FOURIER ANALYSIS METHODS FOR EVOLUTIONARY PARTIAL DIFFERENTIAL EQUATIONS.

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Abstract. Fourier analysis methods have known a growing importance recently in the study of linear and nonlinear PDE's. In particular, techniques based on Littlewood-Paley decomposition and paradifferential calculus have proved to be very efficient for tackling problems in the whole space or in the torus.

In these lectures, we aim at presenting those techniques, having in mind to solve classical systems of equations arising in fluid mechanics. We shall in particular prove the existence and uniqueness of strong solutions for the incompressible and the compressible Navier-Stokes equations in so-called critical functional spaces. In passing, we shall provide a bunch of useful estimates for the linear heat and transport equations.

The first part of those notes (that is the first two sections) presents rather classical matter : Fourier transform, Sobolev spaces, functional analysis, with applications to the incompressible Navier-Stokes equations.

In the second part (sections 3 to 5), we present more elaborate tools with applications to more complicated systems or situations.

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1. Reminders on Fourier transform and Sobolev spaces

1.1. The Fourier transform.

1.1.1. *The Fourier transform on L^1.* The *Fourier transform* is defined on $L^1(\mathbb{R}^d)$ by

$$\mathcal{F}f(\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^d}e^{-i(x|\xi)}f(x)\,dx \tag{1}$$

where $(x|\xi)$ denotes the *inner product* on $\mathbb{R}^d$. It is a continuous linear map from $L^1(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$ because obviously, $|\widehat{f}(\xi)|\leq\|f\|_{L^1}$. It is also clear that for any function $\phi\in L^1$ and automorphism L over $\mathbb{R}^d$, we have

$$\mathcal{F}(\varphi\circ L)=\frac{1}{|\det L|}\widehat{\varphi}\circ{}^tL^{-1} \tag{2}$$

and that

$$\forall(f,g)\in L^1(\mathbb{R}^d)\times L^1(\mathbb{R}^d),\ \mathcal{F}(f\star g)=\mathcal{F}f\,\mathcal{F}g. \tag{3}$$

For any L^1 function with Fourier transform in L^1, the following *inversion formula* is available:

$$\forall x\in\mathbb{R}^d,\ f(x)=\frac{1}{(2\pi)^d}\int e^{i(x|\xi)}\mathcal{F}f(\xi)\,d\xi=\frac{1}{(2\pi)^d}\check{\mathcal{F}}\mathcal{F}f(x). \tag{4}$$

1.1.2. *The Schwartz space, and its dual: the space of tempered distributions.* The *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ is the basic set for extending the Fourier transform to a very large class of distributions over $\mathbb{R}^d$. Let us introduce first the following notation. If α is a *multi-index* (i.e. an element of $\mathbb{N}^d$), x an element of $\mathbb{R}^d$ and f a smooth function of $\mathbb{R}^d$, then the *length* $|\alpha|$ of α is defined by $|\alpha|\stackrel{\text{def}}{=}\alpha_1+\cdots+\alpha_d$. Moreover we state $\partial^\alpha f\stackrel{\text{def}}{=}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}f$ and $x^\alpha\stackrel{\text{def}}{=}x^{\alpha_1}\cdots x^{\alpha_d}$.

Definition 1.1. *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of smooth functions u over $\mathbb{R}^d$ such that, for any $k\in\mathbb{N}$ we have*

$$\|u\|_{k,\mathcal{S}}\stackrel{def}{=}\sup_{\substack{|\alpha|\leq k\\ x\in\mathbb{R}^d}}(1+|x|)^k|\partial^\alpha u(x)|<\infty.$$

Equipped with the family of semi-norms $(\|\cdot\|_{k,\mathcal{S}})_{k\in\mathbb{N}}$, the set $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space and the space $\mathcal{D}(\mathbb{R}^d)$ of smooth compactly supported functions on $\mathbb{R}^d$ is dense in $\mathcal{S}(\mathbb{R}^d)$.

Proposition 1.1. *For any (u,θ) in $\mathcal{S}\times\mathcal{S}$, $\lambda\in\mathbb{R}\setminus\{0\}$ and $(a,\omega)\in\mathbb{R}^d\times\mathbb{R}^d$, we have*[1]

$$(i\partial)^\alpha\widehat{u}=\mathcal{F}(x^\alpha u)\,,\quad (i\xi)^\alpha\widehat{u}=\mathcal{F}(\partial^\alpha u)\,, \tag{5}$$

$$e^{-i(a|\xi)}\widehat{u}=\mathcal{F}(\tau_a f)\,,\quad \tau_\omega\widehat{f}=\mathcal{F}(e^{i(x|\omega)}f)\,,\quad \lambda^{-d}\widehat{f}(\lambda^{-1}\xi)=\mathcal{F}(f(\lambda x)) \tag{6}$$

$$\text{and}\quad \mathcal{F}(u\star\theta)=\widehat{\theta}\,\widehat{u}. \tag{7}$$

[1]Below, the notation τ_a stands for the *translation operator* $\tau_a:f\mapsto f(\cdot-a)$.

Theorem 1.1. *The Fourier transform maps continuously $\mathcal{S}$ into $\mathcal{S}$: for any integer k, a constant C and an integer N exist such that*

$$\forall \phi \in \mathcal{S}\,,\ \|\widehat{\phi}\|_{k,\mathcal{S}} \leq C\|\phi\|_{N,\mathcal{S}}.$$

Proof. Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with length k. According to the above proposition, we have for any ϕ in $\mathcal{S}$,

$$(i\partial)^\alpha \widehat{f}(\xi) = \mathcal{F}(x^\alpha \phi)(\xi) \quad \text{and} \quad (i\xi)^\alpha \widehat{\phi}(\xi) = \mathcal{F}(\partial^\alpha \phi)(\xi). \tag{8}$$

From this, we deduce that

$$\begin{aligned} \left|\xi^\beta \partial^\alpha \widehat{\phi}(\xi)\right| &\leq \left|\mathcal{F}(\partial^\beta(x^\alpha\phi))(\xi)\right|, \\ &\leq \|\partial^\beta(x^\alpha\phi)\|_{L^1}, \\ &\leq c_d\|(1+|x|)^{d+1}\partial^\beta(x^\alpha\phi)\|_{L^\infty}. \end{aligned}$$

Hence, by definition of the semi-norms, we have $\|\widehat{\phi}\|_{k,\mathcal{S}} \leq C\|\phi\|_{k+d+1,\mathcal{S}}$. □

Definition 1.2. *A tempered distribution over $\mathbb{R}^d$ is any continuous linear functional*[2] *on $\mathcal{S}(\mathbb{R}^d)$. The set of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$.*

A sequence $(u_n)_{n\in\mathbb{N}}$ of tempered distributions is said to converge to u in $\mathcal{S}'(\mathbb{R}^d)$ if

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d)\,,\ \lim_{n\to\infty} \langle u_n, \phi\rangle = \langle u, \phi\rangle.$$

Remark 1.1. *The link with distributions on $\mathbb{R}^d$ is the following: if T is a distribution on $\mathbb{R}^d$ such that for some integer k and positive real C we have*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d)\,,\ |\langle T, \varphi\rangle| \leq C\|\varphi\|_{k,\mathcal{S}},$$

then, as $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, the linear functional T may be uniquely extended into a continuous linear functional. Moreover, if T belongs to $\mathcal{S}'(\mathbb{R}^d)$ then the restriction of T on $\mathcal{D}(\mathbb{R}^d)$ defines a distribution on $\mathbb{R}^d$ because, for any positive R and for any function φ in $\mathcal{D}(B(0,R))$,

$$|\langle T, \varphi\rangle| \leq C\|\varphi\|_{k,\mathcal{S}} \leq C(1+R)^k \sup_{|\alpha|\leq k} \|\partial^\alpha \varphi\|_{L^\infty}. \tag{9}$$

Thus the set of distributions T on $\mathbb{R}^d$ which satisfy (9) *may be identified with $\mathcal{S}'(\mathbb{R}^d)$.*

L. Schwartz' idea of duality to define operators on the space of tempered distributions, is based on the following proposition.

Proposition 1.2. *Let A be a linear continuous map from $\mathcal{S}$ into*[3] *$\mathcal{S}$. Then the formula*

$$\langle {}^tAu, \phi\rangle \stackrel{def}{=} \langle u, A\phi\rangle$$

defines a tempered distribution. Moreover, ${}^tA$ is linear and continuous in the sense that, if $(u_n)_{n\in\mathbb{N}}$ is a sequence of distributions which converges to u in $\mathcal{S}'(\mathbb{R}^d)$, then $({}^tAu_n)_{n\in\mathbb{N}}$ converges to ${}^tAu$.

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[2]That is, u is a tempered distribution if there exist a constant C and an integer k such that $|\langle u, \phi\rangle| \leq C\|\phi\|_{k,\mathcal{S}}$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

[3]That is, for any integer k, a constant C and an integer N exist such that $\|A\phi\|_{k,\mathcal{S}} \leq C\|\phi\|_{N,\mathcal{S}}$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

In order to extend the definition of the Fourier transform to tempered distributions, let us consider a L^1 function f. By Fubini's theorem and by definition of the Fourier transform on L^1, we have, for all $\phi \in \mathcal{S}$,

$$\begin{aligned} \langle {}^t\mathcal{F} f, \phi\rangle &= \int_{\mathbb{R}^d} f(x)\widehat{\phi}(x)\,dx \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)e^{-i(x|\xi)}\phi(\xi)\,dx\,d\xi \\ &= \langle \widehat{f}, \phi\rangle. \end{aligned}$$

In other words, operator ${}^t\mathcal{F}$ restricted to L^1 functions coincides with the Fourier transform of functions. It will be thus also denoted by $\mathcal{F}$ in all that follows.

Remark 1.2. *The formulae in Proposition 1.1 extend to any u in $\mathcal{S}'$.*

1.1.3. *The Fourier transform on L^2.* By using the density of $\mathcal{S}$ in L^2 (the proof of which is left as an exercise), and the inversion formula, one may extend the definition of $\mathcal{F}$ to L^2, as an isometry from L^2 to itself, up to some explicit constant:

Theorem 1.2 (Fourier-Plancherel)**.** *The Fourier transform is an automorphism of L^2 with inverse $(2\pi)^{-d}\check{\mathcal{F}}$. Moreover, for any couple (f,g) of functions in L^2, we have the following* Parseval formula*:*

$$\int \widehat{f}\,\overline{\widehat{g}}\,d\xi = (2\pi)^d \int f\,\overline{g}\,dx.$$

Proof. On $\mathcal{S}$, we have $\mathcal{F}\check{\mathcal{F}} = \check{\mathcal{F}}\mathcal{F} = (2\pi)^d\mathrm{Id}$. Next, using that for any function φ in L^1, we have $\overline{\mathcal{F}\varphi} = \check{\mathcal{F}}(\overline{\varphi})$ and taking advantage of inverse Fourier formula, we get for any function φ in $\mathcal{S}$,

$$\|\mathcal{F}\varphi\|_{L^2}^2 = \langle \mathcal{F}\varphi, \overline{\mathcal{F}\varphi}\rangle = \langle \varphi, \mathcal{F}\check{\mathcal{F}}\overline{\varphi}\rangle = (2\pi)^d\|\varphi\|_{L^2}^2.$$

Using the density of $\mathcal{S}$ in L^2 enables us to conclude the proof. □

1.2. **Sobolev spaces.**

1.2.1. *nonhomogeneous Sobolev spaces.*

Definition 1.3 (**Espace de Sobolev** $H^s(\mathbb{R}^d)$)**.** *Soit s un réel, on dit qu'une distribution tempérée u appartient à* l'espace de Sobolev *d'indice s, noté $H^s(\mathbb{R}^d)$ (ou simplement H^s en l'absence d'ambiguïté) si*

$$\widehat{u}(\xi) \in L^2(\mathbb{R}^d; (1+|\xi|^2)^s\,d\xi).$$

On pose alors

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s|\widehat{u}(\xi)|^2\,d\xi\right)^{\frac12}. \tag{10}$$

On adoptera par la suite la notation du *crochet japonais*:

$$\langle\xi\rangle \stackrel{\text{def}}{=} \sqrt{1+|\xi|^2}.$$

Proposition 1.3. *Pour tout réel s, l'ensemble H^s muni de la norme $\|\cdot\|_{H^s}$ est un espace de Hilbert.*

Proof. Le fait que l'ensemble H^s soit un espace vectoriel résulte de l'inégalité

$$|a+b|^2 \leq 2|a|^2 + 2|b|^2.$$

Il est alors facile de vérifier que $\|\cdot\|_{H^s}$ est bien une norme et provient du produit scalaire

$$(u|v)_{H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi\cdot$$

Démontrons que H^s est complet. Soit $(u_n)_{n\in\mathbb{N}}$ une suite de Cauchy de H^s. Par définition de la norme, la suite $(\widehat{u}_n)_{n\in\mathbb{N}}$ est de Cauchy dans l'espace de Hilbert $L^2(\mathbb{R}^d;\langle\xi\rangle^{2s}\, d\xi)$. Donc il existe une fonction $\widetilde{u}$ appartenant à l'espace $L^2(\mathbb{R}^d;\langle\xi\rangle^{2s}\, d\xi)$ telle que

$$\lim_{n\to\infty} \|\widehat{u}_n - \widetilde{u}\|_{L^2(\mathbb{R}^d;\langle\xi\rangle^{2s}\, d\xi)} = 0. \tag{11}$$

En particulier, la suite $(\widehat{u}_n)_{n\in\mathbb{N}}$ tend vers $\widetilde{u}$ dans l'espace $\mathcal{S}'$ des distributions tempérées. Soit $u = \mathcal{F}^{-1}\widetilde{u}$. Comme la transformée de Fourier est un isomorphisme de $\mathcal{S}'$ dans lui-même, la suite $(u_n)_{n\in\mathbb{N}}$ tend vers u dans l'espace $\mathcal{S}'$, mais aussi dans H^s d'après (11). □

L'échelle de Sobolev mesure la décroissance de la transformée de Fourier de u, et donc de la régularité de u en vertu de (5). La proposition suivante établit ce lien de manière plus explicite pour s entier.

Proposition 1.4. *Si m un entier positif, alors $H^m(\mathbb{R}^d)$ coïncide avec l'espace vectoriel des fonctions u de L^2 dont toutes les dérivées d'ordre inférieur ou égal à m sont des distributions appartenant à L^2. De plus,*

$$\widetilde{\|u\|}_{H^m} \stackrel{def}{=} \sqrt{\sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^2}^2}$$

est une norme hilbertienne sur H^m, équivalente à la norme $\|\cdot\|_{H^m}$.

Proof. Le fait que

$$\widetilde{\|u\|}_{H^m}^2 = \widetilde{(u|u)}_{H^m} \quad \text{avec} \quad \widetilde{(u|v)}_{H^m} \stackrel{\text{def}}{=} \sum_{|\alpha|\leq m} \int_{\mathbb{R}^d} \partial^\alpha u(x)\overline{\partial^\alpha v(x)}\, dx$$

assure que la norme $\widetilde{\|\cdot\|}_{H^m}$ provient d'un produit scalaire. De plus, il existe une constante C telle que

$$\forall \xi\in\mathbb{R}^d,\ C^{-1}\sum_{|\alpha|\leq m} |\xi|^{2|\alpha|} \leq \langle\xi\rangle^{2m} \leq C\sum_{|\alpha|\leq m} |\xi|^{2|\alpha|}. \tag{12}$$

Notons que (5) implique que pour tout multi-indice α de $\mathbb{N}^d$ on a

$$\partial^\alpha u\in L^2 \Longleftrightarrow \xi^\alpha\widehat{u}\in L^2.$$

Donc, on en déduit que

$$u\in H^m \Longleftrightarrow \forall|\alpha|\leq m\,,\ \partial^\alpha u\in L^2.$$

Enfin, l'inégalité(12) assure l'équivalence des normes puisque la transformation de Fourier est une isométrie L^2 -à une constante près. □

Nous collectons dans la proposition suivante certaines propriétés standard des espaces de Sobolev:

Proposition 1.5. *Soit s un réel quelconque.*

i) *L'espace $\mathcal{D}(\mathbb{R}^d)$ des fonctions test est dense dans $H^s(\mathbb{R}^d)$.*

ii) *Pour $s<t$, $H^t\subset H^s$ et on a l'inégalité d'interpolation:*

$$\forall\theta\in[0,1],\quad \|u\|_{H^{\theta s+(1-\theta)t}}\leq\|u\|_{H^s}^{\theta}\|u\|_{H^t}^{1-\theta}. \tag{13}$$

iii) *La multiplication par une fonction de $\mathcal{S}$ est une fonction continue de H^s dans lui-même.*

Remark 1.3. *Par i), on aurait ainsi pu définir $H^s(\mathbb{R}^d)$ comme le complété hilbertien de $\mathcal{D}(\mathbb{R}^d)$ pour la norme* (10)*.*

L'espace H^s étant un espace de Hilbert, il est isomorphe à son dual topologique $(H^s)'$ via le produit scalaire sur H^s par le théorème de représentation de Riesz-Fréchet. Utiliser plutôt le produit scalaire L^2 pour caractériser les éléments de $(H^s)'$ et les théorèmes de prolongement des applications linéaires continues conduit au résultat suivant :

Proposition 1.6. *Soit $s\in\mathbb{R}$ et $f\in\mathcal{S}'$ telle que $\widehat{f}\in L^2_{loc}(\mathbb{R}^d)$. Alors $f\in H^{-s}$ si et seulement si*

$$M_f\stackrel{def}{=}\sup_{\substack{\varphi\in\mathcal{S}\\ \|\varphi\|_{H^s}\leq 1}}\big|\langle f,\varphi\rangle_{\mathcal{S}'\times\mathcal{S}}\big|<\infty.$$

De plus, pour $f\in H^{-s}$, la forme linéaire L_f définie sur $\mathcal{S}$ par $L_f(\varphi)=\langle f,\varphi\rangle_{\mathcal{S}'\times\mathcal{S}}$ se prolonge sur H^s en une forme linéaire continue $\langle f,\cdot\rangle_{H^{-s}\times H^s}$ et l'on a

$$\|f\|_{H^{-s}}=(2\pi)^d\sup_{\substack{\varphi\in\mathcal{S}\\ \|\varphi\|_{H^s}\leq 1}}\big|\langle f,\varphi\rangle_{\mathcal{S}'\times\mathcal{S}}\big|=(2\pi)^d\sup_{\substack{\varphi\in H^s\\ \|\varphi\|_{H^s}\leq 1}}\big|\langle f,\varphi\rangle_{H^{-s}\times H^s}\big|.$$

Enfin, l'application $f\mapsto(2\pi)^d\langle f,\cdot\rangle_{H^{-s}\times H^s}$ est un isomorphisme isométrique de H^{-s} dans $(H^s)'$.

1.2.2. *Homogeneous Sobolev spaces.*

Definition 1.4. *Let s be in $\mathbb{R}$. The homogeneous Sobolev space $\dot H^s(\mathbb{R}^d)$ (also denoted by $\dot H^s$) is the set of tempered distributions u over $\mathbb{R}^d$ with $\widehat{u}$ in $L^1_{loc}(\mathbb{R}^d)$ and*

$$\|u\|^2_{\dot H^s}\stackrel{def}{=}\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{u}(\xi)|^2\,d\xi<\infty.$$

Spaces $\dot H^s$ and $\dot H^{s'}$ *cannot* be compared for the inclusion. Nevertheless, we have the following proposition:

Proposition 1.7. *Let $s_0\leq s\leq s_1$. Then $\dot H^{s_0}\cap\dot H^{s_1}$ is included in $\dot H^s$ and we have*

$$\|u\|_{\dot H^s}\leq\|u\|_{\dot H^{s_0}}^{1-\theta}\|u\|_{\dot H^{s_1}}^{\theta}\quad\textit{with}\quad s=(1-\theta)s_0+\theta s_1.$$

Using Parseval formula, we observe that $L^2=\dot H^0$ and that if s is a positive integer then $\dot H^s$ is the subset of tempered distributions with locally integrable Fourier transforms and such that $\partial^\alpha u$ belongs to L^2 for all α in $\mathbb{N}^d$ of length s.

In the case when s is a negative integer, the Sobolev space $\dot H^s$ is described by the following proposition.

Proposition 1.8. *Let k be a positive integer. The space $\dot H^{-k}(\mathbb{R}^d)$ is the set of distributions which are the sum of derivatives of order k of $L^2(\mathbb{R}^d)$ functions.*

Proposition 1.9. *The space $\dot H^s(\mathbb{R}^d)$ is a Hilbert space if and only if $s<\dfrac{d}{2}$.*

Proposition 1.10. *If $s < d/2$ then the space $\mathcal{S}_0(\mathbb{R}^d)$ of functions of $\mathcal{S}(\mathbb{R}^d)$ the Fourier transform of which vanishes near the origin is* dense *in $\dot{H}^s$.*

Proof. Let us consider u in $\dot{H}^s$ such that

$$\forall \phi \in \mathcal{S}_0(\mathbb{R}^d)\,,\ (u|\phi)_{\dot{H}^s} = \int_{\mathbb{R}^d} |\xi|^{2s} \widehat{u}(\xi)\overline{\widehat{\phi}(\xi)}\, d\xi = 0.$$

This implies that the L^1_{loc} function $\widehat{u}$ vanishes on $\mathbb{R}^d \setminus \{0\}$. Thus $\widehat{u} = 0$, whence $u = 0$. As we are in the case when $\dot{H}^s$ is a Hilbert space, we deduce that $\mathcal{S}_0(\mathbb{R}^d)$ is dense in $\dot{H}^s$. □

Proposition 1.11. *If $|s| < d/2$ then the bilinear functional*

$$\mathcal{B} : \left\{ \begin{array}{ccc} \mathcal{S}_0 \times \mathcal{S}_0 & \to & \mathbb{C} \\ (\phi, \varphi) & \mapsto & \displaystyle\int_{\mathbb{R}^d} \phi(x)\varphi(x)\, dx \end{array} \right.$$

can be extended in a continuous bilinear functional on $\dot{H}^{-s} \times \dot{H}^s$. Moreover, if L is a continuous linear functional on $\dot{H}^s$, a unique tempered distribution u exists in $\dot{H}^{-s}$ such that

$$\forall \phi \in \dot{H}^s\,,\ \langle L, \phi \rangle = \mathcal{B}(u, \phi) \quad \textit{and} \quad \|L\|_{(\dot{H}^s)'} = \|u\|_{\dot{H}^{-s}}.$$

1.2.3. *Sobolev embedding.* In this paragraph, we want to compare Lebesgue and Sobolev spaces. We observe that both family of spaces have some invariance properties with respect to dilation of the space variable. More precisely, if v is a smooth function over $\mathbb{R}^d$, and v_λ stands for the function $v_\lambda(x) \stackrel{\text{def}}{=} v(\lambda x)$ then we have

$$\|v_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}} \|v\|_{L^p} \quad \text{and} \quad \|v_\lambda\|_{\dot{H}^s} = \lambda^{-\frac{d}{2}+s} \|v\|_{\dot{H}^s}.$$

Now, if an inequality of the type

$$\|v\|_{L^p} \leq C \|v\|_{\dot{H}^s}$$

is true for any smooth function v, it is also true for v_λ for any λ. Hence we must have

$$-s + \frac{d}{2} = \frac{d}{p}, \quad \text{that is} \quad p = \frac{2d}{d-2s} \in [2, \infty[. \tag{14}$$

Below we state that this inequality is indeed true if p and s are related through (14).

Lemma 1.1 (Injection de Sobolev homogène). *Soit $0 < s < \frac{d}{2}$ et p_c donné par* (14). *Alors:*

$$\forall f \in \mathcal{S}(\mathbb{R}^d), \quad \|f\|_{L^{p_c}(\mathbb{R}^d)} \leq C_s \|f\|_{\dot{H}^s(\mathbb{R}^d)} \tag{15}$$

où la norme de Sobolev homogène est donnée par:

$$\|f\|_{\dot{H}^s} \stackrel{def}{=} \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2\, d\xi \right)^{\frac{1}{2}}.$$

Proof. La démonstration de (15) repose sur la méthode dite d'*interpolation réelle* dont la portée dépasse largement le cadre de ce cours, et qui est une première illustration de la puissance de l'analyse de Fourier et des découpages hautes/basses fréquences.

Découpons donc f en basses et hautes fréquences en posant

$$f = f_{1,A} + f_{2,A} \quad \text{avec} \quad f_{1,A} = \mathcal{F}^{-1}(\mathbf{1}_{B(0,A)} \widehat{f}) \quad \text{et} \quad f_{2,A} = \mathcal{F}^{-1}(\mathbf{1}_{^cB(0,A)} \widehat{f}), \tag{16}$$

où la fréquence de coupure A est pour l'instant un paramètre libre.

Comme le support de la transformée de Fourier de $f_{1,A}$ est compact, la fonction $f_{1,A}$ est bornée et, plus précisément,

$$\begin{aligned} \|f_{1,A}\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{f_{1,A}}\|_{L^1} \leq (2\pi)^{-d}\int_{B(0,A)} |\xi|^{-s}|\xi|^s|\widehat{f}(\xi)|\,d\xi \\ &\leq (2\pi)^{-d}\biggl(\int_{B(0,A)} |\xi|^{-2s}\,d\xi\biggr)^{\frac12}\|f\|_{\dot H^s} \leq C_sA^{\frac d2-s}\|f\|_{\dot H^s}. \end{aligned} \tag{17}$$

Or, d'après l'inégalité triangulaire, on a, pour tout réel strictement positif A,

$$\{|f|>\lambda\}\subset\{|f_{1,A}|>\lambda/2\}\cup\{|f_{2,A}|>\lambda/2\}.$$

L'inégalité (17) ci-dessus implique que

$$A\leq A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda}{4C_s}\right)^{\frac pd} \Longrightarrow \left|\left\{|f_{1,A}|>\frac{\lambda}{2}\right\}\right|=0.$$

On en déduit donc par le principe de Cavalieri:

$$\|f\|_{L^p}^p\leq p\int_0^\infty \lambda^{p-1}\left|\left\{|f_{2,A_\lambda}|>\frac\lambda2\right\}\right|d\lambda.$$

Afin de majorer l'intégrand, nous allons faire intervenir une norme L^2. Pour cela, on écrit (c'est l'inégalité de Bienaymé-Tchebychev) que

$$\left|\left\{|f_{2,A_\lambda}|>\frac\lambda2\right\}\right| = \int_{\{|f_{2,A_\lambda}|>\frac\lambda2\}} dx \leq \int_{\{|f_{2,A_\lambda}|>\frac\lambda2\}} \frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2}\,dx\leq 4\frac{\|f_{2,A_\lambda}\|_{L^2}^2}{\lambda^2}.$$

Il en résulte donc que l'on a

$$\|f\|_{L^p}^p\leq 4p\int_0^\infty\lambda^{p-3}\|f_{2,A_\lambda}\|_{L^2}^2\,d\lambda. \tag{18}$$

Mais, on sait que la transformée de Fourier est (à une constante près) une isométrie de L^2, donc:

$$\|f_{2,A_\lambda}\|_{L^2}^2=(2\pi)^{-d}\int_{\{|\xi|\geq A_\lambda\}}|\widehat f(\xi)|^2\,d\xi.$$

D'après l'inégalité(18), il vient

$$\|f\|_{L^p}^p\leq 4p(2\pi)^{-d}\int_{\mathbb{R}_+\times\mathbb{R}^d}\lambda^{p-3}\mathbf{1}_{\{(\lambda,\xi)\,/\,\,|\xi|\geq A_\lambda\}}(\lambda,\xi)|\widehat f(\xi)|^2\,d\xi\,d\lambda.$$

Or, par définition de A_λ, on a

$$|\xi|\geq A_\lambda\Longleftrightarrow\lambda\leq C_\xi\stackrel{\text{def}}{=}4C_s|\xi|^{\frac dp}.$$

D'après le théorème de Fubini, on a pour $p>2$,

$$\begin{aligned}\|f\|_{L^p}^p &\leq 4p(2\pi)^{-d}\int_{\mathbb{R}^d}\biggl(\int_0^{C_\xi}\lambda^{p-3}\,d\lambda\biggr)|\widehat f(\xi)|^2\,d\xi\\ &\leq 4\frac{p(2\pi)^d}{p-2}(4C_s)^{p-2}\int_{\mathbb{R}^d}|\xi|^{\frac{d(p-2)}{p}}|\widehat f(\xi)|^2\,d\xi.\end{aligned}$$

Comme $2s=\dfrac{d(p-2)}{p}$, (15) est démontré. □

Theorem 1.3 (**Injection de Sobolev**). *Soit s un nombre réel positif.*

i) *Si* $s > \frac{d}{2}$ *alors* $H^s(\mathbb{R}^d)$ *est une algèbre qui s'injecte continûment dans l'espace* $\mathcal{C}_0(\mathbb{R}^d)$ *des fonctions continues qui tendent vers 0 à l'infini.*

ii) *Si* $0 \leq s < \frac{d}{2}$, *soit l'exposant critique* p_c *défini par (14). Alors* $\dot{H}^s(\mathbb{R}^d)$ *s'injecte continûment dans* $L^{p_c}(\mathbb{R}^d)$ *et, pour tout* $p \in [2, p_c]$, $H^s(\mathbb{R}^d)$ *s'injecte continûment dans* $L^p(\mathbb{R}^d)$.

iii) *Pour* $s = \frac{d}{2}$, $H^s(\mathbb{R}^d)$ *s'injecte continûment dans* $L^p(\mathbb{R}^d)$ *pour tout* $2 \leq p < \infty$.

Proof. Pour le point *i*), on raisonne comme suit. Comme $s > \frac{d}{2}$, la fonction $\langle \cdot \rangle^{-s}$ appartient à L^2. Or, d'après l'inégalité de Cauchy-Schwarz, on a

$$\|\widehat{u}\|_{L^1} \leq \left(\int \langle \xi \rangle^{-2s} \, d\xi \right)^{\frac{1}{2}} \left(\int \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \leq C\|u\|_{H^s}.$$

On conclut maintenant grâce à la formule d'inversion de Fourier et en utilisant le fait que la transformée de Fourier d'une fonction L^1 est bornée et continue (par convergence dominée) et tend vers 0 à l'infini (en vertu du lemme de Riemann-Lebesgue). Pour la démonstration du fait que H^s est une algèbre, on renvoie à la suite du cours.

Les points *ii*) et *iii*) du théorème ci-dessus sont des conséquences immédiates de (15). En effet, si $0 < s < \frac{d}{2}$ alors (15) assure la continuité de l'injection de H^s dans L^{p_c}. Or $H^s \subset L^2$ par définition, et donc par Hölder, $H^s \subset L^p$ pour tout $p \in [2, p_c]$.

Pour $s = \frac{d}{2}$, soit $2 \leq p < +\infty$, alors

$$\sigma = \frac{d}{2} - \frac{d}{p} \;\text{ vérifie }\; 0 \leq \sigma < \frac{d}{2} = s \;\text{ et }\; p_c(\sigma) = p$$

et donc

$$H^s(\mathbb{R}^d) \subset H^\sigma(\mathbb{R}^d) \subset L^p(\mathbb{R}^d).$$

□

Theorem 1.4 (Injection duale de Sobolev). *Si* $p \in (1, 2]$ *alors* $L^p(\mathbb{R}^d)$ *s'injecte continûment dans* $H^{-s}(\mathbb{R}^d)$ *avec* $s = d/p - d/2$.

Proof. Par densité, il suffit de démontrer l'existence d'une constante C telle que pour tout $u \in \mathcal{S}$, on ait

$$\|u\|_{H^{-s}} \leq C\|u\|_{L^p}. \tag{19}$$

D'après la proposition 1.6,

$$\|u\|_{H^{-s}} = (2\pi)^d \sup_{\substack{\varphi \in \mathcal{S} \\ \|\varphi\|_{H^s} \leq 1}} \int u\varphi \, dx.$$

Donc d'après l'inégalité de Hölder,

$$\|u\|_{H^{-s}} \leq (2\pi)^d \sup_{\substack{\varphi \in \mathcal{S} \\ \|\varphi\|_{H^s} \leq 1}} \|u\|_{L^p} \|\varphi\|_{L^{p'}}.$$

Mais comme $d/p' = d/2 - s$, les injections de Sobolev ci-dessus assurent l'existence d'une constante C telle que pour toute fonction φ de $\mathcal{S}$ on ait

$$\|\varphi\|_{L^{p'}} \leq C\|\varphi\|_{H^s}.$$

Comme $\mathcal{S}$ est dense dans H^s, cela achève la démonstration. □

Remark 1.4. *Vu que* (15) *ne met en jeu que des normes de Sobolev* homogènes*, on peut remplacer* $\|u\|_{H^{-s}}$ *par* $\|u\|_{\dot H^{-s}}$ *dans* (19)*.*

Le deuxième corollaire concerne les inégalités d'interpolation de Gagliardo-Nirenberg qui joueront un rôle fondamental dans notre étude des problèmes de minimisation en dimension infinie.

Corollary 1.1 (Inégalité d'interpolation de Gagliardo-Nirenberg). *Soit*[4]

$$2^* = \begin{cases} +\infty & \text{pour } d = 1,2, \\ \frac{2d}{d-2} & \text{pour } d \geq 3, \end{cases}.$$

Soit $2 \leq p < 2^*$*, alors*

$$\forall u \in H^1(\mathbb{R}^d), \quad \|u\|_{L^p} \leq C\|u\|_{L^2}^{1-\sigma}\|\nabla u\|_{L^2}^{\sigma} \quad \text{avec} \quad \sigma = \frac{d(p-2)}{2p}\cdot \tag{20}$$

Proof. D'après (15), on a

$$\|u\|_{L^p} \leq C\|u\|_{\dot H^\sigma}.$$

En remarquant que la contrainte $2 \leq p < 2^*$ assure $\sigma \in [0,1[$ et en procédant comme pour la preuve de (13) mais avec $|\xi|$ au lieu de $\langle\xi\rangle$, on constate que

$$\|u\|_{\dot H^\sigma} \leq \|u\|_{L^2}^{1-\sigma}\|u\|_{\dot H^1}^{\sigma},$$

d'où le résultat. □

1.2.4. *Compact embeddings.*

Theorem 1.5 (Compacité locale de l'injection H^s). *Soit* $d \geq 1$*,* $s > 0$ *et*

$$p_c = \begin{cases} \frac{2d}{d-2s} & \text{pour } s < \frac{d}{2}, \\ +\infty & \text{sinon.} \end{cases}$$

Alors l'injection

$$H^s(\mathbb{R}^d) \hookrightarrow L^p_{loc}(\mathbb{R}^d) \quad \text{est compacte pour} \quad 1 \leq p < p_c.$$

De manière équivalente, pour toute suite $(f_n)_{n\in\mathbb{N}}$ *bornée dans* $H^s(\mathbb{R}^d)$*, on peut trouver* $f \in H^s(\mathbb{R}^d)$ *et une sous-suite* $(f_{\phi(n)})_{n\in\mathbb{N}}$ *telles que:*

$$\begin{array}{lll} f_{\varphi(n)} \rightharpoonup f & \text{dans} & H^s(\mathbb{R}^d), \\ f_{\varphi(n)} \to f & \text{dans} & L^p_{loc}(\mathbb{R}^d), \quad \forall 1 \leq p < p_c. \end{array}$$

Pour $s > \frac{d}{2}$*, la convergence est uniforme sur tout compact de* $\mathbb{R}^d$*.*

Proof. Supposons $0 < s \leq \frac{d}{2}$, $2 \leq p < p_c$. Considérons $\zeta \in \mathcal{D}(\mathbb{R}^d)$, $\zeta \geq 0$ avec

$$\zeta(x) = \begin{cases} 1 & \text{pour } |x| \leq 1, \\ 0 & \text{pour } |x| \geq 2 \end{cases} \quad \text{et} \quad \int_{\mathbb{R}^d} \zeta(x)dx = 1, \tag{21}$$

et la famille, dite *approximation de l'identité*

$$\zeta_\varepsilon(x) = \frac{1}{\varepsilon^d}\zeta\left(\frac{x}{\varepsilon}\right)\cdot \tag{22}$$

Soit l'opérateur de convolution

$$T_\varepsilon(f) = \zeta_\varepsilon \star f.$$

[4] 2^* est la notation consacrée pour l'exposant critique intervenant dans l'injection de Sobolev homogène $\dot H^1(\mathbb{R}^d) \subset L^{2^*}(\mathbb{R}^d)$ en dimension $d \geq 3$.

Il est classique[5] que

$$\forall f \in L^2(\mathbb{R}^d), \quad T_\varepsilon(f) \to f \quad \text{dans} \quad L^2(\mathbb{R}^d) \quad \text{quand} \quad \varepsilon \to 0.$$

Nous allons montrer qu'une borne H^s rend cette convergence uniforme. Plus précisément, en utilisant Parseval, on montre facilement que

$$\sup_{\|f\|_{H^s} \le 1} \|T_\varepsilon f - f\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{quand} \quad \varepsilon \to 0 \quad \text{si} \quad 0 < s \le \frac{d}{2}. \tag{23}$$

Soit $R > 0$ et $\overline{B}_R = \{x \in \mathbb{R}^d, \quad |x| \le R\}$. Alors par (23) l'application identité

$$\text{Id} : (H^s(\mathbb{R}^d), \|\cdot\|_{H^s(\mathbb{R}^d)}) \to L^2(\overline{B}_R, \|\cdot\|_{L^2(\overline{B}_R)})$$

est limite uniforme des applications T_ε. Or T_ε un opérateur de convolution par une fonction régulière, et est donc *compact* de $(L^2(\mathbb{R}^d), \|\cdot\|_{L^2(\mathbb{R}^d)})$ dans $\mathcal{C}(\overline{B}_R, \|\cdot\|_{L^\infty(\overline{B}_R)})$. Il est donc a fortiori compact de $(H^s(\mathbb{R}^d), \|\cdot\|_{H^s(\mathbb{R}^d)})$ dans $L^2(\overline{B}_R, \|\cdot\|_{L^2(\overline{B}_R)})$. Donc

$$\text{Id} : (H^s(\mathbb{R}^d), \|\cdot\|_{H^s(\mathbb{R}^d)}) \to L^2(\overline{B}_R, \|\cdot\|_{L^2(\overline{B}_R)}) \quad \text{est compacte} \tag{24}$$

comme limite d'applications compactes.

Soit alors $(f_n)_{n\in\mathbb{N}}$ une suite bornée dans H^s, alors par compacité faible de la boule unité de H^s, il existe $f \in H^s(\mathbb{R}^d)$ et une extraction $\psi(n)$ telles que

$$f_{\psi(n)} \rightharpoonup f \quad \text{dans} \quad H^s(\mathbb{R}^d). \tag{25}$$

En prenant $R = m$, on construit par récurrence sur $m \ge 1$ par (24) des extractions $(\phi_m)_{m\in\mathbb{N}}$ telles que

$$\forall m \ge 1, \quad f_{\psi\circ\phi_1\circ\cdots\circ\phi_m(n)} \to f \quad \text{dans} \quad L^2(\overline{B}_m) \quad \text{quand} \quad m \to +\infty.$$

Notons que le fait que la limite forte locale soit nécessairement donnée par f est une conséquence immédiate de la convergence faible (25). La suite extraite $f_{\phi(n)}$ où ϕ est la suite diagonale

$$\phi(n) = \psi \circ \phi_1 \circ \cdots \circ \phi_n(n)$$

vérifie maintenant par construction:

$$f_{\phi(n)} \rightharpoonup f \quad \text{dans} \quad H^s(\mathbb{R}^d), \quad f_{\phi(n)} \to f \quad \text{dans} \quad L^2_{loc}(\mathbb{R}^d), \tag{26}$$

et donc par inégalité de Hölder appliquée sur tout compact de $\mathbb{R}^d$:

$$f_{\phi(n)} \to f \quad \text{dans} \quad L^p_{loc}(\mathbb{R}^d) \quad \text{pour} \quad 1 \le p \le 2.$$

Soit alors $2 \le p < p_c$ et $0 < \alpha \le 1$ tels que

$$\frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{p_c},$$

et soit K compact de $\mathbb{R}^d$. Alors par inégalités de Hölder et injection de Sobolev:

$$\begin{aligned} \|f_{\phi(n)} - f\|_{L^p(K)} &\le \|f_{\phi(n)} - f\|^{\alpha}_{L^2(K)} \|f_{\phi(n)} - f\|^{1-\alpha}_{L^{p_c}(K)} \\ &\le C_p \|f_{\phi(n)} - f\|^{\alpha}_{L^2(K)} (\|f_{\phi(n)}\|^{1-\alpha}_{H^s(\mathbb{R}^d)} + \|f\|^{1-\alpha}_{H^s(\mathbb{R}^d)}) \\ &\le C \|f_{\phi(n)} - f\|^{\alpha}_{L^2(K)}. \end{aligned}$$

Par (26), le terme de droite tend vers 0 quand n tend vers $+\infty$.

[5] et immédiat par Parseval et convergence dominée

Le cas $s > d/2$ repose sur la propriété que

$$\sup_{\|f\|_{H^s}\leq 1} \|T_\varepsilon f - f\|_{L^\infty(\mathbb{R}^d)} \to 0 \quad \text{quand} \quad \varepsilon \to 0 \quad \text{si} \quad s > \frac{d}{2}\cdot \tag{27}$$

Les détails sont laissés au lecteur. □

Remark 1.5. *Arguing by duality, we see as well that the embedding of $L^p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ in $H^{-s}(\mathbb{R}^d)$ is* locally compact *whenever $s > d/p - d/2$.*

2. Incompressible Navier-Stokes equations: an elementary approach

This lecture is devoted to the mathematical study of the *Navier-Stokes system* for incompressible fluids evolving in the whole space[6] $\mathbb{R}^d$ with $d \geq 2$. Denoting by $u \in \mathbb{R}^d$ the *velocity field*, by $P \in \mathbb{R}$ the *pressure* function and by $\nu > 0$ the *kinematic viscosity*, the Cauchy problem for the incompressible Navier-Stokes system can be written as follows:

$$\left\{\begin{array}{rcl} \partial_t u + u \cdot \nabla u - \nu \Delta u &=& -\nabla P \\ \operatorname{div} u &=& 0 \\ u_{|t=0} &=& u_0, \end{array}\right.$$

where

$$\operatorname{div} u = \sum_{j=1}^d \partial_j u^j\,, \quad u \cdot \nabla = \sum_{j=1}^d u^j \partial_j \quad \text{and} \quad \Delta = \sum_{j=1}^d \partial_j^2.$$

The first part of this section is devoted to the proof of existence of *global finite energy weak solutions* to the above system : the celebrated Leray theorem.

In the second section, we show that for a large class of "generalized" Navier-Stokes systems, local-in-time well posedness holds true in $\dot H^{\frac d2 -1}$ (and even global-in-time existence for small data). Next we give some qualitative properties of those "strong" solutions in the particular case of the "true" Navier-Stokes system.

In the third section, we establish a general "weak-strong uniqueness" result. We show that it implies that, in dimension 2, any finite energy solution is strong and unique and that, in dimension 3, Fujita-Kato and Leray solutions are the same as long as the Fujita-Kato solution is defined.

2.1. The Leray theorem.

To start with, let us introduce the *weak formulation* for the Navier-Stokes system. From the Leibniz formula, it is clear that, when the vector field u is smooth and divergence free, we have

$$u \cdot \nabla u = \operatorname{div}(u \otimes u) \quad \text{where} \quad \operatorname{div}(u \otimes u)^j \stackrel{\text{def}}{=} \sum_{k=1}^d \partial_k(u^j u^k) = \operatorname{div}(u^j u),$$

so that the Navier-Stokes system may be alternately written

$$(NS_\nu) \qquad \left\{\begin{array}{rcl} \partial_t u + \operatorname{div}(u \otimes u) - \nu \Delta u &=& -\nabla P \\ \operatorname{div} u &=& 0 \\ u_{|t=0} &=& u_0. \end{array}\right.$$

The advantage of this formulation is that it makes sense for more singular vector fields than the previous one, a fact which will be used extensively in what follows.

Based on this observation, let us now define what a *weak solution* of (NS_ν) is:

[6] which means that boundary effects are neglected.

Definition 2.1. *A divergence-free time-dependent vector field u with components in the space $L^2_{loc}([0,T]\times\mathbb{R}^d)$ is a weak solution of (NS_ν) if for any smooth compactly supported time dependent divergence free vector field Ψ, we have*

$$\int_{\mathbb{R}^d} u(t,x)\cdot\Psi(t,x)\,dx+\int_0^t\int_{\mathbb{R}^d}\Big(\nu\nabla u:\nabla\Psi-u\otimes u:\nabla\Psi-u\cdot\partial_t\Psi\Big)(t',x)\,dx\,dt'$$
$$=\int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)\,dx.$$

Let us now derive formally the *energy estimate* associated to (NS_ν). First, taking the $(L^2(\mathbb{R}^d))^d$ scalar product of the system with the solution u gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+(u\cdot\nabla u|u)_{L^2}-\nu(\Delta u|u)_{L^2}=-(\nabla P|u)_{L^2}.$$

Using formal integrations by parts, we may write that

$$\begin{aligned}(u\cdot\nabla u|u)_{L^2} &= \sum_{1\leq j,k\leq d}\int_{\mathbb{R}^d} u^j(\partial_j u^k)u^k\,dx\\ &= \frac{1}{2}\sum_{1\leq j\leq d}\int_{\mathbb{R}^d} u^j\partial_j(|u|^2)\,dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}(\operatorname{div} u)|u|^2\,dx\\ &= 0.\end{aligned}$$

Moreover,

$$-\nu(\Delta u|u)_{L^2}=\nu\|\nabla u\|_{L^2}^2.$$

Again (formal) integrations by parts yield

$$\begin{aligned}-(\nabla P|u)_{L^2} &= -\sum_{j=1}^d\int_{\mathbb{R}^d} u^j\partial_j P\,dx\\ &= \int_{\mathbb{R}^d} P\operatorname{div} u\,dx\\ &= 0.\end{aligned}$$

So we eventually get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2+\nu\|\nabla u(t)\|_{L^2}^2=0,$$

whence, by time integration

$$\|u(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla u(t')\|_{L^2}^2\,dt'=\|u_0\|_{L^2}^2. \tag{28}$$

It follows that the natural assumption for the initial data u_0 is that it is square integrable and divergence free.

In the whole space setting, there is an explicit formula giving the pressure in terms of the velocity field. Indeed, in Fourier variable, the Leray projector $\mathcal{P}$ on divergence free vector fields reads

$$\begin{aligned}\mathcal{F}(\mathcal{P}f)^j(\xi) &= \widehat{f}^j(\xi) - \frac{1}{|\xi|^2}\sum_{k=1}^d \xi_j\xi_k\widehat{f}^k(\xi)\\ &= \sum_{k=1}^d(\delta_{j,k}-1)\frac{\xi_j\xi_k}{|\xi|^2}\widehat{f}^k(\xi)\end{aligned}\tag{29}$$

where $\partial_{jk}=1$ if $j=k$ and 0 if $j\neq k$.

Therefore, applying the Leray projector to the Navier-Stokes system and denoting by Q_{NS} the bilinear operator defined by

$$Q_{NS}(v,w)\stackrel{\text{def}}{=}-\frac{1}{2}\mathcal{P}\big(\mathrm{div}\,(v\otimes w)+(\mathrm{div}\,w\otimes v)\big)$$

yields

$$\left\{\begin{aligned}\partial_t u-\nu\Delta u &= Q_{NS}(u,u)\\ u_{|t=0} &= u_0.\end{aligned}\right.$$

Formally, the divergence free condition is satisfied by u whenever $\mathrm{div}\,u_0=0$, and u does satisfy the original system (NS_ν). In order to make (28) rigorous, let us prove the following lemma:

Lemma 2.1. *Let u and v be two time dependent divergence free vector fields over $\mathbb{R}^d$. If u and v belong to $L^4([0,T];L^4)\cap L^2([0,T];H^1)$ then we have*

$$\int_0^t \big\langle Q_{NS}(u(t'),v(t')),v(t')\big\rangle\,dt'=0.$$

Proof. We know that

$$Q^j_{NS}(u,v)\stackrel{\text{def}}{=}-\mathrm{div}\,(v^ju)-\sum_{1\leq k,\ell\leq d}\partial_j(-\Delta)^{-1}\partial_k\partial_\ell(u^kv^\ell).$$

Note that all the terms of right-hand side are in $L^2([0,T];\dot{H}^{-1})$. Therefore,

$$\begin{aligned}\langle Q_{NS}(u,v),v\rangle &= -\sum_{1\leq j\leq d}\int_{\mathbb{R}^d}v^j\mathrm{div}\,(v^ju)\,dx\\ &\qquad+\sum_{1\leq j,k,\ell\leq d}\int_{\mathbb{R}^d}v^j\partial_j(\Delta^{-1}\partial_k\partial_\ell(u^kv^\ell))\,dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}(\mathrm{div}\,u)|v|^2\,dx\\ &\qquad-\sum_{1\leq k,\ell\leq d}\int_{\mathbb{R}^d}(\mathrm{div}\,v)\Delta^{-1}\partial_k\partial_\ell(u^kv^\ell)\,dx.\end{aligned}\tag{30}$$

As $\mathrm{div}\,u=\mathrm{div}\,v=0$, this completes the proof of the lemma. □

This lemma will enable us to get the following statement which has been first proved by J. Leray in 1934.

Theorem 2.1 (Leray)**.** *Let u_0 be a divergence free vector field in $L^2(\mathbb{R}^d)$. Then (NS_ν) has a weak solution u in the energy space*

$$L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; \dot{H}^1)$$

such that the energy inequality holds, namely

$$\forall t \in \mathbb{R},\ \|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(t')\|_{L^2}^2\, dt' \leq \|u_0\|_{L^2}^2. \tag{31}$$

Proof. Proving Leray theorem relies on a nowadays standard compactness method that we now present.

Step 1. Construction of smooth approximate solutions.

Let J_n be the Friedrichs spectral truncation operator defined on $L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(J_n v)(\xi) \stackrel{\text{def}}{=} 1_{B(0,n)}(\xi)\mathcal{F}(\mathcal{P}v)(\xi).$$

It is clear that J_n is a L^2 orthogonal projector, with range in $H^\infty \stackrel{\text{def}}{=} \cup_{s\in\mathbb{R}} H^s$.

We consider the following ordinary differential equation in L^2:

$$\frac{d}{dt} u = F_n(u), \qquad u_n|_{t=0} = J_n u_0 \tag{32}$$

with $F_n(u) \stackrel{\text{def}}{=} \nu J_n \Delta u - J_n Q_{NS}(J_n u, J_n u)$.

Because F_n is time independent and locally Lipschitz on L^2, Cauchy-Lipschitz theorem ensures that there exists a unique maximal solution $u_n \in \mathcal{C}^1([0,T_n); L^2)$ to (32). As $J_n u_n$ also satisfies (32), uniqueness implies that $u_n = J_n u_n$. Therefore $u_n \in \mathcal{C}^1([0,T_n); H^\infty)$ and $\mathcal{P}u_n = u_n$ (that is $\operatorname{div} u_n = 0$).

Step 2. Global-in-time uniform estimates.

Because u_n is smooth, one may write that

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 = \Bigl(u_n\Big|\frac{d}{dt}u_n\Bigr)_{L^2} \quad \text{and} \quad -\langle \Delta u_n, u_n\rangle = \|\nabla u_n\|_{L^2}^2.$$

Moreover, because J_n is an L^2 orthogonal projector, for all $t \in [0, T_n)$,

$$\bigl\langle J_n Q_{NS}(u_n(t), u_n(t)), u_n(t)\bigr\rangle = \bigl\langle Q_{NS}(u_n(t), u_n(t)), u_n(t)\bigr\rangle.$$

Hence Lemma 2.1 ensures that

$$\int_0^t \bigl\langle Q_{NS}(u_n(t'), u_n(t')), u_n(t')\bigr\rangle\, dt' = 0.$$

After integrating in time, one may thus conclude that

$$\forall t \in [0,T_n),\ \|u_n(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_n\|_{L^2}^2\, d\tau = \|J_n u_0\|_{L^2}^2 \leq \|u_0\|_{L^2}^2. \tag{33}$$

In particular $\|u_n(t)\|_{L^2} \leq \|u_0\|_{L^2}$ for all $t \in [0, T_n)$ and the standard continuation criterion for ODEs thus ensures that $T_n = +\infty$. In addition, Inequality (33) guarantees that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$.

Step 3. Compactness.

Here we assume that $d = 2, 3$ for simplicity. We claim that $(F_n(u_n))_{n\in\mathbb{N}}$ is bounded in $L^p_{loc}(\mathbb{R}_+; H^{-1})$ for some $p > 1$. This is obviously the case for the first term of $F_n(u_n)$ (for which one may take any $p \leq 2$). For the second term, we write

$$\|J_n Q_{NS}(u_n, u_n)\|_{H^{-1}} \leq C\|u_n \otimes u_n\|_{L^2} \leq C\|u_n\|_{L^4}^2.$$

Now, Gagliardo-Nirenberg inequality ensures that

$$\|u_n\|_{L^4}^2 \leq C \times \begin{cases} \|u_n\|_{L^2}\|\nabla u_n\|_{L^2} & \text{if } d=2, \\ \|u_n\|_{L^2}^{\frac12}\|\nabla u_n\|_{L^2}^{\frac32} & \text{if } d=3. \end{cases}$$

Because u_n and ∇u_n are uniformly bounded in $L^\infty(\mathbb{R}_+;L^2)$ and $L^2(\mathbb{R}_+;L^2)$, respectively (see (33)), it is easy to conclude the proof of our claim : one may take $p=2$ if $d=2$, and $p=4/3$ if $d=3$.

Let us put together all the bounds that we proved hitherto:

– u_n is bounded in $\mathcal{C}_b(\mathbb{R}_+;L^2)\cap L^2(\mathbb{R}_+;\dot H^1)$ (and thus also in $L^2([0,T];H^1)$ for all $T>0$),

– there exists $\alpha\in(0,1)$ so that u_n is bounded in $\mathcal{C}^\alpha([0,T];H^{-1})$ for all $T>0$.

Consequently, using that the embedding from L^2 to H^{-1} is locally compact (see the remark that follows Theorem 1.5), one may conclude by combining weak compactness of bounded sets in Hilbert spaces, Ascoli theorem and Cantor diagonal process, to the existence of some vector field $u\in L^\infty(\mathbb{R}_+;L^2)\cap L^2(\mathbb{R}_+;\dot H^1)$ such that, up to extraction,

– u satisfies the energy inequality (31),

– $u_n \rightharpoonup u$ in $L^2([0,T];H^1)$ for all $T>0$,

– $u_n\to u$ in $\mathcal{C}([0,T];H^{-1}_{loc})$ for all $T>0$,

– $u_n(t)\rightharpoonup u(t)$ in L^2 for all $t\geq 0$.

Step 4. Checking that u is a weak solution.

This is just a matter of passing to the limit in the weak formulation of Definition 2.1. The only difficulty lies in justifying that

$$\forall T\in\mathbb{R}_+,\ \int_0^T\int_{\mathbb{R}^d} u_n\otimes u_n:\nabla\Psi\,dx\,dt \longrightarrow \int_0^T\int_{\mathbb{R}^d} u\otimes u:\nabla\Psi\,dx\,dt$$

for all divergence free test vector field Ψ.

As $u_n\rightharpoonup u$ in $L^2([0,T];H^1)$ and $u_n\to u$ strongly in $\mathcal{C}([0,T];H^{-1}_{loc})$, this is obvious.

□

2.2. **The Fujita-Kato theorem.** In contrast with the approach leading to Leray's theorem, the approach that we propose here holds true for a large family of Navier-Stokes like systems. More precisely, we shall consider systems of the form

$$(GNS_\nu):\begin{cases}\partial_t u-\nu\Delta u &=& Q(u,u)\\ u_{|t=0} &=& u_0\end{cases}$$

where Q is any bilinear map defined by

$$Q^j(u,v)\stackrel{\text{def}}{=}\sum_{k,\ell,m} q^{j,m}_{k,\ell}\partial_m(u^k v^\ell),$$

where $q^{j,m}_{k,\ell}$ are Fourier multipliers of the form

$$q^{j,m}_{k,\ell}a\stackrel{\text{def}}{=}\sum_{n,p}\alpha^{j,m,n,p}_{k,\ell}\mathcal{F}^{-1}\Big(\frac{\xi_n\xi_p}{|\xi|^2}\widehat a(\xi)\Big)$$

and $\alpha^{j,m,n,p}_{k,\ell}$ are real numbers.

The incompressible Navier-Stokes system corresponds to the case where $Q = Q_{NS}$.

Let $B(u,v)$ be the solution to the heat equation

$$\begin{cases} \partial_t B(u,v) - \nu\Delta B(u,v) &= Q(u,v) \\ B(u,v)_{|t=0} &= 0. \end{cases}$$

Solving (GNS_ν) amounts to finding a fixed point for the map

$$u \longmapsto e^{\nu t\Delta}u_0 + B(u,u). \tag{34}$$

For that, we shall use the following smoothing properties of the heat flow.

Lemma 2.2. *Let v be the solution in $\mathcal{C}([0,T];\mathcal{S}'(\mathbb{R}^d))$ of the Cauchy problem*

$$\begin{cases} \partial_t v - \nu\Delta v &= f \\ v_{|t=0} &= v_0 \end{cases}$$

with f in $L^2([0,T];\dot H^{s-1})$ and v_0 in $\dot H^s(\mathbb{R}^d)$. Then

$$v \in \Big(\bigcap_{p=2}^{\infty} L^p([0,T];\dot H^{s+\frac{2}{p}})\Big)\bigcap \mathcal{C}([0,T];\dot H^s).$$

Moreover, we have the following estimates:

$$\begin{aligned} \|v(t)\|_{\dot H^s}^2 + 2\nu\int_0^t \|\nabla v(t')\|_{\dot H^s}^2\,dt' &= \|v_0\|_{\dot H^s}^2 + 2\int_0^t \langle f(t'),v(t')\rangle_s\,dt'\,, \\ \Big(\int_{\mathbb{R}^d}|\xi|^{2s}\Big(\sup_{0\le t'\le t}|\widehat v(t',\xi)|\Big)^2 d\xi\Big)^{\frac12} &\le \|v_0\|_{\dot H^s} + \frac{1}{(2\nu)^{\frac12}}\|f\|_{L^2_T(\dot H^{s-1})}, \\ \|v(t)\|_{L^p_T(\dot H^{s+\frac{2}{p}})} &\le \frac{1}{\nu^{\frac1p}}\Big(\|v_0\|_{\dot H^s} + \frac{1}{\nu^{\frac12}}\|f\|_{L^2_T(\dot H^{s-1})}\Big) \end{aligned}$$

with $\langle a,b\rangle_s \stackrel{def}{=} \int |\xi|^{2s}\widehat a(\xi)\overline{\widehat b(\xi)}\,d\xi\,.$

Proof. The first estimate is nothing more than the energy estimate. The proof of the second one consists mainly in writing Duhamel's formula in Fourier space, namely

$$\widehat v(t,\xi) = e^{-\nu t|\xi|^2}\widehat v_0(\xi) + \int_0^t e^{-\nu(t-t')|\xi|^2}\widehat f(t',\xi)\,dt'.$$

The Cauchy-Schwarz inequality implies that

$$\sup_{0\le t'\le t}|\widehat v(t',\xi)| \le |\widehat v_0(\xi)| + \frac{1}{\sqrt{2\nu|\xi|^2}}\|\widehat f(\cdot,\xi)\|_{L^2([0,t])}.$$

Then taking the L^2 norm with respect to $|\xi|^{2s}\,d\xi$ allows to conclude that

$$\begin{aligned} V(t) &\stackrel{\text{def}}{=} \biggl(\int_{\mathbb{R}^d}\Bigl(\sup_{0\leq t'\leq t}|\widehat{v}(t',\xi)|\Bigr)^2|\xi|^{2s}\,d\xi\biggr)^{\frac12}\\ &\leq \|v_0\|_{\dot H^s}+\frac{1}{(2\nu)^{\frac12}}\biggl(\int_{\mathbb{R}^d}\|\widehat{f}(\cdot,\xi)\|^2_{L^2([0,t])}|\xi|^{2s-2}\,d\xi\biggr)^{\frac12}\\ &\leq \|v_0\|_{\dot H^s}+\frac{1}{(2\nu)^{\frac12}}\biggl(\int_{[0,t]\times\mathbb{R}^d}|\widehat{f}(t',\xi)|^2|\xi|^{2s-2}\,d\xi\,dt'\biggr)^{\frac12}\\ &\leq \|v_0\|_{\dot H^s}+\frac{1}{(2\nu)^{\frac12}}\|f\|_{L^2([0,t];\dot H^{s-1})}. \end{aligned}$$

Since, for almost all fixed $\xi\in\mathbb{R}^d$, the map $t\mapsto\widehat{v}(t,\xi)$ is continuous over $[0,T]$, the Lebesgue dominated convergence theorem ensures that $v\in\mathcal{C}([0,T];\dot H^s)$.

Finally, the last inequality follows by interpolation. □

2.2.1. *An abstract fixed point lemma and scaling invariance spaces.* We claim that one may find some functional space in which (GNS_ν) reformulated as in (34) may be solved by means of the following abstract lemma:

Lemma 2.3. *Let X be a Banach space and $\mathcal{B}:X\times X\to X$ a continuous bilinear map with norm M. Then there exists a unique solution v in $B(0,2\|v_0\|_X)$ to*

$$(E)\qquad v=v_0+\mathcal{B}(v,v)$$

whenever

$$(35)\qquad 4M\|v_0\|_X<1.$$

Proof. Denoting $F:v\to v_0+\mathcal{B}(v,v)$, we see that

$$\|F(v)-v_0\|_X\leq M\|v\|_X^2.$$

Hence if (35) is satisfied and $\|v\|_X\leq 2\|v_0\|_X$ then F maps the closed ball $\bar B(0,2\|v_0\|_X)$ into itself.

Next, considering v_1 and v_2 in this closed ball, we see that

$$\|F(v_2)-F(v_1)\|_X\leq M(\|v_1\|_X+\|v_2\|_X)\|v_2-v_1\|_X\leq 4M\|v_0\|_X\|v_2-v_1\|_X.$$

Hence Condition (35) ensures that F is strictly contracting, and applying the classical fixed point theorem for complete metric spaces yields the result. □

Assume in addition that there exists a one-parameter family $(T_\lambda)_{\lambda>0}$ acting on X and which leaves (E) invariant, that is:

$$v=v_0+\mathcal{B}(v,v)\iff T_\lambda v=T_\lambda v_0+\mathcal{B}(T_\lambda v,T_\lambda v)\quad\text{for all}\ \ \lambda>0.$$

Then the smallness condition (35) recasts in

$$4M\|T_\lambda v_0\|_X<1\quad\text{for all}\ \ \lambda>0.$$

In other words, either the problem may be solved for any data in X, or the norm in X has to be invariant (up to an irrelevant constant) by T_λ for all λ. If so then we shall call X *a scaling invariant space* for (E).

In the applications, a dimension analysis often allows to find such a family $(T_\lambda)_{\lambda>0}$. This is the case for instance if considering evolutionary equations such as the nonlinear Schrödinger, wave or heat equations with a power-like nonlinearity. As regards the (generalized) Navier-Stokes equations (GNS_ν), we notice that v is a solution if and only if $T_\lambda v$ is a solution (for all $\lambda>0$) with

$$T_\lambda v(t,x) := \lambda v(\lambda^2 t, \lambda x). \tag{36}$$

Hence one may tempt to solve (GNS_ν) in spaces X with norm invariant by the above transformation. The spaces below obviously meet those two conditions:

$$L^\infty(\mathbb{R}^+;L^d);\ L^\infty(\mathbb{R}^+;\dot H^{\frac d2-1});\ L^4(\mathbb{R}^+;\dot H^{\frac{d-1}{2}});$$
$$L^\infty(\mathbb{R}^+;\dot H^{\frac d2-1})\cap L^2(\mathbb{R}^+;\dot H^{\frac d2}).$$

When $d=2$, the energy space itself $L^\infty(\mathbb{R}^+;L^2)\cap L^2(\mathbb{R}^+;\dot H^1)$ is scaling invariant, which will enable us to prove that any solution generated by a L^2 divergence free data is strong, global and unique.

In the case when $d=3$ however, the energy space is *below* the regularity of the scaling invariant space $\dot H^{\frac12}$. In other words, in dimension $d=2$, stating the global existence of regular solutions for Navier-Stokes system is a critical problem whereas in dimension $d=3$, this can be interpreted as a supercritical problem. This is the core of the difficulty. As we shall see, being able to use the special structure of the equation *in a scaling invariant framework* is one of the challenges for solving the global well-posedness issue.

2.2.2. *The Fujita-Kato theorem.*

Theorem 2.2. *Let u_0 be in $\dot H^{\frac d2-1}(\mathbb{R}^d)$ with $d\geq 2$. There exists a positive time T such that System (GNS_ν) has a unique solution u in $L^4([0,T];\dot H^{\frac{d-1}{2}})$ which also belongs to*

$$\mathcal{C}([0,T];\dot H^{\frac d2-1})\cap L^2([0,T];\dot H^{\frac d2}).$$

Let T_{u_0} denote the maximal time of existence of such a solution. Then

- *there exists a constant c such that*
$$\|u_0\|_{\dot H^{\frac d2-1}}\leq c\nu \Longrightarrow T_{u_0}=\infty,$$
with, moreover, $\|u(t)\|_{\dot H^{\frac d2-1}}\leq 2c\nu$ for any time t.
- *if T_{u_0} is finite, then*

$$\int_0^{T_{u_0}} \|u(t)\|^4_{\dot H^{\frac{d-1}{2}}}\,dt=\infty. \tag{37}$$

Proof. We claim that the map

$$u\longmapsto e^{\nu t\Delta}u_0+B(u,u)$$

has a unique fixed point in the space $L^4([0,T];\dot H^{\frac{d-1}{2}})$ for an appropriate T. This is a consequence of the following inequality

$$\|B(u,v)\|_{L^4_T(\dot H^{\frac{d-1}{2}})}\leq \frac{C}{\nu^{\frac34}}\|u\|_{L^4_T(\dot H^{\frac{d-1}{2}})}\|v\|_{L^4_T(\dot H^{\frac{d-1}{2}})} \tag{38}$$

that we admit for a while.

Now, according to Lemma 2.3, if

$$\|e^{\nu t\Delta}u_0\|_{L^4_T(\dot H^{\frac{d-1}{2}})}\leq\frac{\nu^{\frac34}}{4C_0},\tag{39}$$

with $C_0>C$, then there exists a unique solution of (GNS_ν) in the ball of center 0 and radius $(\frac{\nu^{\frac34}}{2C_0})$ of the space $L^4([0,T];\dot H^{\frac{d-1}{2}})$.

Let us now investigate when Condition (39) is satisfied. Applying the last inequality of Lemma 2.2 with $s=d/2-1$ and $p=4$ yields for any positive time T:

$$\|e^{\nu t\Delta}u_0\|_{L^4_T(\dot H^{\frac{d-1}{2}})}\leq\frac{1}{\nu^{\frac14}}\|u_0\|_{\dot H^{\frac d2-1}}.\tag{40}$$

Thus, if $\|u_0\|_{\dot H^{\frac d2-1}}\leq(4C_0)^{-1}\nu$, the smallness Condition (39) is satisfied and we have a global solution.

Let us now consider the case of a *large* initial data u_0 in $\dot H^{\frac d2-1}$. We shall split u_0 into a small part in $\dot H^{\frac d2-1}$ and a large part with compactly supported Fourier transform. For that, we fix some positive real number ρ_{u_0} such that

$$\Bigl(\int_{|\xi|\geq\rho_{u_0}}|\xi|^{d-2}|\widehat u_0(\xi)|^2\,d\xi\Bigr)^{\frac12}\leq\frac{\nu}{8C_0}.$$

Using (40) and denoting $u_0^\flat\overset{\text{def}}{=}\mathcal F^{-1}(\mathbf 1_{B(0,\rho_{u_0})}\widehat u_0)$, we get

$$\|e^{\nu t\Delta}u_0\|_{L^4_T(\dot H^{\frac{d-1}{2}})}\leq\frac{\nu^{\frac34}}{8C_0}+\|e^{\nu t\Delta}u_0^\flat\|_{L^4_T(\dot H^{\frac{d-1}{2}})}.$$

Let us notice that

$$\begin{aligned}\|e^{\nu t\Delta}u_0^\flat\|_{L^4_T(\dot H^{\frac{d-1}{2}})}&\leq\rho_{u_0}^{\frac12}\|e^{\nu t\Delta}u_0^\flat\|_{L^4_T(\dot H^{\frac d2-1})}\\&\leq(\rho_{u_0}^2T)^{\frac14}\|u_0\|_{\dot H^{\frac d2-1}}.\end{aligned}$$

Thus, if

$$T\leq\biggl(\frac{\nu^{\frac34}}{8C_0\rho_{u_0}^{\frac12}\|u_0\|_{\dot H^{\frac d2-1}}}\biggr)^4,\tag{41}$$

then we have the existence of a unique solution in the ball of center 0 and radius $\frac{\nu^{\frac34}}{2C_0}$ of the space $L^4([0,T];\dot H^{\frac{d-1}{2}})$.

Finally, if u is a solution of (GNS_ν) in $L^4([0,T];\dot H^{\frac{d-1}{2}})$ then, by Lemma 2.4, $Q(u,u)$ belongs to $L^2([0,T];\dot H^{\frac d2-2})$. Hence, Lemma 2.2 implies that the solution u belongs to

$$\mathcal C([0,T];\dot H^{\frac d2-1})\cap L^2([0,T];\dot H^{\frac d2}).$$

The uniqueness of a solution in the existence space is a trivial consequence of the stability estimates stated in Proposition 2.1.

Let us finally prove the blow up criterion. Let us assume that we have a solution u of (GNS_ν) on a time interval $[0,T[$ such that

$$\int_0^T\|u(t)\|^4_{\dot H^{\frac{d-1}{2}}}\,dt<\infty.$$

We claim that the lifespan T_{u_0} of u is greater than T. Indeed, thanks to Lemmas 2.4 and 2.2, we have

$$\int_{\mathbb{R}^d} |\xi|^{d-2}\Big(\sup_{t\in[0,T[}|\widehat{u}(t,\xi)|\Big)^2\,d\xi<\infty.$$

Thus a positive number ρ exists such that

$$\forall t\in[0,T[\,,\ \int_{|\xi|\geq\rho}|\xi|^{d-2}|\widehat{u}(t,\xi)|^2\,d\xi<\frac{c\nu}{2}\cdot$$

Now, Condition (41) implies that for any $t\in[0,T[$, the lifespan for a solution of (GNS_ν) with initial data $u(t)$ is bounded from below by a positive real number τ *independent* of t. Thus $T_{u_0}>T$. □

For completeness, let us justify (38). It just stems from a combination of the last item of Lemma 2.2 and of the following lemma, which is nothing more than a variation about Sobolev embedding.

Lemma 2.4. *A constant C exists such that*

$$\|Q(a,b)\|_{\dot H^{\frac d2-2}}\leq C\|a\|_{\dot H^{\frac{d-1}{2}}}\|b\|_{\dot H^{\frac{d-1}{2}}}.$$

Proof. We focus on the case $d=2,3,4$ where the result may be proved by elementary arguments.

So let us start with the case $d=2$. Then one may use Sobolev embedding to write

$$\begin{aligned}\|Q(a,b)\|_{\dot H^{-1}}&\leq C\|ab\|_{L^2}\\&\leq C\|a\|_{L^4}\|b\|_{L^4}\\&\leq C\|a\|_{\dot H^{\frac12}}\|b\|_{\dot H^{\frac12}}.\end{aligned}$$

Next, if $d=3$ then we have by definition of Q,

$$\|Q(a,b)\|_{\dot H^{-\frac12}}\leq C\sup_{k,\ell}\Big(\|a^k\partial b^\ell\|_{\dot H^{-\frac12}}+\|b^\ell\partial a^k\|_{\dot H^{-\frac12}}\Big).$$

Thanks to the dual Sobolev embedding, and to the Sobolev embedding, we have

$$\begin{aligned}\|Q(a,b)\|_{\dot H^{-\frac12}}&\leq C\sup_{k,\ell}\Big(\|a^k\partial b^\ell\|_{L^{\frac32}}+\|b^\ell\partial a^k\|_{L^{\frac32}}\Big)\\&\leq C\Big(\|a\|_{L^6}\|\nabla b\|_{L^2}+\|\nabla a\|_{L^2}\|b\|_{L^6}\Big)\\&\leq C\|a\|_{\dot H^1}\|b\|_{\dot H^1}.\end{aligned}$$

If $d=4$ then we write that

$$\begin{aligned}\|Q(a,b)\|_{L^2}&\leq C\sup\nolimits_{k,\ell}\Big(\|a^k\partial b^\ell\|_{L^2}+\|b^\ell\partial a^k\|_{L^2}\Big)\\&\leq C\Big(\|a\|_{L^8}\|\nabla b\|_{L^{\frac83}}+\|b\|_{L^8}\|\nabla a\|_{L^{\frac83}}\Big)\\&\leq C\|a\|_{\dot H^{\frac32}}\|b\|_{\dot H^{\frac32}}.\end{aligned}$$

This proves the lemma. □

2.2.3. *A stability result.* The following stability estimate obviously implies the uniqueness part of Fujita-Kato's theorem.

Proposition 2.1. *If u and v are two solutions of (GNS_ν) then*

$$\|u(t)-v(t)\|^2_{\dot H^{\frac d2-1}}+\nu\int_0^t\|u(t')-v(t')\|^2_{\dot H^{\frac d2}}\,dt'\leq\|u_0-v_0\|^2_{\dot H^{\frac d2-1}}\times\exp\Bigl(\frac{C}{\nu^3}\int_0^t\bigl(\|u(t')\|^4_{\dot H^{\frac{d-1}{2}}}+\|v(t')\|^4_{\dot H^{\frac{d-1}{2}}}\bigr)\,dt'\Bigr).$$

Proof. The difference w between the two solutions u and v satisfies

$$\begin{cases}\partial_t w-\nu\Delta w &= Q(w,u+v)\\ w_{|t=0} &= w_0\stackrel{\text{def}}{=}u_0-v_0.\end{cases}$$

Thus, by energy estimate in $\dot H^{\frac d2-1}$ (see Lemma 2.2), we have

$$\begin{aligned}\Delta_{w(t)} &\stackrel{\text{def}}{=} \|w(t)\|^2_{\dot H^{\frac d2-1}}+2\nu\int_0^t\|\nabla w(t')\|^2_{\dot H^{\frac d2-1}}\,dt'\\ &= \|w_0\|^2_{\dot H^{\frac d2-1}}+2\int_0^t\bigl\langle Q(w(t'),u(t')+v(t')),w(t')\bigr\rangle_{\frac d2-1}\,dt'.\end{aligned}$$

Taking advantage of Lemma 2.4, we get

$$\begin{aligned}\bigl\langle Q(w,u+v),w\bigr\rangle_{\frac d2-1} &\leq \|Q(w,u+v)\|_{\dot H^{\frac d2-2}}\|\nabla w\|_{\dot H^{\frac d2-1}}\\ &\leq \|w\|_{\dot H^{\frac{d-1}{2}}}\|u+v\|_{\dot H^{\frac{d-1}{2}}}\|\nabla w\|_{\dot H^{\frac d2-1}}.\end{aligned}$$

Thereforem setting $N(t)\stackrel{\text{def}}{=}\|u(t)\|_{\dot H^{\frac{d-1}{2}}}+\|v(t)\|_{\dot H^{\frac{d-1}{2}}}$, we discover that

$$\Delta_w(t)\leq\|w_0\|^2_{\dot H^{\frac d2-1}}+C\int_0^t\|w(t')\|_{\dot H^{\frac{d-1}{2}}}N(t')\|\nabla w(t')\|_{\dot H^{\frac d2-1}}\,dt'.$$

Interpolating between $\dot H^{\frac d2-1}$ and $\dot H^{\frac d2}$, we infer that

$$\Delta_w(t)\leq\|w_0\|^2_{\dot H^{\frac d2-1}}+C\int_0^t\|w(t')\|^{\frac12}_{\dot H^{\frac d2-1}}N(t')\|\nabla w(t')\|^{\frac32}_{\dot H^{\frac d2-1}}\,dt'.$$

Using the convexity inequality $ab\leq\frac14a^4+\frac34b^{\frac43}$, we deduce that

$$\Delta_w(t)\leq\|w_0\|^2_{\dot H^{\frac d2-1}}+\frac{C}{\nu^3}\int_0^t\|w(t')\|^2_{\dot H^{\frac d2-1}}N^4(t')\,dt'+\nu\int_0^t\|\nabla w(t')\|^2_{\dot H^{\frac d2-1}}\,dt'.$$

By definition of Δ_w, this can be written

$$\|w(t)\|^2_{\dot H^{\frac d2-1}}+\nu\int_0^t\|\nabla w(t')\|^2_{\dot H^{\frac d2-1}}\,dt'\leq\|w_0\|^2_{\dot H^{\frac d2-1}}+\frac{C}{\nu^3}\int_0^t\|w(t')\|^2_{\dot H^{\frac d2-1}}N^4(t')\,dt'.$$

Using Gronwall lemma completes the proof of the proposition. □

2.3. **Weak strong uniqueness, and dimension two.**

Theorem 2.3. *Let $d \geq 2$ and u be a Leray solution of (NS_ν) being in addition in $L^4_{loc}(\mathbb{R}_+; L^4)$. Then $u \in \mathcal{C}(\mathbb{R}_+; L^2)$ and the energy equality (28) is fulfilled.*

Proof. We just have to apply Lemmas 2.1 and 2.2 once noticed that

$$\partial_t u - \nu\Delta u = -Q_{NS}(u,u) \in L^2_{loc}(\mathbb{R}_+; \dot{H}^{-1}).$$

□

Corollary 2.1. *In dimension $d = 2$ the Leray solution is unique in the energy space, continuous from $\mathbb{R}_+$ to L^2, and the energy equality (28) is fulfilled.*

Proof. By interpolation, we see that any Leray solution is in $L^4(\mathbb{R}_+; \dot{H}^{\frac12})$. Hence, in dimension two, Sobolev embedding yields $u \in L^4(\mathbb{R}_+; L^4)$, and the above theorem thus ensures that $u \in \mathcal{C}(\mathbb{R}_+; L^2)$. Applying Proposition 2.1 ensures uniqueness. □

Theorem 2.4. *Assume $d = 3$. Let u be a Leray solution of (NS_ν) with initial data u_0. If v is a Leray solution corresponding to the data v_0 and satisfying $v \in L^4_{loc}(\mathbb{R}_+; \dot{H}^1)$ then there exists some constant C so that for all $t \in \mathbb{R}_+$,*

$$\|(v-u)(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla(v-u)\|^2_{L^2} \leq \|v_0 - u_0\|^2_{L^2} \exp\biggl(\frac{C}{\nu^3}\int_0^t \|v\|^4_{L^6}\, d\tau\biggr)\cdot$$

Proof. Arguing by density, we see that one may extend the identity in the definition of weak solutions (see Definition 2.1) to any test function $\Psi \in \mathcal{C}([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1)$ with $\partial_t\Psi \in L^2([0,T]; H^{-1})$ and $\nabla\Psi \in L^4([0,T]; L^2)$. Indeed, any weak solution is in $L^{\frac83}(\mathbb{R}_+; \dot{H}^{\frac34})$ hence, owing to Sobolev embedding in dimension three, in $L^{\frac83}(\mathbb{R}_+; L^4)$, and this ensures that $u \otimes u \in L^{\frac43}(\mathbb{R}_+; L^2)$, which is precisely the dual space of $L^4(\mathbb{R}_+; L^2)$.

In our case, we see from the equation and dual Sobolev embedding that we do have $\partial_t v \in L^2_{loc}(\mathbb{R}_+; \dot{H}^{-1})$. Hence v may be taken as a test function, and we thus have for all $t \in \mathbb{R}_+$,

$$\langle u(t), v(t)\rangle_0 + \nu\int_0^t \langle\nabla u, \nabla v\rangle_0\, d\tau - \int_0^t \langle u\otimes u, \nabla v\rangle\, d\tau - \int_0^t \langle\partial_t v, u\rangle\, d\tau = \langle u_0, v_0\rangle_0. \tag{42}$$

Now, the difference $w \stackrel{\text{def}}{=} u - v$ satisfies

$$\|w(t)\|^2_{L^2} + 2\nu\int_0^t \|\nabla w\|^2_{L^2} = \|v(t)\|^2_{L^2} + \|u(t)\|^2_{L^2} - 2\langle u(t), v(t)\rangle_0$$
$$+2\nu\int_0^t \|\nabla v\|^2_{L^2} + 2\nu\int_0^t \|\nabla u\|^2_{L^2} - 4\nu\int_0^t \langle\nabla v, \nabla u\rangle_0\, d\tau.$$

Hence using the fact that both u and v satisfy the energy inequality, and (42), we get

$$\|w(t)\|^2_{L^2} + 2\nu\int_0^t \|\nabla w\|^2_{L^2}\, d\tau \leq \|w_0\|^2_{L^2} + 2\int_0^t \bigl(\langle v\otimes v, \nabla u\rangle + \langle u\otimes u, \nabla v\rangle\bigr)\, d\tau. \tag{43}$$

If u and v are smooth and divergence green then we may write

$$\langle v\otimes v, \nabla u\rangle + \langle u\otimes u, \nabla v\rangle = \langle u\otimes u, \nabla u\rangle - \langle u\otimes u, \nabla w\rangle + \langle v\otimes v, \nabla w\rangle + \langle v\otimes v, \nabla v\rangle.$$

According to Lemma 2.1, $\langle u\otimes u, \nabla u\rangle = \langle v\otimes v, \nabla v\rangle = 0$. Hence

$$\langle v\otimes v, \nabla u\rangle + \langle u\otimes u, \nabla v\rangle = -\langle u\otimes w, \nabla w\rangle - \langle w\otimes v, \nabla w\rangle.$$

Using again Lemma 2.1, we thus get

$$\langle v\otimes v,\nabla u\rangle+\langle u\otimes u,\nabla v\rangle=-\langle w\otimes v,\nabla w\rangle. \tag{44}$$

In order to justify that Identity (44) still holds under our assumptions, it suffices to argue by density, noticing that the trilinear map

$$(a,b,c)\mapsto\langle a\otimes b,\nabla c\rangle$$

is continuous:

- on $L^4([0,T];H^1)\times L^4([0,T];H^1)\times L^2([0,T];\dot H^1)$,
- on $L^{\frac83}([0,T];L^4)\times L^{\frac83}([0,T];L^4)\times L^4([0,T];\dot H^1)$,
- on $L^4([0,T];L^3)\times L^4([0,T];L^6)\times L^2([0,T];\dot H^1)$.

Resuming to (43), we thus get

$$\|w(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla w\|_{L^2}^2\,d\tau\leq\|w_0\|_{L^2}^2-2\int_0^t\langle w\otimes v,\nabla w\rangle\,d\tau.$$

Finally, combining Hölder and Gagliardo-Nirenberg inequalities allows to write that

$$-\langle w\otimes v,\nabla w\rangle\leq\|\nabla w\|_{L^2}\|w\|_{L^3}\|v\|_{L^6}\leq C\|\nabla w\|_{L^2}^{\frac32}\|w\|_{L^2}^{\frac12}\|v\|_{L^6}.$$

Hence, by using Young inequality,

$$-2\int_0^t\langle Q(w,v),w\rangle\,d\tau\leq\nu\int_0^t\|\nabla w\|_{L^2}^2+\frac{C}{\nu^3}\int_0^t\|v\|_{L^6}^4\|w\|_{L^2}^2\,d\tau,$$

and the desired inequality thus follows from Gronwall lemma. □

Corollary 2.2. *In dimension $d=3$, any Leray solution coincides with Fujita-Kato's solution whenever it is defined.*

Proof. In dimension $d=3$, Fujita-Kato's solution is in $L^4([0,T];\dot H^1)$, and the result thus follows from the previous theorem. □

2.4. **More qualitative properties.**

2.4.1. *A Lyapunov functional.*

Proposition 2.2. *Let u_0 be in the ball of center 0 and radius $c\nu$ of the space $\dot H^{\frac d2-1}(\mathbb{R}^d)$ and u be the corresponding solution of (GNS_ν). Then the function $t\longmapsto\|u(t)\|_{\dot H^{\frac d2-1}}$ is nonincreasing.*

Proof. Denote for $0\leq t_1\leq t_2$,

$$U(t_1,t_2)\stackrel{\text{def}}{=}\|u(t_2)\|_{\dot H^{\frac d2-1}}^2+2\nu\int_{t_1}^{t_2}\|\nabla u(t')\|_{\dot H^{\frac d2-1}}^2\,dt.$$

Because u is a solution of the equation

$$\partial_t u-\nu\Delta u=Q(u,u)\quad\text{with}\quad Q(u,u)\in L^2(\mathbb{R}_+;\dot H^{\frac d2-2}),$$

Lemma 2.2 guarantees that

$$U(t_1,t_2)=\|u(t_1)\|_{\dot H^{\frac d2-1}}^2+2\int_{t_1}^{t_2}\langle Q(u(t),u(t)),u(t)\rangle_{\frac d2-1}\,dt.$$

Using Lemma 2.4 and interpolation inequality, we thus get

$$\begin{aligned}U(t_1,t_2) &\leq \|u(t_1)\|^2_{\dot H^{\frac d2-1}} + C\int_{t_1}^{t_2} \|u(t')\|^2_{\dot H^{\frac{d-1}{2}}}\|\nabla u(t')\|_{\dot H^{\frac d2-1}}\,dt'\\ &\leq \|u(t_1)\|^2_{\dot H^{\frac d2-1}} + C\int_{t_1}^{t_2} \|u(t')\|_{\dot H^{\frac d2-1}}\|\nabla u(t')\|^2_{\dot H^{\frac d2-1}}\,dt'.\end{aligned}$$

By Theorem 2.2, we know that $u(t)$ remains in the ball of center 0 and of radius $2c\nu$ of the space $\dot H^{\frac d2-1}(\mathbb{R}^d)$. Thus, if c is small enough, we get that

$$\|u(t_2)\|^2_{\dot H^{\frac d2-1}} + \nu\int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot H^{\frac d2-1}}\,dt' \leq \|u(t_1)\|^2_{\dot H^{\frac d2-1}}.$$

This proves the proposition. □

2.4.2. *About the structure of the set of data leading to a global solution.*

Theorem 2.5. *Let u be a global solution of (NS_ν) in $L^4_{loc}(\mathbb{R}^+;\dot H^1)$. Then we have*

$$\lim_{t\to\infty}\|u(t)\|_{\dot H^{\frac12}} = 0 \quad\textit{and}\quad \int_0^\infty \|u(t)\|^4_{\dot H^1}\,dt<\infty.$$

Proof. We know that if $\|u_0\|_{\dot H^{\frac12}}$ satisfies the smallness condition of Theorem 2.2 then the global solution associated to the Cauchy data u_0 belongs to the space $L^4(\mathbb{R}^+;\dot H^1)$. Hence, it suffices to prove that $\lim_{t\to\infty}\|u(t)\|_{\dot H^{\frac12}}=0$.

For fixed given $\rho>0$, we decompose the initial data u_0 into

$$u_0 = u_{0,h}+u_{0,\ell} \quad\text{with}\quad u_{0,\ell}\stackrel{\text{def}}{=}\mathcal{F}^{-1}(\mathbf{1}_{B(0,\rho)}\widehat u_0).$$

Let ε be any positive real number. We can choose ρ such that

$$\|u_{0,\ell}\|_{\dot H^{\frac12}}\leq \min\Big\{c\nu,\ \frac\varepsilon2\Big\}.$$

Let us denote by u_ℓ the global solution of (NS_ν) given by Theorem 2.2 for the initial data $u_{0,\ell}$. Thanks to Proposition 2.2, we have

$$\forall t\in\mathbb{R}^+,\ \|u_\ell(t)\|_{\dot H^{\frac12}}\leq \min\Big\{c\nu,\ \frac\varepsilon2\Big\}. \tag{45}$$

Let us define $u_h\stackrel{\text{def}}{=}u-u_\ell$. It satisfies

$$\begin{cases}\partial_t u_h-\nu\Delta u_h &= Q_{NS}(u,u_h)+Q_{NS}(u_h,u_\ell),\\ \quad u_{h|t=0} &= u_{0,h}.\end{cases}$$

Obviously, $u_{0,h}$ belongs to L^2 (with an L^2 norm which does depend on ρ and thus on ε). Moreover, both $Q_{NS}(u,u_h)$ and $Q_{NS}(u_h,u_\ell)$ belong to the space $L^2_{loc}(\mathbb{R}^+;\dot H^{-1})$. Applying Lemma 2.2 and Lemma 2.1 gives

$$\|u_h(t)\|^2_{L^2}+2\nu\int_0^t\|\nabla u_h(t')\|^2_{L^2}\,dt' = \|u_{0,h}\|^2_{L^2}+2\int_0^t\langle Q_{NS}(u_h(t'),u_\ell(t')),u_h(t')\rangle_{\dot H^{-1}\times\dot H^1}\,dt'.$$

From Sobolev embedding, we infer that

$$\begin{aligned}\Big|\langle Q_{NS}(u_h(t),u_\ell(t)),u_h(t)\rangle_{\dot H^{-1}\times\dot H^1}\Big| &\leq C\|u_h(t)u_\ell(t)\|_{L^2}\|\nabla u_h(t)\|_{L^2}\\ &\leq C\|u_h(t)\|_{L^6}\|u_\ell(t)\|_{L^3}\|\nabla u_h(t)\|_{L^2}\\ &\leq C\|u_\ell(t)\|_{\dot H^{\frac12}}\|\nabla u_h(t)\|^2_{L^2}.\end{aligned}$$

Then we deduce that

$$\|u_h(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla u_h(t')\|_{L^2}^2\,dt'\leq\|u_{0,h}\|_{L^2}^2+C\varepsilon\int_0^t\|\nabla u_h(t')\|_{L^2}^2\,dt'.$$

Choosing ε small enough ensures that

$$\|u_h(t)\|_{L^2}^2+\nu\int_0^t\|\nabla u_h(t')\|_{L^2}^2\,dt'\leq\|u_{0,h}\|_{L^2}^2.$$

This implies that a positive time t_ε exists such that $\|u_h(t_\varepsilon)\|_{\dot H^{\frac12}}<\varepsilon/2$. Thus $\|u(t_\varepsilon)\|_{\dot H^{\frac12}}\leq\varepsilon$. Theorem 2.2 and Proposition 2.2 allow to conclude the proof. □

Theorem 2.5 has the following interesting consequence:

Corollary 2.3. *The set of initial data u_0 such that the solution u given by Theorem 2.2 is global is an open subset of $\dot H^{\frac12}$.*

Proof. Let u_0 in $\dot H^{\frac12}$ be such that the associated solution is global. Let w_0 be in $\dot H^{\frac12}$. Denote by v the maximal local solution pertaining to the initial data $v_0\stackrel{\text{def}}{=}u_0+w_0$. The function $w\stackrel{\text{def}}{=}v-u$ is solution of

$$\left\{\begin{array}{rcl}\partial_t w-\nu\Delta w&=&Q_{NS}(u,w)+Q_{NS}(w,u)+Q_{NS}(w,w)\\ w_{|t=0}&=&w_0.\end{array}\right.$$

Lemma 2.4 together with interpolation inequality give

$$\begin{array}{rcl}\langle Q_{NS}(u,w)+Q_{NS}(w,u),w\rangle_{\dot H^{\frac12}}&\leq&C\|u\|_{\dot H^1}\|w\|_{\dot H^{\frac12}}^{\frac12}\|\nabla w\|_{\dot H^{\frac12}}^{\frac32},\\ \langle Q_{NS}(w,w),w\rangle_{\dot H^{\frac12}}&\leq&C\|w\|_{\dot H^{\frac12}}\|\nabla w\|_{\dot H^{\frac12}}^2.\end{array}$$

Let us assume that $\|w_0\|_{\dot H^{\frac12}}\leq\dfrac{\nu}{8C}$ and define

$$T_{w_0}\stackrel{\text{def}}{=}\sup\Big\{t\,/\,\max_{0\leq t'\leq t}\|w(t')\|_{\dot H^{\frac12}}\leq\frac{\nu}{4C}\Big\}\cdot$$

Then, from Lemma 2.2 and the convexity inequality $ab\leq\dfrac14a^4+\dfrac34b^{\frac43}$, we infer that, for any $t<T_{w_0}$,

$$\|w(t)\|_{\dot H^{\frac12}}^2+\nu\int_0^t\|\nabla w(t')\|_{\dot H^{\frac12}}^2\,dt'\leq\|w_0\|_{\dot H^{\frac12}}^2+\frac{C}{\nu^3}\int_0^t\|u(t')\|_{\dot H^1}^4\|w(t')\|_{\dot H^{\frac12}}^2\,dt'.$$

Gronwall Lemma and Theorem 2.5 imply that, for any $t<T_{w_0}$,

$$\|w(t)\|_{\dot H^{\frac12}}^2+\nu\int_0^t\|\nabla w(t')\|_{\dot H^{\frac12}}^2\,dt'\leq\|w_0\|_{\dot H^{\frac12}}^2\exp\Big(\frac{C}{\nu^3}\int_0^t\|u(t')\|_{\dot H^1}^4\,dt'\Big).$$

Now, according to Theorem 2.5, u is in $L^4(\mathbb{R}^+;\dot H^1)$. Hence one can conclude that if the smallness condition

$$\|w_0\|_{\dot H^{\frac12}}^2\exp\Big(\frac{C}{\nu^3}\int_0^\infty\|u(t)\|_{\dot H^1}^4\,dt\Big)\leq\frac{\nu^2}{16C^2},$$

is satisfied, then the blow up condition for v is never satisfied. Corollary 2.3 is proved. □

3. Littlewood-Paley decomposition and functional spaces

Here we introduce the Littlewood-Paley decomposition, define Besov spaces, list their most important properties and finally establish product and composition estimates (more details may be found in e.g. [1, 30, 32].

3.1. **Littlewood-Paley decomposition.** The Littlewood-Paley decomposition is a dyadic localization procedure in the frequency space for tempered distributions over $\mathbb{R}^d$. One of the main motivations for using it when dealing with PDEs is that the derivatives act almost as dilations on distributions with Fourier transform supported in a ball or an annulus.

In the L^2 framework, this noticeable property easily follows from Parseval's formula. The *Bernstein inequalities* below state that it is also true in the L^p framework:

Proposition 3.1 (Bernstein inequalities). *For all $0 < r < R$, we have:*

- *Direct Bernstein inequality: a constant C exists so that, for any $k \in \mathbb{N}$, any couple (p,q) in $[1,\infty]^2$ with $q \geq p \geq 1$ and any function u of L^p with $\widehat{u}$ supported in the ball $B(0,\lambda R)$ of $\mathbb{R}^d$ for some $\lambda > 0$, we have*
$$\|D^k u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p};$$
- *Reverse Bernstein inequality: there exists a constant C so that for any $k \in \mathbb{N}$, $p \in [1,\infty]$ and any function u of L^p with $\operatorname{Supp} \widehat{u} \subset \{\xi \in \mathbb{R}^d \,/\, r\lambda \leq |\xi| \leq R\lambda\}$ for some $\lambda > 0$, we have*
$$\lambda^k \|u\|_{L^p} \leq C^{k+1} \|D^k u\|_{L^p}.$$

Proof. Changing variables (set $v(x) = u(\lambda^{-1}x)$) reduces the proof to the case $\lambda = 1$. For proving the first inequality, we fix some smooth ϕ with compact support, and value 1 over $B(0,R)$. One may thus write

$$\widehat{u} = \phi\,\widehat{u} \quad \text{whence} \quad D^k u = (D^k \mathcal{F}^{-1}\phi) \star u.$$

Therefore using convolution inequalities, one may write

$$\|D^k u\|_{L^q} \leq \|D^k \mathcal{F}^{-1}\phi\|_{L^r} \|u\|_{L^p}$$

with $1 + 1/q = 1/p + 1/r$ (here we need $q \geq p$), and we are done.

For proving the second inequality, we now assume that ϕ is compactly supported away from the origin and has value 1 over the annulus $\mathcal{C}(0,r,R)$. We thus have

$$\widehat{u} = \left(-i\frac{\xi}{|\xi|^2}\phi(\xi)\right) \cdot \widehat{\nabla u(\xi)}.$$

Therefore, denoting by g the inverse Fourier transform of the first term in the r.h.s.,

$$\|u\|_{L^p} \leq \|g\|_{L^1} \|\nabla u\|_{L^p}.$$

This gives the result for $k = 1$. The general case follows by induction. □

As solutions to nonlinear PDE's need not be spectrally localized in annuli (even if we restrict to initial data with this property), it is suitable to have a device which allows for splitting any function into a sum of functions with this spectral localization. This is exactly what Littlewood-Paley decomposition does.

To construct it, fix some smooth radial non increasing function χ with $\operatorname{Supp}\chi\subset B(0,\frac43)$ and $\chi\equiv1$ on $B(0,\frac34),$ then set $\varphi(\xi)=\chi(\xi/2)-\chi(\xi)$ so that

$$\chi+\sum_{j\in\mathbb{N}}\varphi(2^{-j}\cdot)=1\ \text{ in }\ \mathbb{R}^d\qquad\text{and}\qquad\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\cdot)=1\ \text{ in }\ \mathbb{R}^d\setminus\{0\}.$$

The homogeneous dyadic blocks $\dot\Delta_j$ are defined by

$$\dot\Delta_j u:=\varphi(2^{-j}D)u:=\mathcal{F}^{-1}(\varphi(2^{-j}D)\mathcal{F}u):=2^{jd}h(2^j\cdot)\star u\quad\text{with}\quad h:=\mathcal{F}^{-1}\varphi.$$

We also introduce the low frequency cut-off operator $\dot S_j$:

$$\dot S_j u:=\chi(2^{-j}D)u:=\mathcal{F}^{-1}(\chi(2^{-j}D)\mathcal{F}u):=2^{jd}\check h(2^j\cdot)\star u\quad\text{with}\quad \check h:=\mathcal{F}^{-1}\chi.$$

The nonhomogeneous dyadic blocks Δ_j are defined by

$$\Delta_j:=\dot\Delta_j\ \text{ if }\ j\geq0,\quad \Delta_{-1}:=\dot S_0=\chi(D)\quad\text{ and }\ \Delta_j=0\ \text{ if }\ j\leq-2,$$

and we set

$$S_j:=\sum_{k\leq j-1}\Delta_k.$$

The homogeneous and nonhomogeneous Littlewood-Paley decompositions for u read

$$u=\sum_j\dot\Delta_j u\quad\text{and}\quad u=\sum_j\Delta_j u. \tag{46}$$

The second equality holds true in the set $\mathcal{S}'$ of tempered distributions. This is not the case of the first one which is true *modulo polynomials only* if no further assumptions on u. A way to overcome this is to restrict to the set $\mathcal{S}'_h$ of tempered distributions u such that

$$\lim_{j\to-\infty}\|\dot S_j u\|_{L^\infty}=0\quad\text{with}\ \ \dot S_j:=\chi(2^{-j}D).$$

Note that loosely speaking, this condition on the low frequencies of u amounts to requiring u to tend to 0 at infinity (in the sense of distributions). Then the first equality (46) holds true whenever u is in $\mathcal{S}'_h$.

Finally, let us emphasize that the support properties of φ and χ entail properties of quasi-orthogonality for the Littlewood-Paley decomposition. With the normalization that we adopted here, namely $\operatorname{Supp}\varphi\subset C(0,3/4,8/3)$ and $\operatorname{Supp}\chi\subset B(0,4/3)$, one may easily check that

$$\dot\Delta_j\dot\Delta_k=0\ \text{ if }\ |j-k|>1\qquad\text{and}\qquad\dot\Delta_k(\dot S_{j-1}u\,\dot\Delta_j v)\equiv0\ \text{ if }\ |k-j|>4.$$

3.2. **Functional spaces.** Many classical norms may be written in terms of the Littlewood-Paley decomposition. This is e.g. the case of homogeneous Sobolev or Hölder norms:

Proposition 3.2. *For any $s\in\mathbb{R}$ there exists a constant $C\geq1$ so that for any tempered distribution u,*

$$C^{-1}\|u\|^2_{\dot H^s}\leq\sum_j 2^{2js}\|\dot\Delta_j u\|^2_{L^2}\leq C\|u\|^2_{\dot H^s}.$$

If $s\in(0,1)$ then we have (assuming in addition that $u=\sum_j\dot\Delta_j u$ for the left inequality)

$$C^{-1}\|u\|_{\dot C^{0,s}}\leq\sup_j 2^{js}\|\dot\Delta_j u\|_{L^\infty}\leq C\|u\|_{\dot C^{0,s}}.$$

Proof. Owing to $\operatorname{Supp}\varphi(2^{-j}\cdot)\cap\operatorname{Supp}(2^{-k}\cdot)=\emptyset$ if $|j-k|>1$, we have

$$\frac{1}{2}\leq\sum_j\varphi^2(2^{-j}\xi)\leq 1\quad\text{for}\quad\xi\neq 0.$$

Hence, using the definition of homogeneous Sobolev norm, of $\dot\Delta_j u$ and Parseval equality[7]

$$\|u\|_{\dot H^s}^2=\int|\xi|^2|\widehat{u}(\xi)|^2\,d\xi\approx\sum_j\int|\xi|^2|\varphi(2^{-j})\widehat{u}(\xi)|^2\,d\xi\approx\sum_j 2^{2js}\|\dot\Delta_j u\|_{L^2}^2.$$

As for the homogeneous Hölder norm, we notice that because h has average 0,

$$\dot\Delta_j u(x)=2^{jd}\int h(2^j(x-y))\,(u(y)-u(x))\,dy\quad\text{for all}\quad j\in\mathbb{Z}.$$

Hence for all $x\in\mathbb{R}^d$ and $j\in\mathbb{Z}$,

$$|\dot\Delta_j u(x)|\leq 2^{-js}\|u\|_{\dot C^{0,s}}\biggl(2^{jd}\int|h(2^j(x-y))|(2^j|x-y|)^s\,dy\biggr)\leq 2^{-js}\|u\|_{\dot C^{0,s}}\||\cdot|^s h\|_{L^1}.$$

Conversely, if $C_s(u):=\sup_j 2^{js}\|\dot\Delta_j u\|_{L^\infty}<\infty$ then we may write for any $N\in\mathbb{Z}$,

$$u(y)-u(x)=\sum_{j<N}(\dot\Delta_j u(y)-\dot\Delta_j u(x))+\sum_{j\geq N}(\dot\Delta_j u(y)-\dot\Delta_j u(x)).$$

Hence

$$|u(y)-u(x)|\leq|y-x|\sum_{j<N}\|\nabla\dot\Delta_j u\|_{L^\infty}+2\sum_{j\geq N}\|\dot\Delta_j u\|_{L^\infty}.$$

Therefore, taking advantage of Bernstein inequality for the terms in the first sum,

$$|u(y)-u(x)|\leq C_s(u)\biggl(|y-x|\sum_{j<N}2^{j(1-s)}+2\sum_{j\geq N}2^{-js}\biggr).$$

Then taking the "best" N yields $\|u\|_{\dot C^{0,s}}\leq CC_s(u)$. □

If looking at those two characterizations, we see that three parameters come into play: the regularity parameter s, the Lebesgue exponent that is used for $\dot\Delta_j u$ and the type of summation that this done over $\mathbb{Z}$. This observation motivates the following definition:

Definition 3.1. *For $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$, we set*

$$\|u\|_{\dot B^s_{p,r}}:=\biggl(\sum_j 2^{rjs}\|\dot\Delta_j u\|_{L^p}^r\biggr)^{\frac1r}\quad\textit{if}\quad r<\infty\quad\textit{and}\quad\|u\|_{\dot B^s_{p,\infty}}:=\sup_j 2^{js}\|\dot\Delta_j u\|_{L^p}.$$

We then define the homogeneous Besov space $\dot B^s_{p,r}$ as the subset of distributions $u\in\mathcal{S}'_h$ such that $\|u\|_{\dot B^s_{p,r}}<\infty$.

Similarly we set

$$\|u\|_{B^s_{p,r}}:=\biggl(\sum_j 2^{rjs}\|\Delta_j u\|_{L^p}^r\biggr)^{\frac1r}\quad\textit{if}\quad r<\infty\quad\textit{and}\quad\|u\|_{B^s_{p,\infty}}:=\sup_j 2^{js}\|\Delta_j u\|_{L^p},$$

and define the nonhomogeneous Besov space $B^s_{p,r}$ as the subset of distributions $u\in\mathcal{S}'$ such that $\|u\|_{B^s_{p,r}}<\infty$.

[7] With the convention that $A\approx B$ means that there exists some harmless positive constant C such that $C^{-1}A\leq B\leq CA$.

According to this definition, the space $\dot B^s_{2,2}$ coincides with the homogeneous Sobolev space $\dot H^s$ and it is true that $\dot B^r_{\infty,\infty}$ is the homogeneous Hölder space $\dot C^{0,r}$ if $r\in(0,1)$.

The following lemma ensures that for spectrally localized series, proving that the sum of the series belongs to some space $B^s_{p,r}$ (or $\dot B^s_{p,r}$) amounts to getting a suitable L^p bound for each term of the series. Consequently, the definition of Besov spaces is *independent* of the choice of $(\Delta_j)_{j\in\mathbb{Z}}$ or $(\dot\Delta_j)_{j\in\mathbb{Z}}$. It also turns out to be very useful for proving nonlinear estimates (see Section 3.4).

Lemma 3.1. *Let* $0<R_1<R_2$. *Let* $s\in\mathbb{R}$ *and* $1\leq p,r\leq\infty$. *Let* $(u_j)_{j\geq-1}$ *be such that* $\operatorname{Supp}\widehat{u_{-1}}\subset B(0,R_2)$ *and* $\operatorname{Supp}\widehat u_j\subset 2^j\mathcal{C}(0,R_1,R_2)$ *for all* $j\in\mathbb{N}$. *Then*

$$\Big\|2^{js}\|u_j\|_{L^p(\mathbb{R}^d)}\Big\|_{\ell^r(\mathbb{N}\cup\{-1\})}<\infty\quad\Longrightarrow\quad u:=\sum_{j\geq-1}u_j\ \text{ is in }B^s_{p,r}$$

and we have $\|u\|_{B^s_{p,r}}\lesssim\Big\|2^{js}\|u_j\|_{L^p(\mathbb{R}^d)}\Big\|_{\ell^r(\mathbb{N}\cup\{-1\})}$.

If $s>0$ *then the result is still true under the weaker assumption that* $\operatorname{Supp}\widehat u_j\subset B(0,2^jR_2)$.

A similar statement holds true in the homogeneous setting if assuming in addition that $\sum_{j<0}u_j$ *converges in* $\mathcal{S}'_h$.

Proof. We notice that one may find some integer N depending only on R_1,R_2 such that for all $k\geq-1$,

$$\Delta_k u=\sum_{|j-k|\leq N}\Delta_k u_j.$$

Therefore

$$\|\Delta_k u\|_{L^p}\leq\sum_{|j-k|\leq N}\|\Delta_k u_j\|_{L^p}\leq C\sum_{|j-k|\leq N}\|u_j\|_{L^p}$$

and we get the result.

If we only have $\operatorname{Supp}\widehat u_j\subset B(0,2^jR_2)$ then we just have for some integer N,

$$\Delta_k u=\sum_{j\geq k-N}\Delta_k u_j.$$

Therefore

$$2^{ks}\|\Delta_k u\|_{L^p}\leq\sum_{j\geq k-N}2^{(k-j)s}\,2^{js}\|u_j\|_{L^p}$$

and the convolution inequality $\ell^1\star\ell^r\to\ell^r$ gives the result if $s>0$. □

3.3. **A few properties of Besov spaces.** The following proposition states that, loosely speaking, having u in $\dot B^s_{p,r}$ means that u has s fractional derivatives in L^p (see the proof in e.g. [1]):

Proposition 3.3 (Characterization by finite differences)**.** *For* $s\in]0,1[$ *and finite* p,r, *we have*

$$\|u\|_{\dot B^s_{p,r}}\approx\biggl(\int_{\mathbb{R}^d}\biggl(\int_{\mathbb{R}^d}\biggl(\frac{|u(y)-u(x)|}{|y-x|^s}\biggr)^p\frac{dy}{|x-y|^d}\biggr)^{\frac rp}\biggr)^{\frac1r}.$$

A similar result holds for p *or* r *infinite.*

For negative indices, homogeneous Besov spaces may be characterized by means of the heat semi-group (see e.g. [1]):

Theorem 3.1. *Let s be a positive real number and $(p,r)\in[1,\infty]^2$. A constant C exists which satisfies*

$$C^{-1}\|u\|_{\dot B^{-2s}_{p,r}}\leq\Big\|\|t^se^{t\Delta}u\|_{L^p}\Big\|_{L^r(\mathbb{R}^+,\frac{dt}{t})}\leq C\|u\|_{\dot B^{-2s}_{p,r}}\quad\textit{for all}\ \ u\in\mathcal{S}'_h.$$

Homogenous Besov norms have the following scaling invariance properties:

Proposition 3.4. *For any $s\in\mathbb{R}$ and $(p,r)\in[1,+\infty]^2$ there exists a constant C such that for all positive λ and $u\in\dot B^s_{p,r}$, we have*

$$C^{-1}\lambda^{s-\frac{d}{p}}\|u\|_{\dot B^s_{p,r}}\leq\|u(\lambda\cdot)\|_{\dot B^s_{p,r}}\leq C\lambda^{s-\frac{d}{p}}\|u\|_{\dot B^s_{p,r}}.$$

Here are some classical embedding properties:

Proposition 3.5.
(1) *For any $p\in[1,\infty]$ we have the following chain of continuous embedding: $\dot B^0_{p,1}\hookrightarrow L^p\hookrightarrow\dot B^0_{p,\infty}$;*
(2) *If $s\in\mathbb{R}$, $1\leq p_1\leq p_2\leq\infty$ and $1\leq r_1\leq r_2\leq\infty$, then $\dot B^s_{p_1,r_1}\hookrightarrow\dot B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$;*
(3) *The space $\dot B^{\frac{d}{p}}_{p,1}$ is continuously embedded in the set $\mathcal{C}_b$ of bounded continuous functions. If $p<\infty$ then it is also embedded in the set of continuous functions going to 0 at infinity.*

Proof. The left embedding of the first property follows from the triangle inequality for the L^p norm applied to

$$u=\sum_j\dot\Delta_ju$$

whereas the right inequality is a consequence of the convolution property $L^1\star L^p\to L^p$ which implies that

$$\|\dot\Delta_ju\|_{L^p}\leq\|2^{jd}h(2^j\cdot)\|_{L^1}\|u\|_{L^p}=\|h\|_{L^1}\|u\|_{L^p}.$$

As for the second property, we just have to use that, owing to Bernstein inequality,

$$\|\dot\Delta_ju\|_{L^{p_2}}\leq C2^{j(\frac{d}{p_1}-\frac{d}{p_2})}\|\dot\Delta_ju\|_{L^{p_1}}.$$

Finally, by combining the first two properties, we see that

$$\dot B^{\frac{d}{p}}_{p,1}\hookrightarrow B^0_{\infty,1}\hookrightarrow L^\infty.$$

Note in particular that this implies that if $u\in\dot B^{\frac{d}{p}}_{p,1}$ then the series $\sum_{j\in\mathbb{Z}}\dot\Delta_ju$ converges *uniformly* to u. As each term of the series is continuous and bounded, the same property holds for u. Finally, if $p<\infty$ then each term $\dot\Delta_ju$ goes to 0 at infinity (because it is in L^p), hence so does u. □

Here is a list of classical (and important) properties of Besov spaces that are of constant use in these notes (see the proof in e.g. [1]):

- The space $\dot B^s_{p,r}$ is complete whenever $s<d/p$ or $s\leq d/p$ and $r=1$, and so does $B^s_{p,r}$ without any condition on (s,p,r).

- Fatou property: if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions of $\dot B^s_{p,r}$ that converges in the sense of tempered distributions to some $u\in\mathcal{S}'_h$ then $u\in\dot B^s_{p,r}$ and $\|u\|_{\dot B^s_{p,r}}\leq C\liminf\|u_n\|_{\dot B^s_{p,r}}.$ A similar result holds in $B^s_{p,r}$ (where $u\in\mathcal{S}'$ is enough).
- Duality: If u is in $\mathcal{S}'_h$ then we have
$$\|u\|_{\dot B^s_{p,r}}\leq C\sup_\phi\langle u,\phi\rangle$$
where the supremum is taken over those ϕ in $\mathcal{S}\cap\dot B^{-s}_{p',r'}$ such that $\|\phi\|_{\dot B^{-s}_{p',r'}}\leq 1.$
- The following real interpolation properties are satisfied for all $1\leq p,r_1,r_2,r\leq\infty,$ $s_1\neq s_2$ and $\theta\in(0,1)$:
$$[\dot B^{s_1}_{p,r_1},\dot B^{s_2}_{p,r_2}]_{(\theta,r)}=\dot B^{\theta s_2+(1-\theta)s_1}_{p,r}\quad\text{and}\quad[B^{s_1}_{p,r_1},B^{s_2}_{p,r_2}]_{(\theta,r)}=B^{\theta s_2+(1-\theta)s_1}_{p,r}.$$
- For any smooth homogeneous of degree m function F on $\mathbb{R}^d\setminus\{0\}$ the Fourier multiplier $F(D)$ maps $\dot B^s_{p,r}$ in $\dot B^{s-m}_{p,r}$. In particular, the gradient operator maps $\dot B^s_{p,r}$ in $\dot B^{s-1}_{p,r}$.
- For any smooth function G on $\mathbb{R}^d$, which is homogeneous of degree m outside some ball centered at the origine, the Fourier multiplier $G(D)$ maps $B^s_{p,r}$ in $B^{s-m}_{p,r}$. In particular, the gradient operator maps $B^s_{p,r}$ in $B^{s-1}_{p,r}$.

3.4. **Nonlinear estimates.** The basic questions that we address are: let u and v belong to two Besov spaces:

- Does uv make sense ?
- If so, where does uv lie ?
- If F is a suitably smooth function, what can be said of $F(u)$?

Formally, the product of two distributions u and v may be decomposed into

$$uv=T_uv+R(u,v)+T_vu \tag{47}$$

with

$$T_uv:=\sum_j S_{j-1}u\Delta_jv\quad\text{and}\quad R(u,v):=\sum_j\sum_{|j'-j|\leq1}\Delta_ju\,\Delta_{j'}v.$$

The above operator T is called "paraproduct" whereas R is called "remainder". The decomposition (47) has been first introduced by J.-M. Bony in [2]. However, the discrete version of it that we shall present below is due to P. Gérard and J. Rauch in [17]. As we shall see below, it is of particular efficiency inasmuch as the three terms of (47) may be treated separately. In particular, as T_uv is a sum of product of functions with different spectral localizations, it is always defined (because in Fourier variables, the sum is locally finite and Lemma 3.1 applies). At the same time, it cannot be smoother than what is given by high frequencies, namely v. As for the remainder, it may be not defined (e.g. product of Dirac masses at the same point). However, if it is defined then it may be smoother than the paraproduct term. All this is detailed in the proposition below.

Proposition 3.6. *For any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ and $t<0$ we have*[8]
$$\|T_uv\|_{B^s_{p,r}}\lesssim\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}\quad\textit{and}\quad\|T_uv\|_{B^{s+t}_{p,r}}\lesssim\|u\|_{B^t_{\infty,\infty}}\|v\|_{B^s_{p,r}}.$$
For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R}\times[1,\infty]^2$ we have

[8]The sign $\lesssim$ means that the l.h.s. is bounded by the r.h.s. up to a harmless multiplicative constant.

- *if* $s_1+s_2>0,\ 1/p:=1/p_1+1/p_2\leq 1$ *and* $1/r:=1/r_1+1/r_2\leq 1$ *then*

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}}\lesssim \|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}};$$

- *if* $s_1+s_2=0,\ 1/p:=1/p_1+1/p_2\leq 1$ *and* $1/r_1+1/r_2\geq 1$ *then*

$$\|R(u,v)\|_{B^0_{p,\infty}}\lesssim \|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.$$

Similar results hold in homogeneous Besov spaces.

Proof. We just prove the first result of continuity for T and R. Both are consequences of Lemma 3.1. We first notice that the general term of T_uv is supported in dyadic annuli whereas that of $R(u,v)$ is only supported in dyadic balls. Now, we see that

$$\|S_{j-1}u\Delta_j u\|_{L^p}\leq \|S_{j-1}u\|_{L^\infty}\|\Delta_j u\|_{L^p}\leq C\|u\|_{L^\infty}\|\Delta_j v\|_{L^p},$$

and thus

$$\|(2^{js}\|S_{j-1}u\Delta_j v\|_{L^p})\|_{\ell^r}\leq C\|u\|_{L^\infty}\|(2^{js}\|\Delta_j v\|_{L^p})\|_{\ell^r}$$

hence Lemma 3.1 gives the result.

For proving the first continuity result for R, we may write that

$$2^{j(s_1+s_2)}\|\Delta_j u\widetilde{\Delta_j}v\|_{L^p}\leq (2^{js_1}\|\Delta_j u\|_{L^{p_1}})\,(2^{js_2}\|\widetilde{\Delta_j}v\|_{L^{p_2}})$$

and use the last part of Lemma 3.1 together with convolution inequalities. □

Putting together decomposition (47) and the above results of continuity, one may deduce a number of continuity results for the product of two functions. For instance, one may get the following *tame estimate* which depends *linearly* on the highest norm of u and v:

Corollary 3.1. *Let u and v be in $L^\infty\cap B^s_{p,r}$ for some $s>0$ and $(p,r)\in[1,\infty]^2$. Then there exists a constant C depending only on d, p and s and such that*

$$\|uv\|_{B^s_{p,r}}\leq C\big(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}+\|v\|_{L^\infty}\|u\|_{B^s_{p,r}}\big).$$

Proof. We proceed as follows:

1. Write Bony's decomposition $uv=T_uv+T_vu+R(u,v)$;
2. Use $T:L^\infty\times B^s_{p,r}\to B^s_{p,r}$;
3. Use $R:B^0_{\infty,\infty}\times B^s_{p,r}\to B^s_{p,r}$ if $s>0$;
4. Take advantage of $L^\infty\hookrightarrow B^0_{\infty,\infty}$.

This completes the proof. □

Remark 3.1. *Consequently, because $B^{d/p}_{p,1}$ and $\dot B^{d/p}_{p,1}$ are embedded in L^∞, we deduce that* both spaces $B^{d/p}_{p,1}$ and $\dot B^{d/p}_{p,1}$ are (quasi) Banach algebra *if* $p<\infty$.

Let us finally state a composition result.

Proposition 3.7. *Let $F:\mathbb{R}\to\mathbb{R}$ be a smooth function with $F(0)=0$. Then for all $(p,r)\in[1,\infty]^2$ and all $s>0$, there exists a constant C such that for all $u\in B^s_{p,r}\cap L^\infty$ we have $F(u)\in B^s_{p,r}$ and*

$$\|F(u)\|_{B^s_{p,r}}\leq C\|u\|_{B^s_{p,r}}$$

with C depending only on $\|u\|_{L^\infty}$, F, s, p and d.

Proof. We use Meyers's first linearization method:

$$F(u)=\sum_j F(S_{j+1}u)-F(S_ju)=\sum_j \underbrace{\Delta_j u\int_0^1 F'(S_ju+\tau\Delta_j u)\,d\tau}_{u_j}.$$

We note that

$$\|u_j\|_{L^p}\leq C\|\Delta_j u\|_{L^p}.$$

However, $\mathcal{F}u_j$ is not localized in a ball of size 2^j hence Lemma 3.1 does not apply. Nevertheless, one may show that u_j behaves as if it were as regards differentiation: we have

$$\|D^k u_j\|_{L^p}\leq C_k 2^{jk}\|\Delta_j u\|_{L^p}\quad\text{for all}\quad k\in\mathbb{N}.$$

This family of inequalities allows to complete the proof (see the details in e.g. [1]). □

4. Fourier analysis for a few linear PDEs, and applications

Here we give examples of estimates for different types of linear PDEs that are frequently encountered when dealing with fluid mechanics models. For simplicity, we just consider the heat and transport equations. More examples may be found in e.g. [1].

4.1. **Parabolic equations and incompressible Navier-Stokes equations.**

4.1.1. *The linear heat equation.* Consider the heat equation

$$\partial_t u-\Delta u=f,\qquad u_{|t=0}=u_0. \tag{48}$$

We wonder if there exists a functional space X in which one may "gain two full derivatives with respect to the data", getting

$$\|u\|_{L^\infty(X)}+\|\partial_t u,D^2u\|_{L^1(X)}\leq C\big(\|u_0\|_X+\|f\|_{L^1(X)}\big). \tag{49}$$

Proposition 3.3 gives some insight, but not the whole panorama when X is a homogeneous Besov space. That gain of two derivatives compared to the source term when performing a L^1-in-time integration turns out to be the key to a number of well-posedness results in a critical functional framework for models arising in fluid mechanics.

From the standard parabolic maximal regularity, one knows that if $r\in(1,\infty)$ and $X=L^q$ or $\dot W^{s,q}$ for some $s\in\mathbb{R}$ and $q\in(1,\infty)$ then

$$\|\partial_t u,D^2u\|_{L^r(X)}\leq C\|f\|_{L^r(X)}.$$

However those inequalities fail for the endpoint case $r=1$, and, more generally, in any reflexive Banach space X.

J.-Y. Chemin in [6] noticed that (49) is true for Besov spaces *with third index* 1. This is stated in the following theorem.

Theorem 4.1. *Estimates* (49) *hold true for any* $p\in[1,\infty]$, $\sigma\in\mathbb{R}$ *and* $s\in\mathbb{R}$ *if* $X=\dot B^s_{p,1}$.

Proving the theorem relies on the following:

Lemma 4.1. *There exist two positive constants* c *and* C *such that for any* $j\in\mathbb{Z}$, $p\in[1,\infty]$ *and* $\lambda\in\mathbb{R}^+$, *we have*

$$\|e^{\lambda\Delta}\dot\Delta_j\|_{\mathcal{L}(L^p;L^p)}\leq Ce^{-c\lambda 2^{2j}}.$$

Proof. If $p=2$ this is a mere consequence of Fourier-Plancherel theorem.

In the general case, one may first reduce the proof to the case $j=0$ (just perform a suitable change of variable) then consider a function ϕ in $\mathcal{D}(\mathbb{R}^d\setminus\{0\})$ with value 1 on a neighborhood of the support of φ so as to write

$$\begin{aligned} e^{\lambda\Delta}\dot\Delta_0 u &= \mathcal{F}^{-1}\Big(\phi e^{-\lambda|\cdot|^2}\widehat{\dot\Delta_0 u}\Big),\\ &= g_\lambda\star\dot\Delta_0 u \quad\text{with}\quad g_\lambda(x):=(2\pi)^{-d}\int e^{i(x|\xi)}\phi(\xi)e^{-\lambda|\xi|^2}d\xi. \end{aligned}$$

If it is true that

$$\|g_\lambda\|_{L^1}\leq Ce^{-c\lambda} \tag{50}$$

then using the convolution inequality $L^1\star L^p\to L^p$ implies that

$$\|e^{\lambda\Delta}\dot\Delta_0 u\|_{L^p}\leq\|g_\lambda\|_{L^1}\|\dot\Delta_0 u\|_{L^p}\leq Ce^{-c\lambda}\|\dot\Delta_0 u\|_{L^p},$$

and we are done.

Proving (50) follows from integration by parts: we have

$$g_\lambda(x)=(1+|x|^2)^{-d}\int_{\mathbb{R}^d}e^{i(x|\xi)}(\mathrm{Id}\,-\Delta_\xi)^d\Big(\phi(\xi)e^{-\lambda|\xi|^2}\Big)d\xi.$$

Therefore, combining Leibniz and Faá-di-Bruno's formulae, we may conclude that

$$|g_\lambda(x))|\leq C(1+|x|^2)^{-d}e^{-c\lambda},$$

which obviously implies (50). □

Proof of Theorem 4.1 : If u satisfies (48) then for any $j\in\mathbb{Z}$,

$$\partial_t\dot\Delta_j u-\Delta\dot\Delta_j u=\dot\Delta_j f.$$

Hence, according to Duhamel's formula

$$\dot\Delta_j u(t)=e^{t\Delta}\dot\Delta_j u_0+\int_0^t e^{(t-\tau)\Delta}\dot\Delta_j f(\tau)\,d\tau.$$

Taking advantage of Lemma 4.1, we thus have

$$\|\dot\Delta_j u(t)\|_{L^p}\lesssim e^{-c2^{2j}t}\|\dot\Delta_j u_0\|_{L^p}+\int_0^t e^{-c2^{2j}(t-\tau)}\|\dot\Delta_j f(\tau)\|_{L^p}\,d\tau. \tag{51}$$

Multiplying by 2^{js} and summing up over j yields

$$\sum_j 2^{js}\|\dot\Delta_j u(t)\|_{L^p}\lesssim\sum_j e^{-c2^{2j}t}2^{js}\|\dot\Delta_j u_0\|_{L^p}+\int_0^t e^{-c2^{2j}(t-\tau)}\sum_j\|\dot\Delta_j f(\tau)\|_{L^p}\,d\tau$$

whence

$$\|u\|_{L^\infty_t(\dot B^s_{p,1})}\lesssim\|u_0\|_{\dot B^s_{p,1}}+\|f\|_{L^1_t(\dot B^s_{p,1})}.$$

Note that integrating (51) with respect to time also yields

$$2^{2j}\|\dot\Delta_j u\|_{L^1_t(L^p)}\lesssim\Big(1-e^{-c2^{2j}t}\Big)\Big(\|\dot\Delta_j u_0\|_{L^p}+\|\dot\Delta_j f\|_{L^1_t(L^p)}\Big).$$

Therefore, multiplying by 2^{js} and summing up over j yields

$$\|u\|_{L^1_t(\dot B^{s+2}_{p,1})}\lesssim\sum_j\Big(1-e^{-c2^{2j}t}\Big)2^{js}\Big(\|\dot\Delta_j u_0\|_{L^p}+\|\dot\Delta_j f\|_{L^1_t(L^p)}\Big), \tag{52}$$

which is slightly better than what we wanted to prove.[9] □

Remark 4.1. *Let us point out that starting from* (51) *and using more general convolution inequalities gives a whole family of estimates for the heat equation. However, as time integration has been performed* before *summation over* j, *the norms that naturally appear are not those of the spaces* $L^a(0,t;\dot{B}^\sigma_{b,c})$ *but rather*

$$\|\cdot\|_{\widetilde{L}^a_t(\dot{B}^\sigma_{b,c})} := \Big\|2^{j\sigma}\|\cdot\|_{L^a_t(L^b)}\Big\|_{\ell^c}$$

where $\|\cdot\|_{L^a_t(L^b)} := \|\cdot\|_{L^a((0,t);L^b(\mathbb{R}^d))}$.

Therefore (51) *implies that*

$$\|u\|_{\widetilde{L}^{\rho_1}_t(\dot{B}^{s+\frac{2}{\rho_1}}_{p,r})} \lesssim \|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{\widetilde{L}^{\rho_2}_t(\dot{B}^{s-2+\frac{2}{\rho_2}}_{p,r})} \quad \text{for } \ 1\leq\rho_2\leq\rho_1\leq\infty.$$

Of course those quantities may be compared with the norm in $L^a(0,t;\dot{B}^\sigma_{b,c})$ *according to Minkowski inequality: we have*

$$\begin{aligned}\|\cdot\|_{\widetilde{L}^a_t(\dot{B}^\sigma_{b,c})} &\leq \|\cdot\|_{L^a_t(\dot{B}^\sigma_{b,c})} \quad \text{if } \ a\leq c\\ \|\cdot\|_{L^a_t(\dot{B}^\sigma_{b,c})} &\leq \|\cdot\|_{\widetilde{L}^a_t(\dot{B}^\sigma_{b,c})} \quad \text{if } \ a\geq c.\end{aligned}$$

4.1.2. *An application: the incompressible Navier-Stokes equations in critical Besov spaces.* The results that we want to present here hold not only for the incompressible Navier-Stokes equations (NS_ν) but also for generalized Navier-Stokes equations:

$$(GNS_\nu) \qquad \partial_t u + Q(u,u) - \nu\Delta u = 0$$

introduced in Section 2.

Recall that all the coefficients entering in the definition of $Q(u,u)$ are first order homogeneous Fourier multipliers. From the point of view of homogeneous Besov spaces, the action of such multipliers is exactly the same as that of the gradient operator.

We prove below an existence statement in *scaling invariant homogeneous Besov spaces*[10].

Theorem 4.2. *Let* $u_0\in\dot{B}^{\frac{d}{p}-1}_{p,r}$ *with* $\operatorname{div} u_0=0$. *Assume that* p *is finite. There exist two positive constants* c *and* C *such that if*

$$\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} \leq c\nu$$

then (GNS_ν) *has a unique global solution* u *in the space*[11]

$$X := \widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,r}) \cap \widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,r})$$

satisfying

$$\|u\|_X := \|u\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}-1}_{p,r})} + \nu\|u\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}+1}_{p,r})} \leq C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}}.$$

[9] As obviously $(1-e^{-c2^{2j}t})$ is bounded by 1 and tends to 0 when t goes to 0^+.

[10] Proposition 3.4 ensures that the norms in Theorem 4.2 are indeed invariant by (36).

[11] see the definition of tilde norms in Remark 4.1.

Proof. We want to apply the abstract lemma with $X = \widetilde{L}^\infty(\mathbb{R}^+; \dot{B}^{\frac{d}{p}-1}_{p,r}) \cap \widetilde{L}^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}+1}_{p,r})$,

$$v_0(t) := e^{\nu t\Delta} u_0 \quad \text{and} \quad \mathcal{B}(u,v)(t) := -\int_0^t e^{\nu(t-\tau)\Delta} Q(u,v)\, d\tau.$$

Heat estimates (see Theorem 4.1 and Remark 4.1) imply that

$$\|v_0\|_X \le C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}}.$$

That $\mathcal{B} : X \times X \to X$ is a consequence of embedding properties in Besov spaces, and of continuity results for paraproduct and remainder (see Proposition 3.6). Indeed we have

$$\|Q(u,v)\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}-1}_{p,r})} \le C\|u \otimes v\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}}_{p,r})}.$$

Hence, using the fact that[12]

$$\begin{aligned} \|R(u,v)\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}}_{p,r})} &\lesssim \|u\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}-1}_{p,r})} \|v\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}+1}_{p,r})} \\ \|T_u v\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}}_{p,r})} &\lesssim \|u\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,\infty})} \|v\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}+1}_{p,r})} \end{aligned}$$

and a similar inequality for $T_v u$, and because $\dot{B}^{\frac{d}{p}-1}_{p,r} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$, we eventually find that, for some $C = C(d,p,Q)$:

$$\|Q(u,v)\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}-1}_{p,r})} \le C\nu^{-1}\|u\|_X\|v\|_X.$$

Hence

$$\|\mathcal{B}(u,v)\|_X \le C'\nu^{-1}\|u\|_X\|v\|_X.$$

Therefore $\mathcal{B}$ satisfies (35) provided $\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} \le c\nu$ with c small enough. □

Remark 4.2. *By implementing the fixed point argument in a neighborhood of the free solution* $e^{\nu t\Delta}u_0$*, one may establish a local-in-time result of strong solutions for general* u_0 *in critical Besov spaces. The details are left to the reader.*

4.2. **Estimates for the linear transport equation.** In this paragraph, we focus on the proof of a priori estimates for the following *transport equation* that plays a fundamental role in fluid mechanics:

$$(T_\lambda) \qquad \begin{cases} \partial_t a + v\cdot\nabla a + \lambda a = f \\ a_{|t=0} = a_0 \end{cases}$$

where λ is a given nonnegative parameter.

[12] The product laws for $\widetilde{L}^\rho(\dot{B}^\sigma_{p,r})$ work the same as the usual ones, the time Lebesgue exponent just behaves according to Hölder inequality. Note that for the remainder we need that $(d/p-1)+(d/p+1) > 0$ hence $p < \infty$.

4.2.1. *Estimates in Besov spaces.* Roughly, if v is a Lipschitz time-dependent vector-field, and if $a_0\in X$ and $f\in L^1(0,T;X)$, with X a Banach space then we expect (T) to have a unique solution $a\in\mathcal{C}([0,T);X)$ satisfying (if $\lambda=0$):

$$\|a(t)\|_X\leq e^{CV(t)}\biggl(\|a_0\|_X+\int_0^t e^{-CV(\tau)}\|f(\tau)\|_X\,d\tau\biggr)$$

$$\text{with (say)}\ \ V(t):=\int_0^t\|\nabla v(\tau)\|_{L^\infty}\,d\tau. \tag{53}$$

This is quite obvious if X is the Hölder space $C^{0,\varepsilon}$ with $\varepsilon\in(0,1)$ as (in the case $f\equiv0$ to simplify) the solution to (T_0) is given by

$$a(t,x)=a_0(\psi_t^{-1}(x))$$

where ψ_t stands for the flow of v at time t.

Therefore,

$$\begin{aligned}|a(t,x)-a(t,y)| &= |a_0(\psi_t^{-1}(x))-a_0(\psi_t^{-1}(y))|,\\ &\leq \|a_0\|_{\dot C^{0,\varepsilon}}|\psi_t^{-1}(x)-\psi_t^{-1}(y)|^\varepsilon,\\ &\leq \|a_0\|_{\dot C^{0,\varepsilon}}\|\nabla\psi_t^{-1}\|_{L^\infty}^\varepsilon|x-y|^\varepsilon.\end{aligned}$$

As $\|\nabla\psi_t^{-1}\|_{L^\infty}\leq\exp(V(t))$, we get the result in this particular case.

Littlewood-Paley decomposition will enable us to prove a similar result in a much more general framework.

Theorem 4.3. *Assume that*

$$1\leq p\leq p_1\leq\infty,\quad 1\leq r\leq\infty,\quad -\min\Bigl(\frac{d}{p_1},\frac{d}{p'}\Bigr)<s<1+\frac{d}{p_1}\cdot$$

Then any smooth enough solution to (53) *satisfies the a priori estimate*

$$\|a\|_{\widetilde L_t^\infty(\dot B^s_{p,r})}+\lambda\|a\|_{\widetilde L_t^1(\dot B^s_{p,r})}\leq e^{CV(t)}\Bigl(\|a_0\|_{\dot B^s_{p,r}}+\|f\|_{\widetilde L_t^1(\dot B^s_{p,r})}\Bigr)$$

with

$$V(t)=\int_0^t\|\nabla v(\tau)\|_{\dot B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}\,d\tau.$$

If $r=1$ (resp. $r=\infty$) then the case $s=1+d/p_1$ (resp. $s=-\min\bigl(\frac{d}{p_1},\frac{d}{p'}\bigr)$) with $V(t)=\int_0^t\|\nabla v(\tau)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\,d\tau$ also works.

If $\operatorname{div}v=0$ then the lower bound for s is $s>-1-\min(d/p_1,d/p')$, with the convention that $v\cdot\nabla a\overset{def}{=}\operatorname{div}(va)$.

In the nonhomogeneous framework, there is no upper bound for s. However, we have to take $V(t)=\int_0^t\|\nabla v(\tau)\|_{B^{s-1}_{p_1,r}}\,d\tau$ if $s>1+d/p_1$.

Proof. Applying $\dot\Delta_j$ to (T) gives

$$\partial_t\dot\Delta_ja+v\cdot\nabla\dot\Delta_ja+\lambda\dot\Delta_ja=\dot\Delta_jf+\dot R_j\quad\text{with}\quad \dot R_j:=[v\cdot\nabla,\dot\Delta_j]a. \tag{54}$$

In the case $p\in(1,\infty)$, multiplying both sides by $|\dot\Delta_j a|^{p-2}\dot\Delta_j a$ and integrating over $\mathbb{R}^d$ yields

$$\frac{1}{p}\frac{d}{dt}\|\dot\Delta_j a\|_{L^p}^p+\lambda\|\dot\Delta_j a\|_{L^p}^p+\frac{1}{p}\int v\cdot\nabla|\dot\Delta_j a|^p\,dx=\int\bigl(\dot\Delta_j f+R_j\bigr)|\dot\Delta_j a|^{p-2}\dot\Delta_j a\,dx.$$

Therefore

$$\|\dot\Delta_j a(t)\|_{L^p}+\lambda\|\dot\Delta_j a\|_{L^1_t(L^p)}\leq\|\dot\Delta_j a_0\|_{L^p}+\int_0^t\Bigl(\|\dot\Delta_j f\|_{L^p}+\|\dot R_j\|_{L^p}+\frac{\|\mathrm{div}\,v\|_{L^\infty}}{p}\|\dot\Delta_j a\|_{L^p}\Bigr)\,d\tau. \tag{55}$$

Having p tend to 1 or ∞ implies that (55) also holds if $p=1$ or $p=\infty$.

For simplicity, let us just consider the case $-\min(d/p_1,d/p')<s<1+d/p_1$. Then the remainder term $\dot R_j$ satisfies

$$\|\dot R_j(t)\|_{L^p}\leq Cc_j(t)2^{-js}\|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}\|a(t)\|_{\dot B^s_{p,r}}\quad\text{with}\quad\|(c_j(t))\|_{\ell^r}=1. \tag{56}$$

This may be proved by taking advantage of Bony's decomposition. Indeed we have (with the summation convention over repeated indices):

$$\dot R_j=[T_{v^k},\dot\Delta_j]\partial_k a+T_{\partial_k\dot\Delta_j a}v^k-\dot\Delta_j T_{\partial_k a}v^k+R(v^k,\partial_k\dot\Delta_j a)-\dot\Delta_j R(v^k,\partial_k a).$$

Let us just explain how to bound the first term which is the only one where having a commutator improves the estimates (bounding the other terms mainly stems from Lemma 3.1 or Proposition 3.6). Owing to the properties of spectral localization, we have

$$[T_{v^k},\dot\Delta_j]\partial_k a=\sum_{|j-j'|\leq4}[\dot S_{j'-1}v^k,\dot\Delta_j]\partial_k\dot\Delta_{j'}a.$$

Let $h:=\mathcal{F}^{-1}\varphi$. Remark that

$$[\dot S_{j'-1}v^k,\dot\Delta_j]\partial_k\dot\Delta_{j'}a(x)=2^{jd}\int_{\mathbb{R}^d}h(2^j(x-y))\bigl(\dot S_{j'-1}v^k(x)-\dot S_{j'-1}v^k(y)\bigr)\partial_k\dot\Delta_{j'}a(y)\,dy.$$

Hence, according to the mean value formula,

$$[\dot S_{j'-1}v^k,\dot\Delta_j]\partial_k\dot\Delta_{j'}a(x)=2^{jd}\int_{\mathbb{R}^d}\int_0^1 h(2^j(x-y))\bigl((x-y)\cdot\nabla\dot S_{j'-1}v^k(y+\tau(x-y))\bigr)\partial_k\dot\Delta_{j'}a(y)\,d\tau\,dy.$$

So finally,

$$\|[T_{v^k},\dot\Delta_j]\partial_k a\|_{L^p}\lesssim 2^{-j}\|\nabla v\|_{L^\infty}\sum_{|j'-j|\leq4}\|\partial_k\dot\Delta_{j'}a\|_{L^p}\lesssim\|\nabla v\|_{L^\infty}\sum_{|j'-j|\leq4}\|\dot\Delta_{j'}a\|_{L^p}.$$

Let us resume to (54). Using (55) and (56), multiplying by 2^{js} then summing up over j yields

$$\|a\|_{\widetilde L^\infty_t(\dot B^s_{p,r})}+\lambda\|a\|_{\widetilde L^1_t(\dot B^s_{p,r})}\leq\|a_0\|_{\dot B^s_{p,r}}+\int_0^t\|f\|_{\dot B^s_{p,r}}\,d\tau+C\int_0^t V'\|a\|_{\dot B^s_{p,r}}\,d\tau$$

with $\|a\|_{\widetilde L^\infty_t(\dot B^s_{p,r})}:=\bigl\|2^{js}\|\dot\Delta_j a\|_{L^\infty_t(L^p)}\bigr\|_{\ell^r}$ and $\|a\|_{\widetilde L^1_t(\dot B^s_{p,r})}:=\bigl\|2^{js}\|\dot\Delta_j a\|_{L^1_t(L^p)}\bigr\|_{\ell^r}$.

Then applying Gronwall's lemma yields the desired inequality for a.

In the divergence free case, $\dot{R}_j$ may be decomposed as follows:

$$\dot{R}_j = [T_{v^k}, \dot{\Delta}_j]\partial_k a + T_{\partial_k \dot{\Delta}_j a} v^k - \dot{\Delta}_j T_{\partial_k a} v^k + \partial_k R(v^k, \dot{\Delta}_j a) - \dot{\Delta}_j \partial_k R(v^k, a).$$

which allows to decrease by 1 the lower bound for s, by virtue of the continuity results for the remainder.

The proof in the nonhomogeneous case is in the same spirit. □

One may wonder if, for certain choices of norms, one may avoid the exponential term (due to our using Gronwall lemma) in the estimates for the solution to the transport equation. The answer is yes if $X = L^p$ if in addition $\operatorname{div} v = 0$. In the case $\lambda = 0$, we just have

$$\|a(t)\|_{L^p} \leq \|a_0\|_{L^p} + \int_0^t \|f\|_{L^p}\, d\tau.$$

Does this still work in the spaces $\dot{B}^0_{p,r}$ which are very close to L^p ? In order to solve the problem, we shall follow the *dynamic interpolation method* of T. Hmidi and S. Keraani [22]. The starting point is that the linearity of the transport equation implies that $a = \sum_j a_j$ with

$$\partial_t a_j + v \cdot \nabla a_j = 0, \qquad a_{j|t=0} = \dot{\Delta}_j a_0.$$

Even though a_j is spectrally localized at time $t = 0$, there is no reason why this should be so for $t \neq 0$. Therefore (we focus on estimates in $\dot{B}^0_{p,1}$ for simplicity, $\dot{B}^0_{p,r}$ works the same) one may just write that

$$\|a(t)\|_{\dot{B}^0_{p,1}} \leq \sum_j \|a_j(t)\|_{\dot{B}^0_{p,1}} \leq \sum_{j,k} \|\dot{\Delta}_k a_j(t)\|_{L^p}.$$

Fix some integer N. Then we may write

$$\|a\|_{\dot{B}^0_{p,1}} \leq \sum_{|j-k|\leq N} \|\dot{\Delta}_k a_j\|_{L^p} + \sum_{|j-k|>N} \|\dot{\Delta}_k a_j\|_{L^p}.$$

The first sum may be bounded by $CN\|a_j\|_{L^p}$ and we know that $\|a_j(t)\|_{L^p} = \|\dot{\Delta}_j a_0\|_{L^p}$. For the second sum, we use the fact that according to Theorem 4.3,

$$\|a_j(t)\|_{\dot{B}^{\pm\frac12}_{p,\infty}} \leq \|\dot{\Delta}_j a_0\|_{\dot{B}^{\pm\frac12}_{p,\infty}} e^{C\int_0^t \|\nabla v\|_{L^\infty}\, d\tau}.$$

Coming back to the definition of $\dot{B}^0_{p,\infty}$, this yields

$$\|\dot{\Delta}_k a_j(t)\|_{L^p} \leq 2^{\pm(\frac{j-k}{2})} \|\dot{\Delta}_j a_0\|_{L^p} e^{C\int_0^t \|\nabla v\|_{L^\infty}\, d\tau}.$$

Choosing $\pm$ for each couple (j,k), in such way that we get the exponent $-\frac{|j-k|}{2}$, we eventually discover that

$$\|a(t)\|_{\dot{B}^0_{p,1}} \leq C\|a_0\|_{\dot{B}^0_{p,1}} \Big(N + 2^{-N/2} e^{C\int_0^t \|\nabla v\|_{L^\infty}\, d\tau} \Big).$$

Taking N so that $C2^{-N/2}e^{C\int_0^t \|\nabla v\|_{L^\infty}\, d\tau} \approx 1$ implies

$$\|a(t)\|_{\dot{B}^0_{p,1}} \leq \|a_0\|_{\dot{B}^0_{p,1}} \Big(1 + C\int_0^t \|\nabla v\|_{L^\infty}\, d\tau \Big). \tag{57}$$

Inequality (57) is of particular interest in the study of the lifespan of solutions to critical nonlinear PDEs. We shall see an example in the next paragraph.

4.2.2. *An application to the 2D incompressible Euler equation.* We use the vorticity formulation

$$(E) \qquad \partial_t \omega + \operatorname{div}(v\,\omega) = 0$$

with $v \stackrel{\text{def}}{=} B(\omega)$ given by the Biot-Savart law:

$$v \stackrel{\text{def}}{=} BS(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,dy \quad \text{with} \quad z^\perp \stackrel{\text{def}}{=} (-z_2, z_1). \tag{58}$$

Note that Biot-Savart law also writes $v = -\nabla^\perp(-\Delta)^{-1}\omega$ with $\nabla^\perp = (-\partial_2, \partial_1)$.

We shall use repeatedly the following lemma:

Lemma 4.2. *Let $v = BS(\omega)$.*

1. *If $\omega \in L^1 \cap L^\infty$ then $v \in L^\infty$ and*
$$\|v\|_{L^\infty} \leq C\sqrt{\|\omega\|_{L^1}\|\omega\|_{L^\infty}}.$$
2. *If $\omega \in B^s_{p,r}$ with $1 < p < \infty$ then $\nabla v \in B^s_{p,r}$ and*
$$\|\nabla v\|_{B^s_{p,r}} \leq C\|\omega\|_{B^s_{p,r}}.$$
3. *If $p \in \{1, \infty\}$ then we have $\|\nabla v\|_{B^s_{p,r}} \leq C\|\omega\|_{B^s_{p,r}\cap L^1}$.*
4. *There exists a constant C such that for any $\varepsilon \in (0,1)$,*
$$\|\nabla v\|_{L^\infty} \leq C\bigg(\|\omega\|_{L^1} + \frac{1}{\varepsilon}\|\omega\|_{L^\infty}\log\bigg(e + \frac{\|\omega\|_{C^\varepsilon}}{\|\omega\|_{L^\infty}}\bigg)\bigg).$$

Proof. For the first item, we just have to split the integral in (58) into $|x-y| \leq R$ and $|x-y| > R$, and to optimize R.

Proving the second item relies on the fact that $\nabla\nabla^\perp(-\Delta)^{-1}$ is homogeneous of degree 0 and thus associated to a Calderon-Zygmund operator. Ditto for the third item except that we first have to use Bernstein inequality and the first item to bound $\|\nabla\Delta_{-1}v\|_{L^p}$.

Proving the last item is based on Littlewood-Paley decomposition : we write for any $N > 0$,

$$\nabla v = \Delta_{-1}\nabla v + \sum_{0\leq j<N}\Delta_j\nabla v + \sum_{j\geq N}\Delta_j\nabla v.$$

Hence taking the L^∞ norm and using Bernstein inequality for the first term,

$$\|\nabla v\|_{L^\infty} \leq C\|v\|_{L^\infty} + \sum_{0\leq j<N}\|\Delta_j\nabla\nabla^\perp(-\Delta)^{-1}\omega\|_{L^\infty} + \sum_{j\geq N}\|\Delta_j\nabla\nabla^\perp(-\Delta)^{-1}\omega\|_{L^\infty}.$$

Hence

$$\|\nabla v\|_{L^\infty} \leq C\bigg(\|\omega\|_{L^1\cap L^\infty} + \sum_{0\leq j<N}\|\Delta_j\omega\|_{L^\infty} + 2^{-N\varepsilon}\sum_{j\geq N}2^{j\varepsilon}\|\Delta_j\omega\|_{L^\infty}\bigg).$$

Therefore

$$\|\nabla v\|_{L^\infty} \leq C\big(\|\omega\|_{L^1\cap L^\infty} + N\|\omega\|_{L^\infty} + 2^{-N\varepsilon}\|\omega\|_{C^\varepsilon}\big),$$

and choosing N so that $2^{-N\varepsilon}\|\omega\|_{C^\varepsilon} \approx \|\omega\|_{L^\infty}$ gives the desired inequality. □

Theorem 4.4. *Let ω_0 be in $L^1 \cap B^s_{p,r}$ with $B^s_{p,r} \hookrightarrow L^\infty$. Then (E) with data ω_0 admits a unique global solution $\omega \in L^\infty_{loc}(\mathbb{R}; B^s_{p,r})) \cap \mathcal{C}(\mathbb{R}; L^1)$ with constant L^q norms for all $q \in [1,\infty]$ and which is also in $\mathcal{C}(\mathbb{R}; B^s_{p,r})$ if $r < \infty$.*

Proof. As in the equation is time-reversible, we concentrate on positive times.

In order to prove the local in time existence, the easiest way is to use the following iterative scheme:

$$\left\{\begin{array}{l}\partial_t\omega^{n+1}+\mathrm{div}\,(v^n\omega^{n+1})=0, \qquad v^n\stackrel{\mathrm{def}}{=}BS(\omega^n),\\[1ex] \omega^{n+1}|_{t=0}=\omega_0,\end{array}\right. \tag{59}$$

where we agree that $\omega^0(t)\equiv\omega_0$ for all $t\in\mathbb{R}$.

As $\omega_0\in B^s_{p,r}$, Theorem 4.3 and induction ensure that ω^{n+1} is uniquely and globally defined in $L^\infty_{loc}(\mathbb{R};B^s_{p,r})$ with, for some constant $C=C(s,p)$ and all $t\geq0$ (for simplicity):

$$\|\omega^{n+1}(t)\|_{B^s_{p,r}}\leq C\exp\Bigl(C\int_0^t\|\nabla v^n\|_{B^s_{p,r}}\,d\tau\Bigr)\|\omega_0\|_{B^s_{p,r}}.$$

As $\mathrm{div}\,v^n=0$ by construction, it is also clear that

$$\|\omega^{n+1}(t)\|_{L^q}=\|\omega_0\|_{L^q}\quad\text{for all}\quad q\in[1,\infty]. \tag{60}$$

Hence, using Lemma 4.2, we get

$$\|\omega^{n+1}(t)\|_{Y^s}\leq C\exp\Bigl(C\int_0^t\|\omega^n\|_{Y^s}\,d\tau\Bigr)\|\omega_0\|_{Y^s}\quad\text{with}\quad Y^s\stackrel{\mathrm{def}}{=}L^1\cap B^s_{p,r}. \tag{61}$$

A standard bootstrap argument thus ensures that

$$\|\omega^{n+1}(t)\|_{Y^s}\leq 2C\|\omega_0\|_{Y^s}\quad\text{whenever}\quad 2Ct\|\omega_0\|_{Y^s}\leq\log 2. \tag{62}$$

That estimate is thus satisfies on $[0,T_0]$ with

$$T_0=c\|\omega_0\|_{Y^s}^{-1}\quad\text{and}\quad c=(\log 2)/(2C). \tag{63}$$

We next want to prove that the sequence converges on $[0,T_0]$. To this end, we set $\delta\omega^n\stackrel{\mathrm{def}}{=}\omega^{n+1}-\omega^n$, $\delta v^n\stackrel{\mathrm{def}}{=}v^{n+1}-v^n$ and observe that $\delta\omega^n(0)=0$ and that

$$\partial_t\delta\omega^n+\mathrm{div}\,(v^n\delta\omega^n)=-\mathrm{div}\,(\delta v^{n-1}\omega^n). \tag{64}$$

Owing to the r.h.s., one can perform estimates at most in $B^{s-1}_{p,r}$. For technical reasons, it is in fact easier to estimate $\delta\omega^n$ in $B^{-1}_{1,\infty}$ (of course ω^n is in $\mathcal{C}([0,T];L^1)$ hence $\delta\omega^n$, too, and we have obviously $L^1\hookrightarrow B^{-1}_{1,\infty}$).

Now, applying Theorem 4.3 (in the divergence free case) yields

$$e^{-C\int_0^t\|\nabla v^n\|_{B^0_{\infty,1}}\,d\tau}\|\delta\omega^n(t)\|_{B^{-1}_{1,\infty}}\leq\int_0^t e^{-C\int_0^\tau\|\nabla v^n\|_{B^0_{\infty,1}}\,d\tau'}\|\mathrm{div}\,(\delta v^{n-1}\omega^n)\|_{B^{-1}_{1,\infty}}\,d\tau. \tag{65}$$

Now, decomposing δv^{n-1} in $\Delta_{-1}\delta v^{n-1}+\widetilde{\delta v}^{n-1}$, we can write

$$\|\mathrm{div}\,(\delta v^{n-1}\omega^n)\|_{B^{-1}_{1,\infty}}\leq C\bigl(\|\Delta_{-1}\delta v^{n-1}\,\omega^n\|_{B^0_{1,\infty}}+\|\widetilde{\delta v}^{n-1}\,\omega^n\|_{B^0_{1,\infty}}\bigr).$$

For the first stem, we juste have to notice that, by virtue of Lemmas 3.1 and 4.2, and of Bernstein inequality,

$$\begin{aligned}\|\Delta_{-1}\delta v^{n-1}\,\omega^n\|_{B^0_{1,\infty}}&\lesssim\sup_{j\geq-1}\|\Delta_{-1}\delta v^{n-1}\,\Delta_j\omega^n\|_{L^1}\\&\lesssim\|\Delta_{-1}\delta v^{n-1}\|_{L^\infty}\sup_{j\geq-1}\|\Delta_j\omega^n\|_{L^1}\\&\lesssim\|\Delta_{-1}\delta\omega^{n-1}\|_{L^1}\|\omega^n\|_{L^1}.\end{aligned}$$

For the second term, we have from Bony decomposition and the fact that operator $(\mathrm{Id}-\Delta_{-1})\nabla^\perp(-\Delta)^{-1}$ is homogeneous of degree -1 away from a neighborhood of the origine:

$$\begin{aligned}\|\widetilde{\delta v}^{n-1}\,\omega^n\|_{B^0_{1,\infty}} &\leq C\|\widetilde{\delta v}^{n-1}\|_{B^0_{1,\infty}}\|\omega^n\|_{B^s_{p,r}}\\ &\leq C\|\delta\omega^{n-1}\|_{B^{-1}_{1,\infty}}\|\omega^n\|_{B^s_{p,r}}.\end{aligned}$$

Resuming to (65) and using (62) we get for all $t\in[0,T_0]$,

$$e^{-Ct\|\omega_0\|_{B^s_{p,r}}}\|\delta\omega^n(t)\|_{B^{-1}_{1,\infty}}\leq C\int_0^t e^{-C\tau\|\omega_0\|_{B^s_{p,r}}}\|\omega_0\|_{B^s_{p,r}}\|\delta\omega^n\|_{B^{-1}_{1,\infty}}\,d\tau.$$

Summing up over $n\geq 1$ eventually leads to

$$\begin{aligned}e^{-Ct\|\omega_0\|_{Y^s}}\sum_{n\geq1}\|\delta\omega^n(t)\|_{B^{-1}_{1,\infty}}\leq C\int_0^t e^{-C\tau\|\omega_0\|_{Y^s}}\|\omega_0\|_{B^s_{p,r}}\sum_{n\geq1}\|\delta\omega^n\|_{B^{-1}_{1,\infty}}\,d\tau\\ +C\|\omega_0\|_{B^{-1}_{1,\infty}}\|\omega_0\|_{B^s_{p,r}}\int_0^t e^{-C\tau\|\omega_0\|_{Y^s}}\,d\tau.\end{aligned}$$

Gronwall lemma thus yields

$$\sum_{n\geq1}\|\delta\omega^n(t)\|_{B^{s-1}_{p,r}}\leq\frac12\Big(e^{2Ct\|\omega_0\|_{Y^s}}-1\Big)\|\omega_0\|_{B^{-1}_{1,\infty}}.$$

Hence ω^n is a Cauchy sequence in $\mathcal{C}([0,T_0];B^{-1}_{1,\infty})$ and thus converges to some function $\omega\in\mathcal{C}([0,T_0];B^{-1}_{1,\infty})$. Taking advantage of the bounds (60) and (62), we eventually get a local-in-time solution with the desired properties.

Let us turn to the proof of the global existence in the supercritical case $s>d/p$. We denote by T^* the lifespan of the solution generated by ω_0.

For simplicity, we just consider the case where $B^s_{p,r}=C^s$ with $s\in(0,1)$. Then Theorem 4.3 ensures that for all $t\in[0,T^*)$,

$$\|\omega(t)\|_{B^s_{p,r}}\leq\|\omega_0\|_{B^s_{p,r}}\exp\Big(C\int_0^t\|\nabla v\|_{L^\infty}\,d\tau\Big). \tag{66}$$

Now, combining the last item of Lemma 4.2 with (60), we find out that

$$\|\nabla v(t)\|_{L^\infty}\leq C\bigg(\|\omega_0\|_{L^1}+\|\omega_0\|_{L^\infty}\log\Big(e+\frac{\|\omega(t)\|_{C^s}}{\|\omega_0\|_{L^\infty}}\Big)\bigg).$$

Hence, inserting Inequality (66) and performing obvious calculations,

$$\|\nabla v(t)\|_{L^\infty}\leq D_0+C\|\omega_0\|_{L^\infty}\int_0^t\|\nabla v\|_{L^\infty}\,d\tau,$$

where D_0 depending only on the norm of the data.

Therefore Gronwall's lemma implies that

$$\|\nabla v(t)\|_{L^\infty}\leq D_0e^{Ct\|\omega_0\|_{L^\infty}} \tag{67}$$

and resuming to (66) implies that $\omega\in L^\infty([0,T^*);C^s\cap L^1)$ if $T^*<\infty$. Remembering of (63), this enables us to continue ω beyond T^*, which contradicts the maximality of T^*. Hence $T^*=+\infty$ and (67) holds globally. □

We end this paragraph with the proof of the global existence in the case where $\omega_0 \in L^1 \cap B^0_{\infty,1}$. Then (57) (adapted to the nonhomogeneous framework) leads to

$$\|\omega(t)\|_{B^0_{\infty,1}\cap L^1} \leq C\|\omega_0\|_{B^0_{\infty,1}\cap L^1}\biggl(1+\int_0^t \|\nabla u\|_{L^\infty}\,d\tau\biggr)\cdot$$

Now, Lemma 4.2 together with the embedding $B^0_{\infty,1} \hookrightarrow L^\infty$ allows us to bound $\|\nabla u\|_{L^\infty}$, and we end up with

$$\|\omega(t)\|_{B^0_{\infty,1}\cap L^1} \leq C\|\omega_0\|_{B^0_{\infty,1}\cap L^1}\biggl(1+\int_0^t \|\omega\|_{B^0_{\infty,1}\cap L^1}\,d\tau\biggr)\cdot$$

which, according to Gronwall lemma leads to

$$\forall t\in\mathbb{R},\ \|\omega(t)\|_{B^0_{\infty,1}\cap L^1} \leq C\|\omega_0\|_{B^0_{\infty,1}\cap L^1} e^{C\|\omega_0\|_{B^0_{\infty,1}\cap L^1}|t|}.$$

This implies global existence. Note that in contrast with what is given by (67), we here get exponential growth of Besov norm, instead of double exponential one.

5. The compressible Navier-Stokes equations in critical spaces

This section is devoted to proving local well-posedness results for the compressible Navier-Stokes equations governing the evolution of the density $\rho = \rho(t,x) \in \mathbb{R}^+$ and of the velocity field $u = u(t,x) \in \mathbb{R}^d$ of a barotropic viscous compressible fluid:

$$\left\{\begin{array}{l}\partial_t\rho + \mathrm{div}\,(\rho u) = 0,\\ \partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) - \mathrm{div}\,(\mu(\rho)(Du+\nabla u)) - \nabla(\lambda(\rho)\mathrm{div}\,u) + \nabla(P(\rho)) = 0,\\ \rho|_{|t=0} = \rho_0, \qquad u|_{|t=0} = u_0.\end{array}\right. \tag{68}$$

Above x belongs to the whole space $\mathbb{R}^d$ and the time variable t is nonnegative. The notation Du designates the *Jacobian matrix* of u (that is $(Du)_{ij} := \partial_j u^i$) while ∇u stands for the transposed matrix of Du (therefore $Du+\nabla u$ is twice the deformation tensor). The pressure P is a given smooth function of the density. The viscosity coefficients λ and μ are smooth functions of the density and satisfy the conditions

$$\alpha := \min\Bigl(\inf_{\rho>0}(\lambda(\rho)+2\mu(\rho)), \inf_{\rho>0}\mu(\rho)\Bigr) > 0, \tag{69}$$

which ensures the second order operator in the momentum equation of (68) to be uniformly elliptic.

In order to solve (68) locally, we use the "standard" approach for nonlinear hyperbolic systems. As for the incompressible Euler equations, we first prove uniform a priori estimates for the "high norm" of the solution, and next stability estimates for a lower norm.

5.1. **The local existence theory.** This paragraph is dedicated to solving (68) locally in time, in critical spaces. To simplify, we focus on small perturbations of the constant density state $\bar\rho = 1$. Hence, setting $\rho = 1+a,\ \lambda = \lambda(1),\ \mu = \mu(1)$ and $\mathcal{A} = \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$, System (68) rewrites

$$\left\{\begin{array}{l}\partial_t a + u\cdot\nabla a = -(1+a)\mathrm{div}\,u,\\[1ex] \partial_t u - \mathcal{A}u = -u\cdot\nabla u - \nabla(G(a))\\ \qquad\qquad +I(a)(Du+\nabla u)\cdot\nabla a + J(a)\mathrm{div}\,u\,\nabla a + K(a)\Delta u + L(a)\nabla\mathrm{div}\,u\end{array}\right. \tag{70}$$

with

$$I(a)=(1+a)^{-1}\mu'(1+a),\quad J(a)=(1+a)^{-1}\lambda'(1+a),\quad K(a)=(1+a)^{-1}\mu(1+a)-\mu(1),$$
$$L(a)=(1+a)^{-1}(\lambda(1+a)+\mu(1+a))-\lambda(1)-\mu(1)\ \text{ and }\ G'(a)=(P'(1+a))/(1+a).$$

Note that up to a change of the pressure law, the barotropic compressible Navier-Stokes equations are invariant by the rescaling

$$a(t,x)\to a(\lambda^2t,\lambda x),\qquad u(t,x)\to\lambda u(\lambda^2t,\lambda x).$$

In the homogeneous Besov spaces scale, this induces to take

$$a_0\in\dot B^{\frac{d}{p_1}}_{p_1,r_1}\quad\text{and}\quad u_0\in\dot B^{\frac{d}{p_2}-1}_{p_2,r_2}.$$

However, it is not clear that one may solve (70) in such spaces for any choice of p_1,p_2,r_1,r_2. Indeed, first, in order to preclude vacuum (and keep ellipticity of the second order operator in the velocity equation), an L^∞ control for a is needed. In the scale of Besov spaces $\dot B^{\frac{d}{p_1}}_{p_1,r_1}$, having $r_1=1$ is the only choice which ensures (continuous) inclusion in L^∞ As for the velocity equation, a gain of two derivatives is required to handle the term $J(a)\mathcal{A}u$ (as $\mathcal{A}$ is second order). According to Theorem 4.1(that easily extends to operator $\mathcal{A}$ instead of Δ in the whole space setting), we thus have to take $r_2=1$. This is all the more appropriate that this will ensure that $u\in L^1_T(\dot B^{\frac{d}{p_2}+1}_{p_2,1})$, hence that $u\in L^1_T(C^{0,1})$, a property that is needed to transport the initial Besov regularity of a. Finally, owing to the coupling between the mass and velocity equations, we take $p_1=p_2=p$ for simplicity.

So, in short, we want to investigate the well-posedness issue of (70) for data

$$a_0\in\dot B^{\frac{d}{p}}_{p,1}\quad\text{and}\quad u_0\in\dot B^{\frac{d}{p}-1}_{p,1}.$$

According to Theorems 4.1 and 4.3, we expect a to be in $\mathcal{C}([0,T];\dot B^{\frac{d}{p}}_{p,1})$, and u to be in the space

$$E_p(T):=\big\{u\in\mathcal{C}([0,T];\dot B^{\frac{d}{p}-1}_{p,1}),\ \partial_tu,\nabla^2u\in L^1(0,T;\dot B^{\frac{d}{p}-1}_{p,1})\big\}.$$

We shall endow $E_p(T)$ with the norm

$$\|u\|_{E_p(T)}:=\|u\|_{L^\infty_T(\dot B^{\frac{d}{p}-1}_{p,1})}+\|\partial_tu,\nabla^2u\|_{L^1_T(\dot B^{\frac{d}{p}-1}_{p,1})}.$$

Let us now state our first local well-posedness existence result in critical spaces.

Theorem 5.1. *Assume that $a_0\in\dot B^{\frac{d}{p}}_{p,1}$ and that $u_0\in\dot B^{\frac{d}{p}-1}_{p,1}$ with $1\le p<2d$. If in addition $1+a_0$ is bounded away from 0 then (70) has a local-in-time solution (a,u) with a in $\mathcal{C}([0,T];\dot B^{\frac{d}{p}}_{p,1})$ and u in $E_p(T)$.*

To simplify the presentation, we only treat the case where

$$\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\le c \tag{71}$$

for a small enough constant $c=c(p,d,G)$, and prove the uniqueness only in the range $p\in[1,d]$. The reader may refer to [14] for the proof of the statement in its full generality.

The proof mostly relies on the estimates for the transport equation and for the Lamé system that have been presented so far. The important fact is that, if one restricts oneself to local-in-time results for the Cauchy problem, then one may just resort to independent estimates for the density and the velocity.

Before going further into the proof of existence however, let us emphasize that one cannot expect to reduce System (70) to Lemma 2.3. This is due to the hyperbolic nature of the transport equation which entails a loss of one derivative in the Lipschitz-type stability estimates. Hence, existence will rather stem from bounds in high norm for the solution and stability in low norms or, alternately, from Schauder-Tikhonov type fixed point arguments.

5.1.1. *Uniform estimates in large norm.* We assume that (70) with data $a_0 \in \dot B^{\frac dp}_{p,1}$ and $u_0 \in \dot B^{\frac dp-1}_{p,1}$ has a solution (a,u) such that

$$a\in\mathcal{C}([0,T];\dot B^{\frac dp}_{p,1})\quad\text{and}\quad u\in\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1})\cap L^1([0,T];\dot B^{\frac dp+1}_{p,1}).$$

We claim that if a_0 satisfies (71) then this solution may be bounded in terms of the data.

Let $U(T):=\|\nabla u\|_{L^1_T(\dot B^{\frac dp}_{p,1})}$. Estimates for the transport equation imply that

$$\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\leq e^{CU(T)}\biggl(\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\int_0^T e^{-CU}\|(1+a)\mathrm{div}\,u\|_{\dot B^{\frac dp}_{p,1}}\,d\tau\biggr).$$

From product laws in Besov spaces, we have:

$$\|(1+a)\mathrm{div}\,u\|_{\dot B^{\frac dp}_{p,1}}\lesssim(1+\|a\|_{\dot B^{\frac dp}_{p,1}})\|\nabla u\|_{\dot B^{\frac dp}_{p,1}}.$$

Inserting this in the above inequality and applying Gronwall's lemma, we thus get

$$\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\leq e^{CU(T)}\|a_0\|_{\dot B^{\frac dp}_{p,1}}+e^{CU(T)}-1.$$

Hence, for any $\eta>0$, if $e^{CU(T)}-1\leq\eta$ then

$$\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\leq 2\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\eta.$$

Let us now prove estimates for the velocity. From Theorem 4.1, we get

$$\begin{aligned}\|u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})\cap L^1_T(\dot B^{\frac dp+1}_{p,1})}&\lesssim\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\int_0^T\|u\cdot\nabla u+\nabla(G(a))\|_{\dot B^{\frac dp-1}_{p,1}}\,dt\\&+\int_0^T\|I(a)(Du+\nabla u)\cdot\nabla a+J(a)\mathrm{div}\,u\,\nabla a+K(a)\Delta u+L(a)\nabla\mathrm{div}\,u\|_{\dot B^{\frac dp-1}_{p,1}}\,dt.\end{aligned}$$

Product and composition laws in Besov spaces yield if $d>1$ and $1\leq p<2d$,

$$\begin{aligned}\|u\cdot\nabla u\|_{\dot B^{\frac dp-1}_{p,1}}&\lesssim\|u\|_{\dot B^{\frac dp-1}_{p,1}}\|u\|_{\dot B^{\frac dp+1}_{p,1}},\\\|K(a)\Delta u\|_{\dot B^{\frac dp-1}_{p,1}}+\|L(a)\nabla\mathrm{div}\,u\|_{\dot B^{\frac dp-1}_{p,1}}&\lesssim\|a\|_{\dot B^{\frac dp}_{p,1}}\|u\|_{\dot B^{\frac dp+1}_{p,1}},\\\|I(a)(Du+\nabla u)\cdot\nabla a+J(a)\mathrm{div}\,u\,\nabla a\|_{\dot B^{\frac dp-1}_{p,1}}&\lesssim(1+\|a\|_{\dot B^{\frac dp}_{p,1}})\|\nabla u\|_{\dot B^{\frac dp}_{p,1}}\|\nabla a\|_{\dot B^{\frac dp-1}_{p,1}},\\\|\nabla(G(a))\|_{\dot B^{\frac dp-1}_{p,1}}&\lesssim\|a\|_{\dot B^{\frac dp}_{p,1}}.\end{aligned}$$

Hence

$$\|u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+\|u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}\lesssim\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\int_0^T\|u\|_{\dot B^{\frac dp-1}_{p,1}}\|u\|_{\dot B^{\frac dp+1}_{p,1}}\,dt+\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\|u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}+T\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}.$$

The last-but-one term may be absorbed by the left hand-side if $\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}$ is small. According to (71), this may be ensured if $\|a_0\|_{\dot B^{\frac dp}_{p,1}}$ and η are small enough. Note also that applying Gronwall lemma shows that the term with the integral may be eliminated if $U(T)$ is small enough. From this, we conclude that

$$\|u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+\|u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}\leq C\big(\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+T(\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\eta)\big).$$

It is clear that $U(T)$ tends to 0 for T going to 0. However, in order to implement an iterative scheme for solving (70) and get a fixed time interval on which all the terms of the sequences satisfy the above estimates, it is suitable to exhibit a lower bound for T in terms of the data. To achieve it, one may split u into $u_L+\widetilde u$ with u_L solution to

$$\partial_t u_L-\mathcal{A}u_L=0,\qquad u_L|_{t=0}=u_0.$$

We have

$$U(T)\leq\|u_L\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}+\|\widetilde u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}.$$

The first term goes to 0 for T tending to 0 with a decay that may be described according to (52). We expect the second term to be small for T small as $\bar u(0)=0$. In order to get a more accurate information, one may use the fact that $\widetilde u$ satisfies

$$\partial_t\widetilde u-\mathcal{A}\widetilde u=-\widetilde u\cdot\nabla u-u_L\cdot\nabla\widetilde u-u_L\cdot\nabla u_L-\nabla(G(a))+I(a)(Du+\nabla u)\cdot\nabla a+J(a)\mathrm{div}\,u\,\nabla a+K(a)\Delta u+L(a)\nabla\mathrm{div}\,u.$$

By combining Theorem 4.1 and the product laws in Besov spaces, we get

$$\|\widetilde u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}+\|\widetilde u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim\int_0^T\|\widetilde u\|_{\dot B^{\frac dp-1}_{p,1}}\|u\|_{\dot B^{\frac dp+1}_{p,1}}\,d\tau+\|u_L\|_{L^2_T(\dot B^{\frac dp}_{p,1})}\|\widetilde u\|_{L^2_T(\dot B^{\frac dp}_{p,1})}+\|u_L\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}\|u_L\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\|u\|_{L^1_T(\dot B^{\frac dp-1}_{p,1})}+T\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}.\tag{72}$$

Arguing by interpolation yields for any $\beta>0$,

$$\|u_L\|_{L^2_T(\dot B^{\frac dp}_{p,1})}\|\widetilde u\|_{L^2_T(\dot B^{\frac dp}_{p,1})}\leq\beta\|u_L\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}\|\widetilde u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}+C\beta^{-1}\|u_L\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}\|\widetilde u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}.$$

Note that $\|u_L\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}\leq C\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}$. Therefore, taking β small enough, using Gronwall lemma and (71), we conclude by a standard bootstrap argument that the l.h.s. of (72) may be made smaller than any given ε for all $t\in[0,T]$ if, for some α (depending on ε) we have

$$\max(T,\|u_L\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})})\leq\alpha.$$

Hence, according to (52), it suffices to choose $T\in]0,\alpha]$ so that

$$\sum_j\Big(1-e^{-c2^{2j}T}\Big)2^{j(\frac dp-1)}\|\dot\Delta_ju_0\|_{L^p}\lesssim\alpha.$$

This gives a (non so) explicit lower bound for the time interval on which the norm of the solution (a,u) may be bounded in terms of the initial data.

5.1.2. *Stability estimates in small norm.* Consider two solutions (a^1,u^1) and (a^2,u^2) of (70) with the above regularity. The difference $(\delta a,\delta u):=(a^2-a^1,u^2-u^1)$ satisfies

$$\left\{\begin{array}{l}\partial_t\delta a+u^2\cdot\nabla\delta a=\sum_{i=1}^3\delta F_i,\\ \partial_t\delta u-\mathcal A\delta u=\sum_{i=1}^5\delta G_i,\end{array}\right.\tag{73}$$

with $\quad\delta F_1:=-\delta u\cdot\nabla a^1,\qquad \delta F_2:=-\delta a\,\mathrm{div}\,u^2,\qquad \delta F_3:=-(1+a^1)\mathrm{div}\,\delta u,$

$$\delta G_1:=\big(J(a^1)-J(a^2)\big)\mathcal Au^2,\quad \delta G_2:=-J(a^1)\,\mathcal A\delta u,\quad \delta G_3:=-\nabla(G(a^2)-G(a^1)),$$
$$\delta G_4:=-u^2\cdot\nabla\delta u,\qquad \delta G_5:=-\delta u\cdot\nabla u^1.$$

Owing to the hyperbolic nature of the mass equation, one loses one derivative in the stability estimates: indeed, δF_1 has (at most) the same regularity as ∇a^1. This induces also a loss of one derivative for δu (look at δG_1 for instance). Hence, we expect to be able to prove stability estimates only in

$$F_T:=\mathcal C([0,T];\dot B^{\frac dp-1}_{p,1})\times\big(\mathcal C([0,T];\dot B^{\frac dp-2}_{p,1})\cap L^1_T(\dot B^{\frac dp}_{p,1})\big)^d.$$

The most obvious unpleasant effect of this loss of one derivative is that when applying composition and product laws in Besov spaces for bounding the norms of δF_i and δG_j, one has to impose *stronger* conditions on p and on d. Indeed, Proposition 3.6 applied to our situation forces us to assume that

$$d>2\quad\text{and}\quad 1\le p<d$$

then, after a few computation, we conclude that if T and $\|a^1\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}$ are small enough then we have the following stability estimate:

$$\|\delta a\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+\|\delta u\|_{L^\infty_T(\dot B^{\frac dp-2}_{p,1})}+\|\delta u\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\lesssim\|\delta a(0)\|_{\dot B^{\frac dp-1}_{p,1}}+\|\delta u(0)\|_{\dot B^{\frac dp-2}_{p,1}},$$

which implies uniqueness.

The limit case $d=2$ or $p=d$ is more involved as, for instance,

$$\Big(\delta a\in\dot B^0_{d,1}\ \text{ and }\ \mathcal Au^2\in\dot B^0_{d,1}\Big)\Longrightarrow\delta a\mathcal Au^2\in\dot B^{-1}_{d,\infty}\quad\text{only.}\tag{74}$$

Hence estimates have to be performed in a Besov space with third index ∞. On the one hand, this is not a trouble for δa as applying Theorem 4.3 and product laws implies

$$\|\delta a\|_{L^\infty_T(\dot B^0_{d,\infty})}\le\Big(\|\delta a(0)\|_{\dot B^0_{d,\infty}}+(1+\|a^1\|_{L^\infty_T(\dot B^1_{d,1})})\|\delta u\|_{L^1_T(\dot B^1_{d,1})}\Big)e^{\|u^2\|_{L^1_T(\dot B^2_{d,1})}}.$$

On the other hand, owing to (74), one has to generalize Theorem 4.1 to Besov spaces with third index ∞. This is not quite possible. In fact one may only get a control on the following norm:

$$\|\delta u\|_{\widetilde L^1_T(\dot B^1_{d,\infty})}:=\sup_j 2^j\|\dot\Delta_j\delta u\|_{L^1_T(L^d)},$$

which is definitely weaker than $\|\delta u\|_{L^1_T(\dot B^1_{d,1})}$. However, one may prove the following logarithmic interpolation inequality (see [12]):

$$\|\delta u\|_{L^1_T(\dot B^1_{d,1})} \lesssim \|\delta u\|_{\widetilde L^1_T(\dot B^1_{d,\infty})} \log\biggl(e + \frac{\|\delta u\|_{\widetilde L^1_T(\dot B^0_{d,\infty})} + \|\delta u\|_{\widetilde L^1_T(\dot B^2_{d,\infty})}}{\|\delta u\|_{\widetilde L^1_T(\dot B^1_{d,\infty})}}\biggr). \tag{75}$$

Inserting this logarithmic inequality in the estimate for δa and using Osgood lemma, we end up with

$$\|\delta a\|_{L^\infty_t(\dot B^0_{d,\infty})} + \|\delta u\|_{L^\infty_t(\dot B^{-1}_{d,\infty})\cap \widetilde L^1_t(\dot B^1_{d,\infty})} \lesssim \Bigl(\|\delta a(0)\|_{\dot B^0_{d,\infty}} + \|\delta u(0)\|_{\dot B^{-1}_{d,\infty}}\Bigr)^{\exp(-\int_0^t \alpha\, d\tau)}$$

where α is in $L^1(0,T)$ and depends only on the high norms of the two solutions.

From those a priori estimates, it is not difficult to construct an iterative scheme for proving rigorously the local-in-time existence of a solution. We get a sequence of smooth solutions which satisfies (uniformly) the bounds in high norm of the first step, on some fixed time interval. Then resorting to compactness arguments and the Fatou property for Besov spaces allows to prove that this sequence satisfies (70) and belongs to the required space. In the case $p \in [1, d]$, uniqueness follows from the above stability estimates in small norm.

5.2. **The global existence theory.** The linearized barotropic compressible Navier-Stokes equations about the state $(a, u) = (0, 0)$ read

$$\begin{cases} \partial_t a + \operatorname{div} u = 0, \\ \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla a = 0 \end{cases} \quad \text{with} \quad \alpha := P'(1). \tag{76}$$

Applying operators $\mathcal{P}$ and $\mathcal{Q}$ to the second equation and setting $\nu := \lambda + 2\mu$, System (76) translates into

$$\begin{cases} \partial_t a + \operatorname{div} \mathcal{Q}u = 0, \\ \partial_t \mathcal{Q}u - \nu\Delta\mathcal{Q}u + \alpha\nabla a = 0, \\ \partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = 0. \end{cases} \tag{77}$$

In the homogeneous Besov spaces setting, it is equivalent to bound $\mathcal{Q}u$ or $v := |D|^{-1}\operatorname{div} u$, the advantage of the latter quantity being that it is real valued. So we are led to considering

$$\begin{cases} \partial_t a + |D|v = 0, \\ \partial_t v - \nu\Delta v - \alpha|D|a = 0, \\ \partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = 0. \end{cases}$$

Note that the vorticity part of the velocity field $\mathcal{P}u$ satisfies a mere heat equation with constant diffusion, the property of which is well described by Theorem 4.1. So we have to concentrate on the first two equations, namely

$$\begin{cases} \partial_t a + |D|v = 0, \\ \partial_t v - \nu\Delta v - \alpha|D|a = 0. \end{cases} \tag{78}$$

Taking the Fourier transform with respect to the space variable yields

$$\frac{d}{dt}\begin{pmatrix} \widehat a \\ \widehat v \end{pmatrix} = A(\xi)\begin{pmatrix} \widehat a \\ \widehat v \end{pmatrix} \quad \text{with} \quad A(\xi) := \begin{pmatrix} 0 & -|\xi| \\ \alpha|\xi| & -\nu|\xi|^2 \end{pmatrix}.$$

The characteristic polynomial of $A(\xi)$ is $X^2+\nu|\xi|^2X+\alpha|\xi|^2$, the discriminant of which is

$$\delta(\xi):=|\xi|^2(\nu^2|\xi|^2-4\alpha).$$

If $\alpha<0$ then there is one positive eigenvalue hence the linear system is unstable. Therefore we assume from now that $\alpha>0$ (i.e. $P'(1)>0$), that is we focus on the case where the pressure law is increasing in some neighborhood of the reference density. Note also that changing v into $v/\sqrt{\alpha}$ reduces the study to the case $\alpha=1$, an assumption that we shall make from now on.

The low frequency regime $\nu|\xi|<2$. There are two distinct *complex conjugated* eigenvalues:

$$\lambda_\pm(\xi)=-\frac{\nu|\xi|^2}{2}(1\pm iS(\xi))\quad\text{with}\quad S(\xi):=\sqrt{\frac{4}{\nu^2|\xi|^2}-1},$$

and after some computations we find that

$$\begin{aligned}\widehat{a}(t,\xi)=e^{t\lambda_-(\xi)}\left(\frac12\Big(1+\frac{i}{S(\xi)}\Big)\widehat{a}_0(\xi)-\frac{i}{\nu|\xi|S(\xi)}\widehat{v}_0(\xi)\right)&\\ +e^{t\lambda_+(\xi)}\left(\frac12\Big(1-\frac{i}{S(\xi)}\Big)\widehat{a}_0(\xi)+\frac{i}{\nu|\xi|S(\xi)}\widehat{v}_0(\xi)\right)&,\end{aligned}$$

$$\begin{aligned}\widehat{v}(t,\xi)=e^{t\lambda_-(\xi)}\left(\frac{i}{\nu|\xi|S(\xi)}\widehat{a}_0(\xi)+\frac12\Big(1-\frac{i}{S(\xi)}\Big)\widehat{v}_0(\xi)\right)&\\ +e^{t\lambda_+(\xi)}\left(-\frac{i}{\nu|\xi|S(\xi)}\widehat{a}_0(\xi)+\frac12\Big(1+\frac{i}{S(\xi)}\Big)\widehat{v}_0(\xi)\right)&.\end{aligned}$$

For $\xi\to0$, we have

$$\begin{aligned}\widehat{a}(t,\xi)&\sim\tfrac12e^{t\lambda_-(\xi)}\left(\widehat{a}_0(\xi)-i\widehat{v}_0(\xi)\right)+\tfrac12e^{t\lambda_+(\xi)}\left(\widehat{a}_0(\xi)+i\widehat{v}_0(\xi)\right),\\ \widehat{v}(t,\xi)&\sim\tfrac12e^{t\lambda_-(\xi)}\left(i\widehat{a}_0(\xi)+\widehat{v}_0(\xi)\right)+\tfrac12e^{t\lambda_+(\xi)}\left(-i\widehat{a}_0(\xi)+\widehat{v}_0(\xi)\right).\end{aligned}$$

Hence, the low frequencies of a and v have a similar behavior. As $|e^{t\lambda_\pm(\xi)}|=e^{-\nu t|\xi|^2/2}$ applying Parseval's formula gives

$$\|(\dot\Delta_ja,\dot\Delta_jv)(t)\|_{L^2}\leq Ce^{-c\nu t2^{2j}}\|(\dot\Delta_ja_0,\dot\Delta_jv_0)\|_{L^2}\ \text{whenever}\ \ 2^j\nu\leq1.\tag{79}$$

In other words, in the L^2 framework, the low frequencies of (a,u) behave as if satisfying the heat equation with diffusion ν. This is no longer true in L^p with $p\neq2$ however, for the eigenvalues have a nonzero imaginary part.

The high frequency regime $\nu|\xi|>2$. There are two distinct real eigenvalues:

$$\lambda_\pm(\xi):=-\frac{\nu|\xi|^2}{2}(1\pm R(\xi))\quad\text{with}\quad R(\xi):=\sqrt{1-\frac{4}{\nu^2|\xi|^2}}$$

and we find that

$$\begin{aligned}\widehat{a}(t,\xi)=e^{t\lambda_-(\xi)}\left(\frac12\Big(1+\frac{1}{R(\xi)}\Big)\widehat{a}_0(\xi)-\frac{1}{\nu|\xi|R(\xi)}\widehat{v}_0(\xi)\right)&\\ +e^{t\lambda_+(\xi)}\left(\frac12\Big(1-\frac{1}{R(\xi)}\Big)\widehat{a}_0(\xi)+\frac{1}{\nu|\xi|R(\xi)}\widehat{v}_0(\xi)\right)&,\end{aligned}$$

$$\widehat{v}(t,\xi)=e^{t\lambda_-(\xi)}\left(\frac{1}{\nu|\xi|R(\xi)}\widehat{a}_0(\xi)+\frac{1}{2}\Big(1-\frac{1}{R(\xi)}\Big)\widehat{v}_0(\xi)\right)$$
$$+e^{t\lambda_+(\xi)}\left(-\frac{1}{\nu|\xi|R(\xi)}\widehat{a}_0(\xi)+\frac{1}{2}\Big(1+\frac{1}{R(\xi)}\Big)\widehat{v}_0(\xi)\right).$$

For $|\xi|\to\infty$, we have $R(\xi)\to 1$ and $1-R(\xi)\sim 2/(\nu\xi)^2$. Hence $\lambda_+(\xi)\sim-\nu|\xi|^2$ and $\lambda_-(\xi)\sim-\frac{1}{\nu}\cdot$ In other words, a parabolic and a damped mode coexist and the asymptotic behavior of (a,v) for $|\xi|\to\infty$ is given by

$$\widehat{a}(t,\xi)\sim e^{-\frac{t}{\nu}}\left(\widehat{a}_0(\xi)-(\nu|\xi|)^{-1}\widehat{v}_0(\xi)\right)+e^{-\nu t|\xi|^2}\left(-(\nu|\xi|)^{-2}\widehat{a}_0(\xi)+(\nu|\xi|)^{-1}\widehat{v}_0(\xi)\right),$$
$$\widehat{v}(t,\xi)\sim e^{-\frac{t}{\nu}}\left((\nu|\xi|)^{-1}\widehat{a}_0(\xi)-(\nu|\xi|)^{-2}\widehat{v}_0(\xi)\right)+e^{-\nu t|\xi|^2}\left(-(\nu|\xi|)^{-1}\widehat{a}_0(\xi)+\widehat{v}_0(\xi)\right).$$

At first, one would expect the damped mode to dominate as $e^{-\nu t|\xi|^2}$ is negligible compared to $e^{-\frac{t}{\nu}}$ for ξ going to infinity. This is true as far as a is concerned. This is not quite the case for v however owing to the negative powers of $\nu|\xi|$ in the formula. More precisely, by taking advantage of Parseval formula, we get

Lemma 5.1. *There exist two positive constants c and C such that for any $j\in\mathbb{Z}$ satisfying $2^j\nu\geq 3$ and $t\in\mathbb{R}^+$, we have*

$$\|\dot\Delta_j a(t)\|_{L^2}\leq Ce^{-\frac{t}{2\nu}}\Big(\|\dot\Delta_j a_0\|_{L^2}+(2^j\nu)^{-1}\|\dot\Delta_j v_0\|_{L^2}\Big),$$
$$\|\dot\Delta_j v(t)\|_{L^2}\leq C\Big((2^j\nu)^{-1}e^{-\frac{t}{2\nu}}\|\dot\Delta_j a_0\|_{L^2}+\big(e^{-c\nu t2^{2j}}+(\nu2^j)^{-2}e^{-\frac{t}{2\nu}}\big)\|\dot\Delta_j v_0\|_{L^2}\Big).$$

The same inequalities hold true for any $p\in[1,\infty]$. Indeed, arguing as in the proof of Lemma 4.1 yields

$$\dot\Delta_j a(t)=h_1^j(t)*\dot\Delta_j a_0+h_2^j(t)*(\nu|D|)^{-1}\dot\Delta_j v_0+h_3^j(t)*(|\nu|D|)^{-2}\dot\Delta_j a_0+h_4^j(t)*(\nu|D|)^{-1}\dot\Delta_j v_0,$$
$$\dot\Delta_j v(t)=k_1^j(t)*(|\nu D|^{-1}\dot\Delta_j a_0)+k_2^j(t)*(\nu|D|)^{-2}\dot\Delta_j v_0+k_3^j(t)*(|\nu D|^{-1}\dot\Delta_j a_0)+k_4^j(t)*\dot\Delta_j v_0$$

with

$$\|h_1^j(t)\|_{L^1}+\|h_2^j(t)\|_{L^1}+\|k_1^j(t)\|_{L^1}+\|k_2^j(t)\|_{L^1}\ \leq\ Ce^{-\frac{t}{2\nu}},$$
$$\|h_3^j(t)\|_{L^1}+\|h_4^j(t)\|_{L^1}+\|k_3^j(t)\|_{L^1}+\|k_4^j(t)\|_{L^1}\ \leq\ Ce^{-c\nu t2^{2j}}.$$

This implies that

$$\|\dot\Delta_j a\|_{L_t^\infty(L^p)}+\nu^{-1}\|\dot\Delta_j a\|_{L_t^1(L^p)}\ \lesssim\ \|\dot\Delta_j a_0\|_{L^p}+\|(\nu|D|)^{-1}\dot\Delta_j v_0\|_{L^p},$$
$$\|\dot\Delta_j v\|_{L_t^\infty(L^p)}+\nu2^{2j}\|\dot\Delta_j v\|_{L_t^1(L^p)}\ \lesssim\ \|\nu|D|\dot\Delta_j a_0\|_{L^p}+\|\dot\Delta_j v_0\|_{L^p}.$$

In other words, even at the linear level, we see that it is suitable to work with the same regularity for ∇a and v. For low frequencies however, one has to work with a and v, a fact which does not follow from our scaling considerations for (70).

Putting together all the estimates for the dyadic blocks and using Duhamel's formula, we get the following proposition:

Proposition 5.1. *Let $(s,s')\in\mathbb{R}^2$ and $p\in[1,+\infty]$. Assume that (a,u) satisfies*

$$(LPH)\qquad\left\{\begin{array}{l}\partial_t a+\operatorname{div}u=F,\\[1ex] \partial_t u-\mathcal{A}u+\nabla a=G.\end{array}\right.$$

Then we have for the low frequencies:

$$\|(a,u)\|^{\ell}_{\widetilde{L}^\infty_t(\dot B^{s'}_{2,1})}+\nu\|(a,u)\|^{\ell}_{L^1_t(\dot B^{s'+2}_{2,1})}\lesssim\|(a_0,u_0)\|^{\ell}_{\dot B^{s'}_{2,1}}+\|(F,G)\|^{\ell}_{L^1_t(\dot B^{s'}_{2,1})}$$

and for the high frequencies:

$$\nu\|a\|^h_{\widetilde{L}^\infty_t(\dot B^{s+1}_{p,1})}+\|a\|^h_{L^1_t(\dot B^{s+1}_{p,1})}+\|u\|^h_{\widetilde{L}^\infty_t(\dot B^{s}_{p,1})}+\nu\|u\|^h_{L^1_t(\dot B^{s+2}_{p,1})}$$
$$\lesssim\nu\|a_0\|^h_{\dot B^{s+1}_{p,1}}+\|u_0\|^h_{\dot B^{s}_{p,1}}+\nu\|F\|^h_{L^1_t(\dot B^{s+1}_{p,1})}+\|G\|^h_{L^1_t(\dot B^{s}_{p,1})},$$

where the index ℓ (resp. h) means that only low (resp. high) frequencies have been taken into account when computing the norm[13].

5.3. **Global existence for small perturbations of a stable equilibrium state.** This section is devoted to the proof of global-in-time results for the barotropic Navier-Stokes equations for small perturbations of a stable constant equilibrium $(\bar\rho,0)$ (that is we assume that the reference positive density $\bar\rho$ is such that $P'(\bar\rho)>0$). The estimates that have been proved in the previous section will play a crucial role. In particular we have to keep in mind that, in low frequency, some eigenvalues have nonzero imaginary part. Hence we are stuck to the L^2 framework for the low frequencies of the solution. At the same time, at the linear level there is no obstacle to use a L^p type framework for high frequencies.

Theorem 5.2. *Let $p\in[2,2d)\cap[2,4]$. Assume that $P'(1)>0$, $a_0\in\dot B^{\frac dp}_{p,1}$ and $u_0\in\dot B^{\frac dp-1}_{p,1}$ and that in addition a_0^ℓ and u_0^ℓ are in $\dot B^{\frac d2-1}_{2,1}$. There exist two constants c and M depending only on d, and on the parameters of the system such that if*

$$\|(a_0,u_0)\|^{\ell}_{\dot B^{\frac d2-1}_{2,1}}+\nu\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\|u_0\|^h_{\dot B^{\frac dp-1}_{p,1}}\leq c\nu$$

then (70) has a unique global-in-time solution (a,u) with

$$(a,u)^\ell\in\mathcal{C}_b(\mathbb{R}^+;\dot B^{\frac d2-1}_{2,1})\cap L^1(\mathbb{R}^+;\dot B^{\frac d2+1}_{2,1}),\quad a^h\in\mathcal{C}_b(\mathbb{R}^+;\dot B^{\frac dp}_{p,1})\cap L^1(\mathbb{R}^+;\dot B^{\frac dp}_{p,1}),$$
$$u^h\in\mathcal{C}_b(\mathbb{R}^+;\dot B^{\frac dp-1}_{p,1})\cap L^1(\dot B^{\mathbb{R}^+;\frac dp+1}_{p,1}).$$

Remark 5.1. *The smallness condition is satisfied for small densities and large highly oscillating velocities: take $u_0^\varepsilon:x\mapsto\phi(x)\sin(\varepsilon^{-1}x\cdot\omega)\,n$ with ω and n in $\mathbb{S}^{d-1}$ and $\phi\in\mathcal{S}(\mathbb{R}^d)$. Then*

$$\|u_0^\varepsilon\|_{\dot B^{\frac dp-1}_{p,1}}\leq C\varepsilon^{1-\frac dp}\quad\text{if}\ \ p>d.$$

Hence such data with small enough ε generate global unique solutions.

We concentrate on the proof of existence. This is mainly a matter of proving global a priori estimates for small enough data. The quantity that we have to bound reads

$$X(t):=\|(a,u)\|^{\ell}_{L^\infty_t(\dot B^{\frac d2-1}_{2,1})}+\nu\|(a,u)\|^{\ell}_{L^1_t(\dot B^{\frac d2+1}_{2,1})}$$
$$+\nu\|a\|^h_{L^\infty_t(\dot B^{\frac dp}_{p,1})}+\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}+\|u\|^h_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}+\nu\|u\|^h_{L^1_t(\dot B^{\frac dp+1}_{p,1})}.$$

In order to bound the low frequency part of the norm, we shall take advantage of Proposition 5.1 (where parabolic smoothing in the L^2 framework are available for both a and u). As

[13] We mean that $\|z\|^\ell_{\dot B^\sigma_{p,1}}=\sum_{2^j\nu\leq1}2^{j\sigma}\|\dot\Delta_jz\|_{L^p}$ and that $\|z\|^h_{\dot B^\sigma_{p,1}}=\sum_{2^j\nu>1}2^{j\sigma}\|\dot\Delta_jz\|_{L^p}$.

pointed out in the previous section, in high frequency, the fundamental observations are that, at the linear level:

- $\mathcal{P}u$ satisfies a heat equation (hence parabolic smoothing in any Besov space);
- The compressible parabolic mode tends to be collinear to $\mathcal{Q}u+\nu^{-1}G'(0)(-\Delta)^{-1}\nabla a$;
- The density a is damped.

In order to make those observations more accurate, let us rewrite the nonlinear system (70) as follows:

$$\begin{cases} \partial_t a + u\cdot\nabla a + (1+a)\mathrm{div}\,\mathcal{Q}u = 0,\\ \partial_t \mathcal{Q}u + \mathcal{Q}(u\cdot\nabla u) - \nu\Delta\mathcal{Q}u + \nabla(G(a)) = -\mathcal{Q}(J(a)\mathcal{A}u),\\ \partial_t\mathcal{P}u + \mathcal{P}(u\cdot\nabla u) - \mu\Delta\mathcal{P}u = -\mathcal{P}(J(a)\mathcal{A}u). \end{cases}$$

The last equation is a heat equation with quadratic terms. Hence one may expect that parabolic smoothing for $\mathcal{P}u$ holds in any (critical) Besov space. To handle the first two equations, our linear analysis motivates us to introducing the *effective velocity* $w := \mathcal{Q}u + \nu^{-1}G'(0)(-\Delta)^{-1}\nabla a$ which is expected to satisfy also parabolic type estimates in the high frequency regime.

First step: Bounds for the effective velocity. The effective velocity w satisfies the heat equation:

$$\partial_t w - \nu\Delta w = -\mathcal{Q}(u\cdot\nabla u) - \mathcal{Q}(J(a)\mathcal{A}u) \\ +(G'(0)-G'(a))\nabla a - \nu^{-1}G'(0)(-\Delta)^{-1}\nabla\mathrm{div}\,((1+a)u).$$

All the terms of the right-hand side (but the last one) are at least quadratic hence expected to be small if we start with small data. The last term has a linear part that is lower order. One can make this heuristics more accurate by combining the regularity estimates for the heat equation and product and composition estimates in Besov spaces. Applying a high frequency cut-off to the equation for w, we readily get for any $p\in[2,2d)$

$$\|w\|^h_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})} + \nu\|w\|^h_{L^1_t(\dot B^{\frac dp+1}_{p,1})} \lesssim \|w_0\|^h_{\dot B^{\frac dp-1}_{p,1}} + \nu^{-1}\|\mathcal{Q}u\|^h_{L^1_t(\dot B^{\frac dp-1}_{p,1})} \\ +\big(\|u\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})} + \|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\big)\|\nabla u\|_{L^1_t(\dot B^{\frac dp}_{p,1})} + \|a\|^2_{L^2_t(\dot B^{\frac dp}_{p,1})} + \nu^{-1}\|au\|^h_{L^1_t(\dot B^{\frac dp-1}_{p,1})}.$$

The terms with $\mathcal{Q}u$ and au do not have the right scaling. However, they be made arbitrarily small if setting the threshold between low and high frequencies high enough. More precisely, if this threshold is at $|\xi|=2^{j_0}$ with j_0 s.t. $1\ll 2^{j_0}\nu$ then

$$\nu^{-1}\|\mathcal{Q}u\|^h_{L^1_t(\dot B^{\frac dp-1}_{p,1})} \leq \nu^{-1}2^{-2j_0}\|\mathcal{Q}u\|^h_{L^1_t(\dot B^{\frac dp+1}_{p,1})} \ll \nu\|\mathcal{Q}u\|^h_{L^1_t(\dot B^{\frac dp+1}_{p,1})}. \tag{80}$$

A similar inequality may be proved for $\|au\|^h_{L^1_t(\dot B^{\frac dp-1}_{p,1})}$. Therefore, because $\mathcal{Q}u = w - \nu^{-1}G'(0)(-\Delta)^{-1}\nabla a$, we end up with

$$\|w\|^h_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})} + \nu\|w\|^h_{L^1_t(\dot B^{\frac dp+1}_{p,1})} \lesssim \|w_0\|^h_{\dot B^{\frac dp-1}_{p,1}} + \nu^{-2}G'(0)\|a\|^h_{L^1_t(\dot B^{\frac dp-2}_{p,1})} + \nu^{-1}X^2(t). \tag{81}$$

Arguing as in (80), we see that the term involving a is very small compared to $\|a\|^h_{L^1(\dot B^{\frac dp}_{p,1})}$.

Second step: Parabolic estimates for $\mathcal{P}u$.

$$\partial_t\mathcal{P}u+\mathcal{P}(u\cdot\nabla u)-\mu\mathcal{P}u=-\mathcal{P}(J(a)\mathcal{A}u),$$

we readily have

$$\|\mathcal{P}u\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}+\mu\|\mathcal{P}u\|_{L^1_t(\dot B^{\frac dp+1}_{p,1})}\lesssim\|\mathcal{P}u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\nu^{-1}X^2(t).$$

Third step: Decay estimates for a. We notice that

$$\partial_ta+u\cdot\nabla a+\nu^{-1}G'(0)a=-a\,\mathrm{div}\,u-\mathrm{div}\,w.$$

Given that $G'(0)>0$, Theorem 4.3 implies (if $\|\nabla u\|_{L^1_t(\dot B^{\frac dp}_{p,1})}$ is small enough) that

$$\|a\|^h_{L^\infty_t(\dot B^{\frac dp}_{p,1})}+\nu^{-1}\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}\leq\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\|\mathrm{div}\,w\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}+\nu^{-1}X^2(t).\tag{82}$$

Now, according to (81),

$$\nu\|\mathrm{div}\,w\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}\lesssim\|w_0\|^h_{\dot B^{\frac dp-1}_{p,1}}+(\nu2^{j_0})^{-2}\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}+\nu^{-1}X^2(t).\tag{83}$$

Hence plugging (82) in (83) and taking j_0 large enough, we deduce that

$$\|w\|^h_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}+\nu\|w\|^h_{L^1_t(\dot B^{\frac dp+1}_{p,1})}+\nu\|a\|^h_{L^\infty_t(\dot B^{\frac dp}_{p,1})}+\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}\lesssim\|w_0\|^h_{\dot B^{\frac dp-1}_{p,1}}+\nu\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\nu^{-1}X^2(t).$$

Of course, as $\mathcal{Q}u=w-\nu^{-1}G'(0)(-\Delta)^{-1}\nabla a$, one may replace w by $\mathcal{Q}u$ in the above inequality.

Fourth step: Low frequency estimates. As explained before, we have to restrict ourselves to Besov spaces of type $\dot B^s_{2,1}$. Now, by taking advantage of Proposition 5.1, we get (for some function k vanishing at 0):

$$\begin{aligned}\|(a,u)\|^\ell_{L^\infty_t(\dot B^{\frac d2-1}_{2,1})}+\nu\|(a,u)\|^\ell_{L^1_t(\dot B^{\frac d2+1}_{2,1})}&\lesssim\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|\mathrm{div}\,(au)\|^\ell_{L^1_t(\dot B^{\frac d2-1}_{2,1})}\\&+\|u\cdot\nabla u\|^\ell_{L^1_t(\dot B^{\frac d2-1}_{2,1})}+\|J(a)\mathcal{A}u)\|^\ell_{L^1_t(\dot B^{\frac d2-1}_{2,1})}+\|k(a)\nabla a\|^\ell_{L^1_t(\dot B^{\frac d2-1}_{2,1})}.\end{aligned}$$

At this point, in order to bound the nonlinear terms by $\nu^{-1}X^2(t)$, the assumption that $p\leq4$ is needed. This is due to the fact that product between high frequencies generate low frequencies. The latter have to be bounded in a space related to L^2 whereas only L^p type estimates are available for high frequencies. Heuristically, this corresponds to the fact that in order that the product of two functions in L^p belongs to L^q for some $q\leq2$ then we need $p\leq4$. To simplify the presentation, we skip the details. The reader may refer to the paper by B. Haspot [21].

Last step: Global estimate. Putting all the previous estimates together, we get for any $p\in[2,4]\cap[2,2d)$, the inequality

$$X(t)\leq C\big(X(0)+\nu^{-1}X^2(t)\big).\tag{84}$$

Now it is clear that as long as

$$2CX(t)\leq\nu,\tag{85}$$

Inequality (84) ensures that

$$X(t)\leq2CX(0).\tag{86}$$

Using a bootstrap argument, one may conclude that if $X(0)$ is small enough with respect to ν then (85) is satisfied as long as the solution exists. Hence (86) holds globally in time.

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