section 0 *before* the start of the lecture series, this would be helpful. I don't expect people to have checked every single detail or memorized every single definition!

Sources for general material on projectivity, injectivity, amenability etc

This is an incomplete list which will probably be updated during the lectures; it is not meant to be comprehensive. Full bibliographic details can be found in the list of references.

Books

- F. F. Bonsall and J. Duncan, Complete Normed Algebras.
  (Chapter I for modules, Chapter VI for cohomology and amenability.)
- A. Paterson, Amenability.
  (Only a small part of Chapter 1, which gives a quick overview of amenability for Banach algebras and C*-algebras)
- V. Runde, Lectures on Amenability.
- A. Ya. Helemskii, The Homology of Banach and Topological Algebras.

Articles or monographs

- P. C. Curtis (Jr.) and R. J. Loy, The structure of amenable Banach algebras. (1989)
- N. Grønbæk, Various notions of amenability: a survey of problems. (1998)
- B. E. Johnson, Approximate diagonals and cohomology of certain annihilator Banach algebras. (1972)
- B. E. Johnson, Cohomology in Banach algebras. (1972)
- M. C. White, Injective modules for uniform algebras. (1996)

Since there are many small choices one has to make in setting up the core background, the route chosen in the lectures will deviate in various places from these sources.

Contents

General notation and other conventions

Given a Banach space E, we denote by $\mathsf{ball}_r(E)$ its closed unit ball of radius r. Usually we shall abbreviate $\mathsf{ball}_1(E)$ to $\mathsf{ball}(E)$. The dual of E is denoted by E^* and the canonical embedding of E in its double dual is denoted by $\kappa_E : E \to E^{**}$.

If E and F are Banach spaces, then $\mathcal{B}(E, F)$ denotes the space of bounded linear maps from E to F. As usual we abbreviate $\mathcal{B}(E, E)$ to $\mathcal{B}(E)$. More generally, given Banach spaces $E_1, \dots, E_n$ and F, we denote by $\mathcal{B}(E_1, \dots, E_n; F)$ denotes the space of bounded multilinear maps $E_1 \times \cdots \times E_n \to F$. Unless stated otherwise, we always equip it with the norm

$$\|T\| = \sup\{|T(x_1, \dots, x_n)| \colon x_i \in \mathsf{ball}(E_i) \text{ for each } i = 1, \dots, n\}$$

The usual tensor product of vector spaces is denoted by $\otimes$. The projective tensor product of Banach spaces (which we will define during the lectures) is denoted by $\widehat{\otimes}$.

If $E \times F \to \mathbb{C}$ is a pairing of Banach spaces, we denote this by $\langle \cdot, \cdot \rangle_{E-F}$. Usually we omit the subscripts, if it is clear from context which spaces are involved.

If A is a Banach algebra, we will frequently write $\pi : A \widehat{\otimes} A \to A$ for the unique bounded linear map that satisfies $\pi(a \otimes b) = ab$ for all $a, b \in A$. If A has an identity element it will usually be denoted by $\mathtt{1}_A$.

We denote by $A^\sharp$ the so-called "forced unitization" of A; that is, $A^\sharp = A \oplus_1 \mathbb{C}$ with the product $(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda, \mu)$. Usually we will denote the identity element of $A^\sharp$ by $\mathtt{1}$.

The set of invertible elements in a unital algebra A is denoted by $A_{\rm inv}$.

The positive cone of a C*-algebra B is denoted by B^+, and the unitary group of a unital C*-algebra B is denoted by $\mathcal{U}(B)$. Unless stated otherwise, H denotes a Hilbert space, not necessarily separable. $\mathbb{M}_n$ denotes $M_n(\mathbb{C})$: usually if we use this notation we are equipping $\mathbb{M}_n$ with its canonical C*-algebra structure.

Other notation will be introduced as and when needed. From time to time I will use basic category-theoretic terminology. Most of this will be introduced and explained as we go along; the audience should merely need some familarity with the language of morphisms and functors, rather than any of the results of category theory.

Finally, despite my best efforts in this document, it is almost certain that I will be inconsistent in my usage of ℓ_p or ℓ^p during the lectures. Apologies for any inconvenience or annoyance this may cause.

0 Background: group algebras, and Banach modules

I assume everyone attending has taken a course that defines Banach algebras, gives a few examples, and proves basic core results (spectral radius formula, some basic holomorphic functional calculus, Gelfand representation, etc).

Audience participation. Since I do not know which Banach algebras and Banach spaces the audience will be familiar or unfamiliar with, I will not devote a separate (sub)section to listing all the ones which will occur. Therefore I *strongly encourage* the audience to stop me as soon as they see a Banach algebra or Banach space whose definition they are unsure of.

0.1 L^1-group algebras and measure algebras

One class of examples has proved so important in the history and the development of the theory of amenable Banach algebras, that we shall give the relevant definitions in some detail. If this is all familiar to you then I apologize for the repetition.

Given a locally compact group G, and a fixed choice of left Haar measure on G (we shall write integrals with respect to this Haar measure as $\int_G f(x)\,dx$), we define the following **convolution product** on $L^1(G)$:

$$(f * g)(t) := \int_G f(s)g(s^{-1}t)\,ds = \int_G f(ts)g(s^{-1})\,ds \qquad \text{(a.e. } t \in G). \tag{0.1}$$

(Strictly speaking, to check this is well-defined requires an application of Fubini's theorem.) You should hopefully have seen this in some form for the cases $G = \mathbb{Z}$, $\mathbb{T}$ and $\mathbb{R}$. Note that if G is discrete then we can define convolution on $\ell^1(G)$ in a more abstract way: it is the unique continuous extension of the obvious way to multiply point masses, $\delta_x * \delta_y := \delta_{xy}$.

Equipped with convolution, $L^1(G)$ becomes a Banach algebra, which we call the **L^1-group algebra** or **L^1-convolution algebra** of G. When G is commutative this algebra is semisimple, and the Gelfand transform is given by the Fourier transform $L^1(G) \to C_0(\widehat{G})$.

In fact $L^1(G)$ has extra structure: it can be equipped with an isometric algebra involution given by

$$f^*(s) := \overline{f(s^{-1})}\Delta(s^{-1})$$

where $\Delta : G \to (0,\infty)$ is the **modular function of G**.

As soon as we try to work with $L^1(G)$ when G is non-discrete, another algebra inevitably turns up: this is the **measure algebra** of G. This algebra is obtained by taking the space $M(G)$ of finite Radon measures on G (which by the Riesz representation theorem can be identified with the dual of $C_0(G)$) and using duality to define an algebra product as follows:

$$\langle \mu * \nu,\, h\rangle_{M(G)-C_0(G)} := \int_{G\times G} h(xy)d(\mu \times \nu)(x,y)\ . \tag{0.2}$$

By taking Radon–Nikodym derivatives, we may identify $L^1(G)$ with the subspace of $M(G)$ consisting of all finite Radon measures that are absolutely continuous with respect to (our choice of) Haar measure: this subspace is a closed ideal in $M(G)$, and by abuse of notation we will identify $L^1(G)$ with this ideal. If $f, g \in L^1(G)$ and $h \in C_0(G)$, then Fubini's theorem yields

$$\int_{G\times G} h(xy)f(x)g(y)\,dx\,dy = \int_G h(t)(f * g)(t)\,dt\ ;$$

and so the identification of $L^1(G)$ with this ideal *is an isomorphism of Banach algebras*. That is: our notion of convolution of measures really does generalize the definition we gave for convolution of L^1-functions.

REMARK 0.1. We have cheated here slightly. Strictly speaking, the *proper* definition of convolution is (0.2), and then one defines convolution on $L^1(G)$ using the embedding $L^1(G) \hookrightarrow M(G)$, before using Fubini to justify the pointwise a.e. formula (0.1). However, the formula (0.1) may be more familiar from existing contexts. Moreover, in many arguments with L^1-group algebras, one can get away with checking things on the dense subset $C_c(G) \subset L^1(G)$, in which case (0.1) is a perfectly good way to think of convolution.

Reassurance. All of this may give the misleading impression that the lectures will involve lots of technical harmonic analysis. This is far from the case, and we are only stating things in this level of generality for sake of completeness. If you keep in mind the cases of discrete noncommutative groups and Lie groups such as $\mathbb{T}$, $\mathbb{R}$, $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{R})$ and

$$\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a,b \in \mathbb{R}; a > 0 \right\}$$

then this should suffice. This leads on to the final remark of this subsection.

Haar measure: technical issues? Careful readers may be concerned about issues concerned with the Baire σ-algebra versus the Borel σ-algebra, or with the use of Fubini's theorem, since I did not assume G is σ-compact. The *extremely* thorough book of Hewitt & Ross [20] explains why we can gloss over these technical details when dealing with convolution of measures. You can also see some discussion at

`http://mathoverflow.net/questions/28865/`

Since all the locally compact groups that we explicitly consider will either be discrete or σ-compact, I will ignore these measure-theoretic issues. However, any results that I have stated as being true for locally compact groups *should* be true in the generality stated: if you think that I am using second-countability or something similar and have forgotten to assume this, please let me know.

0.2 The projective tensor product of Banach spaces

I assume the audience is familar with the basic properties of tensor products of vector spaces. We will mostly use the universal property with respect to multilinear maps, rather than any particular explicit construction.

To develop a theory of Banach modules in parallel with the classical theory of modules over $\mathbb{C}$-algebras, one needs a way to form tensor products of Banach spaces and Banach modules. If one wishes to work with arbitrary Banach spaces and modules, then in some sense there is only one natural choice (as category theorists might say). Fittingly, then, the key properties of this so-called "projective tensor product" were established by Grothendieck.

We will take a slightly non-standard route to the definition (similar to that of [4]), which takes advantage of duality and the Hahn-Banach theorem.

Given Banach spaces $E_1, \dots, E_n$ and F, recall that $\mathcal{B}(E_1, \dots, E_n; F)$ denotes the space of bounded multilinear maps $E_1 \times \dots \times E_n \to F$, equipped with the "obvious" norm.

$$\|T\| = \sup\{|T(x_1, \dots, x_n)| : x_i \in \mathsf{ball}(E_i) \text{ for each } i = 1, \dots, n\}.$$

By the universal property of the usual algebraic tensor product, there is a natural linear map of vector spaces

$$\theta_n : E_1 \otimes \dots E_n \to \mathcal{B}(E_1, \dots, E_n; \mathbb{C})^*$$

which sends an elementary tensor $x_1 \otimes \dots \otimes x_n$ to the functional "evaluate on the n-tuple $(x_1, \dots, x_n)$. Moreover, θ_n is injective.

Definition 0.2 (The projective tensor product). Let $E_1, \dots, E_n$ be Banach spaces. The **projective tensor norm** on $E_1 \otimes \dots \otimes E_n$ is defined to be the subspace norm inherited from the Banach space $\mathcal{B}(E_1, \dots, E_n; \mathbb{C})^*$. We define the **projective tensor product** to be the completion of $E_1 \otimes \dots \otimes E_n$ with respect to the projective tensor norm.

We denote the projective tensor product of two Banach spaces E and F by $E \widehat{\otimes} F$. It can be shown that the projective tensor norm is a so-called cross-norm on $E \otimes F$: that is, $\|x \otimes y\| = \|x\|_E \|y\|_F$ for all $x \in E$ and $y \in F$. We will not discuss tensor norms and Banach space tensor products in any detail, but let us merely note some properties that we will use.

The universal property for bounded bilinear maps. Let E_1, E_2 and G be Banach spaces, and let $\beta \in \mathcal{B}(E_1, E_2; G)$. Then there is a unique bounded linear map $\widetilde{\beta} : E_1 \widehat{\otimes} E_2 \to G$ making the following diagram commute

$$\begin{tikzcd} E_1 \times E_2 \arrow[d] \arrow[dr, "\beta"] & \\ E_1 \widehat{\otimes} E_2 \arrow[r, "\widetilde{\beta}"'] & G \end{tikzcd}$$

Moreover, $\|\widetilde{\beta}\| = \|\beta\|$.

This result has the following consequence, which a category theorist might call "Hom-tensor duality" or "closed monoidal structure" for the category of Banach spaces and bounded linear maps.

Theorem 0.3. *For Banach spaces E, F and G there is a natural isometric isomorphism of Banach spaces*

$$\mathcal{B}(E \widehat{\otimes} F, G) \cong \mathcal{B}(E, \mathcal{B}(F, G))$$

In particular $(E \widehat{\otimes} F)^* \cong \mathcal{B}(E, F^*) \cong \mathcal{B}(F, E^*)$ isometrically.

The following result will be quoted without proof.

Theorem 0.4 (Grothendieck). *Let (Ω_1, μ_1) and (Ω_2, μ_2) be measure spaces. Then the natural map*

$$L^1(\Omega_1, \mu_1) \widehat{\otimes} L^1(\Omega_2, \mu_2) \to L^1(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$$

is an isometric isomorphism of Banach spaces.

Remark 0.5 (Associativity of $\widehat{\otimes}$). As one might expect, $\widehat{\otimes}$ is associative, in the sense that one has natural isometric identifications

$$(E_1 \widehat{\otimes} E_2) \widehat{\otimes} E_3 \cong E_1 \widehat{\otimes} (E_2 \widehat{\otimes} E_3)$$

and both are isometrically isomorphic to the projective tensor product of the triple (E_1, E_2, E_3), which we therefore denote by $E_1 \widehat{\otimes} E_2 \widehat{\otimes} E_3$. By induction, the same works for any finite number of spaces and their projective tensor product.

0.3 Banach modules: first definitions

The material we need on Banach modules is covered in [4, 19]. (But see Remark 0.14 below.)

Definition 0.6. Let A be a Banach algebra. A **left Banach A-module** is a Banach space X which is also a left A-module (in the sense of algebra), such that the action $A \times X \to X$ is a bounded bilinear map. (If this bilinear map has norm ≤ 1 we say that the module X is **contractive**.)

A **morphism of left Banach A-modules**, or a (left) **A-module map** for short, is defined to be exactly what you would guess it to be. The space of all morphisms between two given left Banach A-modules E and F will be denoted by ${}_A\mathrm{Hom}(E, F)$.

Note that even if A has an identity element, we are not assuming that $1_A \cdot x = x$ for all $x \in X$. If this is the case, then we say X is **unit-linked**.

The unit-linked part of a module If A has an identity element, then we can decompose X as a direct sum of closed submodules: $X \cong 1_A \cdot X \oplus X_0$, where $X_0 := \{x : 1_A \cdot x = 0\}$.

Making modules unit-linked by changing the algebra. Every left Banach A-module becomes a unit-linked, left $A^\sharp$-module in a natural way. (This is "natural" in the sense of category theory.) However, $A^\sharp$ is not always a natural object to introduce. (This is "natural" in the colloquial sense.) For instance, sometimes we would prefer to use $M(G)$ rather than $L^1(G)^\sharp$, or the Stone-Cech compactification rather than the one-point compactification, or the *multiplier algebra* of a C*-algebra rather than its unitization.

Right Banach A-modules, and Banach A-bimodules These are defined just as for left Banach A-modules, and the definition of unit-linked applies in the obvious way. We omit the details.

0.4 Some standard examples and constructions of modules

Example 0.7 (Duals of modules). If $X \in A\text{-}\mathsf{mod}$ then its Banach space dual X^* becomes a right Banach A-module in a natural way:

$$\langle \psi \cdot a, x \rangle := \langle \psi, a \cdot x \rangle \qquad (\psi \in X^*, a \in A, x \in X).$$

Moreover, if $X \in A\text{-}\mathsf{bimod}$ then X^* is also a Banach A-bimodule.

Example 0.8 (Tensoring modules over an algebra). Given $E \in \mathsf{mod}\text{-}A$ and $F \in A\text{-}\mathsf{mod}$ we define $E \widehat{\otimes}_A F$ to be the quotient of $E \widehat{\otimes} F$ by the closed subspace generated by the set

$$\{(x \cdot a) \otimes y - x \otimes (a \cdot y) : a \in A, x \in E, y \in F\}$$

Note that if $E, F \in A\text{-}\mathsf{bimod}$ then so is $E \widehat{\otimes} F$.

Clearly A is an example of a Banach A-(bi)module, so as a special case of the previous example we get natural Banach A-(bi)module structures on A^*, A^{**}, etc.

There is an important "Hom-tensor duality": we may identify $(E \widehat{\otimes}_A F)^*$ with either $\mathrm{Hom}_A(E, F^*)$ or ${}_A\mathrm{Hom}(F, E^*)$. We leave the details to the reader.

Remark 0.9 (For those who have seen Arens products). If A is unital, then $A^* \widehat{\otimes}_A A \cong A^*$ as Banach A-bimodules. Taking adjoints once again and using Hom-tensor duality gives an isomorphism of Banach A-bimodules $A^{**} \cong \mathrm{Hom}_A(A^*)$. *This gives us one of the Arens products on A^{**}.*

Example 0.10 (Modules given by characters). Let A be a Banach algebra and $\varphi : A \to \mathbb{C}$ a **character** on A (i.e. a multiplicative linear functional that is not identically zero). We immediately get two examples of Banach A-(bi)modules that deserve attention: the one-dimensional module $\mathbb{C}_\varphi$ where the action of A is just given by applying the homomorphism φ; and the two-sided, closed ideal $\ker \varphi$.

Historically, modules given by characters were mostly considered for unital, commutative Banach algebras — presumably motivated by questions concerning function algebras. However there are noncommutative and non-unital examples where properties of the point modules relate to interesting properties of the algebras. The following examples are particularly relevant to the historical development of amenability, and .

Definition 0.11 (The augmentation character and augmentation ideal). Given a semigroup S, the **augmentation character** is the homomorphism $\varepsilon : S \to \{1\}$. This gives rise to a character on $\ell^1(S)$ (in the sense of Banach algebras) which we also denote by ε.

In the case of a locally compact group G, we may regard ε as a character on the measure algebra $M(G)$, given by $\varepsilon(\mu) := \mu(G)$, and hence by restriction as a character on $L^1(G)$, given by $\varepsilon(f) := \int_G f(x)\,dx$.

The **augmentation ideal** (in $\ell^1(S)$, or $M(G)$, or $L^1(G)$ depending on context) is defined to be $\ker \varepsilon$.

The following classes of modules, although seeming rather trivial or artificial, are crucial to the notions of projectivity and injectivity that we will consider in Section 1.

Definition 0.12 (Free modules). Let A be a Banach algebra and let E be a Banach space. Then $A^\sharp \widehat{\otimes} E$ becomes a left Banach A-module, with the action defined in the obvious way. We call this the **free left Banach A-module generated by E**. Similarly, $E \widehat{\otimes} A^\sharp$ is the free right Banach A-module generated by E; and the free Banach A-bimodule generated by E is $A^\sharp \widehat{\otimes} E \widehat{\otimes} A^\sharp$.

Definition 0.13 (Co-free modules). With A and E as before: $\mathcal{B}(A^\sharp, E)$ may be viewed as a left Banach A-module or as a right Banach A-module; we call it the (left or right) **co-free Banach A-module generated by E.** The co-free Banach A-bimodule is $\mathcal{B}(A^\sharp \widehat{\otimes} A^\sharp, E)$.

0.5 Neo-unital modules and Cohen's factorization theorem

Remark 0.14 (Remarks on some of the literature). The book [4] of Bonsall & Duncan takes great care to work with Banach algebras that may or may not have identity elements, and hence discusses what to do with Banach modules over such algebras. This is in keeping with Johnson's own approach in [22], which has a concise but instructive discussion of how to pass from "neo-unital" Banach $L^1(G)$-modules to "Banach G-modules" and back again. In fact, Johnson shows how to do something similar for any Banach algebra with a bounded approximate identity.

Helemskii's book [19] has a quicker treatement which emphasizes functorial aspects, but in many places it uses unitizations to reduce to the case of unit-linked Banach modules over unital Banach algebras, and sometimes this is undesirable if one wants to do certain explicit constructions. (It also becomes a headache if you want to study the cohomology of the tensor product of two non-unital algebras.) However, unitizations are easier to work with when setting up the general theory.

Definition 0.15 (One-sided and two-sided bounded approximate identities). Let A be a Banach algebra. A **bounded right approximate identity** for A is a bounded net $(f_j) \subset A$ such that $\|af_j - a\| \to 0$ for each $a \in A$. The definition of a **bounded left approximate identity** is similar. A **bounded approximate identity** for A is a net that is simultaneously a BRAI and a BLAI.

Remark 0.16 (A standard trick; see e.g. [4, Prop. 11.4]). Suppose we merely had a net (f_j) – not necessarily bounded — such that $af_j \to a$ in the $\sigma(A, A^*)$ topology for each $a \in A$. Then 0 belongs to the $\sigma(A, A^*)$-closure of the convex hull of $\{f_j\}$, hence by Mazur's theorem 0 belongs to the norm closure of the convex hull. In paricular, if the original net was bounded, then A has a BRAI. The corresponding statements for left approximate identities and two-sided approximate identities are left to the reader.

The following result, based on due originally to Cohen and refined by various authors, is an indispensable tool when working with non-unital algebras that have bounded approximate identities. Our version comes from [4, Theorem 11.10].

Theorem 0.17 (Cohen's factorization theorem – the module version). *Let A be a Banach algebra and $X \in A$-mod. Suppose there exists a bounded net $(e_i) \subset A$ with $\sup_i \|e_i\| \le C$, such that*

$e_i \cdot x \to x$ for all $x \in X$. Then each $y \in X$ can be written as $y = az$ for some $a \in A$ and $z \in X$; moreover, we can take $\|a\| \le C$ and take $\|z - y\|$ to be as small as desired.

COROLLARY 0.18 (Cohen's factorization theorem – original version). *Let A be a Banach algebra with a b.a.i. Then for each $a \in A$ we can find $b_1, b_2 \in A$ with $a = b_1 b_2$.*

In the standard approach to amenability, via derivations, it is both traditional and natural to work with **neo-unital modules** and the **essential part** of a module. While they are not so important in this series of lectures, we give the definitions here in case we need to refer to them later. To save time we state the results for bimodules, but it should be clear that by restricting the proofs one obtains versions for one-sided modules.

DEFINITION 0.19 (Neo-unital modules). Let $X \in A\text{-}\mathsf{bimod}$. We say that X is **neo-unital** if for each $x \in X$ there exist $a_1, a_2 \in A$ and $y_1, y_2 \in X$ such that $a_1 y_1 = x = y_2 a_2$.

The following result is not too difficult to prove although it relies on a trick.

PROPOSITION 0.20 (The essential part of an A-bimodule). *Suppose A has a b.a.i. (e_i) and let $X \in A\text{-}\mathsf{bimod}$. Then $X_{\rm ess} := \{a \cdot x \cdot b \colon a, b \in A; x \in X\}$ is a closed sub-A-bimodule of X. Moreover, $X_{\rm es}$ is neo-unital.*

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1 Projective, injective, flat

1.1 Motivation from algebra

Up to isomorphism, there are precisely two commutative, unital $\mathbb{C}$-algebras:

$A_0 := \mathbb{C} \oplus \mathbb{C}$ and $A_1 := \mathbb{C}[\epsilon]/(\epsilon^2)$.

They behave very differently. For example, every finite-dimensional A_0-module is the direct sum of irreducible A_0-modules. On the other hand, A_1 has a two-dimensional, indecomposable representation

$$a + b\epsilon \mapsto \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

and this representation is not irreducible.

Algebras which behave more like A_0 than A_1 are: $\mathbb{C}^n$; $\mathbb{M}_n$; and $\mathbb{C}G$ for a finite group G. For each of these three algebras, indecomposable representations/modules are irreducible. To prove this we need to show that submodules of a given module "split off" as module summands. **This will be done in lectures.**

1.2 The Banach versions

We are following [19] except for one small change in terminology.

DEFINITION 1.1 (Non-standard terminology!). Let $\mathcal{C}$ be one of the categories $A\text{-}\mathsf{mod}$, $\mathsf{mod}\text{-}A$ or $A\text{-}\mathsf{bimod}$. A morphism $f : M \to N$ in $\mathcal{C}$ is a **Ban-split epimorphism** if there exits a bounded linear map $c : N \to M$ such that $cf = \iota$. A morphism $g : M \to N$ in $\mathcal{C}$ is said to be a **Ban-split monomorphism** if there exits a bounded linear map $r : N \to M$ such that $rg = \iota$.

(The r stands for "retract" and c for "co-retract".)

REMARK 1.2. Helemskii uses the adjective "admissible" where we have used "Ban-split", and this has become the standard terminology. However, in my view it is needlessly vague, and not very descriptive, to say "admissible". See also [37] for some remarks on terminology.

Definition 1.3 (Projective modules). Let $P \in \mathcal{C}$. We say that P is **projective in $\mathcal{C}$** if it has the following "lifting property": whenever $f : M \to N$ is a Ban-split epimorphism in $\mathcal{C}$, then for each morphism $h : P \to N$ there exists a morphism $\widetilde{h} : P \to M$ making the following diagram commute:

$$\begin{tikzcd} & M \arrow[d, "f"] \\ P \arrow[ur, dotted, "\exists\, \widetilde{h}"] \arrow[r, "h"'] & N \end{tikzcd}$$

Proposition 1.4 (Free modules are projective). *Let E be a Banach space. Then $A^\sharp \widehat{\otimes} E$ is projective in A-mod, $E \widehat{\otimes} A^\sharp$ is projective in mod-A, and $A^\sharp \widehat{\otimes} E \widehat{\otimes} A^\sharp$ is projective in A-bimod.*

Since this result serves as a prototype and "crutch" for later calculations, we give a detailed proof for the case of A-mod, and leave the others to the audience.

Proof. Let $f : M \to N$ be a Ban-split epimorphism in A-mod — so there exists a bounded linear map $c : N \to M$ such that $fc = \iota$. Note that M and N become **unit-linked** Banach $A^\sharp$-modules in a way that extends the existing A-module structures (we just let the identity element 1 of $A^\sharp$ act on M and on N as the identity map); and f is then an $A^\sharp$-module map.

Let $h : A^\sharp \widehat{\otimes} E \to M$ be a left A-module map (and hence a left $A^\sharp$-module map). We define

$$\widetilde{h}(a^\sharp \otimes x) := a^\sharp \cdot ch(1 \otimes x) \qquad (a^\sharp \in A^\sharp, x \in E). \tag{1.1}$$

It is clear that h is an A-module map. We now do the following calculation:

$$\begin{aligned} f\widetilde{h}(a^\sharp \otimes x) &= a^\sharp \cdot f(ch1 \otimes x) && \text{(since } f \text{ is an } A^\sharp\text{-module map)} \\ &= a^\sharp \cdot h(1 \otimes x) && \text{(since } fc = \iota\text{)} \\ &= h(a^\sharp \otimes x) && \text{(since } h \text{ is an } A^\sharp\text{-module map)} \end{aligned}$$

Thus $f\widetilde{h} = h$, as required. ☐

Exercise. Suppose A is unital. By adapting the argument above, convince yourself that $A \widehat{\otimes} E$ is projective in A-mod – the point is that we do not assume A-modules are unit-linked in the definition of "projective in A-mod". Similarly, $E \widehat{\otimes} A$ is projective in mod-A and $A \widehat{\otimes} E \widehat{\otimes} A$ is projective in A-bimod. *In particular: when A is unital, $A \widehat{\otimes} A$ is projective in A-bimod.*

Definition 1.5 (Injective modules). Let $W \in \mathcal{C}$. We say that W is **injective in $\mathcal{C}$** if it has the following "extension property": whenever $g : M \to N$ is a Ban-split monomorphism in $\mathcal{C}$, then for each morphism $h : M \to W$ there exists a morphism $\widetilde{k} : N \to W$ satisfying $\widetilde{k}g = k$.

$$\begin{tikzcd} N \arrow[dr, dotted, "\exists\, \widetilde{k}"] & \\ M \arrow[u, "g"] \arrow[r, "k"'] & W \end{tikzcd}$$

Example 1.6 (Co-free modules are injective). Let E be a Banach space. Then $\mathcal{B}(A^\sharp, E)$ is injective in A-mod and in mod-A, and $\mathcal{B}(A^\sharp \widehat{\otimes} A^\sharp, E)$ is injective in A-bimod.

Exercise.

(i) Suppose A is unital. Show that for any Banach space E, $\mathcal{B}(A, E)$ is injective in A-mod and in mod-A, and $\mathcal{B}(A \widehat{\otimes} A, E)$ is injective in A-bimod.

(ii) Suppose that A has a bounded approximate identity, and let $E = (E_*)^*$ be a *dual Banach space*. Show that the conclusions of part (i) still hold. In particular, when A has a bounded approximate identity, $(A \widehat{\otimes} A)^*$ is injective in A-bimod.

The next definition is not the original one but is equivalent for our purposes.

DEFINITION 1.7. Let $E \in A\text{-mod}$. We say that E is a **flat left A-module**, or **flat in A-mod**, if E^* is injective in mod-A.

REMARK 1.8. These notions could more accurately be called relatively projective, relatively injective, and relatively flat, because we are always working relative to a special class of monomorphisms and epimorphisms. A good explanation of the terminology is given in [37].

Informally: direct summands of projectives/injectives are projective/injective. The precise statement is given in the following exercise.

EXERCISE. Let $X \in \mathcal{C}$ and let $V \in \mathcal{C}$ be a closed sub-(bi)module of X. Suppose there exists a morphism $p : X \to V$ in $\mathcal{C}$ which is left inverse to the inclusion $V \hookrightarrow X$. If X is projective in $\mathcal{C}$, then so is V. If X is injective in $\mathcal{C}$, then so is V.

The definitions of projective and injective require one to test over all Ban-split epimorphisms and monomorphisms, respectively. It is often more convenient to use the following characterizations, which only require one to test for a particular canonical epimorphism/monomorphism.

LEMMA 1.9.

(i) *Let $X \in A\text{-mod}$. Then X is projective in A-mod if and only if there exists $\sigma_L : X \to A^\sharp \widehat{\otimes} X$ which is left inverse to the canonical map $A^\sharp \widehat{\otimes} X \to X$.*

(ii) *Let $X \in \text{mod-}A$. Then X is projective in A-mod if and only if there exists $\sigma_R : X \to X \widehat{\otimes} A^\sharp$ which is left inverse to the canonical map $X \widehat{\otimes} A^\sharp \to X$.*

(iii) *Let $X \in A\text{-bimod}$. Then X is projective in A-bimod if and only if there exists $\sigma_2 : X \to A^\sharp \widehat{\otimes} X \widehat{\otimes} A^\sharp$ which is left inverse to the canonical map $A^\sharp \widehat{\otimes} X \widehat{\otimes} A^\sharp \to X$.*

LEMMA 1.10.

(i) *Let $Y \in \text{mod-}A$. Then Y is injective in mod-A if and only if there exists $\rho_R : \mathcal{B}(A^\sharp, Y) \to Y$ that is right inverse to the canonical inclusion $Y \hookrightarrow \mathcal{B}(A^\sharp, Y)$.*

(ii) *Similarly for injectivity in A-mod.*

(iii) *Let $Y \in A\text{-bimod}$. Then Y is injective in A-bimod if and only if there exists $\rho_2 : \mathcal{B}(A^\sharp \widehat{\otimes} A^\sharp, Y)$ which is right inverse to the canonical inclusion $Y \hookrightarrow \mathcal{B}(A^\sharp \widehat{\otimes} A^\sharp, Y)$.*

In many examples we will deal with, one can avoid use of the unitization when checking if various modules are projective, injective or flat.

REMARK 1.11. Let A be unital and let $\mathcal{C}$ be one of the categories A-mod, mod-A or A-bimod. Then to test projectivity of X in $\mathcal{C}$, one can replace $A^\sharp$ with A in Lemma 1.9.

REMARK 1.12. Suppose A has a b.a.i., and let $\mathcal{C}$ be as in the previous remark. Let $X \in \mathcal{C}$ and put $Y = X^*$. Then to test flatness of X in $\mathcal{C}$, one can replace $A^\sharp$ with A in Lemma 1.10.

1.3 Projective and injective 1-dimensional modules

In the next two propositions, φ is a character on A (that is, a multiplicative linear functional which is not identically zero).

PROPOSITION 1.13. *$\mathbb{C}_\varphi$ is projective in A-mod if and only if there exists $b_0 \in A$ satisfying $\varphi(b_0) = 1$ and $ab_0 = \varphi(a)b_0$ for all $a \in A$. When A has an identity element, this is equivalent to requiring that $\ker(\varphi)$ has a right identity element.*

PROPOSITION 1.14. *$\mathbb{C}_\varphi$ is flat in A-mod if and only if there exists a bounded net $(b_i) \subset A$ satisfying $\varphi(b_i) = 1$ for all i and $ab_i - \varphi(a)b_i \to 0$ for all $a \in A$. When A has an identity element, this is equivalent to requiring that $\ker(\varphi)$ has a bounded right approximate identity.*

COROLLARY 1.15. *Let A be a commutative Banach algebra and let $\varphi : A \to \mathbb{C}$ be a character. Let $\mathbb{C}_\varphi$ be the corresponding 1-dimensional A-module. Then $\mathbb{C}_\varphi$ is projective in A-mod $\iff$ φ is an isolated point of the Gelfand spectrum.*

EXAMPLE 1.16. Let G be a locally compact group and suppose $\mathbb{C}_\varepsilon$ is injective in $L^1(G)$-mod. Then there exists $\Psi \in \mathcal{B}(L^\infty(G), \mathbb{C}_\varepsilon) = L^1(G)^{**}$ such that $\Psi(\mu \cdot h) = (\int_G d\mu)\Psi(h)$ for every $\mu \in L^1(G)$ and $h \in L^\infty(G)$. From this, standard tools allow us to eventually construct a positive, G-invariant functional $m : C_b(G) \to \mathbb{C}$ such that $m(\mathbf{1}) = 1$ — and this is one of the known characterizations of **amenability** for a locally compact group.

REMARK 1.17. In recent years several authors have looked at A-bimodules X where the action on the right is given by a character, seemingly with the aim of generating new papers doing variations on known results for amenability, (bi)projectivity, etc. However, such a bimodule is nothing but a tensor product of a left module with a 1-dimensional right module: that is, $X \cong X_L \otimes \mathbb{C}_\varphi$. Therefore, it is possible that some of the "new" definitions and results of these authors could be recovered quickly from the existing definitions and results for Banach modules.

1.4 A quick look at some function algebras

There is a lot that could be said here: we will just consider some basic, prototypical examples. We start with the disc algebra $A(\mathbb{D})$ and the convolution algebra $\ell^1(\mathbb{Z}_+)$, since these are natural completions of the polynomial ring $\mathbb{C}[z]$. It is convenient to think of $\ell^1(\mathbb{Z}_+)$ as the algebra of absolutely convergent Taylor series, with multiplication given by the usual product of power series; from this viewpoint, $\ell^1(\mathbb{Z}_+)$ embeds continuously as a dense unital subalgebra of $A(\mathbb{D})$.

EXAMPLE 1.18 (Point modules for $A(\mathbb{D})$ and $\ell^1(\mathbb{Z}_+)$). We can construct an explicit b.a.i. in $\ker(\mathrm{ev}_1 : \ell^1(\mathbb{Z}_+) \to \mathbb{C})$ (which is therefore a b.a.i. in the corresponding maximal ideal of $A(\mathbb{D})$). **Details to be given in lectures.** Thus the corresponding point module $\mathbb{C}_1$ is flat in both $\ell^1(\mathbb{Z}_+)$-mod and $A(\mathbb{D})$-mod; and by symmetry the same is true for point modules given by any other point of $\partial\mathbb{D} = \mathbb{T}$.

On the other hand, if $w \in \mathbb{D}$, then $\ker(\mathrm{ev}_w : A(\mathbb{D}) \to \mathbb{C})$ does not have a b.a.i. (so *a fortiori*, the corresponding maximal ideal of $\ell^1(\mathbb{Z}_+)$ also has no b.a.i.).

EXAMPLE 1.19 (Maximal ideals in $A(\mathbb{D})$ and $\ell^1(\mathbb{Z}_+)$). For both these algebras, flatness of $\mathbb{C}_\varphi$ when φ is evaluation at a boundary point implies that the corresponding maximal ideal is also flat.

What about the maximal ideals corresponding to interior points of $\mathbb{D}$? First consider the case of $A(\mathbb{D})$. By applying a suitable automorphism of $A(\mathbb{D})$ it suffices to consider the ideal $zA(\mathbb{D}) \lhd A(\mathbb{D})$. But this is clearly isomorphic in $A(\mathbb{D})$-mod to $A(\mathbb{D})$, and so is projective in $A(\mathbb{D})$-mod. (Or one could see this directly using Lemma 1.9.)

The case of $\ell^1(\mathbb{Z}_+)$ is similar, although note that we have to consider each point in $\mathbb{D}$ since we can't use the same trick of "moving everything to the origin". For an application of this kind of calculation to results in Hochschild cohomology of point modules, see [8].

Let us adopt some (not entirely standard) terminology. If $\mathcal{C}$ be one of the categories A-mod, mod-A or A-bimod. An object $E \in \mathcal{C}$ is said to be **Hilbertian** if it is isomorphic as a Banach space to a Hilbert space.

EXAMPLE 1.20 ($L^2(\mathbb{T})$ is $A(\mathbb{D})$-injective). We will show in lectures that $L^2(\mathbb{T})$ is injective in $A(\mathbb{D})$-mod, using an explicit averaging argument.

Exercise. Show that the Hardy space $H^2(\mathbb{D})$ is not $A(\mathbb{D})$-injective. (Hint: show that as a module, $H^2(\mathbb{D})$ is isomorphic to its submodule $zH^2(\mathbb{D})$; then show that there is no submodule of $H^2(\mathbb{D})$ complementary to $zH^2(\mathbb{D})$.)

The following result will tell us (once we have defined amenability) that amenable uniform algebras are not interesting as uniform algebras. We include it here because the proof is a lovely application of Fuglede's theorem, and shows the power of abstraction.

Theorem 1.21 (Sheinberg, paraphrased). *Let $A \subseteq C(\Omega)$ be a closed unital subalgebra. Suppose all Hilbertian A-modules are injective. Then A is self-adjoint (i.e. closed under conjugation).*

The hypotheses of Theorem 1.21 may seem to be too strong, or too vague. In fact, we will see in due course that if Ω is a compact Hausdorff space then all dual Banach $C(\Omega)$-modules are injective.

Remark 1.22. The study of Hilbertian modules over uniform algebras was pursued in several papers in the 1980s and 1990s but seems to have dwindled away, due to a combination of negative results and technical obstacles. (However, studying Hilbert spaces that are modules over natural commutative algebras such as $\mathbb{C}[z_1, z_2]$ is an area of ongoing work; see work of various authors such as X.-M. Chen and K.Y. Guo, or R. G. Douglas and G. Misra.)

1.5 Possible further topics

White's paper [37] studies how one might obtain injective modules over "nice" uniform algebras, in particular the following question:

> if A is a uniform algebra with Shilov boundary ∂A, we can show that dual Banach $C(\partial A)$-modules are injective in $C(\partial A)$-mod; but when can we say that they are injective in A-mod?

The results of [37] are stated in the language of strictly flat inclusions $A \to C(\partial A)$. Depending on how quickly we get through this section, I may say some things about how the arguments work, and what kinds of uniform algebras they apply to.

2 Biprojective and biflat algebras

In most accounts of amenability these are seen as "lesser variants". One aim of the lectures is to indicate that these concepts have their own intrinsic worth, and are not just obtained by taking a known characterization of amenability and weakening it in an *ad hoc* manner.

2.1 Definitions and general results

Definition 2.1. We say that a Banach algebra A is **biprojective** if it is projective in A-bimod, and **biflat** if it is flat in A-bimod.

Lemma 2.2. *Let A be a Banach algebra and let $\pi_L : A^\sharp \widehat{\otimes} A$ be the "multiplication map".*

(i) *Suppose A is biprojective. Then there exists an A-bimodule map $\sigma_L : A \to A^\sharp \widehat{\otimes} A$ satisfying $\pi_L \sigma_l = \iota$.*

(ii) *Suppose A is biflat. Then there exists an A-bimodule map $\rho_L : (A^\sharp \widehat{\otimes} A)^* \to A^*$ satisfying $\rho_L \pi_L^* = \iota$.*

Remark 2.3. With some more work, one can strengthen this result to obtain the following: if A is biprojective then any such σ_L has range contained in $A \widehat{\otimes} A$; similarly, if A is biflat then any such ρ_L descends to a functional on $A \widehat{\otimes} A$. Full proofs can be found in [19] or [33], but to save time we will probably omit this from the lectures.

The following results gives some indication why one might be interested in showing that particular algebras are biprojective or biflat. In particular: if you know about Fourier algebras, then these results (or rather, the natural cb analogues) explain *why* we might wish to know when the Fourier algebra of a group is "operator biflat".

Proposition 2.4 ("Induced" modules over biprojective algebras). *Let A be a biprojective Banach algebra and let $X \in A$-mod. Let $X_1 := A \widehat{\otimes}_A X$. Then X_1 is projective in A-mod.*

Proposition 2.5 ("Induced" modules over biflat algebras). *Let A be a biflat Banach algebra and let $X \in A$-mod. Let $X_1 := A \widehat{\otimes}_A X$. Then X_1 is flat in A-mod.*

The module $A \widehat{\otimes}_A X$ is not always easy to describe, but in many cases one uses the following.

Lemma 2.6. *Suppose A has a b.a.i. and let $X \in A$-mod be neo-unital. Then the natural map $A \widehat{\otimes}_A X \to X$ is an isomorphism in A-mod.*

Corollary 2.7. *If A is biprojective and has a b.a.i. then all neo-unital, left Banach A-modules are projective in A-mod.*

The corresponding corollary for biflat algebras with a b.a.i. is important enough that we extend it slightly and then elevate it to a theorem.

Theorem 2.8 ("That which we call a rose / by another word would smell as sweet"). *Suppose A is biflat and has a bounded approximate identity. Then for any $X \in A$-mod, X^* is injective in mod-A. Consequently, if*

$$0 \to E \to F \to G \to 0$$

is a short exact sequence in A-mod such that the dual sequence

$$0 \to G^* \to F^* \to E^* \to 0$$

is Ban-split, then G^* splits off as a module summand of F^*.

2.2 Some corollaries

Corollary 2.9. *If A is biprojective and has a b.a.i. then all left Banach A-modules are projective in A-mod.*

Corollary 2.10. *Let A be a biprojective, commutative Banach algebra. Then the Gelfand spectrum of A is either empty, or discrete.*

Remark 2.11 (For those doing LCQG). If one sets up the "cb-version" of this theory, then the cb-version of the previous corollary gives a very quick proof that "operator biprojectivity" of $\mathrm{A}(G)$ forces G to be discrete.

2.3 Some examples of biprojective and biflat algebras

Recall that in Section 1.1 we considered three classes of finite-dimensional algebras: $\mathbb{C}^n$, $\mathbb{M}_n$, and $\mathbb{C}G$ for G a finite group. In each case we found that all modules are injective (although we didn't state it that way at the time). Moreover, the proofs all worked for the same reason: if A denotes any of those algebras, then we used the existence of some element $M \in A \otimes A$ satisfying $\pi(M)a = a$ and $a \cdot M = M \cdot a$ for each $a \in A$.

Definition 2.12. Let A be a Banach algebra. A **diagonal for A** is an element $\Delta \in A \widehat{\otimes} A$ such that $\pi(\Delta)$ is an identity element for A and $a \cdot \Delta = \Delta \cdot a$ for all $a \in A$.

Exercise. Suppose A has a diagonal. Show that A is biprojective (and hence is biflat).

There is no example known of an infinite-dimensional Banach algebra which has a diagonal. Thankfully, there are more examples of biprojective and biflat algebras.

Example 2.13. Let E be a Banach space and equip $E \widehat{\otimes} E^*$ with the product

$$(x_1 \otimes \psi_1)(x_2 \otimes \psi_2) := \psi_1(x_2) x_1 \otimes \psi_2 .$$

Then $E \widehat{\otimes} E^*$ is a biprojective Banach algebra. In particular, the algebra of trace-class operators on Hilbert space is biprojective (when given the trace-class norm).

Example 2.14. Let $\mathbb{I}$ be any indexing set and consider $c_0(\mathbb{I})$ and $\ell^1(\mathbb{I})$ as Banach algebras with respect to pointwise product. Then both of them are biprojective.

Theorem 2.15 (Johnson; Helemskii). *Let G be a compact group. Then $L^1(G)$ is biprojective.*

3 Introducing amenable Banach algebras

3.1 The core definitions and equivalences

We do not use Johnson's original definition from [22], but instead the equivalent characterization from his subsequent paper [21].

Definition 3.1. Let A be a Banach algebra and let $\pi : A \widehat{\otimes} A \to A$ be the linearized multiplication map. A **virtual diagonal for A** is an element $M \in (A \widehat{\otimes} A)^{**}$ with the following two properties:

(i) $a \cdot M = M \cdot a$ for all $a \in A$;

(ii) $a \cdot \pi^{**}(M) = \kappa_A(a)$ for all $a \in A$.

If A has a virtual diagonal, we say that A is **amenable**. The **amenability constant** of A is the infimum of the norms of all possible virtual diagonals for A, with the convention that this is defined to be $+\infty$ if A does not have a virtual diagonal for A.

As Johnson himself remarks in the introduction to [21]: "the element M plays the same role in the theory of amenable Banach algebras that the invariant mean does for amenable groups"

Remark 3.2. If A is amenable and unital then every virtual diagonal M satisfies $\pi^{**}(M) = \kappa_A(1_A)$.

Definition 3.3. Let A be a Banach algebra and $\pi : A \widehat{\otimes} A \to A$ as before. A **bounded approximate diagonal for A** (sometimes abbreviated to b.a.d.) is a bounded net $(m_\alpha) \subset A \widehat{\otimes} A$ that satisfies: $\lim_\alpha (a \cdot m_\alpha - m_\alpha \cdot a) = 0$ and $\lim_\alpha a\pi(m_\alpha) = a$, for all $a \in A$.

Example 3.4 (Motivating example). For $\ell^1(\mathbb{Z})$ we can write down an explicit bounded approximate diagonal:

$$d_n := \frac{1}{2n+1} \sum_{j=-n}^{n} \delta_j \otimes \delta_{-j}$$

Exercise: show that for each $a \in \ell^1(\mathbb{Z})$, we have $a \cdot d_n - d_n \cdot a \to 0$ as $n \to \infty$. (Hint: first do the case where $a = \delta_k$ for some $k \in \mathbb{Z}$.)

Remark 3.5. Note that since $\ell^1(\mathbb{Z})$ embeds continuously into $C(\mathbb{T})$ with dense range, the sequence (d_n) also serves as a bounded approximate diagonal for $C(\mathbb{T})$. This shows very directly that $C(\mathbb{T})$ is amenable.

The following theorem links these definitions with our previous discussion of biflat algebras, injective modules, and so on.

THEOREM 3.6 (Characterizations of amenable Banach algebras). *Let A be a Banach algebra. The following are equivalent:*

(i) *A has a virtual diagonal;*

(ii) *A has a bounded approximate diagonal;*

(iii) *$A^\sharp$ has a bounded approximate diagonal;*

(iv) *$A^\sharp$ has a virtual diagonal;*

(v) *$A^\sharp$ is biflat;*

(vi) *A has a bounded approximate identity and is biflat.*

REMARK 3.7. If one wishes to "use" amenability of a Banach algebra then bounded approximate diagonals are usually more tractable and convenient. Trying to do the same things with virtual diagonals can lead into dangerous waters, e.g. trying to take iterated limits of multilinear maps that are separately weak-star continuous, or constructing functions whose measurability is open to initial doubt.

However, if you want to prove that a particular Banach algebra is amenable, it may be easier to aim for a virtual diagonal. (For those who are interested in amenability of groups: a similar philosophy applies when considering Reiter nets versus invariant means.)

REMARK 3.8. We can use bounded approximate diagonals to give a more explict construction of the splitting maps in Theorem 2.8. (This is the approach taken in [13].)

3.2 Revisiting some of our earlier results

Let us restate some of the earlier results from Section 2 in forms closer to those usually found in the literature.

THEOREM 3.9 (Sheinberg). *Let A be an amenable uniform algebra. Then $A = C_0(K)$ for some locally compact Hausdorff space K.*

PROPOSITION 3.10 (Maximal ideals in unital amenable CBAs). *Let A be a unital, amenable, commutative Banach algebra and let M be a maximal ideal in A. Then M has a bounded approximate identity (e_α) satisfying* $\sup_\alpha \|1_A - e_\alpha\| \leq \mathrm{AM}(A)$.

The converse of this proposition does not hold. Here are two instructive examples, although we will have to wait until Section 5.3 for the full story.

EXAMPLE 3.11. Consider $AC[0,1]$, with norm $\|f\|_{\mathrm{AC}} := \|f\|_\infty + \|f'\|_1$. Every maximal ideal has a b.a.i. (e_α) satisfying $\sup_\alpha \|1 - e_\alpha\| \leq 2$. But this algebra is not weakly amenable (to be defined in Section 5.3) so it is not amenable.

EXAMPLE 3.12. Let G be a compact group and consider the **Fourier algebra** $\mathrm{A}(G)$. We may think of this as the space of all $f \in C(G)$ for which

$$\|f\|_{\mathrm{A}} := \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_1$$

is finite. Then every maximal ideal in $\mathrm{A}(G)$ has a b.a.i. (e_α) satisfying $\sup_\alpha \|1 - e_\alpha\| \leq 1$. Once again this algebra is not weakly amenable so it is not amenable.

Example 3.13 (Important non-examples). Two algebras which are very far from being amenable are $\ell^1(\mathbb{Z}_+)$ and $A(\mathbb{D})$. In both cases, to prove non-amenability it suffices to find a dual Banach module that is not injective. We could use $H^2(\mathbb{D}) \hookrightarrow L^2(\mathbb{T})$, with the same argument as before. A simpler example is to consider the character ev_0 corresponding to the point $0 \in \overline{\mathbb{D}}$, and regard the corresponding module $\mathbb{C}_0$ as a submodule of the dual of the algebra.

Remark 3.14. The semigroup $\mathbb{Z}_+$ is amenable but $\ell^1(\mathbb{Z}_+)$ is very far from being amenable. This is not a problem! Amenability of Banach algebras has proved to be important because it is a functional analytic version of something very natural in algebra, not because it is some property that detects amenability of a (semi)group.

3.3 L^1-convolution algebras of amenable groups

This is the class of examples which motivated Johnson to call certain Banach algebras "amenable".

Theorem 3.15. *Let G be a locally compact amenable group. Then $L^1(G)$ is amenable.*

We have already seen how to do this when G is compact (because for such G $L^1(G)$ is biprojective, so we can use (iv) $\Longrightarrow$ (i) in Theorem 3.6). For G discrete and amenable, one can use Følner sets to build an explicit bounded approximate diagonal.

For G which are amenable, but neither discrete nor compact, it is not so easy to build a bounded approximate diagonal for $L^1(G)$. At this level of generality, the simplest way I know of is a construction of R. Stokke in his PhD thesis, published in [35].

Outline of Stokke's construction. This relies on two ingredients:

- when G is amenable there is a bounded net $(f_i) \subset L^1(G)$ satisfying $\|\delta_x * f_i - f_i\|_1 \to 0$ uniformly on compact subsets of G (a so-called **Reiter P_1 net**);
- when G is amenable there is a b.a.i. $(e_j) \subset L^1(G)$ satisfying $\|\delta_x * e_j - e_j * \delta_x\|_1 \to 0$ uniformly on compact subsets of G (a so-called **quasi-central b.a.i.** — this part of the result is due to work of Losert and Rindler, but is not hard to prove directly).

One then defines $m_{i,j}(s,t) := f_i(s)e_j(st)$ and shows that the net $(m_{i,j}) \subset L^1(G \times G)$ is indeed a bounded approximate diagonal for $L^1(G)$.

Remark 3.16. One might try to bypass Stokke's construction and go directly for a virtual diagonal for $L^1(G)$. This approach requires more work than one might first guess: it is possible to generalize the approach used for the compact case, replacing an invariant integral over G with a choice of left invariant mean $\Lambda : L^\infty(G) \to \mathbb{C}$, but extra care is needed. For instance, it is clear that if one tries to define a virtual diagonal $\Psi \in L^\infty(G \times G)$ directly as

$$\Psi(h) := \Lambda(s \mapsto h(s, s^{-1})$$

then this is not well-defined, since $\{(s, s^{-1})\}$ may have measure zero. One has to instead

- define Ψ on a certain subspace $V \subseteq L^\infty(G \times G)$ satisfying $V \subseteq C_b(G \times G)$,
- use left invariance of Λ to get $\delta_g \cdot (\Psi|_V) = (\Psi|_V) \cdot \delta_g$ for all $g \in G$,
- use arguments with Bochner integrals or the "strict topology" on $M(G)$ to deduce $\mu \cdot (\Psi|_V) = (\Psi|_V) \cdot \mu$ for all $\mu \in L^1(G)$,
- and finally use the original choice of V to extend Ψ in a suitable way to a genuine virtual diagonal $\widetilde{\Psi} \in L^\infty(G \times G)^*$.

Those in the audience who want to see further details should look at the proof of [35, Corollary 1.3] and some of the references cited there.

Remark 3.17. The converse of Theorem 3.15 is true: if $L^1(G)$ is amenable then G is amenable. Johnson's proof (which he actually credits to Ringrose) involved taking certain derivations. Given our earlier approach, we could instead argue as follows: if $L^1(G)$ is amenable then it is biflat, so the point module $\mathbb{C}_\varepsilon$ is (left or right) injective in $L^1(G)$-mod, and now we refer to Example 1.16. (However, note that in Example 1.16 we swept some work under the carpet.)

My personal view is that the lesson to draw from all this is not "amenability of the Banach algebra detects amenability of the group". Instead, the point is that we can do things with L^1 of an amenable group that we cannot do for arbitrary L^1-group algebras: Banach-algebraic amenability can be seen as a way to unify some of this good behaviour, in a Banach-algebraic concept that can be applied to other kinds of Banach algebras. Similarly for the Fourier algebra $\mathrm{A}(G)$ when G is amenable: the paper [14] uses "operator amenability" of $\mathrm{A}(G)$ in such cases to get information on the behaviour of closed ideals in $\mathrm{A}(G)$.

3.4 Hereditary properties and some examples

The following two results are very useful, and the proofs are almost trivial given the way we have defined amenability.

Proposition 3.18 (Amenability passes to closures of homomorphic images). *Let $\phi : A \to B$ be a continuous homomorphism of Banach algebras which has dense range. If A is amenable then so is B.*

Note that this proposition fails if we replace the word "amenable" with "biflat".

Proposition 3.19 (The tensor product of amenable algebras is amenable). *If A and B are amenable Banach algebras then so is $A \widehat{\otimes} B$.*

To see how we can start to use these results, we can finally give a quick proof of a fundamental example. (We note that this is an example that Helemskii originally proved in restricted cases, using explicit constructions of bounded approximate diagonals based on partitions of unity. The trick we will use, due to Johnson, completely bypasses this.)

Corollary 3.20. *Let K be a locally compact Hausdorff space. Then $C_0(K)$ is amenable.*

Combining these results, we see that if K is a locally compact Hausdorff space and A is an amenable Banach algebra, then $C_0(K,A)$ is also amenable. In particular, the algebras $C_0(K,\mathbb{M}_n)$ and $C_0(K,\mathcal{K}(\mathsf{H}))$ are amenable for every locally compact Hausdorff K.

Warning. It is tempting to leap from these examples to the guess that $\ell^\infty(\mathbb{N},A)$ is amenable when A is amenable. However, this is false for e.g. $A = \mathcal{K}(\mathsf{H})$ or $A = \ell^1(\mathbb{T}_d)$, although the proofs rely on hard(er) results. The point is, of course, that the previous observations apply to $C(\beta\mathbb{N},A) = \ell^\infty(\mathbb{N})\check{\otimes}A$, which in general can be a lot smaller than $\ell^\infty(\mathbb{N},A)$.

Proposition 3.21. *Let A be an amenable Banach algebra and suppose J is a closed ideal in A. If J has a b.a.i. then it is amenable.*

To be done later. One important result, whose proof we defer until Section 5.1, says that if J is a closed ideal in a Banach algebra A, and both J and A/J are amenable, then so is A. Attempting to prove this with bounded approximate diagonals is surprisingly fiddly.

The last result of this subsection gives a partial converse to Proposition 3.19. Surprisingly it was not included in [22].

Proposition 3.22 (Johnson, [25, Prop. 3.4]). *Let A and B be Banach algebras such that $A \widehat{\otimes} B$ is amenable. Suppose there exists $b_0 \in B$ such that b_0 is not in the closure of the set $\{bb_0 - b_0 b \colon b \in B\}$. Then A is amenable.*

Exercise. Show that B satisfies the assumption in Proposition 3.22 in each of the following cases:

(i) $Z(B) \neq \{0\}$;

(ii) B has a non-zero tracial functional;

(iii) B contains a non-zero idempotent.

3.5 Extending "tracial" functionals from submodules

In their entertaining book [5], Brown and Ozawa have a discussion of "amenable traces" on unital C*-algebras. The usefulness of this notion seems to depend quite heavily on things particular to C*-algebras (e.g. Stinespring dilation for ucp maps, GNS constructions, etc) but we can at least discuss some of the "easy" aspects in the general setting of amenable Banach algebras.

Theorem 3.23 (Extension of tracial functionals). *Let A be an amenable Banach algebra and let $X \in A\text{-}\mathsf{bimod}$. Let $Y \subset X$ be a closed sub-A-bimodule, and let $\tau \in Y^*$ satisfy $\tau(ay) = \tau(ya)$ for all $a \in A$ and $y \in Y$. Then there exists $\phi \in X^*$, extending τ, such that $\phi(ax) = \phi(xa)$ for all $a \in A$ and $x \in X$.*

The particular corollary that Brown and Ozawa have in mind, and which motivates their name of "amenable trace", is as follows.

Corollary 3.24. *Let A be a unital C*-algebra with a tracial state τ. Suppose A is amenable, and regard it as a unital $*$-subalgebra of some containing copy of $\mathcal{B}(\mathsf{H})$. Then there exists a state ϕ on $\mathcal{B}(\mathsf{H})$ satisfying $\phi(aT) = \phi(Ta)$ for all $a \in A$ and all $T \in \mathcal{B}(\mathsf{H})$.*

This was applied by Bunce to reduced C*-algebras of discrete groups (see Theorem 7.9 below. A similar principle is the *starting point* for Connes's proof that II$_1$ factors which are "amenable as von Nuemann algebras" are in fact semidiscrete, see [12].

4 Some more examples of (non-)amenable algebras

4.1 Symmetric amenability

Symmetric amenability was introduced by Johnson in [25] and has received relatively little attention. In my view there is more to be done on clarifying how it fits into the broader picture; Johnson himself noted that "*symmetric amenability does not seem to have a concise homological characterisation.*"

Definition 4.1. Let A be a Banach algebra and let $\sigma : A \widehat{\otimes} A \to A \widehat{\otimes} A$ be the "flip map" $a \otimes b \mapsto b \otimes a$. We say A is **symmetrically amenable** if it has a bounded approximate diagonal (d_α) satisfying $\sigma(d_\alpha) = d_\alpha$ for all α.

Johnson proved that symmetric amenability enjoys some of the same hereditary properties as "usual" amenability: see [25, Thm 3.1]. For instance, it is clear that Propositions 3.18 and 3.19 remain true with "amenable" replaced by "symmetrically amenable". The same goes for Proposition 3.21, once one inspects the proof.

Proposition 4.2. *If A is a symmetrically amenable, unital Banach algebra, then there exist $\psi \in A^*$ such that $\psi(1) = 1$ and $\psi(ab) = \psi(ba)$ for all $a, b \in A$.*

Corollary 4.3. *Let E be a Banach space which is isomorphic to its square, i.e. $E \cong E \oplus E$. Then $\mathcal{B}(E)$ is not symmetrically amenable.*

Finally: Johnson also showed [25, Theorem 4.1] that if G is amenable then $L^1(G)$ is not just amenable, it is symmetrically amenable. For G discrete this is very easy; but for non-discrete groups more work is required.

4.2 A quick proof that $\mathcal{A}(\ell^p)$ is amenable

The following argument is given in [22].

Proposition 4.4. *Let $1 < p < \infty$. There is an amenable discrete group G and a bounded (in fact, contractive) algebra homomorphism $\theta : \ell^1(G) \to \mathcal{A}(\ell^p)^\sharp$ with dense range.*

Corollary 4.5. *$\mathcal{A}(\ell^p)$ is (symmetrically) amenable for $1 < p < \infty$. Moreover it has a symmetric b.a.d. of norm 1.*

4.3 Amenability for $\mathcal{A}(E)$ and $\mathcal{B}(E)$

Property $\mathbb{A}$ of [GJW] In [17] Grønbæk, Johnson and Willis studied amenability of $\mathcal{A}(E)$ (strictly speaking, of $\mathcal{K}(E)$) in more detail. They came up with a technical condition on a Banach space — which they called property $\mathbb{A}$ — that allows one to run more careful versions of the arguments used for $\mathcal{A}(\ell^p)$.

The following description of property $\mathbb{A}$, which is slightly more concise than the original one, is taken from Blanco's recent article [3].

Definition 4.6. Let E be a Banach space. We say that E has **property $\mathbb{A}$** if there exists a bounded net (p_α) of finite-rank projections and a constant $K > 0$, such that the following conditions are satisfied:

(i) $p_\alpha \to I_E$ in the strong operator topology of $\mathcal{B}(E)$;

(ii) $p_\alpha^* \to I_{E^*}$ in the strong operator topology of $\mathcal{B}(E^*)$;

(iii) for each α there is a finite group $G_\alpha \subset \mathsf{ball}_K(\mathcal{B}(p_\alpha E))$ such that $\operatorname{lin}(G_\alpha) = \mathcal{B}(p_\alpha E)$.

Exercise. Show that if E has property $\mathbb{A}$ (with constant K) then $\mathcal{A}(E)$ has a symmetric b.a.d. of norm $\leq K$.

Examples of Banach spaces with property $\mathbb{A}$ include: $L^p(\Omega, \mu)$ for $1 \leq p < \infty$; $C(\Omega)$ for Ω compact Hausdorff; and certain reflexive sequence spaces.

Remark 4.7. It was shown in [17] that property $\mathbb{A}$ passes to preduals. The Banach space E of Argyros and Haydon which solves the "scalar plus compact" problem is a predual of ℓ^1. Hence it has the approximation property and property $\mathbb{A}$, and so $\mathcal{B}(E) = \mathbb{C}I + \mathcal{A}(E)$ is amenable. This was the first known example of an infinite-dimensional Banach space X for which $\mathcal{B}(X)$ is amenable; it had been conjectured by several authors that no such space could exist.

Runde has shown (building on ideas of various authors) that $\mathcal{B}(\ell^p)$ is non-amenable. However, discussion of this result is beyond the scope of these lectures course. Note that it is extremely easy to show that $\mathcal{B}(\ell^p)$ is not *symmetrically* amenable.

We can at least give some examples of E where it is easy to prove that $\mathcal{B}(E)$ is non-amenable. The following is an observation of G. A. Willis, see [16].

Theorem 4.8. *Let X and Y be Banach spaces such that $\mathcal{A}(X, Y) = \mathcal{B}(X, Y)$ but $\mathcal{A}(Y, X) \neq \mathcal{A}(Y, X)$. Then $\mathcal{B}(X \oplus Y)$ has a non-amenable quotient, and hence is non-amenable.*

By Pitt's theorem this applies to $X = \ell^p$ and $Y = \ell^q$ when $\infty > p > q \geq 1$.

4.4 Digression: the existence of an amenable radical Banach algebra

For a long time this was an open question, until Runde [32] produced a (non-explicit) example.

It had been known for a long time that if one could find an amenable group G such that $L^1(G)$ had certain kinds of closed ideals, then the corresponding quotient algebras would be radical (because of the properties of the ideals) and also amenable (because $L^1(G)$ is). What Runde discovered is an unpublished result of Boidol from the 1980s, stating that such ideals do indeed exist for the L^1-convolution algebra of the amenable group $G_{4,9}(0) = \mathbb{H} \rtimes_\phi \mathbb{R}$, where

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

and $\phi : \mathbb{R} \to \mathrm{Aut}(\mathbb{H})$ is defined by

$$\phi_t \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & e^{-t}x & z \\ 0 & 1 & e^t y \\ 0 & 0 & 1 \end{pmatrix}$$

However, it seems that those who knew of Boidol's result did not realize at the time that it had an application to the world of amenable Banach algebras... The paper [32] gives a complete proof, using work of Leptin and Poguntke to avoid relying on unpublished results.

(Subsequently, C. J. Read used completely different methods to construct *commutative* radical amenable Banach algebras. An exposition of this result, with entertaining commentary, can be found in [33].)

4.5 Possible further topics

There are two nice results of Blanco [3] that complement some of the previous discussion.

THEOREM 4.9 ([3, Theorem 3.1]). *If $\mathcal{A}(E)$ is amenable then it is symmetrically amenable.*

THEOREM 4.10 ([3, Theorem 4.4]). *Let $\mathcal{A}$ be a finite-dimensional Banach algebra. The following are equivalent:*

(i) *$\mathcal{A}$ is amenable and* $\mathrm{AM}(\mathcal{A}) = 1$*;*

(ii) *there exists a group $G \subset \mathsf{ball}(\mathcal{A})$ such that* $\mathrm{lin}(G) = \mathcal{A}$.

(It is not known if we can always take the group in part (ii) of the theorem to be finite.)

Modulo some results in Banach space theory, the proofs are accessible; but they require some work. If there is time and interest I may say more about the proofs.

5 Amenability and derivations

5.1 Johnson's original definition/characterization

As stated before, Johnson introduced the definition of an amenable Banach algebra in [22]. His original motivation was the study of derivations on $L^1(G)$, and so when he came to define amenable Banach algebras he did so in terms of derivations, rather than virtual diagonals or approximate diagonals. My *personal* belief is that derivations are over-emphasized in introductions to amenability, but in several instances they allow one to get simpler or cleaner proofs.

DEFINITION 5.1. Let X be a Banach A-bimodule. A bounded linear map $D : A \to X$ is said to be a **derivation** if it satisfies the "Leibniz identity"

$$D(ab) = a \cdot D(b) + D(b) \cdot a \quad \text{for all } a, b \in A$$

An **inner derivation** from A to X is a linear map of the form $\mathrm{ad}_x : a \mapsto a \cdot x - x \cdot a$. (You should quickly check that ad_x is always a derivation!)

NOTATION. Let $\mathcal{Z}^1(A, X)$ denote the space of all bounded derivations $A \to X$. Let $\mathcal{B}^1(A, X)$ denote the subspace (not necessarily closed!) consisting of inner derivations. Let $\mathcal{H}^1(A, X) := \mathcal{Z}^1(A, X)/\mathcal{B}^1(A, X)$ – this is called the **1st-degree Hochschild cohomology group.**

The following theorem of Singer and Wermer has an attractive proof using Gelfand theory and Liouville's theorem, which can be found in many standard texts. A clever, elementary proof by Sinclair is given in [4, §18]

THEOREM 5.2 (Singer–Wermer). *Let A be a commutative, semisimple Banach algebra. Then $\mathcal{Z}^1(A, A) = \{0\}$.*

It is this result, more than any other, which highlights the differences between continuous Hochschild cohomology of Banach algebras and the "usual" Hochschild cohomology of $\mathbb{C}$-algebras. Of course, the analogous result is also false for Fréchet algebras: consider the usual derivative map on the function algebra $\mathcal{O}(\mathbb{D})$.

THEOREM 5.3 (BEJ but algebraically just Hochschild). *Let A be a Banach algebra and let $X_\pi = \ker(A_+ \widehat{\otimes} A_+ \to A_+)$. The following are equivalent:*

(i) *A is amenable;*

(ii) $\mathcal{H}^1(A, X_\pi^{**}) = 0$*;*

(iii) $\mathcal{H}^1(A, X^*) = 0$ *for all* $X \in A\text{-}\mathsf{bimod}$.

REMARK 5.4. If A already has an identity element, then we can replace X_π in this theorem by $\ker(\pi : A \widehat{\otimes} A \to A)$.

We can use this to give alternative proofs of some of the hereditary properties of amenability. Let A be a Banach algebra and J a closed ideal in A. Then:

- if A is amenable, so is A/J;
- if J has a bounded approximate identity and A is amenable then J is amenable.

(These were, of course, Johnsons's original proofs in [22].) We can also now finally give a proof of the following result, which is surprisingly fiddly if you try to do it with bounded approximate diagonals.

THEOREM 5.5 (Amenability is preserved by extensions). *Let A be a Banach algebra and J a closed ideal in A. If J and A/J are amenable, then A is amenable.*

REMARK 5.6 (Symmetric amenability is preserved by extensions). If one writes down *how* bounded approximate diagonals imply derivations are inner, then one can rewrite this proof to construct a bounded approximate diagonal for A given b.a.d.s for J and A/J. If J and A/J are actually symmetrically amenable, then one ends up after some work with a proof that A is sym. amenable: see [25, Theorem 3.1] for the details.

5.2 Digression: the link between derivations and homomorphisms

The following result can be found in various algebra textbooks, but does not seem to be as well known in Banach-algebraic circles as it should be.

Exercise (Derivations as infinitesimal deformations of homomorphisms). Let $\phi : A \to B$ be a $\mathbb{C}$-algebra homomorphism and let $f : A \to B$ be a $\mathbb{C}$-linear map. Suppose that $f(A)^2 = 0$, i.e. $f(a_1)f(a_2) = 0$ for all $a_1, a_2 \in A$. Show that $\phi + f : A \to B$ is an algebra homomorphism if and only if $f : A \to B$ is a derivation.

Another way to look at this is the slogan "derivations arise as the corners of homomorphisms". To be more precise: given a Banach algebra A and a contractive A-bimodule X we can build a new Banach algebra $A \oplus_{\ltimes} X$, which is defined to be the Banach space $A \oplus_1 X$ equipped with the multiplication

$$(a, x)(b, y) := (ab, ay + xb)$$

Intuitively we identify (a, x) with the block matrix $\begin{bmatrix} a & x \\ 0 & a \end{bmatrix}$ and the product in $A \oplus_{\ltimes} X$ is just what one gets from usual matrix multiplication.

Let $q_A : A \oplus_{\ltimes} X \to A$ be the canonical quotient homomorphism ("restrict to the top-left corner"). Then the previous exercise shows, after a little more work:

> *there is a bijection between the set of derivations $A \to X$, and the set of homomorphisms $A \to A \oplus_{\ltimes} X$ that are right inverse to q_A.*

We mention this construction here because the papers [11, 15] show how one can use this idea as a useful technical device when dealing with derivations. (Once again, this is part of a deeper story in category theory, but that would take us too far off-topic.)

Remark 5.7. The algebra $A \oplus_{\ltimes} X$ seems to have been re-invented independently on several occasions by people in the world of Banach algebras; the actual idea (without functional-analytic conditions and adjectives) goes back at least as far as Hochschild's work in the 1940s. Several papers by people seeking something to do have asked when $A \oplus_{\ltimes} X$ is amenable or weakly amenable or biprojective or ... etc. By now you should have guessed that the present author does not think this is particularly worthwhile. (Hint: think of $A = X = \mathbb{C}$.)

5.3 Using derivations to define variants of amenability

While on the topic of derivations, we should *briefly* say something about weak amenability for commutative Banach algebras.

Theorem 5.8 (Bade–Curtis–Dales, [1]). *Let A be a commutative Banach algebra and let X_π^{ab} be the quotient of X_π by the closure of the "commutator submodule" $[A, X_\pi]$. Then the following are equivalent:*

(i) $\mathcal{H}^1(A, A^*) = 0$.

(ii) $\mathcal{H}^1(A, X_\pi^{\mathrm{ab}}) = 0$;

(iii) $\mathcal{H}^1(A, X) = 0$ *for all* ***symmetric*** $X \in A$-bimod.

If A satisfies any of these equivalent conditions, it is said to be ***weakly amenable****.*

Remark 5.9. See Remark 5.4. In fact, for the unital setting, what we have here is — apparently unrecognized by the authors of [1] — the natural Banach-algebraic analogue of the **Kähler module of differentials** for a commutative $\mathbb{C}$-algebra. As far as I know, this "Banach Kähler module" has not been studied systematically, except for some work in [31].

REMARK 5.10. It is customary to mention that this has absolutely nothing to do with the notion of "weakly amenable group" that was introduced and studied in work of Cowling and Haagerup. Indeed, the term "weakly amenable Banach algebra" is highly unsatisfactory in many respects, but it seems this is too well-established to be changed now.

REMARK 5.11. There have been attempts to define a "new version of amenability", using somewhat artificial module actions and derivations, such that for certain semigroups S, the convolution algebra $\ell^1(S)$ has the "modified version of amenability" if and only if S is amenable. In my view this is extremely misguided: amenability of S is already captured by a homological property of $\ell^1(S)$, namely that the augmentation character $\varepsilon : S \to \mathbb{C}$ gives rise to a 1-dimensional flat module.

EXAMPLE 5.12 (Revisiting some examples of function algebras). We said earlier that $AC[0,1]$ and $\mathrm{A}(G)$ for G compact were not weakly amenable. In both cases one can write down explicit non-zero derivations from the algebra into its dual. For instance, define $D : AC[0,1] \to AC[0,1]^*$ by

$$D(f)(g) = \int_0^1 f'(t)g(t)\,dt \qquad (f,g \in AC[0,1]).$$

For the Fourier algebra of $\mathrm{SU}(2,\mathbb{C})$, one can show that if one chooses a suitable directional derivative ∂_ϕ, then

$$D_0(f)(g) = \int_{\mathrm{SU}(2,\mathbb{C})} (\partial_\phi f)(x)g(x)\,dx$$

extends to a bounded derivation from the algebra to its dual. This was first shown by Johnson [24] with a somewhat indirect proof; a more direct approach is sketched in [10].

6 Unitarizable representations of (discrete) groups

The material in this section is not, strictly speaking, part of the main narrative about Banach modules and amenable Banach algebras. However, it forms part of the back-story to the paper [9] and the results of Section 8.

REMARK 6.1. In many sources one sees the phrase: "we restrict to the discrete case for simplicity". To my mind, it would be more honest to say: "we restrict to the discrete case to avoid dealing with technical issues and problems for the general case". New phenomena can arise when dealing with SOT-continuous representations of non-discrete groups, especially in the non-compact and non-abelian settings.

6.1 An approach via fixed points

Consider: a discrete group Γ and a unital C^*-algebra A, and a homomorphism (=representation) $\theta : \Gamma \to A_{\mathrm{inv}}$. We say the representation is **bounded** if $\sup_{x\in\Gamma} \|\theta(x)\| < \infty$.

DEFINITION 6.2. A (bounded) representation $\theta : \Gamma \to A_{\mathrm{inv}}$ is **A-unitarizable**, or **similar in A to a $*$-representation**, if there exists $s \in A_{\mathrm{inv}}$ such that

$$s\theta(x)s^{-1} \in \mathcal{U}(A) \quad \text{for all } x \in \Gamma.$$

We say that s is a **similarity element** for θ.

The following notation is non-standard but useful.

DEFINITION 6.3 (Actions on the invertible positive elements). Given a (bounded) representation (θ, Γ, A), let $A^+_{\mathrm{inv}} = A_{\mathrm{inv}} \cap A^+$. Then Γ acts on A^+_{inv}, as follows;

$$\theta^+(x) : h \mapsto \theta(x)h\theta(x)^* \,.$$

LEMMA 6.4. *Let Γ be a group, $\mathcal{A}$ a unital* C**-algebra and $\theta : \Gamma \to \mathcal{A}_{\rm inv}$ a bounded representation. The following are equivalent:*

(i) *θ is unitarizable inside A;*

(ii) *the associated action $\mathrm{Ad}^*_\theta : \Gamma \curvearrowright \mathcal{A}^+_{\rm inv}$ has a fixed point.*

PROPOSITION 6.5 (various authors, implicitly). *Let Γ be an amenable (discrete) group, let $\mathcal{M}$ be a von Neumann algebra; and let $\theta : \Gamma \to \mathcal{M}_{\rm inv}$ be a bounded representation. Then $\mathrm{Ad}^*_\theta : \Gamma \curvearrowright \mathcal{M}^+_{\rm inv}$ has a fixed point. Consequently, θ is $\mathcal{M}$-unitarizable.*

Proposition 6.5 is essentially due to Dixmier (1950) and Day (1950) and no doubt known to other authors independently (although most of these sources just take $\mathcal{M} = \mathcal{B}(\mathsf{H})$ for some Hilbert space H). However, instead of looking for fixed points of Ad^*_θ they construct a Γ-invariant inner product, since that approach generalizes more easily to non-discrete amenable groups. If time permits in the lectures I will say more about this: see Section 6.2 below.

The following example shows that simultaneous unitarization is much rarer. It will be used in Section 8.

EXAMPLE 6.6. Let $\varepsilon > 0$ and consider the two involutions

$$x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad y = \begin{bmatrix} 1 & \varepsilon \\ 0 & -1 \end{bmatrix}.$$

These give a pair of representations $\mathbb{Z}/2\mathbb{Z} \to (\mathbb{M}_2)_{\rm inv}$.

One can check (exercise!) that the corresponding actions $\mathrm{Ad}^*_x : \mathbb{Z}/2\mathbb{Z} \curvearrowright (\mathbb{M}_2)^+_{\rm inv}$ and $\mathrm{Ad}^*_y : \mathbb{Z}/2\mathbb{Z} \curvearrowright (\mathbb{M}_2)^+_{\rm inv}$ have no common fixed point. Hence there is no $s \in (\mathbb{M}_2)_{\rm inv}$ which *simultaneously* unitarizes the original representations.

In some settings, when the codomain of the representation is well-behaved, we can dispense with the assumption of amenability. The following result is easy, but will prove useful in Section 8.

EXAMPLE 6.7 (Bounded subgroups of products of matrix groups). Let $n(1) \leq n(2) \leq \dots$ be an increasing sequence of positive integers and let $\mathcal{M} := \prod_{k=1}^\infty \mathbb{M}_{n(k)}$. Let G be any group and let $\theta : G \to \mathcal{M}_{\rm inv}$ be a bounded representation. Then $\mathrm{Ad}^*_\theta : G \curvearrowright \mathcal{M}^+_{\rm inv}$ has a fixed point. Consequently, θ is unitarizable inside $\mathcal{M}$.

REMARK 6.8. In fact this result can be substantially generalized: it remains true if $\mathcal{M}$ is replaced by any finite von Neumann algebra (without any assumptions on the group G). This stronger result is due to Vasilescu and Szido: their proof relies on the Ryll-Nardzewski fixed point theorem.

6.2 Comments on the non-discrete setting

The theorem of Dixmier and Day (and others) is usually stated in the following more general form:

> Let G be an amenable locally compact group and let $\theta : G \to \mathcal{B}(\mathsf{H})_{\rm inv}$ be a SOT-continuous, bounded representation. Let $\mathcal{M}$ be the von Neumann algebra geneated by $\theta(G)$. Then θ is similar inside $\mathcal{M}$ to a $*$-representation.

Since I will probably not have time to discuss the proof in lectures, a quick sketch is included below. The underlying idea is to adapt the classical proof of Maschke's theorem for representations of finite groups.

Outline of the proof. Fix a left-invariant mean $\Lambda : C_b(G) \to \mathbb{C}$ and define a new sesquilinear form $(\cdot \mid \cdot)_G : \mathsf{H} \times \mathsf{H} \to \mathbb{C}$ by

$$(\xi \mid \eta)_G := \Lambda\left(x \mapsto (\theta(x)\xi \mid \theta(x)\eta)\right).$$

One can check that this defines a $\theta(G)$-invariant inner product on H that is equivalent to the original one. An argument with the Riesz–Fischer theorem then yields $R \in \mathcal{B}(\mathsf{H})^{+}_{\rm inv}$ such that $(R\xi \mid R\eta) = (\xi \mid \eta)_G$. It follows from $\theta(G)$-invariance that $R\theta(x)R^{-1}$ is unitary for every $x \in G$; and finally we show that $R \in \mathcal{M}$ using von Neumann's double commutant theorem. □

Remark 6.9 (Future work?). It would be desirable to have versions of Definition 6.2, Definition 6.3 and Lemma 6.4 which were suitable for, say, nondiscrete amenable groups.

One immediate point is that the "right" notion of continuity for group representations on Hilbert space is SOT-continuity, not continuity with respect to the norm on $\mathcal{B}(\mathsf{H})$. However, each SOT-continuous bounded representation $\theta : G \to \mathcal{B}(\mathsf{H})_{\rm inv}$ can be integrated to give a norm-continuous homomorphism $L^1(G) \to \mathcal{B}(\mathsf{H})$; so perhaps the right analogue of Definition 6.2 would be in the setting of bounded algebra homomorphisms $L^1(G) \to A$ where A is a C*-algebra.

It seems like it would be worthwhile to pursue this line of thought more systematically; I might make some comments on this during the actual lectures.

6.3 Unitarizable groups and derivations

(This topic could merit a lecture course on its own, so what follows is only a very small glimpse. The Wikipedia entry `http://en.wikipedia.org/wiki/Uniformly_bounded_representation` seems to currently be quite good.)

An old question of Dixmier asks if every non-amenable locally compact group has a bounded representation in $\mathcal{B}(\mathsf{H})_{\rm inv}$ that is *not* $\mathcal{B}(\mathsf{H})$-unitarizable. The question is still open, even for the class of discrete groups. However, the following partial result is sufficiently striking that it seems worth mentioning, although we will not say anything about the proof.

Theorem 6.10 (Pisier, [29]). *Let Γ be a discrete, non-amenable group. Then there exists a von Neumann algebra $\mathcal{M}$ and a bounded representation $\theta : \Gamma \to \mathcal{M}_{\rm inv}$ that is not $\mathcal{M}$-unitarizable.*

Going back to Dixmier's original question: one can use non-vanishing cohomology to obtain non-unitarizable representations. Note the theme of "the derivation in the corner" which we briefly mentioned in Section 5.2.

Lemma 6.11. *Let $\pi : \Gamma \to \mathcal{U}(\mathsf{H}_\pi)$ be a unitary representation, so that $\mathcal{B}(\mathsf{H}_\pi)$ becomes a Banach $\ell^1(\Gamma)$-bimodule. Suppose $D : \ell^1(\Gamma) \to \mathcal{B}(H_\pi)$ is a non-inner derivation. Then*

$$\theta_D(x) := \begin{bmatrix} \pi(x) & D(x) \\ 0 & \pi(x) \end{bmatrix}$$

defines a bounded representation $\theta_D : \Gamma \to \mathcal{B}(\mathsf{H}_\pi \oplus \mathsf{H}_\pi)$ that is not $\mathcal{B}(\mathsf{H}_\pi \oplus \mathsf{H}_\pi)$-unitarizable.

It can be shown that $\Gamma = \mathbb{F}_\infty$ satisfies the conditions of this lemma, and so $\mathbb{F}_\infty$ has a bounded representation in $\mathcal{B}(\mathsf{H})$ that is not $\mathcal{B}(\mathsf{H})$-unitarizable. From this, a little more work shows that any locally compact group containing a discrete copy of $\mathbb{F}_2$ also has a bounded representation in $\mathcal{B}(\mathsf{H})$ that is not $\mathcal{B}(\mathsf{H})$-unitarizable.

Theorem 6.12 (Monod–Ozawa, [26]). *Let Γ be a discrete, non-amenable group. Then there is an abelian group A and a bounded representation of the **wreath product** $A \wr \Gamma \to \mathcal{B}(\mathsf{H})_{\rm inv}$ that is not $\mathcal{B}(\mathsf{H})$-unitarizable.*

Lemma 6.11 also plays a small role in the proof of Theorem 6.12, but deep results are needed in order to obtain non-inner derivations. Pisier has written a very clear and accessible exposition of Monod and Ozawa's result in `http://arxiv.org/abs/1109.1863`

6.4 Possible further topics

We have already seen that, when working in a dual Banach space (or module), a bounded sequence or net that "asymptotically satisfies some algebraic condition" will often give rise to an element which satisfies the condition exactly, just by taking a suitable cluster point in the weak-star topology.

For certain C*-algebras that are not von Neumann algebras, but are still "big enough", one can perform a similar passage from "asymptotic/approximate solutions" to "genuine solutions", provided that there is some countability or separability in the set of conditions that need to be satisfied. This vague philosophy seems to have grown out of a series of results in the 1970s by G. K. Pedersen and his coauthors, and in recent years has been given a precise form in work of Farah and others.

The techniques of Farah and his coauthors can be applied to ultraproducts of a sequence of unital C*-algebras, and also certain "corona algebras" such as $\prod_{k=1}^{\infty} \mathbb{M}_{n(k)} / \bigoplus_{k=1}^{\infty} \mathbb{M}_{n(k)}$. While a proper discussion of this is beyond the scope of these lectures (or, indeed, beyond the lecturer's knowledge), we may if time permits say something about the following result, which is a special case of a theorem in [9].

Theorem. Let Γ be an amenable countable group. Let $n(1) \leq n(2) \leq \dots$ be an increasing sequence of positive integers and let

$$\mathcal{M} := \prod_{k=1}^{\infty} \mathbb{M}_{n(k)} \quad , \quad \mathcal{J} := \bigoplus_{k=1}^{\infty} \mathbb{M}_{n(k)} \quad , \quad \mathcal{C} := \mathcal{M}/\mathcal{J} .$$

If $\theta : \Gamma \to \mathcal{C}_{\rm inv}$ is a bounded representation, then $\mathrm{Ad}_\theta^* : \Gamma \curvearrowright \mathcal{C}_{\rm inv}^+$ has a fixed point.

7 A look at amenable C*-algebras

7.1 Where angels fear to tread

It is known, by combining work of various authors over a number of years, that a C*-algebra is amenable if and only if it has the CPAP. (This is usually stated as: amenability is equivalent to nuclearity for C*-algebras; I prefer to avoid this since the original definition of nuclearity is in terms of tensor norms.)

Note that (as far as I know) it is quite difficult to show directly that the CPAP passes to quotients! But for amenability this is almost trivial.

Rather than attempt to go into the proof of this deep result — which could easily take a whole lecture course if one were to do things in full detail — I shall instead give an overview of some of the easier proofs of amenability and non-amenability for various interesting examples.

7.2 Strong amenability for C*-algebras

We do not give the original definition of [22], which was in terms of derivations. Instead we use one based on bounded approximate diagonals

Definition 7.1. Let A be a unital C*-algebra. We say that A is **strongly amenable** if it has a bounded approximate diagonal contained in the convex hull of $\{u \otimes u^* : u \in \mathcal{U}(A)\}$.

A non-unital C*-algebra A is said to be strongly amenable if $A^\sharp$ is amenable.

The following result is almost a trivial consequence of what we saw for $\ell^1(\Gamma)$.

THEOREM 7.2 (Johnson, [22]). *Let* Γ *be a discrete amenable group. Then* $\mathrm{C}^*_r(\Gamma)$ *is strongly amenable.*

REMARK 7.3. If G is a locally compact amenable group, then symmetric amenability of $\mathrm{C}^*_r(G)$ is an easy consequence of symmetric amenability of $L^1(G)$. But it does not seem so easy to deduce strong amenability of $\mathrm{C}^*_r(G)$ from its symmetric amenability.

Let us connect strong amenability with some notions and results that have been mentioned in earlier lectures.

THEOREM 7.4. *Every strongly amenable* C^**-algebra is symmetrically amenable.*

It is not known if the converse of Theorem 7.4 holds. We note that recently Ozawa [27] has given a characterization of symmetrically amenable C^*-algebras in terms of C^*-algebraic properties. His proof still relies on many of the same results which are used to show that for C^*-algebras amenability is equivalent to the CPAP.

THEOREM 7.5 (Bunce [6]). *Let* A *be a strongly amenable unital* C^**-algebra. Then every bounded unital homomorphism* $A \to \mathcal{B}(\mathsf{H})$ *is similar to a* $*$*-homomorphism.*

The proof is very similar to the argument used by Day and Dixmier to prove that bounded representations of amenable groups in $\mathcal{B}(\mathsf{H})$ are $\mathcal{B}(\mathsf{H})$-unitarizable (see Section 6.2).

REMARK 7.6. It should be noted that Theorem 7.5 remains true if we relax "strongly amenable" to "amenable", but become significantly harder to prove! Indeed, Bunce's theorem gives an easier approach to some results which are usually proved by saying: "this C^*-algebra is amenable, hence it is nuclear, and nuclear C^*-algebras enjoy Kadison's similarity property."

Unsurprisingly strong amenability passes to quotients, and the (minimal) tensor product of two strongly amenable C^*-algebras is strongly amenable. In [22] more hereditary properties are proved, culminating (via structure theory for postliminal C^*-algebras) in the following result (which we will *not* prove).

THEOREM 7.7 (Johnson, [22]; also follows from Haagerup's later results in [18]).
Let A *be a postliminal* C^**-algebra. Then* A *is strongly amenable.*

A class of C^*-algebras which behave very differently from group C^*-algebras are the Cuntz algebras $\mathcal{O}_n$.

THEOREM 7.8 (Rosenberg [30]). *The algebras* $\mathcal{O}_n$, $2 \leq n \leq \infty$, *are amenable but not symmetrically amenable (and hence not strongly amenable)*

Rosenberg's proof uses the description of $\mathcal{O}_n$ as a crossed product of a UHF algebra by a *semigroup* action. If time permits I will give an outline of his argument.

A result which deserves have an easier proof. The following observation is due to Haagerup [18]:

If A is an amenable C^*-algebra then $A\overline{\otimes}\mathcal{K}(\ell^2)$ is strongly amenable.

However, because he deduces this result as a consequence of harder and more important results in his paper, he goes through the fact that if A is amenable then A^{**} is AFD — widely regarded as one of the hardest results to come out of Connes's work on "injective factors". I do not know of any simpler proof of the quoted result!

7.3 When are group C^*-algebras amenable?

For discrete groups, one gets the expected result.

THEOREM 7.9 (Bunce, [7]). *Let Γ be a discrete group. If $\mathrm{C}^*_r(\Gamma)$ is amenable then Γ is amenable.*

However, unlike the case of L^1-convolution algebras, amenability of group C^*-algebras does not always imply the group is amenable.

THEOREM 7.10 (various). *$\mathrm{C}^*(\mathrm{SL}(n,\mathbb{R}))$ is liminal, and hence by Johnson's results is (strongly) amenable.*

If time permits I will say something about a "soft" proof of Theorem 7.10, which generalizes to many other semisimple Lie groups.

7.4 Possible further topics

Strong amenability for $\mathrm{C}^*_r(G)$? It seems to be generally accepted that if G is a locally compact amenable group then $\mathrm{C}^*_r(G)$ should be strongly amenable. However, this is never actually stated in [22], despite what some of the secondary literature seems to suggest. In one section of his book [28, §1.31] Paterson gives a sketch of a proof but some details remain to be filled in, it seems. Cf. Remark 7.3.

8 Amenable subalgebras of $\mathcal{B}(\mathsf{H})$

8.1 Some motivation and background

Quoting from Grønbæk's survey article [16]:

> ...So the philosophy is this: the category of modules should reflect important properties of the algebra. However, this seems to be contradicted by the fact that the question of amenability of a C^*-algebra, which is a question concerning a category (all Banach modules) that does not reflect properties of the involution, has a C^*-algebra answer (nuclearity as a C^*-algebra). Perhaps this apparent conflict can be resolved by a better understanding of amenable operator algebras...

8.2 Amenable operator algebras that are not isomorphic to C^*-algebras

DEFINITION 8.1. Let A be a unital Banach algebra. We say A is **generated by a bounded group** if there exists $G \subset A_{\mathrm{inv}}$ such that (i) $\sup_{a\in G}\|a\| < \infty$ (ii) $\mathrm{lin}(G)$ is dense in A.

Clearly this property is preserved under isomorphism of Banach algebras.

LEMMA 8.2. *Every unital C^*-algebra is generated by a bounded group.*

THEOREM 8.3 (C.–Farah–Ozawa [9]). *Let $\varepsilon > 0$. There exists a closed unital subalgebra $\mathcal{A} \subset \ell^\infty \otimes \mathbb{M}_2$ which is not generated by a bounded group, but has a symmetric virtual diagonal.*

In fact, the methods of [9] do more: given any $\varepsilon > 0$, one can find an algebra $\mathcal{A}_\varepsilon$ as in Theorem 8.3 which also has the following properties:

- each $\mathcal{A}_\varepsilon$ contains $c_0 \otimes \mathbb{M}_2$ as a closed ideal;
- the quotient of $\mathcal{A}_\varepsilon$ by this ideal is isomorphic to $C(\Omega)$ where $\Omega = \{\pm 1\}^{\aleph_1}$;

- there is a natural exhaustion of $\mathcal{A}_\varepsilon$ by a family of separable subalgebras, each of which is isomorphic to a C*-algebra, and where the Banach-Mazur constants of these isomorphisms can be bounded by $1 + O(\varepsilon)$.

The last of these properties shows that the obstruction to $\mathcal{A}_\varepsilon$ being isomorphic to a C*-algebra is "genuinely global" and not merely a consequence of "local isomorphism with worse and worse constants".

Remark 8.4 (Important remark about history and priority). Strictly speaking, the first examples of amenable operator algebras that are not isomorphic to C*-algebras can be found in the original preprint of Farah and Ozawa. It must be emphasized that all the key ideas needed to prove Theorem 8.3 are present in the earlier preprint, in particular the idea of taking a bounded representation of an abelian group into some kind of corona algebra. However, there are certain technical simplifications that are possible in the setting of Theorem 8.3, essentially because $\ell^\infty \otimes \mathbb{M}_2$ and $(\ell^\infty/c_0) \otimes \mathbb{M}_2$ are easier to handle than $\mathcal{B}(\ell^2)$ and $\mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$.

8.3 Reducing the problem to a question of unitarizability

We will find it useful to fix some new notation. Let $\mathcal{C}(\mathbb{N})$ denote the quotient algebra $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$: this can be identified with $C(\beta\mathbb{N}^*)$ where $\beta\mathbb{N}^* := \beta\mathbb{N} \setminus \mathbb{N}$ is sometimes known as the Stone-Çech remainder of $\mathbb{N}$. Let $q : \ell^\infty(\mathbb{N}) \to \mathcal{C}$ be the quotient homomorphism and let $Q = q \otimes \iota : \ell^\infty(\mathbb{N}, \mathbb{M}_2) \to \mathcal{C}(\mathbb{N}, \mathbb{M}_2)$, which is also a quotient homomorphism with kernel $c_0(\mathbb{N}, \mathbb{M}_2)$.

Let Γ be an amenable discrete group and $\theta : \Gamma \to \mathcal{C}(\mathbb{N}, \mathbb{M}_2)$ a bounded representation. Let $B := \overline{\operatorname{lin}}^{\|\cdot\|}\theta(\Gamma)$, and define $\mathcal{A} = q^{-1}(B)$. We now make the following observations:

- $\ell^1(\Gamma)$ is (symmetrically) amenable — we saw in earlier lectures how to build a (symmetric) bounded approximate diagonal.
- $c_0(\mathbb{N}, \mathbb{M}_2)$ is a closed ideal in $\mathcal{A}$, and is a (symmetrically) amenable Banach algebra.
- $\mathcal{A}/c_0(\mathbb{N}, \mathbb{M}_2) = B$ is the closure of a continuous homomorphic image of $\ell^1(\Gamma)$, so is (symmetrically) amenable.

Since the class of symmetrically amenable Banach algebras is closed under forming extensions, we conclude that $\mathcal{A}$ is symmetrically amenable.

Suppose $\mathcal{A}$ is generated by a bounded group $G \subset \mathcal{A}_{\mathrm{inv}}$. Then by the result of Example 6.7, and by Lemma 6.4, there exists $s \in \ell^\infty(\mathbb{N}, \mathbb{M}_2)^+_{\mathrm{inv}}$ such that sxs^{-1} is unitary for every $x \in G$. Hence $\widetilde{A} := s\mathcal{A}s^{-1}$ is a self-adjoint subalgebra of $\ell^\infty(\mathbb{N}, \mathbb{M}_2)$.

Moreover, $\widetilde{A}$ contains $c_0(\mathbb{N}, \mathbb{M}_2)$ as a closed ideal, and the quotient algebra $\widetilde{B}$ is just $Q(s)BQ(s)^{-1}$.

Now assume Γ is abelian, and let $g \in \Gamma$. Consider $\widetilde{g} := Q(s)\theta(g)Q(s)^{-1} \in \widetilde{B}$. **Since $\widetilde{B}$ is a commutative C*-algebra**, $\widetilde{g}$ is a normal element of $\mathcal{C}(\mathbb{N}, \mathbb{M}_2)$. On the other hand, $\widetilde{g}$ is also a doubly-power bounded element of a unital Banach algebra, so by the spectral radius formula its spectrum is contained in $\mathbb{T}$. Hence, by spectral theory for normal operators, $\widetilde{g}$ is unitary.

Summing up, we have proved the following result:

Proposition 8.5. *Let Γ be an abelian group and $\theta : \Gamma \to \mathcal{C}(\mathbb{N}, \mathbb{M}_2)_{\mathrm{inv}}$ a bounded representation; define B and $\mathcal{A}$ as above. Then $\mathcal{A}$ is amenable. Moreover, if $\mathcal{A}$ is generated by a bounded group, then θ is $\mathcal{C}(\mathbb{N}, \mathbb{M}_2)$-unitarizable.*

8.4 A non-unitarizable representation

In this subsection, we will show that for a suitable cardinal κ there exists a bounded representation $\theta : (\mathbb{Z}/2\mathbb{Z})^{\oplus\kappa} \to \mathcal{C}(\mathbb{N}, \mathbb{M}_2)$ that is not $\mathcal{C}(\mathbb{N}, \mathbb{M}_2)$-unitarizable. The presentation is an expanded version of the one in [9].

Let $\mathcal{F}, \mathcal{G} \subset \mathcal{P}(\mathbb{N})$ be two families of subsets of $\mathbb{N}$, such that

$$(q(1_J))_{J\in\mathcal{F}} \cup (q(1_K))_{K\in\mathcal{G}}$$

is a family of non-zero, pairwise-orthogonal projections in $\mathcal{C}(\mathbb{N})$. In detail: we require that

- every $J \in \mathcal{F}$ and every $K \in \mathcal{G}$ is infinite,
- if $J_1, J_2 \in \mathcal{F}$ and $J_1 \neq J_2$ then $|J_1 \cap J_2| < \infty$,
- if $K_1, K_2 \in \mathcal{G}$ and $K_1 \neq K_2$ then $|K_1 \cap K_2| < \infty$,
- if $J \in \mathcal{F}$ and $K \in \mathcal{G}$ then $|J \cap K| < \infty$.

The existence of such families is well known, but what is crucial for our purposes is that we can impose the following extra condition on our families $\mathcal{F}$ and $\mathcal{G}$.

Proposition 8.6 (A gap "condition"). *We can choose $\mathcal{F}$ and $\mathcal{G}$ to have the following property. For each partition $\mathbb{N} = X \sqcup Y$, either there exists $J \in \mathcal{F}$ such that $X \cap J$ is infinite, or there exists $K \in \mathcal{G}$ such that $Y \cap K$ is infinite.*

I will probably treat this proposition as a black box, but if there is sufficient interest I may try to write up some more details during the lectures.

We also need two involutions $x, y \in \mathbb{M}_2$ with the following property: the corresponding actions $\mathrm{Ad}^*_x, \mathrm{Ad}^*_y : \mathbb{Z}/2\mathbb{Z} \curvearrowright (\mathbb{M}_2)^+_{\mathrm{inv}}$ have no common fixed point. (For instance, see Example 6.6.)

Defining our representation. Let $\Gamma_\mathcal{F} = (\mathbb{Z}/2\mathbb{Z})^{\oplus\mathcal{F}}$ and $\Gamma_\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{\oplus\mathcal{G}}$. For each $J \in \mathcal{F}$ and $K \in \mathcal{G}$, define involutions in $\ell_\infty \otimes \mathbb{M}_2$ by

$$x_J = 1_J \otimes x + 1_{\mathbb{N}\setminus J} \otimes I_2 \quad \text{and} \quad y_K = 1_K \otimes y + 1_{\mathbb{N}\setminus K} \otimes I_2.$$

Crucially, if $J \in \mathcal{F}$ and $K \in \mathcal{G}$ then $J \cap K$ is finite for each J, and so $Q(x_J y_K) = Q(y_K x_J)$. Therefore, if we define $\Theta[x] : \Gamma_\mathcal{F} \to \mathcal{C}(\mathbb{N}, \mathbb{M}_2)$ and $\Theta[y] : \Gamma_\mathcal{G} \to \mathcal{C}(\mathbb{N}, \mathbb{M}_2)$ by

$$\Theta_x(e_J) = Q(x_J) \quad , \quad \Theta_y(e_K) = Q(y_K) \,,$$

these are two bounded representations *whose ranges commute*. Hence

$$\Theta := \Theta[x] \times \Theta[y] : \Gamma_\mathcal{F} \times \Gamma_\mathcal{G} \to \mathcal{C}(\mathbb{N}, \mathbb{M}_2)_{\mathrm{inv}}$$

is a bounded representation. Moreover, Ad^*_Θ has a fixed point if an d only if $\mathrm{Ad}^*_{\Theta[x]}$ and $\mathrm{Ad}^*_{\Theta[y]}$ have a *common fixed point*, and so our task is reduced to showing this does not happen.

Suppose that $\mathrm{Ad}^*_{\Theta[x]}$ and $\mathrm{Ad}^*_{\Theta[y]}$ do have a common fixed point, say $Q(s)$ for some positive invertible $s \in \ell^\infty(\mathbb{N}, \mathbb{M}_2)$. We write $s = (s_n)$ where $s_n \in (\mathbb{M}_2)^+_{\mathrm{inv}}$ for all $n \in \mathbb{N}$. Now we need two lemmas.

Lemma 8.7. $\lim_{n\in J} \mathrm{dist}(s_n, \mathrm{Fix}(\mathrm{Ad}^*_x)) = 0$*, for all* $J \in \mathcal{F}$*; and* $\lim_{n\in K} \mathrm{dist}(s_n, \mathrm{Fix}(\mathrm{Ad}^*_y)) = 0$*, for all* $K \in \mathcal{G}$.

Lemma 8.8. $\inf_n \mathrm{dist}(s_n, \mathrm{Fix}(\mathrm{Ad}^*_x)) + \mathrm{dist}(s_n, \mathrm{Fix}(\mathrm{Ad}^*_y) = \delta > 0$.

Now put

$$X := \{n \in \mathbb{N} : \ \mathrm{dist}(s_n, \mathrm{Fix}(\mathrm{Ad}^*_x)) \geq \delta/2\}$$
$$Y := \{n \in \mathbb{N} : \ \mathrm{dist}(s_n, \mathrm{Fix}(\mathrm{Ad}^*_y)) \geq \delta/2\} \,.$$

By Lemma 8.8, $X \cup Y = \mathbb{N}$. But by Lemma 8.7

$$|X \cap J| < \infty \quad \text{for all } J \in \mathcal{F} \quad , \quad |Y \cap K| < \infty \quad \text{for all } K \in \mathcal{G}.$$

This contradicts the gap condition. So no such s exists. □

9 A look at higher-degree Hochschild cohomology

If one is interested in cohomology, deformations or perturbations for Banach algebras, then amenability can provide a way to solve these problems; and seeing how this works can suggest ways to attack these problems for certain non-amenable Banach algebras.

Warning. This section is much more sketchy than the previous ones, and **it is very likely to be cut, or at least abridged.** If necessary I will circulate some extra material to fill in the context and proofs.

9.1 2-cocycles

DEFINITION 9.1 (2nd-degree Hochschild cohomology). Let $\mathcal{Z}^2(A,X)$ denote the space of all bounded bilinear maps $\psi : A\times A\to X$ that satisfy the following identity:

$$a\cdot\psi(b,c)-\psi(ab,c)+\psi(a,bc)-\psi(a,b)\cdot c=0 \qquad (a,b,c\in A).$$

Let $\mathcal{B}^2(A,X)$ denote the space of all bounded linear maps $A\times A\to X$ that have the form

$$(a,b)\mapsto a\cdot\phi(b)-\phi(ab)+\phi(a)\cdot b \qquad (a,b,\in A).$$

for some bounded linear map $\phi : A\to X$.

Elements of $\mathcal{Z}^2(A,X)$ are called **2-cocycles** (on A with values in X) and elements of $\mathcal{B}^2(A,X)$ are called **2-coboundaries** (on A with values in X). One can check that $\mathcal{B}^2(A,X)\subseteq\mathcal{Z}^2(A,X)$, and the quotient space $\mathcal{Z}^2(A,X)/\mathcal{B}^2(A,X)$ is denoted by $\mathcal{H}^2(A,X)$ and called the **2nd-degree Hochschild cohomology group** of A with coefficients in X.

A nice illustration of what these conditions "really mean" comes from the deformation theory of associative algebras. Let A be an associative $\mathbb{C}$-algebra and let $A_\varepsilon = A\otimes\mathbb{C}[\varepsilon]/(\varepsilon^2)$. (Intuitively, we are just adjoining a free parameter ε that commutes with everything in A and squares to zero.) Given a bilinear map $\psi : A\times A\to A$, we can define a bilinear map $A_\varepsilon\times A_\varepsilon\to A_\varepsilon$ by

$$(a+\varepsilon a_1)\star_\psi(b+\varepsilon b_1) := ab+\varepsilon\left(\psi(a,b)+a_1b+ab_1\right) \qquad (a,b,a_1,b_1\in A). \tag{9.1}$$

When is $\star_\psi$ associative? Well, direct calculation shows that

$$(a+\varepsilon a_1)\star_\psi\left((b+\varepsilon b_1)\star_\psi(c+\varepsilon c_1)\right) = abc+\varepsilon\left(a\psi(b,c)+\psi(a,bc)+a_1bc+ab_1c+abc_1\right),$$

while

$$\left((a+\varepsilon a_1)\star_\psi(b+\varepsilon b_1)\right)\star_\psi(c+\varepsilon c_1) = abc+\varepsilon\left(\psi(ab,c)+\psi(a,b)c+a_1bc+ab_1c+abc_1\right).$$

$$\boxed{\text{Thus } \star_\psi \text{ is associative if and only if } \psi\in\mathcal{Z}^2(A,A).}$$

Moreover, suppose there exists $\phi : A\to A$ linear and consider $\iota-\varepsilon\phi : A\to(A_\varepsilon,\star_\psi)$.

$$(a-\varepsilon\phi(a))\star_\psi(b-\varepsilon\phi(b) = ab+\varepsilon\left(\psi(a,b)-\phi(a)b-a\phi(b)+\psi(a,b)\right),$$

and so $\psi(a,b)=a\phi(b)-\phi(ab)+\phi(a)b$.

$$\boxed{\text{Thus such a } \phi \text{ exists if and only if } \psi\in\mathcal{B}^2(A,A).}$$

A topic we will overlook. We already remarked that Johnson's original motivation for looking at Hochschild cohomology came from studying derivations on $L^1(G)$. Helemskii's original motivation appears to have been the search for a functional analytic version of Wedderburn decomposition. (This had also been studied by Bade and Curtis.) Such questions lead naturally to the study of the 2nd-degree Hochschild cohomology group; indeed, Wedderburn decomposition was one of Hochschild's motivations for introducing "his" cohomology theory.

We will not go into the details of how $\mathcal{H}^2$ parametrizes "Ban-split, square-zero extensions", or acts as an obstruction to realizing such extensions as semi-direct products, since this is well covered elsewhere. For instance, the interested reader could consult [19, Ch. 1, §2] or the monograph [2] which explain clearly how all this works. Alternatively, find an exposition of the "purely algebraic version", such as [36, §9.3], and make the necessary adjustments to obtain the "Banach version".

9.2 Explicit cobounding using a bounded approximate diagonal

In general: the Hochschild cochain complex and the coboundary operator on it are given by

$$\mathcal{C}^n(A,X) := \{n\text{-multilinear maps } A\times\cdots\times A\to X\}$$

and

$$\delta\psi(a_1,\dots,a_{n+1}) := \begin{cases} a_1\cdot\psi(a_2,\dots,a_{n+1}) \\ +\displaystyle\sum_{j=1}^{n}(-1)^j\psi(a_1,\dots,a_ja_{j+1},\dots,a_{n+1}) \\ +(-1)^{n+1}\psi(a_1,\dots,a_n)\cdot a_{n+1} \end{cases}$$

One defines $\mathcal{Z}^n(A,X)$ to be the kernel of $\delta:\mathcal{C}^n(A,X)\to\mathcal{C}^{n+1}(A,X)$, $\mathcal{B}^n(A,X)$ to be the image of $\delta:\mathcal{C}^{n-1}(A,X)\to\mathcal{C}^n(A,X)$, and finally defines $\mathcal{H}^n(A,X):=\mathcal{Z}^n(A,X)/\mathcal{B}^n(A,X)$.

Theorem 9.2 (Amenable algebras are trivial for cohomology with dual coefficients). *Let A be an amenable Banach algebra and let $X=(X_*)^*$ be a dual Banach A-bimodule. Then there exist maps $\alpha:\mathcal{C}^n(A,X)\to\mathcal{C}^{n-1}(A,X)$ such that $\alpha\delta+\delta\alpha$ is the identity. In particular, if $\psi\in\mathcal{Z}^n(A,X)$, then $\psi=\delta\alpha\psi\in\mathcal{B}^n(A,X)$.*

We will show how this can be done using a bounded approximate diagonal (at least in the case where A is unital and X is unit-linked).

Remark 9.3 (The power of Ext). If this had been a lecture series on Hochschild cohomology, then we would have set up the machine of derived functors and projective/ injective resolutions. The payoff for all that effort would have been as follows:

- For any $n\geq 0$ and $M,N\in A\text{-}\mathsf{bimod}$ we can define a vector space ${}_A\operatorname{Ext}^n_A(M,N)$, which vanishes if $n\geq 1$ and M is flat in A-bimod and N is a dual Banach A-bimodule;
- There is a canonical way to identify $\mathcal{H}^n(A,\_\_)$ with ${}_A\operatorname{Ext}^n_A(A^\sharp,\_\_)$.

Now if A is amenable, $A^\sharp$ is flat in A-bimod, and so we obtain (a slightly weaker version) of Theorem 9.2

9.3 Application to AMNM problems

If we cover this topic in lectures then I will provide the necessary definitions.

Theorem 9.4 (Johnson [23]). *Let A be an amenable Banach algebra and let $B=(B_*)^*$ be a dual Banach algebra. Then (A,B) is an AMNM pair.*

9.4 Further topics I probably won't have time to mention.

"Raising bounded subgroups": a result of Bade, Curtis and Sinclair. Building on work of Solovej, Albrecht and Ermert showed that every extension of $C(X)$ by a nilpotent ideal is Ban-split. Using the theory of Hermitian-equivalent elements in Banach algebras, Bade, Curtis and Sinclair gave a simpler proof that applies to a more general class of radical extensions. One step in their proof can be interpreted as setting up a bounded 2-cocycle on $\mathbb{Z}$ and proving that it is a 2-coboundary.

The "Brooks cocycles", discovered by Johnson. These give explicit non-zero elements of H_b^2 for the free group. Note that in usual group cohomology, free groups have cohomological dimension 1.

$\mathcal{H}^2(A(\mathbb{D}), A(\mathbb{D}))$ **is an infinite-dimensional Banach space.** In usual Hochschild cohomology for $\mathbb{C}$-algebras, $\mathbb{C}[z]$ has (co)homological dimension 1. However, Johnson showed in [22, §8] that $\mathcal{H}^2(A(\mathbb{D}), A(\mathbb{D})) \neq \{0\}$. It seems very likely to be folklore that $\mathcal{H}^2(A(\mathbb{D}), A(\mathbb{D}))$ is infinite-dimensional as a $\mathbb{C}$-vector space; but as far as I know, it has never appeared in the literature that $\mathcal{B}^2(A(\mathbb{D}), A(\mathbb{D}))$ is closed in $\mathcal{Z}^2(A(\mathbb{D}), A(\mathbb{D}))$, even though the proof is not too hard once you have the right idea. I may sketch how this works.

Stability" via vanishing cohomology. This was done independently by Johnson and by Raeburn & Taylor in the 1970s, motivated by known ideas in deformations of associative algebras. The conditions are quite strong (using $\mathcal{H}^2(A, A) = 0$ and $\mathcal{H}^3(A, A)$ Hausdorff) and are difficult to verify in practice. Nevertheless, a cb-version of this was used in some recent work of Roydor to prove a "noncommutative Amir–Cambern theorem" for injective von Neumann algebras.

Normalization over amenable subalgebras. If we discuss this topic then we will just state the definitions and results without going into the proofs. (This is done in an explicit way in [34].) Note that the subalgebra doesn't have to be commutative.

Final thoughts

> But the Demon of the Second Kind continued to operate at a speed of three hundred million facts per second, and mile after mile of tape coiled out and gradually buried the Ph.D. pirate beneath its windings, wrapping him, as it were, in a paper web, while the tiny diamond-tipped pen shivered and twitched like one insane, and it seemed to Pugg that any minute now he would learn the most fabulous, unheard-of things, things that would open up to him the Ultimate Mystery of Being, so he greedily read everything that flew out from under the diamond nib, the drinking songs of the Quaidacabondish and the sizes of bedroom slippers available on the continent of Cob, with pompons and without, and the number of hairs growing on each brass knuckle of the skew-beezered flummox, and the average width of the fontanel in indigenous stepinfants, and the litanies of the M'hot-t'ma-hon'h conjurers to rouse the reverend Blotto Ben-Blear, and the inaugural catcalls of the Duke of Zilch, and six ways to cook cream of wheat, and a good poison for uncles with goatees, and twelve types of forensic tickling, and the names of all the citizens of Foofaraw Junction beginning with the letter M, and the results of a poll of opinions on the taste of beer mixed with mushroom syrup...
>
> And it grew dark before his hundred eyes, and he cried out in a mighty voice that he'd had enough, but Information had so swathed and swaddled him in its three hundred thousand tangled paper miles, that he couldn't move and had to read on... Thus was the pirate Pugg severely punished for his inordinate thirst for knowledge.

From *The Cyberiad* by Stanislaw Lem (translation by Michael Kandel).

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