SOME ASPECTS OF THE THEORY OF DYNAMICAL SYSTEMS FROM THE TOPOLOGICAL POINT OF VIEW

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In the notes there are presented some definitions, theorems, several remarks and in some cases the outline of the proofs. There are no examples, because the purpose of these notes is to summarize lectures and not to present the theory in full details. Several examples were presented during the lectures.

1. The basic results from the topological theory of dynamical systems.

Let X be a metric space. By $B(A,\delta)$ we define $\{x \in X : d(x,A) < \delta\}$, where by $d(x,A)$ we mean the distance from a point x to a set A. A triplet $(X,\mathbb{R},\pi)$ is a *dynamical system (a flow)* if $\pi : \mathbb{R} \times X \to X$ is a continuous function with $\pi(0,x) = x$ and $\pi(t,\pi(s,x)) = \pi(t+s,x)$ for every t,s,x; replacing $\mathbb{R}$ by $\mathbb{R}_+$ (i.e. by the interval $[0,+\infty)$) we get the definition of a *semidynamical system.* The set X is called a *phase space.*

Assume that a dynamical system $(X,\mathbb{R},\pi)$ is given. We define the *positive trajectory* (the *positive orbit*) of x as $\pi^+(x) = \pi([0,+\infty),x) = \pi([0,+\infty) \times \{x\})$. By a negative trajectory of x we mean $\pi^-(x) = \pi((-\infty,0],x)$. By a trajectory of x we mean $\pi^-(x) \cup \pi^+(x)$. We put $\pi^+(A)$ as $\bigcup\{\pi^+(x) : x \in A\}$; we define $\pi^-(A)$ and $\pi(A)$ in an analogous way.

A point x is said to be *stationary* if $\pi(t,x) = x$ for every $t \geqslant 0$. A point x is *periodic* if there exists a $t > 0$ such that $\pi(t,x) = x$ and x is not stationary; such t is called a *period* of x. A point x is *regular* if it is neither periodic nor stationary.

It is easy to notice that for a periodic point x there exists a smallest period of x and if T is this period, then the set of all periods of x is equal to $\{kT : k \in \mathbb{Z}_+\}$. For a stationary point we have $\pi^+(x) = \pi(x) = \{x\}$.

It is easy to prove

1.1. Theorem. *The set of all stationary points is closed.*

For the proof, take a sequence of stationary points (x_n) converging to x and consider $\pi(t,x_n)$ for $t > 0$.

A set A is *invariant* if $\pi(A) = A$. A set A is *positively (negatively) invariant* if $\pi^+(A) = A$ ($\pi^-(A) = A$). A set A is *minimal* if it is nonempty, closed, invariant and no proper subset of A has all these properties.

It is easy to prove that if A is invariant then its complement, closure, interior and boundary are also invariant.

We have

1.2. Theorem. *If B is a positively invariant set homeomorphic to the closed ball in $\mathbb{R}^n$, then there is a stationary point contained in B.*

The theorem is proved in [BS], however, for simpler and more general proof see [C-To]. For the proof we notice that the mapping $\pi(\frac{1}{n},\cdot) : B \to B$ has a fixed point for each n (from the Brouwer Fixed Point Theorem). We may assume that $x_n \to x \in B$. Now we take a $t > 0$ and show that for each n there is a k_n with $k_n \leqslant t \cdot n < k_n + 1$. Then $\frac{k_n}{n} \to t$ and $\pi(t, x_n) \to x$.

For a given point x we define *the positive limit set of x* (or *ω-limit set of x*) as $L^+(x) = \{y \in X : \pi(t_n, x) \to y$ for some $t_n \to \infty\}$ and *the negative limit set of x* (or *α-limit set of x*) as $L^-(x) = \{y \in X : \pi(t_n, x) \to y$ for some $t_n \to -\infty\}$. It easy to notice that $L^+(x) = \bigcap\{\overline{\pi^+(\pi(t,x))} : t \geqslant 0\}$ and prove that $L^+(x)$ is closed and invariant (not only positive invariant). The same properties hold for $L^-(x)$.

Generally, limit sets need not to be connected. However, we have

1.3. Theorem. *If X is locally compact and $L^+(x)$ is compact, then it is also connected. If it is not compact, then it does not have a connected component.*

For the proof ot the first part we suppose that $L^+(x)$ can be presented as a disjoint union of two closed sets C and D. These sets are compact and we may find an ε such that $B(C,\varepsilon) \cap B(D,\varepsilon) = \emptyset$. Then we take sequences (t_n) and (s_n) converging to $+\infty$ with $\pi(t_n, x) \to c \in C$ and $\pi(s_n, x) \to d \in D$. Taking a suitable subsequences we may assume that $t_n < s_n < t_{n+1}$ and find a sequence u_n converging to $+\infty$ such that $\pi(u_n, x) \to p$ and p is contained in the sphere centred in C with a radius ε, which is a contradition. The proof of the second part of the theorem is more advanced and uses some topological tools (Lemma 6.1.1. from [E]) and the structure of compactification of dynamical systems.

There are

1.4. Theorem. *A set A is minimal if and only if it is a closure of the trajectory of any of its points.*

1.5. Theorem. *In any nonempty compact invariant set A there is contained a minimal set.*

The proof of 1.4 is simple and elementary. For the proof of 1.5 we use Kuratowski-Zorn Lemma and the Riesz condition for compact sets.

Note that there may exist points not contained in any minimal set.

A very interesting and important example of minimal set is a two-dimensional torus where each trajectory is dense. This is also an important example for the next definition.

A point x is said to be *positively Poisson stable* if $x \in L^+(x)$.

The important concept in dynamical systems is stability.

Assume that a set M is compact. We say that M is *stable* if for every $\varepsilon > 0$ and $x \in M$ there exists a $\delta > 0$ such that $\pi^+(B(x, \delta)) \subset B(M, \varepsilon)$. It is easy to show that in a locally compact space the above condition is equivalent to the following: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\pi^+(B(M, \delta)) \subset B(M, \varepsilon)$.

We define *the region of attraction* $A(M)$ of M as $\{x \in X : L^+(x) \neq \emptyset, L^+(x) \subset M\}$ and *the region of weak attraction* $A_w(M)$ of M as $\{x \in X : L^+(x) \cap M \neq \emptyset\}$. It is easy to prove that in a locally compact space:

$x \in A_w(M) \;\Leftrightarrow\;$ there exists a sequence $(t_n) : t_n \to +\infty, d(\pi(t_n, x), M) \to 0$

$x \in A(M) \;\Leftrightarrow\; d(\pi(t, x), M) \to 0$ as $t \to +\infty$

where by $d(x, M)$ we mean the distance from a point x to a set M.

If $A(M)$ is a neighbourhood of M we say that M is an *attractor*. If $A_w(M)$ is a neighbourhood of M we say that M is a *weak attractor*. A set M which is simultaneously an attractor and a stable set is called to be *asymptotically stable*.

Assume that X is locally compact. It is easy to show that:

if M is an attractor (a weak attractor) then $A(M)$ $(A_w(M))$ is open

if M is an attractor (a weak attractor) then $A(M)$ $(A_w(M))$ is invariant

if M is stable then M is positively invariant.

Note that the assumption of local compactness is not always necessary.

There are examples showing that a stable set need not be an attractor and also that an attractor need not to be stable (as the property of attraction says, in fact, only about the behaviour of the trajectories when t goes to infinity). However, we have

1.6. Theorem. *In a locally compact space, if a set M is stable and is a weak attractor then it is an attractor.*

To prove this it is enough to show that if $x \in A_w(M)$, then $x \in A(M)$. For that, take $\varepsilon > 0$ and a $y \in L^+(x) \cap M$. From stability, there is a δ such that $\pi^+(B(y, \delta)) \subset B(M, \frac{\varepsilon}{2})$. Also, $\pi(t, x) \in B(y, \delta)$ for some t. A simple analysis shows that $L^+(x) \subset B(M, \varepsilon)$. Thus $L^+(x) \subset \bigcap\{B(M, \varepsilon) : \varepsilon > 0\} = \overline{M} = M$.

For a locally compact phase space, M is stable if and only if each component of M is stable. An analogous result does not hold for asymptotically stable sets. For an asymptotically stable set M, its component is stable if and only if it is isolated.

2. Isomorphisms of dynamical systems.

Assume that X is a metric space and $D \subset \mathbb{R} \times X$ is open; moreover, for any $x \in X$ there are $\alpha_x < 0$ and $\omega_x > 0$ such that $D \cap (\mathbb{R} \times \{x\}) = (\alpha_x, \omega_x)$. We define a *local dynamical system* (D, π) if $\pi : D \to X$ is a continuous function with

$\pi(0, x) = x$ for any $x \in X$ and $\pi(t, \pi(s, x)) = \pi(t + s, x)$ for every $t, s \in \mathbb{R}, x \in X$ where those expressions are defined. It is easy to prove that the function $x \mapsto \omega_x$ is lower semicontinuous and the function $x \mapsto \alpha_x$ is upper semicontinuous.

In an analogous way we define a *local semi-dynamical system*; then $D \cap (\mathbb{R}_+ \times \{x\}) = [0, \omega_x)$ for each x.

We say that local dynamical systems (D, π) and (G, ρ) are *isomorphic* ($G \subset \mathbb{R}\times Y$) if there exist a homeomorphism $h : X \to Y$ and a continuous function $\phi : D \to \mathbb{R}$ such that $\phi(0, x) = 0$ for any x, $\phi(\cdot, x)$ is an increasing homeomorphism from (α_x, ω_x) to $(\alpha_{h(x)}, \omega_{h(x)})$ for any x, and $h(\pi(t, x)) = \rho(\phi(t, x), h(x))$ for any $(t, x) \in D$.

If h is equal to identity we will call such an isomorphism a *reparametrization*.

Of course, the definition may be applied also in the case where the systems are not local. In such a case, where for any x the function $\phi(\cdot, x)$ is the identity on $\mathbb{R}$ we speak about a *topological conjugacy*.

There are many results on isomorphisms of dynamical systems. Here, we mention two fundamental results.

In 1972 David H. Carlson published in *Journal of Differential Equations* a paper where he proved that any local dynamical system on a metric space is isomorphic to a global dynamical system and an analogous theorem holds for local semidynamical systems. In fact, Carlson proved much more. The isomorphism he constructed was a reparametrization. Also, he did not limit himself to the consideration of metric spaces. In particular, it is enough to assume that $[0, 1] \times X$ is a normal topological space to get the assertion of the theorem.

One of the most crucial results in the theory of dynamical systems was obtained by Carlos Gutierrez. He proved that if the phase space X is a compact $2-$manifold (the result can be easily extended to $\mathbb{R}^2$), then any dynamical system on X is topologically equivalent to a C^1 dynamical system on X. The paper was published in *Ergodic Theory and Dynamical Systems* in 1986. In the proof there were used many advanced techniques and results. For the systems on $\mathbb{R}$ an analogous theorem was known earlier and it is rather simple. For the systems on $\mathbb{R}^4$ a similar theorem does not hold. There is a counterexample given by William C. Chewning and published in 1974 in the *Bulletin of the American Mathematical Society.* The example is based on a classical topological result by R.H.Bing that there exists a set M which is not a three-dimensional manifold such that $M \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. In the case where $X = \mathbb{R}^3$ the problem is, to the best of the author's knowledge, still open.

3. Persistence in dynamical systems.

An important topic in the theory of dynamical systems is persistence. The theory has its origin in applications, especially in the description of a system of interacting species. Imagine that we describe the evolution of some species in $[0, +\infty)^n$ and the variables denoted by x_k describe the quantities of the particular species. When x_k approaches 0, the population does not survive. Generally, instead of $[0, +\infty)^n$ and its boundary we may investigate a closed set in a metric space and its boundary.

The notion of persistence attempts to represent a model ecosystem where all the components of the ecosystem survive.

Let X be a locally compact metric space with metric d and E be a closed subset of X. We assume that neither $\operatorname{Int} E$ nor ∂E is empty. Let $(E, \mathbb{R}, \pi)$ be a dynamical system on E. We assume that $\operatorname{Int} E$ and ∂E are invariant (note that the system is defined on E, not on X).

We say that $(E, \mathbb{R}, \pi)$ is weakly persistent if $\limsup_{t\to+\infty} d(\pi(t,x), \partial E) > 0$ for any $x \in \operatorname{Int} E$.

We say that $(E, \mathbb{R}, \pi)$ is persistent if $\liminf_{t\to+\infty} d(\pi(t,x), \partial E) > 0$ for any $x \in \operatorname{Int} E$.

We say that $(E, \mathbb{R}, \pi)$ is uniformly weakly persistent if there exists an $\eta > 0$ such that $\limsup_{t\to+\infty} d(\pi(t,x), \partial E) > \eta$ for any $x \in \operatorname{Int} E$.

We say that $(E, \mathbb{R}, \pi)$ is uniformly persistent if there exists an $\eta > 0$ such that $\liminf_{t\to+\infty} d(\pi(t,x), \partial E) > \eta$ for any $x \in \operatorname{Int} E$.

We say that $(E, \mathbb{R}, \pi)$ is dissipative if $L^+(x) \neq \emptyset$ for any $x \in E$ and the set $\Omega = \bigcup\{L^+(x) : x \in E\}$ has a compact closure.

The uniform persistence seems to be the most important condition, as it describes the situation where the species survive "in a good way".

It is immediate that a uniformly persistent system is persistent and a persistent impulsive system is weakly persistent. Also, a uniformly persistent system is weakly uniformly persistent and a weakly uniformly persistent system is weakly persistent. The converse implications do not hold. Also, there is no implication between persistence and week uniform persistence.

We have an important

3.1. Theorem. *The system is uniformly persistent provided it is weakly uniformly persistent and dissipative.*

The idea of the proof is as follows (the full proof can be found in [FM]). We take an ε_0 from the definition of the weak uniform persistence and (from dissipativity) an open neighbourhood G of Ω with $\overline{G}$ compact. For the contrary we suppose that there exists a sequence (p_k) of points of $\operatorname{Int} E$ with $\liminf_{t\to+\infty} d(\pi(t,p_k), \partial E) < \frac{1}{k}$. For large t we have $\pi(t, p_k) \in \overline{G}$, moreover, for each k there is an s_k with $d(\pi(s_k, p_k), \partial E) < \frac{1}{k}$. Using the assumption of the weak uniform persistence we construct the sequence (x_k) of points contained in the trajectories $\pi^+(p_k)$ such that each x_k is contained in the sphere centred in ∂E of radius ε_0 and also in $\overline{G}$, and a sequence (α_k) such that $d(\pi(\alpha_k, x_k), \partial E) < \frac{1}{k}$ and $\pi((0, \alpha_k], x_k) \subset B(\partial E, \varepsilon_0)$ for large k. We may assume that $x_k \to x \in \overline{G}$. From the weak uniform persistence there is a β with $d(\pi(\beta, x), \partial E) > \varepsilon_0$. We have $\pi(\beta, x_k) \to \pi(\beta, x)$ and $\pi(\beta, x_k) \notin \overline{B}(\partial E, \varepsilon_0)$ for large k (by $\overline{B}$ we mean a closed ball). Thus there is a δ such that $\pi([0, \beta]), x_k) \cap \overline{B}(\partial E, \delta) = \emptyset$. However, this shows that $\alpha_k > \beta$ for large k and $\pi(\beta, x_k) \in B(\partial E, \varepsilon_0)$, which is a contradiction.

The same proof works when instead of dissipativity we assume that the boundary of E is compact.

Many results connected with persistence yield to obtaining information about the system from some properties of the behaviour on the boundary. For many situations, of great importance is the so-called Butler-McGehee Lemma.

3.2. Butler-McGehee Lemma. *Assume that a dynamical system is given by a C^1 differential equation and p is an isolated hyperbolic stationary point (i.e. it is hyperbolic equilibrium and it is a unique stationary point in some its neighbourhood). Assume also that $p \in L^+(p)$. Then either $L^+(p) = \{p\}$ or there are $a, b \in L^+(p)$ $(a, b \neq p)$ such that $L^+(a) = \{p\}$ and $L^-(b) = \{p\}$.*

There are many generalizations of this lemma, also for topological systems.

4. Some results on semidynamical systems.

For dynamical systems, $\pi(t, x)$ is defined for all real t. For semidynamical systems, we define "the movement $\pi(\cdot, x)$" only for positive values of t (which is often regarded as a time variable). However, we may ask about "the past" of a given point t.

For semidynamical systems, two kind of problems occur. First, we may investigate the properties of dynamical systems which hold also for semidynamical systems (i.e. they depend only on the possibility of moving forward), however, the proofs need not be automatic generalizations. Second, although the movement is defined only forward, we may ask about "the past" of given points and analyse several properties "backward".

From the point of view of differential equations, many interesting models of dynamical systems appear in finite dimensional systems. For semidynamical systems, natural models from differential equations appear in infinite dimensional systems.

Now we formulate some definitions.

We define the *positive trajectory* (the *positive orbit*) of x as $\pi^+(x) = \pi([0, +\infty), x) = \pi([0, +\infty) \times \{x\})$. For $t \geqslant 0$ and $y \in X$ by $F(t, y)$ we mean $\{z \in X : \pi(t, z) = y\}$ and $F([u, v], x) = \bigcup\{F(t, x) : t \in [u, v]\}$ for $u < v$. In an analogous way we define $F(\Delta, D)$ for $\Delta \subset [0, +\infty)$ and $D \subset X$. A point $x \in X$ is said to be a *start point* if $F(t, x) = \emptyset$ for $t > 0$. By a *negative solution through* x we mean a function $\sigma : \Delta \to M$ (where Δ is an interval equal to $[\alpha, 0]$ or $(\alpha, 0]$, in the second case α may be equal to $-\infty$) such that $\sigma(0) = x$ and $\pi(t, \sigma(u)) = \sigma(t + u)$ for any t, u with $u \in \Delta, t \geqslant 0, t + u \in \Delta$ and a solution σ is maximal (relative to the above properties). The image of the negative solution is called a *negative trajectory through* x.

We say that x is a *start point* if $F(t, x) = \emptyset$ for any $t > 0$. If $F(t, x)$ has at most one element for any $t > 0$ we say that x is a *point of negative unicity*. We define stationary, periodic and regular point in the same way as for dynamical system. We put $L^+(x) = \{y \in M : \pi(t_n, x) \to y$ for some $t_n \to \infty\}$ and $L^-_\sigma(x) = \{y \in M : \sigma(t_n) \to y$ for some $t_n \to -\infty\}$, where σ is a negative solution through x and call it a *negative limit set*. Generally we call all these sets to be *limit sets*. Note that for a given point x, different negative solutions may give different negative limit sets.

Semidynamical systems were introduced in the sixties and then N.P.Bhatia and O.Hajek wrote the first survey book on the subject ([BH]). In 1967 H.Halkin gave a solution of the problem of start points and proved

4.1. Theorem. *A semidynamical system on a manifold does not have start points.*

For the proof we have to show that for any $x_0 \in X$ there are y and $t > 0$ with $\pi(t, y) = x_0$. There is a neighbourhood B of x_0 homeomorphic to a closed ball B with radius ε. Define $f_t(x) = x_0 + x - \pi(t, x)$. There is a t such that for any $x \in B$ the distance between $\pi(t, x)$ and x is smaller than ε. For this t the mapping f_t maps B to B and from the Brouwer Fixed Point Theorem there is a y with $f_t(y) = y$, so $x_0 = \pi(t, y)$.

There are examples showing that generally the set of all start points may be dense.

For semidynamical systems, of importance there is the notion of negative escape time introduced by Roger C. McCann in 1977. The definition presented below is different that the one given by McCann, but equivalent to original.

By a *negative escape time* $N(x)$ of x we mean $N(x) = \inf\{s \in (0, \infty]\colon (-s, 0]$ (or $[-s, 0]$) is a domain of a negative solution through $x\}$.

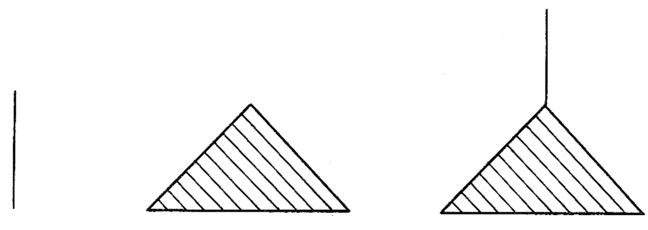
Mc Cann proved that if X is a locally compact metric space and a semidynamical system $(X, \mathbb{R}_+, \pi)$ has no start points then it is isomorphic to a system on X with $N(x) = +\infty$ for each x; moreover, this isomorphism is a reparametrization (note that it is a reparametrization "forward" which give interesting properties "backward").

Now we present some topological properties of the set describing "past" in semidynamical systems on 2−manifolds.

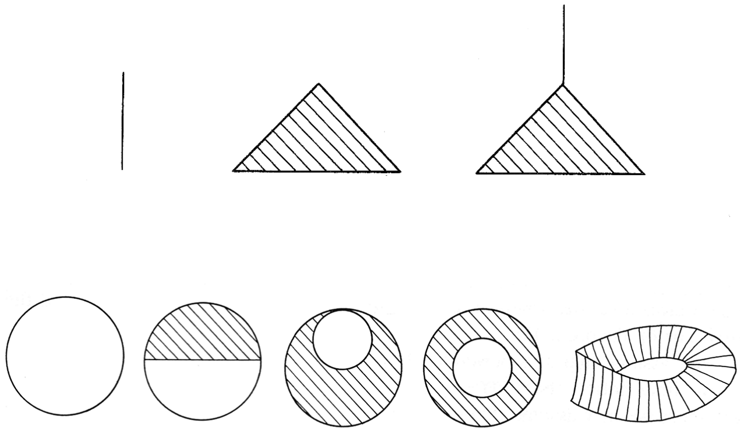
Assume that a semidynamical system on a 2−manifold X is given and $N(y) = +\infty$ for any $y \in X$. Let x be a non-stationary point of X. Then:

4.2. Theorem. *If x is not a point of negative unicity, then there exists an $s \geqslant 0$ such that $F(t, x)$ is a point for $t \leqslant s$ and $F(t, x)$ is an arc for $t > s$.*

4.3. Theorem. *If x is regular and $t < s$, then $F([t, s], x)$ is homeomorphic to one of three sets:*



4.4. Theorem. *If x is periodic and $t < s$, then $F([t, s], x)$ is homeomorphic to one of eight sets:*



The last set, i.e. Möbius strip, may occur only when X is a non-orientable manifold.

Let us summarize briefly the consecutive steps of the proof. The crucial point is the characterization of $F(t, x)$.

We start from the semicontinuity properties for the function $F(\cdot, \cdot)$ in semiflows. These results are proved by methods from the theory of dynamical systems. Then, we come to the characterization of $F(t, x)$. R.Srzednicki proved that if the phase space is a manifold then $F(t, x)$ is connected. He used techniques from algebraic topology, however in the 2−dimensional case this can be shown in an elementary way.

When we know that $F(t, x)$ is connected, we show that it is a point or an arc. The proof uses semicontinuity properties mentioned above and purely topological methods. In particular, the Schönflies Theorem and the characterizations of arcs are applied. Then further theorems follow from the analysis of properties of semiflows and some classical topological results; especially, consequences of Moore's theorem on monotone decomposition of the plane are used. Full proofs can be found in [CO].

For stationary points a similar characterization does not hold. It is possible to give examples of infinitely many non-homeomorphic sets which may be obtained as $F(t, x)$. This is because $F(t, x) = F([0, t], x)$ for a non-stationary point x.

It is natural to ask for a generalization of above theorems for higher dimensions. Here an analogous characterization does not hold. It is possible to construct an example of a semiflow on $\mathbb{R}^3$ with $F(t, x)$ not arcwise connected. Such an example was constructed by Srzednicki.

As a consequence, we get

4.5. Theorem. *The set of points which are not the points of negative unicity is of the first Baire category.*

For the proof split our set onto $A \cup B$, where in A there are nonstationary points and in B there are stationary points. Let P be a countable dense set in X. If $x \in A$ then, according to Theorems 4.3 and 4.4, there is a point $p \in P$ with $x \in \pi^+(p)$ (as we have an open set contained in $F([u,v],x)$ for some u and v). Thus $A \subset \bigcup\{\pi^+([n,n+1],p) : p \in P, n \in \mathbb{N}\}$, so A is of the first category. On the other hand, $\overline{B}$ is contained in the set of all stationary points (as in the case of dynamical systems we can show that the set of all stationary points is closed). Thus $\operatorname{Int} \overline{B}$ is contained in the interior of the set of stationary points. However, a point contained in the last set is in an obvious way a point of negative unicity, so B is nowhere dense and $A \cup B$ is of the first category.

The analogous problem for the semidynamical systems on $3-$manifolds seems to be still open.

5. Sections.

The concept of sections is a fundamental problem in the theory of dynamical systems. In a dynamical system $(X, \mathbb{R}, \pi)$, by a section through x we mean a set S containing x such that for some $\lambda > 0$ the set $U = \pi((-\lambda,\lambda), S)$ is a neighbourhood of x and for every $y \in U$ there is a unique $z \in S$ and a unique $t \in (-\lambda,\lambda)$ with $\pi(t,z) = y$. A section through any nonstationary point x exists according to the Whitney-Bebutov Theorem.

The existence of a section through a given point shows that local parallelizability of the system is fulfilled. This gives a very good tool for solving many problems as it is possible to describe very precisely the behaviour of a system in a neighbourhood of any nonstationary point.

For a semidynamical system the situation is much more complicated, as we do not have negative unicity guaranteed and a trajectory can join another one; also, it may happen that it is impossible to prolongate trajectories in negative direction far enough. Thus in general situation it is impossible to present a neighbourhood of a point as a union of "parallel" segments of trajectories. However, the natural questions arise: is it possible to find a neighbourhood of a given point which can be presented as the union of segments of trajectories going with the same "time length" in the same direction? Is it possible to present a neighbourhood of a given point in a similar way as in a dynamical system (although this presentation must be slightly more complicated)? One of the main problems is stating a definition of section in a semidynamical system.

The definition presented below describe a neighbourhood of a nonstationary point x as a "box" in which the segments of trajectories go from one side of the box to the opposite side by the time interval 2λ. Moreover, any another trajectory does not join the trajectories in this "box", i.e. if any point y belongs to this box then the whole segment of a trajectory with a required time length is contained in the box. Also, these sections have some fundamental properties, in particular the images of sections by time translations through a small time are also sections.

The main idea of the definition is to take a set L containing the point $\pi(\lambda, x)$ with $U = F([0, 2\lambda], L)$ being a neighbourhood of x which guarantees that no another trajectory join the trajectories in U except in the points of the base $F(2\lambda, L)$ of the "box" U. Then, assuming further conditions, we take as a section the set $S = F(\lambda, L)$.

Let us state it in a formal way.

Assume that $(X, \mathbb{R}_+, \pi)$ is a semidynamical system.

A closed set S containing x is called a *section* (a λ-*section*) through x if there is a closed set L such that:

(a) $F(\lambda, L) = S$

(b) $F([0, 2\lambda], L)$ is a neighbourhood of x

(c) $F(\mu, L) \cap F(\nu, L) = \emptyset$ for $0 \leqslant \mu < \nu \leqslant 2\lambda$

If we assume additionally that

(d) $N(y) > 2\lambda$ for every $\lambda \in L$

then $F([0, 2\lambda], L)$ is a parallelizable "box" in which all the segments of the trajectories go from one side to the opposite one in the time interval 2λ and begin their movement in this box at the base $F(2\lambda, L)$ of the box. Without this assumption some trajectories coming to L may have their "beginning" inside $F([0, 2\lambda], L)$. However, also in this case the tube gives the presentation of a neighbourhood of x as the union of "quasi-parallel lines".

We have (see [C-Bu])

5.1. Theorem. *For each nonstationary point x there exists a section through x.*

5.2. Theorem. *Under the additional assumption that there is an $\eta > 0$ and a $\delta > 0$ with $N(y) > \eta$ for all $y \in B(x, \delta)$, there exists a section through x fulfilling the condition (d).*

According to McCann's results, we now may say that in locally compact spaces: for each nonstationary point x there exists a section through x fulfilling the condition (d).

6. The Poincaré-Bendixson Theorem.

The structure of limit sets is very important in the study of the qualitative behaviour of autonomous differential equations and dynamical systems. For the systems on $\mathbb{R}^2$ of particular importance is the Poincaré-Bendixson theorem. It describes very precisely the structure of limit sets in such systems.

First, we formulate the Poincaré-Bendixson Theorem in its classical version.

6.1. Theorem. *Consider a plane autonomous system*

$x' = f(x)$

where $x \in \mathbb{R}^2$

and assume that this system gives a dynamical system. Assume that the positive semiorbit $\pi^+(p)$ through a point $p \in \mathbb{R}^2$ is bounded and that the positive limit set $L^+(p)$ does not contain any stationary point. Then $L^+(p)$ is a periodic orbit.

Moreover, either p is a periodic point or $L^+(p)$ spirals towards a limit cycle of the system.

The analogous result holds for the negative limit set $L^-(p)$.

There are several proofs of this theorem. Generally, they are based on two important facts: the Jordan Curve Theorem and the local parallelizability of a small neighbourhood of a non–stationary point. Here we describe the outline of the proof. For an autonomous system of differential equations which gives a dynamical system (a flow), by *a transversal* we mean a Jordan arc which is not tangent to any orbit of the system in any of its points.

Step 1. Let T be a transversal. Then T is a section according to the definition of section for flows (the most frequently presented proof uses the Implicit Function Theorem).

Step 2. Let T be a transversal and $\pi([s,t],p)$ be a segment of the orbit through p. Then the intersection $T \cap \pi([s,t],p)$ is finite (possibly empty).

Step 3. Let x_1, x_2, x_3 be common points of the transversal T and the orbit $\pi(p)$ of a regular point p. Let $x_i = h(u_i)$ where h is a parametrization of the transversal and let $x_i = \pi(t_i, x)$. Assume that $u_1 < u_2 < u_3$. Then either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$. In other words, for the common points of the transversal and the orbit, the order on the transversal given by its parametrization coincides with the order on the orbit given by the time variable.

Step 4. If T is a transversal and $\pi(p)$ is a periodic orbit, then $T \cap \pi(p)$ has at most one element.

Step 5. If L is a limit set and T is a transversal, then $T \cap L$ has at most one element.

Then it is shown that if any bounded limit set L contains a periodic orbit then L is equal to that periodic orbit. The property that any bounded limit set is compact and invariant is also used.

There are many applications of this theorem. For many purposes there is interesting to know about the existence of the periodic solution. Then, by the analysis of the behaviour of trajectories on the boundary of the domain and the information that there is not a stationary point in the interior (which is usually easy to verify) we may come to the conclusion that there exists a periodic orbit in the investigated domain.

In his four-part memorable paper published in 1881–1886 Henri Poincaré studied celestial mechanics and two–dimensional systems. He made the investigations of the phase portrait of the solutions. He considered only the systems $x' = f(x)$ given by an analytic function f. Theorem 6.1 in the analytic case is due to part III of this work.

In 1901 Ivar Bendixson published in “Acta Mathematica” a paper where he completed the analysis of stationary points by making a more detailed classification. He proved Theorem 6.1 with much weaker assumption on function f. He assumed that

$f = (f_1, f_2)$ is continuous and each of f_1, f_2 has continuous partial derivatives. In this paper he also obtained the classification of isolated stationary points.

Neither Poincaré nor Bendixson investigated limit sets with infinite number of stationary points. However, in 1945 J.K.Solntzev got the following generalization.

6.2. Theorem.
Consider an autonomous system
$x' = f(x)$
where $x \in \mathbb{R}^2$
and assume that this system defines a dynamical system.
Assume that the positive semiorbit $\pi^+(p)$ *through a point* $p \in \mathbb{R}^2$ *is bounded. Then either*
(a) the positive limit set $\omega(p)$ *is a periodic orbit*
or
(b) the set of nonstationary orbits contained in $L^+(p)$ *is at most countable; then for any nonstationary point* q *contained in* $L^+(p)$ *the sets* $L^+(q)$ *and* $L^-(q)$ *contain only stationary points.*

Poincaré and Bendixson considered only planar systems. The obtained theorems can be in an obvious way adopted for the systems on the sphere S^2. Then the natural question arises about the similar results for the systems on other $2-$dimensional compact manifolds.

A famous example of the system on the torus mentioned in Chapter 1 shows that the Poincaré-Bendixson Theorem in its classical form cannot be generalized for all $2-$dimensional manifolds. However, many questions connected with this problems arose. The systems on $2-$manifolds were investigated by many mathematicians.

Poincaré himself posed the question if for a flow on a torus $\mathbb{T}^2$ given by the analytic function f, the only possible minimal sets are points, periodic trajectories and the whole torus $\mathbb{T}^2$. This was proved in 1932 by Arnaud Denjoy. Denjoy showed

6.3. Theorem. *Assume that* $x' = f(x)$ *($x = (x_1, x_2)$ with a suitable identification) is an autonomous system on the torus* $\mathbb{T}^2$ *where* f *is of class* C^2*. Assume that this system gives a dynamical system. Let* M *be a minimal set for this system. Then either* M *is a stationary point or* M *is homeomorphic to the circle (i.e. is a periodic orbit) or* $M = \mathbb{T}^2$.

This theorem was generalized in 1963 by Arthur J. Schwartz who proved

6.4. Theorem. *Assume that* $x' = f(x)$ *is an autonomous system on a compact, connected* $2-$*dimensional manifold* X *of class* C^2 *where* f *is of class* C^2*. Assume that this system gives a dynamical system. Let* M *be a minimal set for this system. Then either* M *is a stationary point or* M *is homeomorphic to the circle (i.e. is a periodic orbit) or* $M = X$ *(i.e. is the whole manifold); in the last case the manifold* X *must be equal to the two–dimensional torus* $\mathbb{T}^2$.

In 1965 Otomar Hajek gave a characterizaton of sections for planar dynamical systems which helped with the generalization of the Poincaré-Bendixson Theorem for planar dynamical systems. This was done by Hajek in 1968. Now, this follows immediately by the result by Gutierrez (see Chapter 2). However, then this theorem was not known; also, it requires much more advanced results and techniques.

Gutierrez in his mentioned above paper generalized also the theorem of Schwartz.

6.5. Theorem. *Let $(X, \mathbb{R}, \pi)$ be a dynamical system on a compact $2-$dimensional manifold X of class C^∞. Then the following conditions are equivalent:*

(1) $(X, \mathbb{R}, \pi)$ is topologically equivalent to a C^2 dynamical system on X

(2) $(X, \mathbb{R}, \pi)$ is topologically equivalent to a C^∞ dynamical system on X

(3) if M is a minimal set in the dynamical system $(X, \mathbb{R}, \pi)$, then either M is a stationary point or M is homeomorphic to the circle (i.e. is a periodic orbit) or $M = X$ (i.e the whole manifold); in the last case the manifold X must be equal to the two–dimensional torus $\mathbb{T}^2$.

A natural question arises about the Poincaré-Bendixson type properties for semidynamical systems on $\mathbb{R}^2$. The Gutierrez theorem about the topological equivalence says only about $2-$dimensional dynamical systems, not semidynamical systems.

In the following results by a limit set Λ we mean either a positive limit set or a negative limit set given by a negative solution through a non–stationary point p. We investigate a semidynamical system $(\mathbb{R}^2, \mathbb{R}_+, \pi)$.

6.6. Theorem. *If a limit set Λ is connected and does not contain stationary points, then Λ is a single trajectory.*

6.7. Theorem. *If a positive trajectory or negative trajectory is bounded, then either the limit set Λ associated with this orbit is a periodic orbit, or any semi-orbit contained in Λ may contain in its limit set only stationary points.*

In the proofs, except of the existence of sections in semidynamical systems, the following properties play an important role:

– any compact section in a semidynamical system on a $2-$manifold X is either a Jordan arc or a Jordan curve.

– for any non–stationary point y contained in a limit set Λ in a semidynamical system on a $2-$manifold and for any $t > 0$ the set $\Lambda \cap F(t, y)$ has precisely one element.

– if a limit set Λ in a semidynamical system on a $2-$manifold X does not contain any stationary point, then the semidynamical system induced from X on Λ is a system with negative unicity and (after an obvious introducing the values of $\pi(t, x)$ for negative t) gives a dynamical system on Λ.

Also, the continuity properties of the function $F(\cdot, \cdot)$ in semidynamical systems play an important role in the proof.

The above theorems show that the Poincaré-Bendixson Theorem is not only purely topological, but in fact it depends only on the continuous movement defined for positive values of the time variable t. Roughly speaking, “the reason” of this theorem is a possibility of a continuous moving forward and we do not need bother about backward direction.

7. Ważewski Retract Theorem.

One of classical famous result in the theory of dynamical systems is the Ważewski Retract Theorem. It was proved by Tadeusz Ważewski in 1947. In 1960 Solomon Lefschetz presented the opinion that Ważewski's retract method was the most original achievement in the theory of ordinary differential equations since the war. In the next years, on the base of this theorem Charles Conley constructed the topological index and started the whole important theory, now known as the Conley index theory, developed widely in many directions. Here, we present the Ważewski Theorem in one of its simplest form, formulated for dynamical systems.

Let $(X, \mathbb{R}, \pi)$ be dynamical system and $U \subset X$ be nonempty and open. A point $a \in \partial U$ is called an *exit point* if there exists an $\varepsilon > 0$ such that $\pi((-\varepsilon, 0), a) \subset U$. An exit point is called a *strong exit point* if, additionally, there exists an $\varepsilon_1 > 0$ such that $\pi((0, \varepsilon_1), a) \cap \overline{U} = \emptyset$. Denote the set of exit points from U by U_e.

We have

7.1. Theorem. *Assume that in a dynamical system $(X, \mathbb{R}, \pi)$ for a nonempty, open set $U \subset X$ each exit point is simultaneously a strong exit point. If U_e is not a retract of $U \cup U_e$ (for example, U_e is not connected and $U \cup U_e$ is connected) then there is a point $x \in U$ such that $\pi^+(x) \subset U$.*

This theorem also allows to get some information about the system inside a particular domain on the base of the behaviour of the boundary.

As an illustration of the theorem, imagine a football match during Euro 2012 such that the whole stadium is fulfilled by football fans. Seen by a helicopter pilot flying above the stadium, the fans form a single, colourful patch. At the end of the match the fans leave through the stadium gates. If even two fans leave through different gates, then the group must break up; the pilot sees the single patch dissolve into a number of smaller patches as the continuity cannot be fulfilled.

For the proof, suppose to the contrary that for any $x \in U \cup U_e$ there is a $t > 0$ with $\pi(t, x) \notin U$. Define $\tau(x)$ as $\inf\{t \geqslant 0 : \pi(t, x) \notin U\}$. Thus we have constructed the function $\tau : U \cup U_e \to [0, +\infty)$. Note that $\tau(x) = 0$ if and only if $x \in U_e$. If we show that τ is continuous, then the function $U \cup U_e \ni x \mapsto \pi(\tau(x), x) \in U_e$ is a retraction, which is a contradiction. However, the continuity of τ follows from the property that each exit point is simultaneously a strong exit point. If $\tau(x) > 0$ then there is an α such that $\pi((\tau(x), \tau(x) + \alpha), x) \cap \overline{U} = \emptyset$. Then for ε with $\varepsilon < \tau(x)$ and $\varepsilon < \alpha$ we may find a neighbourhood V of x with $\pi([0, \tau(x) - \varepsilon), V) \subset U$ and $\pi([\tau(x) + \varepsilon), V) \cap \overline{U} = \emptyset$. Thus for $y \in V$ we have $\tau(y) \in (\tau(x) - \varepsilon, \tau(x) + \varepsilon)$ and we have proved the continuity of τ at x whereas $\tau(x) > 0$. When $\tau(x) = 0$ the proof is similar but simpler.

One of more general form of the theorem with an analogous proof is

7.2. Theorem. *Assume that in a dynamical system $(X, \mathbb{R}, \pi)$ for a nonempty, open set $U \subset X$ each exit point is simultaneously a strong exit point. Let S be nonempty* *subset of $U \cup U_e$. If $S \cap U_e$ is a retract of U_e and $S \cap U_e$ is not a retract of S, then there is a point $x \in S$ such that $\pi^+(x) \subset U$.*

For systems given by differential equations, it is very frequently not difficult to verify the assumptions of the theorem on the boundary of the domain (for example, the boundary is not connected and all the trajectories "strongly exit" from the domain). Then we get the existence of a solution remaining in the domain.

8. Appendix – some biographical notes.

In the lecture, some outstanding Kraków mathematicians, especially those who obtained significant results in the theory of differential equations were mentioned. Below, there are very short notes about them. They are listed in the order given by the dates of their deaths.

Ważewski was a pupil of Zaremba, however, he passed Ph.D. exam in Paris at Sorbona. All other mathematicians mentioned below wrote Ph.D. papers under the supervision of Ważewski.

All of them (except Turowicz who is buried in Tyniec monastery) are buried in Kraków Rakowicki cemetary.

In Kraków, there are streets named by Zaremba, Ważewski, Opial, Szarski, Łojasiewicz. The new building of the Mathematics Department of the Jagiellonian University is at Łojasiewicza street. There are some efforts to give a name of Pelczar to one street in Kraków (at the new university campus area).

Stanisław ZAREMBA (1863-1942) is regarded as the best Polish mathematician at the end of the 19th century and first decades of the 20th century. An author of many outstanding results, mainly in partial differential equations and the applications of mathematics. He made a great influence to the development of Polish mathematics in the beginning of the 20th century and the creation of the strong mathematical centre at the Jagiellonian Univeristy. He spent the years 1886-1900 in Paris, he got Ph.D from Sorbona in 1889. In 1900 he got a Chair at the Jagiellonian University. The first President of the Polish Mathematical Society.

Tadeusz WAŻEWSKI (1896-1972) is regarded as the creator of Kraków School of differential equations. He was a man who played the crucial role on the rebuilding Kraków's mathematics after the Second World War. The President of the Polish Mathematical Society. A member of the Editorial Board of *Journal of Differential Equations* since the publication of the first volume of the journal in 1965 until his death. His mathematical research started from topology (the Ph.D. dissertation concerned dendrites), then he turned to diffferential equations.

Zdzisław OPIAL (1930-1974) was a mathematician with a very broad mathematical interest, an author of very outstanding results in differential equations. He worked also in the history of mathematics and was very active in many aspects of mathematical education. An author of very modern (for those times) textbook on algebra (with many re-editions) and some books about mathematics for young people. The so-called Opial inequality is now a base for many further results.

Jacek SZARSKI (1921-1980) was an author of many important papers on differential equations and differential inequalities, an author of a fundamental textbook on differential inequalities. Probably he was the youngest recipient of Ph.D. in mathematics from the Jagiellonian Univeristy ever. He headed mathematics at the Jagiellonian University in 1957–1977. The President of the Polish Mathematical Society.

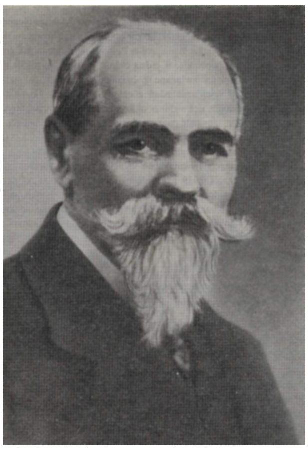
Andrzej TUROWICZ (1904-1989) was a mathematician and a Benedictine and a priest (he took holy orders after the Second World War) and being a priest he lectured mathematics to students. He was also a member of staff of the Mathematical Institute of the Polish Academy of Science. Before the war, he spend some years in Lvov. He was the first person who obtained the official Master's diplomma in mathematics from the Jagiellonian University. He worked in many areas of mathematics, including differential equations, numerical analysis, game theory, logic, control theory, functional analysis and probability.

Andrzej PLIŚ (1929-1991) was one of the most outstanding world specialists in differential equations, his several results led to new directions of research. Known particularly from the creation of many sophisticated and original counterexamples. A member of the Editorial Board of *Journal of Differential Equations* since the publication of the fourth volume of the journal in 1968 until his death.

Stanisław ŁOJASIEWICZ (1926-2002) was one of the most outstanding Polish mathematicians of the second half of the 20th century. The author of the solution of the fundamental problem of the division of distributions by analytic sets, which led to the creation of a new branch of mathematics, i.e. semianalytical geometry. It is said that he was a very strong candidate to obtain the Fields Medal. A member of the Pontifical Academy of Sciences. An author of some advanced texbooks in mathematics.

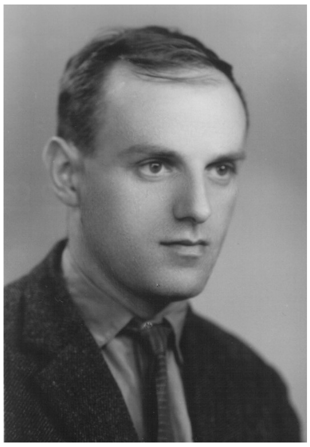
Andrzej LASOTA (1932-2006) was a professor at the universities in Kraków, Katowice and Lublin and made a great influence to the development of mathematics in those universities. An author of outstanding results in differential equations, probability, ergodic theory and fractals. He worked very actively in the applications of mathematics. Together with Maria Ważewska-Czyżewska (the daughter of Ważewski) from Medical University in Kraków he gave a mathematical model of the process of reproductions of blood cells.

Andrzej PELCZAR (1937-2010) specialized in differential equations and dynamical systems, in recent years of his life he worked actively also in history of mathematics. An author of many books on differential equations and dynamical systems. The rector of the Jagiellonian University, the President of the Polish Mathematical Society, a Vice-President of the European Mathematical Society, the President of the Polish Council of Science and Higher Education.



S.Zaremba

T.Ważewski

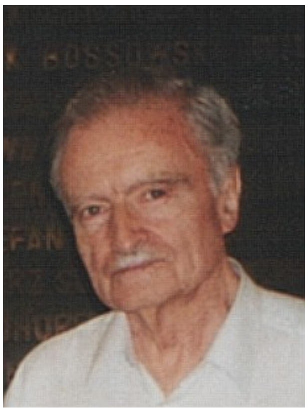


Z.Opial

J.Szarski

A.Turowicz

A.Pliś



S.Łojasiewicz

A.Lasota

A.Pelczar

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