Point processes and stable laws

Lecture notes

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Contents

1 Introduction

Non formal description:

$$\text{point process} = \text{random cloud of points}$$

Examples: Let (X_k) be the iid random variables in $\mathbb{R}^d$.

1) empirical point process: $\mathcal{N}_n = \{X_1, X_2, \dots, X_n\}$;

2) $\mathcal{N}_{n,D} = \mathcal{N}_n \cap D$ for some fixed set $D \subset \mathbb{R}^d$;

3) $\mathcal{N} = \{X_1, \dots, X_\nu\}$ where ν is random variable independent of sequence (X_k) and $\mathrm{P}(\nu \in \mathbb{N}) = 1$.

Simple point process:
Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the probability space, $(E, \mathcal{E})$ — measurable space, called *phase space*. For us:

*Preparing LaTeX version.

E is complete, separable metric space,
$\mathcal{E} = \mathcal{B}_E$ is its Borel σ–algebra,
K is a space of configurations. Formally

$$\mathsf{K} = \big\{\kappa = \{x_i, i \in I\} \subset E, I \subset \mathbb{N} \ : \ \kappa \cap B(0,r) \text{ is finite for any } r > 0\big\}.$$

$$\pi_B : \mathsf{K} \to \overline{\mathbb{R}}, \qquad \pi_B(\kappa) = |\kappa \cap B|, \quad B \in \mathcal{E}.$$

Let $\mathcal{K} = \sigma(\pi_B, B \in \mathcal{E})$ be σ–algebra generated by the family of projections $\{\pi_B, B \in \mathcal{E}\}$.

Definition 1.1. *(Simple) point process* (p.p.) is a measurable application from $(\Omega, \mathcal{F}, \mathrm{P})$ into $(\mathsf{K}, \mathcal{K})$.

Second approach:
Point measure:

$$\mu = \sum_{i \in I} \delta_{x_i}, \qquad I \subset \mathbb{N}.$$

Simple point measure: $\{x_i \in E \ : \ i \in I\}$ is locally finite.

$$\mathbb{M} = \{\mu \ : \ \mu \text{ is a simple point measure}\}, \qquad \mathcal{M} = \sigma(\pi_B, B \in \mathcal{E})$$

where

$$\pi_B : \mathbb{M} \to \overline{\mathbb{R}}, \qquad \pi_B(\mu) = \mu(B), \qquad B \in \mathcal{E}.$$

Definition 1.2. *(Simple) point process (p.p.)* is a measurable application from $(\Omega, \mathcal{F}, \mathrm{P})$ into $(\mathbb{M}, \mathcal{M})$.

Remark 1.3.

- Application $\varphi : \mathsf{K} \to \mathbb{M}$ such that

$$\varphi(\kappa) = \sum_{x \in \kappa} \delta_x \qquad (= \mu)$$

  is bijection and

$$\pi_B(\kappa) = |\kappa \cap B| = \sum_{x \in \kappa} \mathbb{1}_B(x) = \mu(B).$$

- We can consider the family $\{\mathcal{N}(B), b \in \mathcal{E}\}$ as a random process defined on $\mathcal{E}$.

Proposition 1.4. (1) $\iff$ (2) *where:*

(1) $\mathcal{N} : \Omega \to \mathbb{M}$ *is a p.p.*

(2) *for all* $B \in \mathcal{E}$ *the application* $\mathcal{N}(B) : \Omega \to \overline{\mathbb{N}}$ *is a random variable.*

Intensity measure:

Proposition 1.5. *Let* $\mathcal{N}$ *be a p.p. The function* $m : \mathcal{E} \to \overline{\mathbb{R}_+}$,

$$m(B) = \mathbb{E}\left[\mathcal{N}(B)\right]$$

is a measure. We call it intensity measure for $\mathcal{N}$.

Remark 1.6. 1) It is possible that $m(B) = \infty$ for very large subfamily of $\mathcal{E}$.

2) $m(\cdot)$ is a mean value of the process $\{\mathcal{N}(B), B \in \mathcal{E}\}$.

3) Assume that $X_1, \ldots, X_n$ are i.i.d. random variables with distribution $\mathcal{P}$. For $\mathcal{N}_n = \sum_{i=1}^n \delta_{X_i}$ we have $\mathbb{E}[\mathcal{N}_n(B)] = n\mathcal{P}(B)$.

Try to prove 3)-5).

4) $\mathbb{E}[\mathcal{N}_{n,D}(B)] = n\mathcal{P}(B \cap D)$.

5) For $\mathcal{N} = \sum_{i=1}^{\nu} \delta_{X_i}$ we have $\mathbb{E}[\mathcal{N}(B)] = \mathbb{E}[\nu]\,\mathcal{P}(B)$.

Finite dimensional laws:

Theorem 1.7. *Let $\mathcal{N}, \mathcal{N}_1$ be two p.p. in $(E, \mathcal{E})$. The following conditions are equivalent:*

1) $\mathcal{N} \stackrel{\mathcal{L}}{=} \mathcal{N}_1$;

2) *for all $k \in \mathbb{N}$ and $B_1, \ldots, B_n \in \mathcal{E}$*

$$(\mathcal{N}(B_1), \ldots, \mathcal{N}(B_n)) \stackrel{\mathcal{L}}{=} (\mathcal{N}_1(B_1), \ldots, \mathcal{N}_1(B_n))\,; \tag{1}$$

3) *Equality* (1) *takes place for mutually disjoint sequences.*

2 Poisson point process (p.p.p.)

Definition 2.1. Let m be diffuse measure on $(E, \mathcal{E})$. The p.p. $\mathcal{N}$ is called *Poissonian* if

1) for any $B \in \mathcal{E}$ such that $0 < m(B) < \infty$

$$\mathcal{N}(B) \sim Pois(m(B));$$

($\mathcal{N}(B)$ has Poissonian distribution with parameter $m(B)$)

2) for all mutually disjoint sets $B_1, \ldots, B_n \in \mathcal{E}$ random variables $\mathcal{N}(B_1), \ldots \mathcal{N}(B_n)$ are independent.

Remark 2.2. The measure m is exactly intensity measure for $\mathcal{N}$.

Theorem 2.3. *Let m be a finite diffuse measure. Let (X_k) be a sequence of i.i.d. random variables with common distribution $Q = \frac{1}{m(E)} m$. Let ν be a random variable independent of the sequence (X_k) having Poissonian distribution with parameter $m(E)$. Then the p.p.*

$$\mathcal{N} = \sum_{i=1}^{\nu} \delta_{X_i}$$

is p.p.p. with intensity measure m.

Proof. Let $B_1, \ldots, B_k \in \mathcal{E}$ be mutually disjoint sets such that $\bigcup_{i=1}^k B_i = E$. Moreover let $n_1 + \ldots + n_k = n$. Then

$$\begin{aligned}
\mathrm{P}(\mathcal{N}(B_1) = n_1, \ldots, \mathcal{N}(B_k) = n_k) &= \mathrm{P}(\mathcal{N}(B_1) = n_1, \ldots, \mathcal{N}(B_k) = n_k, \nu = n) \\
&= \mathrm{P}(\nu = n) \frac{n!}{n_1! \ldots n_k!} \prod_{i=1}^k Q(B_i)^{n_i} \\
&= e^{-m(E)} \frac{m(E)^n}{n!} \frac{n!}{n_1! \ldots n_k!} \prod_{i=1}^k \left(\frac{m(B_i)}{m(E)} \right)^{n_i} \\
&= \prod_{i=1}^k e^{-m(B_i)} \frac{m(B_i)^{n_i}}{n_i!} = \prod_{i=1}^k \mathrm{P}(\mathcal{N}(B_i) = n_i).
\end{aligned}$$

□

3 Properties of p.p.p.

1) Let $\mathcal{N}$ be p.p.p. with intensity measure m and

$$f : E \longrightarrow E_1$$

be a measurable injection such that for each ball $B \subset E_1$ the set $f^{-1}(B)$ is bounded. Then $\mathcal{N}_1 = \mathcal{N} f^{-1}$ is p.p.p. with intensity $m_1 = m f^{-1}$.

Example 3.1. Homogeneous p.p.p. on $\mathbb{R}^d$ is invariant relatively translations and rotations.

2) Let $\mathcal{N}_1, \mathcal{N}_2$ be two independent p.p.p. with diffuse intensity measures m_1, m_2 respectively. Then $\mathcal{N}_1 + \mathcal{N}_2$ is p.p.p. with intensity measure $m_1 + m_2$.

3) Let $\mathcal{N}_1$ be a p.p.p. with intensity measure m_1 defined on E_1. Let (Y_j) be a sequence of i.i.d. random variables with values in E_2 and with common distribution Q. Then the p.p.

$$\mathcal{N} = \{(x_i, Y_i), x_i \in \mathcal{N}_1\}$$

is p.p.p. on $E_1 \times E_2$ with intensity measure $m = m_2 \times Q$. $\mathcal{N}$ is called *marked* p.p.p. and variables (Y_j) are called *marks*.

Theorem 3.2. *Existence of p.p.p. in general case follows from theorem 2.3.*

Homogeneous p.p.p. in $\mathbb{R}^d$:

Definition 3.3. P.p.p. $\mathcal{N}$ defined on $E = \mathbb{R}^d$ is called *homogeneous* if $m = \alpha \lambda^d$ (λ^d is Lebesgue measure on $\mathbb{R}^d$) and $\alpha > 0$ is called *intensity* of $\mathcal{N}$.

Case $d = 1$: Let π be homogeneous p.p.p. with intensity 1. Consider the process

$$X_t = \pi([0, t]), \qquad t \geq 0.$$

Then

(i) $X(0) = 0$

(ii) $X(t) - X(s) = \pi((s, t]) \sim Pois(t - s)$ for $t > s$

(iii) X has independent increments

(iv) paths of X are right continuous almost surely

It means that X is a standard Poissonian process on $\mathbb{R}_+$.

Theorem 3.4 ("Exponential description" of π)**.** *Let (ε_k) be a sequence of i.i.d. random variables having common exponential distribution with parameter 1. Let $\Gamma_n = \varepsilon_1 + \ldots + \varepsilon_n$. Then*

$$\pi \overset{\mathcal{L}}{=} \sum_{k=1}^{\infty} \delta_{\Gamma_k}.$$

Example 3.5. $\mathcal{N}$ is homogeneous p.p.p. on $\mathbb{R}^d$ with intensity measure λ^d and (Y_j) a sequence of i.i.d. random variables which is independent of $\mathcal{N}$. Let

$$M = \{X_i + Y_i,\ X_i \in \mathcal{N}\}.$$

Then $M \overset{\mathcal{L}}{=} \mathcal{N}$.

Example 3.6 (Process Π_α)**.** Let (γ_k) be i.i.d. random variables, $\gamma_k \sim \mathcal{E}(1)$ and $\Gamma_k = \gamma_1 + \ldots + \gamma_k$ for $k = 1, 2, \ldots$. Let (ε_k) be i.i.d. random variables in $\mathbb{R}^d$, $|\varepsilon_k| = 1$ a.s. Denote by σ common law of ε_k. Let $\alpha > 0$.

Proposition 3.7. *Point process*

$$\Pi_\alpha = \sum_{k=1}^{\infty} \delta_{\varepsilon^k \Gamma_k^{-\frac{1}{\alpha}}}$$

is poissonian with intensity measure m *which in polar representation of* $\mathbb{R}^d \backslash \{0\} = (\mathbb{R}_+ \backslash \{0\}) \times S^{d-1}$ *has the following structure:*

$$m = \mu \times \sigma \qquad \textit{where} \qquad \frac{d\mu}{dt} = \alpha t^{-1-\alpha}.$$

Proof. Let $\mathcal{N} = \sum_{k=1}^{\infty} \delta_{\Gamma_k}$ be homogeneous p.p.p. on $\mathbb{R}_+$. Then $\Pi = \mathcal{N} h^{-1}$ where $h : \mathbb{R}_+ \to \mathbb{R}_+ \backslash \{0\}$, $h(\tau) = \tau^{-\frac{1}{\alpha}}$ is a p.p.p. and $\Pi = \sum_{k=1}^{\infty} \delta_{\Gamma_k^{-\frac{1}{\alpha}}}$ $(\sim \mu)$. Now it is clear that Π_α is a marked p.p.p. constructed by Π and marks (ε_k). □

4 Stability property

We consider only strict stability.

Definition 4.1. Let X be a random variable with values in $\mathbb{R}^d$. X is called *strictly α–stable* if for all $a, b > 0$

$$a^{\frac{1}{\alpha}} X_1 + b^{\frac{1}{\alpha}} X_2 \stackrel{\mathcal{L}}{=} (a+b)^{\frac{1}{\alpha}} X, \tag{2}$$

where X_1, X_2 are independent copies of X.

This class is very important: if the limit distribution for normed sums

$$\frac{Y_1 + \ldots + Y_n}{b_n}$$

exists, then it is strictly α–stable for some $\alpha \in (0, 2]$ and in this case $b_n = nh(n)$, where h is a slowly varying function.

Stability of p.p.:
Consider a p.p. $\mathcal{N}$ in $E = \mathbb{R}^d$. Let $D_a : \mathbb{R}^d \to \mathbb{R}^d$, $D_a x = ax$, for $a > 0$. For point measure μ we use the notation $D_a \circ \mu$ for μD_a^{-1}.

Definition 4.2. P.p. $\mathcal{N}$ is called *α–stable* if for all $a, b > 0$

$$D_{a^{\frac{1}{\alpha}}} \mathcal{N}_1 \oplus D_{b^{\frac{1}{\alpha}}} \mathcal{N}_2 \stackrel{\mathcal{L}}{=} D_{(a+b)^{\frac{1}{\alpha}}} \mathcal{N}, \tag{3}$$

where $\mathcal{N}_1, \mathcal{N}_2$ are 2 independent copies of $\mathcal{N}$.

Proposition 4.3. *The p.p.p.* Π_α *is strictly α–stable.*

Proof. Stability property for Π_α means that for every $a, b > 0$

$$D_{a^{\frac{1}{\alpha}}} \Pi_\alpha^{(1)} \oplus D_{b^{\frac{1}{\alpha}}} \Pi_\alpha^{(2)} \stackrel{\mathcal{L}}{=} D_{(a+b)^{\frac{1}{\alpha}}} \Pi_\alpha, \tag{4}$$

where $\Pi_\alpha^{(1)}, \Pi_\alpha^{(2)}$ are 2 independent copies of Π_α. To prove this equality it is sufficient to state that the intensity measures of the left part and the right part coincide. □

Let

$$f(\mu) = \int_E x\mu(dx)$$

be a "linear" functional on $\mathbb{M}$. Of course

$$f(\mu_1 + \mu_2) = f(\mu_1) + f(\mu_2) \qquad \text{and} \qquad f(a \circ \mu) = af(\mu).$$

We have formally form (4) applying f:

$$a^{\frac{1}{\alpha}} Y_\alpha^{(1)} + b^{\frac{1}{\alpha}} Y_\alpha^{(2)} \stackrel{\mathcal{L}}{=} (a+b)^{\frac{1}{\alpha}} Y_\alpha,$$

where $Y_\alpha^{(1)}, Y_\alpha^{(2)}$ are 2 independent copies of

$$Y_\alpha = f(\Pi_\alpha) = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k.$$

When we can justify this passage?
ANSWER: It is sufficient to understand when the series $\sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k$ converges.

Proposition 4.4. 1) *If $\alpha \in (0,1)$, then the convergence of $\sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k$ takes place almost surely.*

2) *For $\alpha \in [1,2)$ if $\mathbb{E}[\varepsilon_1] = 1$, then also we have this convergence.*

Proof. 1) In this case a.s. the series converges absolutely because $|\varepsilon_k| = 1$ and $\Gamma_k \sim k$ a.s.

2) Easily follows from Kolmogorov's three-series theorem.

□

Terminology:

- The series $\sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k$ is called LaPage series.
- Its sum Y_α is strictly α–stable vector in $\mathbb{R}^d$.
- σ is called *spectral* measure of Y_α.

Comments:

- Case $\alpha = 2$ corresponds to Gaussian law. There is no LePage representation in this case.
- For $\alpha > 2$ there is no stable laws in $\mathbb{R}^d$.

Theorem 4.5. *Let (ε_k) and (Γ_k) be 2 sequences as before. Let (η_k) be a sequence of i.i.d. real random variables, independent on (Γ_k) and (ε_k) and such that*

$$a^\alpha = \mathbb{E}[|\eta_k|^\alpha] < \infty.$$

Then the series

$$Z_\alpha = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \eta_k \varepsilon_k$$

is almost surely convergent and in the case $\eta_k \geq 0$ almost surely

$$Z_\alpha \stackrel{\mathcal{L}}{=} aY_\alpha.$$

Remark 4.6. In general case: Z_α is strictly α–stable and the last relation is fulfilled with $\tilde{Y}_\alpha$ having changed spectral measure $\tilde{\sigma}$:

$$\tilde{\sigma}(B) = \sigma(B)\mathbb{E}\left[\eta_+^\alpha\right] + \sigma(-B)\mathbb{E}\left[\eta_-^\alpha\right].$$

If σ is symmetric, then $\tilde{\sigma} = \sigma$. If η has a symmetric distribution, then

$$\tilde{\sigma}(B) = \frac{\sigma(B) + \sigma(-B)}{2}.$$

Proof of theorem 4.5. It is sufficient to state the equality of intensity measures for two p.p.p.:

$$\mathcal{N}_1 = \{\Gamma_k^{-\frac{1}{\alpha}}\eta_k\} \qquad \text{and} \qquad \mathcal{N}_2 = \{a\Gamma_k^{-\frac{1}{\alpha}}\}.$$

Let compute the intensity measure μ_1 for $\mathcal{N}_1$:

$$\mu_1 = (m \times Q)i^{-1},$$

where Q is common distribution of η_k and

$$i : (0,\infty) \times \mathbb{R} \to \mathbb{R}, \qquad\qquad i(x,t) = tx.$$

We have

$$\begin{aligned}(m \times Q)i^{-1}([\tau,\infty)) &= (m \times Q)\left(\{(x,t)\ :\ xt \geq \tau\}\right) = \int_0^\infty \int_0^\infty \mathbb{1}_{\{xt\geq\tau\}} m(dx)Q(dt) \\ &= \int_0^\infty m\left(\left[\frac{\tau}{t},\infty\right)\right) Q(dt) = \int_0^\infty \left(\frac{\tau}{t}\right)^{-\alpha} Q(dt) \\ &\overset{\text{assuming } \eta_1 \geq 0}{=} a^\alpha m([\tau,\infty)).\end{aligned}$$

Hence $\underline{\mu_1 = a^\alpha m.}$ In particular, for $\eta_k \equiv a$ we obtain the same measure. □

Remark 4.7. In symmetric case, choosing $\eta_k \sim \mathcal{N}(0,1)$ we see that the law of Y_α is a mixture of Gaussian distributions.

Stable Lévy processes:
Let (U_k) be i.i.d. random variables with $[0,1]$–uniform distribution. Let

$$X(t) = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \mathbb{1}_{[0,t]}(U_k)\varepsilon_k, \qquad t \in [0,1].$$

This process possesses the following properties:

1) It has independent, stationary increments.

2) $X(t) - X(s) \overset{\mathcal{L}}{=} (t-s)^{\frac{1}{\alpha}} X(1)$ (increments are strictly α–stable)

3) Paths are right–continuous.

These properties follow directly from definition and theorem 4.5.

5 Convex cones

It is clear that if we want to formulate stability property, it is sufficient to have 2 operations: addition and multiplication on positive solutions. It means that the stability property could be considered in convex cones.

Definition 5.1. Topological space $\mathbb{K}$ is called *convex cone* if two operations (addition and scalar multiplication) satisfy the following conditions:

0) they are continuous;

1) for all $a \geq 0$ and $x, y \in \mathbb{K}$

$$a(x+y) = ax + ay;$$

2) for all $a, b \geq 0$ and $x \in \mathbb{K}$

$$a(bx) = (ab)x$$

3) there exists element 0 such that

$$x + 0 = x, \quad x \in \mathbb{K}, \qquad \text{and} \qquad c \cdot 0 = 0, \quad c \geq 0.$$

Remark 5.2. We do not require $(a+b)x = ax + bx$ (second distributive law).

Examples:

1) $\mathbb{B}$ — Banach space (or subcone of $\mathbb{B}$)

2) Let T be a compact metric space. Take

$$\mathcal{C}_+^{\max}(T) := \{f \geq 0 \ : \ f \text{ is continuous on } T\}$$

with operations

$$(f+g)(t) = \max\{f(t), g(t)\}, \quad (cf)(t) = cf(t), \qquad t \in T, c \geq 0.$$

Remark that $f + f = f$ for all $f \in \mathcal{C}_+^{\max}(T)$!

In particular, when T is singleton, $\mathbb{K} = \mathbb{R}_+^{\max}$, that is $\mathbb{K} = [0, \infty)$, $x + y = x \vee y$ and cx is as usual.

3) $\mathbb{K} = \mathcal{K}(\mathbb{R}^d)$ space of all convex compact subsets of $\mathbb{R}^d$.
$A \oplus B$ — Minkowski addition:

$$A \oplus B = \{x + y \ : \ x \in A, \ \ y \in B\}$$

$cA = \{cx \ : \ x \in A\}$ is a dilatation.

It is supposed that $\mathbb{K}$ is metrisable, d is a metric on $\mathbb{K}$. Denote $|x| = d(0, x)$ and

$$S^1 = \{x \in \mathbb{K} \ : \ |x| = 1\}.$$

Polar representation for $\mathbb{K} \backslash \{0\}$:

$$\mathbb{K} \simeq (0, \infty) \times S^1, \qquad x \longleftrightarrow \left(|x|, \frac{x}{|x|}\right).$$

Stability in $\mathbb{K}$:
Definition of stability in $\mathbb{K}$ is the same as in $\mathbb{R}^d$.

Theorem 5.3. *If the series* $Y_\alpha = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k$ *converges (in this case* (ε_k) *is i.i.d. sequence such that* $|\varepsilon_k| = 1$ *a.s.), then* Y_α *will have a strictly* α*–stable distribution.*

Example 5.4. $\mathbb{K} = \mathbb{R}_+^{\max}$. Then $\varepsilon_k \equiv 1$ and

$$\sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k = \max_k \left\{ \Gamma_k^{-\frac{1}{\alpha}} \right\} = \Gamma_1^{-\frac{1}{\alpha}}.$$

It implies that for any $\alpha > 0$ there exists a strictly α–max–stable law.

6 Convergence of p.p.

If we want to define the weak convergence of p.p., it is necessary before to introduce a convenient topology into space $\mathbb{M}$. Usually one uses the so–called *vague topology.*

We consider $\mathbb{M}$ as a subspace of $\mathbb{M}_+(E)$ — space of all positive locally bounded measures on $\mathcal{E}$. We say that $\mu_n \in \mathbb{M}_+(E)$ *converges vaguely* to μ (denote $\mu_n \overset{\text{vague}}{\Longrightarrow} \mu$) if

$$\int_E f d\mu_n \longrightarrow \int_E f d\mu$$

for each continuous function f having a bounded support. The topology induced by such type of convergence is called *vague topology.*

Laplace functional:

Laplace functional is an analogue of Laplace transform for random vectors and it is very useful in the study of weak convergence of p.p.

- Laplace transform for random variables:
  Let $X \in \mathbb{R}_+$ be a random variable. Laplace transform of X is the function

  $$L(t) = \mathbb{E}\left[e^{-tX}\right], \qquad t \in \mathbb{R}_+.$$

  It is known that Laplace transform defines completely the distribution of X.
  Exercise: Let $X \sim Pois(\alpha)$ be a Poissonian random variable. Then $L(t) = \exp\left(\alpha(e^{-t} - 1)\right)$.

- Laplace transform for random vectors:
  Let $X \in \mathbb{R}_+^d$ be a random vector. Then

  $$L(\bar{t}) = \mathbb{E}\left[e^{-\langle \bar{t}, X\rangle}\right], \qquad t \in \mathbb{R}_+^d.$$

Proposition 6.1. *Let* (X_n) *be a sequence of random variables. Then weak convergence* $X_n \Rightarrow X$ *is equivalent to convergence of Laplace transforms:*

$$L_n(t) \longrightarrow L(t), \qquad t \in \mathbb{R}_+.$$

Proof. "$\Longrightarrow$": is evident.
"$\Longleftarrow$": Suppose that $L_n(t) \to L(t)$, where L is Laplace transform of X. Let $Y_n = e^{-X_n}$, then $Y_n \in [0,1]$ a.s. Hence sequence $\{Y_n\}$ is tight. Let $Y = e^{-X}$. If for $\{n_k\} \subset \mathbb{N}$

$$Y_{n_k} \Rightarrow \tilde{Y},$$

then $\mathbb{E}\left[Y_{n_k}^t\right] \to \mathbb{E}\left[\tilde{Y}^t\right]$ for $t \in \mathbb{R}_+$. Moreover

$$\mathbb{E}\left[Y_{n_k}^t\right] = L_{n_k}(t) \longrightarrow L(t) = \mathbb{E}\left[Y^t\right].$$

Therefore we have $\tilde{Y} \stackrel{\mathcal{L}}{=} Y$. Hence $Y_n \Rightarrow Y$ and it follows that $X_n \Rightarrow X$. □

Definition 6.2. Let $\mathcal{N}$ be a p.p. *Laplace functional* of $\mathcal{N}$ is operator

$$L_{\mathcal{N}}(f) = \mathbb{E}\left[e^{-\mathcal{N}(f)}\right],$$

where $\mathcal{N}(f) = \langle f, \mathcal{N}\rangle = \int_E f d\mathcal{N} = \sum_{x \in \operatorname{supp} \mathcal{N}} f(x)$, f being nonnegative function with bounded support.

By the transfer theorem

$$L_{\mathcal{N}}(f) = \int_{\mathbb{M}} \exp\left(-\int_E f(x)\kappa(dx)\right)\mathcal{P}_{\mathcal{N}}(d\kappa).$$

Proposition 6.3. *$L_{\mathcal{N}}$ uniquely defines $\mathcal{P}_{\mathcal{N}}$.*

Proof. Let $k \geq 1$ and $B_1, \ldots, B_k \in \mathcal{E}$, $\lambda_1, \ldots, \lambda_k \geq 0$. Let $f(x) = \sum_{i=1}^k \lambda_i \mathbb{1}_{B_i}(x)$. Then

$$\mathcal{N}(f) = \sum_{i=1}^k \lambda_i \mathcal{N}(B_i)$$

and $L_{\mathcal{N}}(f) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^k \lambda_i \mathcal{N}(B_i)\right)\right]$ represents the joint Laplace transform of the random variable $(\mathcal{N}(B_1), \ldots, \mathcal{N}(B_k))$. It means that $L_{\mathcal{N}}$ defines all finite-dimensional distributions of $\mathcal{N}$, hence $\mathcal{P}_{\mathcal{N}}$ itself. □

Proposition 6.4. *Let $\mathcal{N}$ be a p.p.p. with intensity measure μ. Then*

$$L_{\mathcal{N}}(f) = \exp\left(-\int_E (1 - e^{-f(x)})\mu(dx)\right).$$

Proof. **Step 1:** $f(x) = c\mathbb{1}_F(x)$. Then $\mathcal{N}(f) = c\mathcal{N}(F)$ and

$$\begin{aligned} L_{\mathcal{N}}(f) &= \mathbb{E}\left[e^{-\mathcal{N}(f)}\right] = \mathbb{E}\left[e^{-c\mathcal{N}(F)}\right] \quad \left[\text{as } \mathcal{N}(F) \sim Pois(\mu(F))\right] \\ &= \exp\left((e^{-c} - 1)\mu(F)\right) = \exp\left(-\int_E (1 - e^{-f(x)})\mu(dx)\right). \end{aligned}$$

Step 2: $f(x) = \sum_{i=1}^k c_i \mathbb{1}_{F_i}(x)$ where sets (F_i) are disjoint and $c_i \geq 0$. Then $\{\mathcal{N}(f_i)\}$ are independent. Hence

$$\begin{aligned} L_{\mathcal{N}}(f) &= \mathbb{E}\left[\exp\left(-\sum_{i=1}^k c_i \mathcal{N}(F_i)\right)\right] = \prod_{i=1}^k \mathbb{E}\left[\exp\left(-c_i \mathcal{N}(F_i)\right)\right] \\ &= \prod_{i=1}^k \exp\left(-\int_E (1 - e^{-c_i \mathbb{1}_{F_i}(x)})\mu(dx)\right) \\ &= \exp\left(-\int_E \sum_{i=1}^k (1 - e^{-c_i \mathbb{1}_{F_i}(x)})\mu(dx)\right) \\ &= \exp\left(-\int_E (1 - e^{-\sum_{i=1}^k c_i \mathbb{1}_{F_i}(x)})\mu(dx)\right) \\ &= \exp\left(-\int_E (1 - e^{-f(x)})\mu(dx)\right). \end{aligned}$$

We used here this obvious functional equality

$$\sum_{i=1}^{k}(1 - e^{-c_i \mathbb{1}_{F_i}(x)}) = 1 - e^{-\sum_{i=1}^{k} c_i \mathbb{1}_{F_i}(x)}.$$

Step 3: Let $f \geq 0$ and (f_n) be non-decreasing sequence of step-functions such that $f_n \nearrow f$. Then $\mathcal{N}(f_n) \to \mathcal{N}(f)$ and

$$L_{\mathcal{N}}(f) = \mathbb{E}[\underbrace{e^{-\mathcal{N}(f_n)}}_{\leq 1}] \longrightarrow \mathbb{E}\left[e^{-\mathcal{N}(f)}\right] = L_{\mathcal{N}}(f).$$

□

Proposition 6.5. *$\mathcal{N}_n \Rightarrow \mathcal{N}$ iff $L_{\mathcal{N}_n}(f) \to L_{\mathcal{N}}(f)$ for all $f \in \mathcal{C}_K^+(E)$.*

Sketch of proof. "$\Longrightarrow$": If $\mathcal{N}_n \Rightarrow \mathcal{N}$ then by continuous mapping theorem

$$\langle f, \mathcal{N}_n \rangle \Rightarrow \langle f, \mathcal{N} \rangle, \qquad f \in \mathcal{C}_K^+(E)$$

and then $\mathbb{E}\left[e^{-\langle f, \mathcal{N}_n \rangle}\right] \to \mathbb{E}\left[e^{-\langle f, \mathcal{N} \rangle}\right]$, that is $L_{\mathcal{N}_n}(f) \to L_{\mathcal{N}}(f)$.
"$\Longleftarrow$": Main point:

> **Lemma.** *$(\mathcal{N}_n)$ is tight in $\mathbb{M}_+(E)$ iff for all $f \in \mathcal{C}_K^+(E)$ the sequence $(\mathcal{N}_n(f))$ is tight in $\mathbb{R}$.*

Suppose $L_{\mathcal{N}_n}(f) \to L_{\mathcal{N}}(f)$. It means that for any $t \in \mathbb{R}_+$

$$\mathbb{E}\left[e^{-t\mathcal{N}_n(f)}\right] \longrightarrow \mathbb{E}\left[e^{-t\mathcal{N}(f)}\right],$$

that is Laplace transform of random variable $\mathcal{N}_n(f)$ converges to Laplace transform of $\mathcal{N}(f)$. Hence

$$\mathcal{N}_n(f) \Rightarrow \mathcal{N}(f)$$

and we see that $(\mathcal{N}_n(f))$ is tight. By lemma we get the tightness of $(\mathcal{N}_n)$. Now to finish the proof it is sufficient to use standard arguments. □

7 Convergence of empirical p.p.

Theorem 7.1. *Let $(X_{k,j})_{j=1,2,\ldots,n}$ be a triangular array of random variables independent and identically distributed in each row. Let $\mathcal{N}_n = \sum_{j=1}^{n} \delta_{X_{n,j}}$. Let $\mathcal{N}$ be p.p.p. with intensity measure μ. Then $\mathcal{N}_n \Rightarrow \mathcal{N}$ iff*

$$n\mathrm{P}(X_{n,1} \in \cdot) \overset{\text{vague}}{\Longrightarrow} \mu. \tag{5}$$

Proof. It is sufficient to study the behaviour of Laplace functional. We have

$$\begin{aligned} L_{\mathcal{N}_n}(f) &= \mathbb{E}\left[\exp\left(-\sum_{j=1}^{n} f(X_{n,j})\right)\right] = \left(\mathbb{E}\left[\exp(-f(X_{n,1}))\right]\right)^n \\ &= \left(1 - \frac{\int_E (1 - e^{-f(x)}) n\mathrm{P}(X_{n,1} \in dx)}{n}\right)^n. \end{aligned}$$

We know that $(1 - \frac{a_n}{n})^n \to e^{-a}$ iff $a_n \to a$. Using this we see that $L_{\mathcal{N}_n}(f) \to L_{\mathcal{N}}(f)$ iff (5) is fulfilled. □

Corollary 7.2. *Let* (X_k) *be a sequence of i.i.d. random variables,* $X_{n,i} = \frac{X_i}{b_n}$*,* $i = 1, 2 \dots, n$*. Then* $\mathcal{N}_n = \sum_{i=1}^n \delta_{\frac{X_i}{b_n}}$ *is empirical p.p. The condition* (5) *now looks like the following*

$$n\mathrm{P}\left(\frac{X_1}{b_n} \in \cdot\right) \overset{\text{vague}}{\Longrightarrow} \mu. \tag{6}$$

It is possible to show that (6) *in case* $E = \mathbb{R}^d$ *is equivalent to*

1) $b_n = n^{\frac{1}{\alpha}} l(n)$*, for some* $\alpha > 0$ *and slowly varying function* $l : \mathbb{N} \to \mathbb{R}$*.*

2) $\mu = m \times \sigma$ *where* $m([\tau, \infty)) = \tau^{-\alpha}$ *for* $\tau > 0$ *and* $\sigma(S^{d-1}) = 1$*.*

3)
$$n\mathrm{P}\left(\frac{X_1}{|X_1|} \in A, |X_1| \geq \tau b_n\right) \longrightarrow \sigma(A)m([\tau, \infty)) \tag{RV}$$
for all $\tau > 0$ *and measurable* $A \subset S^{d-1}$ *such that* $\sigma(\partial A) = 0$*.*

4) $\mathcal{N} = \Pi_\alpha$*.*

Relation (RV) is called *condition of regular variation* for the distribution of X_1. We can define this condition in the same way for any convex cone $\mathbb{K}$. Parameter α is called *tail index* (or *exposant*), σ is called *spectral measure.*

Let (X_i) be i.i.d. random variables such that $X_i \overset{\mathcal{L}}{=} X$. Under (RV) the empirical p.p. $\mathcal{N}_n = \sum_{i=1}^n \delta_{\frac{X_i}{b_n}}$ converges weakly to Π_α.

8 Application to order statistics

Let $\mathbb{K} = \mathbb{R}\backslash\{0\}$ and (X_i) be i.i.d. random variables satisfying (RV). The application $f : \mathbb{K} \to \mathbb{R}$ such that

$$f(\kappa) = \max\{x \ : \ x \in \kappa\}$$

is continuous with respect to vague topology. We get immediately from $\mathcal{N}_n \Rightarrow \Pi_\alpha$ that

$$\frac{M_n}{b_n} := \frac{1}{b_n}\max\{X_1, \dots, X_n\} \implies \max\left\{\Gamma_k^{-\frac{1}{\alpha}}\varepsilon_k \ : \ k \in \mathbb{N}\right\},$$

here

$$\varepsilon_k \in \{-1, +1\}, \qquad \mathrm{P}(\varepsilon_k = \pm 1) = \sigma(\pm 1).$$

If $X_k \geq 0$ then $\sigma(-1) = 0$ and we see $\boxed{\frac{M_n}{b_n} \Rightarrow \Gamma_1^{-\frac{1}{\alpha}}.}$

By the same way we deduce the results for all order statistics.

Now let $\mathbb{K} = \mathbb{R}^d\backslash\{0\}$ and (X_i) be i.i.d. random variables satisfying (RV). We can consider the convex hull of the sample $\{X_1, \dots, X_n\}$ (denoted by $\mathrm{conv}\{\mathrm{X_1, \dots, X_n}\}$) as an analogue of the extremal segment $[\min_{k=1\dots n} X_k, \max_{k=1\dots n} X_k]$ in one–dimensional case. Remind that in $\mathbb{R}^d\backslash\{0\}$ the topology corresponds to the metric

$$d(x, y) = \left|\frac{1}{|x|} - \frac{1}{|y|}\right| + \left|\frac{x}{|x|} - \frac{y}{|y|}\right|.$$

The application f from $\mathbb{R}$ to the space of compact convex subsets of $\mathbb{R}^d$ (notation $\mathcal{K}(\mathbb{R}^d)$) defined

$$f(\kappa) = \mathrm{conv}\{\kappa\}$$

being continuous, we get from convergence $\mathcal{N}_n \Rightarrow \Pi_\alpha$ that

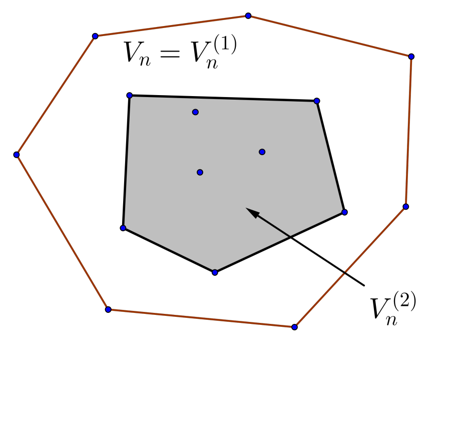
$$\frac{1}{b_n}\mathrm{conv}\{X_1, \dots, X_n\} \Rightarrow \{\Pi_\alpha\}.$$

By the same way we can consider the behaviour of the second convex hull

$$V_n^{(2)} = \mathrm{conv}\{X_i \ : \ X_i \in V_n^0\},$$

the third one, and so on.
The limiting convex sets are:

$$V^{(1)} = \mathrm{conv}\{\Pi_\alpha\}$$
$$V^{(2)} = \mathrm{conv}\{x \in \Pi_\alpha \ : \ x \in [V^{(1)}]^0\}$$
$$\dots$$



A^0 = interior of A

The sequence of convex sets $\{V^{(k)}\}$ is decreasing. We say that it is constructed by **piling** procedure. Is it possible to deduce from the convergence $\mathcal{N}_n \Rightarrow \Pi_\alpha$ the convergence of sums

$$\frac{S_n}{b_n} \Longrightarrow Y_\alpha?$$

Yes: for $\alpha < 1$ it is possible to change the topology of vague convergence in $\mathbb{M}$ replacing it by stronger in such a way that

1) the convergence $\mathcal{N}_n \Rightarrow \Pi_\alpha$ still be true,

2) the application $f(\mu) = \int_E x\mu(dx)$ becomes continuous.

For details see [DaEg].

9 Application to random zonotopes

Definition 9.1. (i) *Zonotope* $= \sum_{i=1}^m \Delta_i$, where Δ_i are segments.

(ii) *Zonoid* $= \lim_n Z_n$, where Z_n are zonotopes.

(iii) $Z = \sum_{i=1}^\infty Z_i$ is called *countable zonotope.*

Consider $\mathbb{K} = \mathcal{K}(\mathbb{R}^d)$ equipped by Hausdorff distance. It is convex cone (with second distributivity law). Let (X_k) be a sequence of i.i.d. random vectors in $\mathbb{R}^d$. We compare the behaviour of normalized sums $\frac{S_n}{b_n}$ with the behaviour of the sums

$$\frac{Z_n}{b_n} = \frac{1}{b_n}\sum_{i=1}^n [0, X_i].$$

Theorem 9.2 ([DPR])**.**

$$\frac{S_n}{b_n} \Rightarrow Y_\alpha \qquad \textit{iff} \qquad \frac{Z_n}{b_n} \Rightarrow Z_\alpha,$$

where $Z_\alpha = \sum_{k=1}^\infty \Gamma_k^{-\frac{1}{\alpha}}[0, \varepsilon_k]$ *and* $Y_\alpha = \sum_{k=1}^\infty \Gamma_k^{-\frac{1}{\alpha}}\varepsilon_k$.

It means that the limit of $\frac{Z_n}{b_n}$ is a countable zonotope which is strictly α–stable in $\mathbb{R}$.

Comments to the proof:
We know that condition (RV) for X_1 gives the convergence $\frac{S_n}{n} \Rightarrow Y_\alpha$ and RV–condition for $[0, X_1]$ gives the convergence of $\frac{Z_n}{n}$ to Z_α. But (RV) for X_1 is equivalent to (RV) for $[0, X_1]$! More exactly, if we have (RV) for X_1 with spectral measure σ, then (RV) for $[0, X_1]$ will be fulfilled with the measure $\tilde{\sigma}$ concentrated on the subset W of S^1 composed by the segments $[0, x]$, $x \in \mathbb{R}^d$, $|x| = 1$, and if

$$\varphi : S^{d-1} \to W, \qquad \varphi(x) = [0, x],$$

then $\tilde{\sigma} = \sigma\varphi^{-1}$. We have now:

1) $\{\frac{S_n}{b_n} \Rightarrow Y_\alpha\} \xrightarrow{\text{known fact}} \{\text{RV for } X_1\} \Rightarrow \{\text{RV for } [0, X_1]\} \Rightarrow \{\frac{X_n}{b_n} \Rightarrow Z_\alpha\}$.

2) $\{\frac{X_n}{b_n} \Rightarrow Z_\alpha\} \Rightarrow \{\frac{S_n}{b_n} \Rightarrow Y_\alpha\}$ we get this directly due to the fact, that the knowledge of Z_n allows us to find all segments $[0, X_i], i = 1, \dots, n,$ and the application

$$Z \longmapsto S_n$$

is continuous.

Info on the structure of the boundary of Z_α. The set of extreme points of Z_α is a Cantor-type set and its Lebesgue measure is equal to 0.

10 Exercises

1. Let $\mathcal{N}$ be a point process. Consider intensity measure $\mu(B) = \mathbb{E}[\mathcal{N}(B)]$, $B \in \mathcal{E}$. Check that $\mu(B)$ is a measure.

2. Suppose $X_1, \dots, X_n \in \mathbb{R}^d$ are i.i.d. random variables and define

$$\mathcal{N}(B) = \sum_{j=1}^{\nu} \mathbb{1}_B(X_j), \quad B \in \mathcal{B}^d, \tag{7}$$

where ν is a random variable independent of $(X_j)_{j=1,2,\dots,n}$. Show that $\mathcal{N}$ is a point process.

3. Let $\mathcal{N}$ be a point process defined by (7). Show that $\mathbb{E}[\mathcal{N}(B)] = \mathbb{E}[\nu] \cdot \mathcal{P}(B)$ if ν is independent of $(X_j)_{j=1,2,\dots,n}$.

4. Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are point processes defined on $(\Omega, \mathcal{F}, \mathrm{P})$ with state spaces $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ respectively. Then independence of

$$\{\mathcal{N}_1(F_i), 1 \le i \le k\} \quad \text{and} \quad \{\mathcal{N}_2(G_j), 1 \le j \le l\}$$

for any k, l; $F_1, \dots, F_k \in \mathcal{E}_1$, $G_1, \dots, G_l \in \mathcal{E}_2$ implies $\mathcal{N}_1$ and $\mathcal{N}_2$ independence.

5. Suppose $\mathcal{N}$ is a point process with state space $\mathbb{R}^d$. If τ is a random vector in $\mathbb{R}^d$, show $\mathcal{N}(\cdot + \tau)$ is a point process. Note, the value of $\mathcal{N}(\cdot + \tau)$ on the set F for realization ω is $\mathcal{N}(\omega, F + \tau(\omega))$, where $F + \tau = \{x + t, x \in F\}$.

6. Two point processes $\mathcal{N}_1$ and $\mathcal{N}_2$ are equal in distribution ($\mathcal{N}_1 \stackrel{\mathcal{L}}{=} \mathcal{N}_2$) iff for each $f \geq 0$ bounded and measurable we have

$$\mathcal{N}_1(f) \stackrel{\mathcal{L}}{=} \mathcal{N}_2(f)$$

as random variables. ($\mathcal{N}(f) = \int_E f d\mathcal{N}$).

7. Let $\mathcal{N}$ be a Poissonian point process (p.p.p.) with intensity measure $\mu(dt) = t^{-1}dt$ on $E = (0, \infty)$. Express $\mathcal{N}$ as a time changed homogeneous process.

8. Let $\mathcal{N}$ be a nonhomogeneous p.p.p. on $\mathbb{R}_+$ with mean measure $\mu(B) = \int_B \lambda(s)ds$, where λ is a locally integrable. Express $\mathcal{N}$ as a function of a homogeneous p.p.p.

9. Let $(\varepsilon_i)_{i=1,2,...}$ be i.i.d. exponential random variables on $\mathbb{R}_+$; $\mathrm{P}(\varepsilon_i > x) = e^{-x}$, $x > 0$. Let $\Gamma_n = \sum_{i=1}^n \varepsilon_i$. Show that $\mathcal{N} = \sum_{n=1}^\infty \delta_{\Gamma_n}$ is a homogeneous p.p.p. on $\mathbb{R}_+$. Here δ_a is δ-measure concentrated in a.

10. Suppose $\mathcal{N}$ be a homogeneous Poissonian point process on $\mathbb{R}_+$ with intensity measure λ, where λ is Lebesgue measure. Suppose $\mathcal{N}$ is an integer number. Show that almost surely

$$\lim_{n\to\infty} \frac{\mathcal{N}([0, n])}{n} = 1.$$

11. (Continuation) Suppose T is a real number, show that

$$\lim_{T\to\infty} \frac{\mathcal{N}([0, T])}{T} = 1.$$

12. Suppose $\mathcal{N}'$ be a Poissonian point process on $\mathbb{R}_+$ such that its intensity measure μ is given by

$$\mu(dx) = \frac{1}{x}dx, \quad x \in \mathbb{R}_+.$$

Show that

$$\lim_{T\to\infty} \frac{\mathcal{N}'([0, T])}{b(T)}$$

exists almost surely if one chooses proper normalization $b(T)$. Find $b(T)$ and prove the result.

13. Suppose $\mathcal{N}$ be a Poissonian point process with intensity measure μ. Denote the restriction of $\mathcal{N}$ on A as $\mathcal{N}\mid_A$, where $\mathcal{N}\mid_A (B) = \mathcal{N}(A \cap B)$. Suppose $A_1, A_2, \ldots, A_m$ are mutually disjoint sets. Show that p.p.p. $\mathcal{N}\mid_{A_1}, \ldots, \mathcal{N}\mid_{A_m}$ are independent.

14. Suppose $A_1, A_2, \ldots, A_m$ are mutually disjoint sets, $A = \bigcup_{j=1}^m A_j$. Show that vector $(\mathcal{N}(A_1), \ldots, \mathcal{N}(A_m))$ has conditional multinomial distribution

$$\begin{aligned} &\mathrm{P}(\mathcal{N}(A_1) = k_1, \ldots \mathcal{N}(A_m) = k_m \mid \mathcal{N}(A) = k_1 + \ldots + k_m) \\ &= \frac{k!}{k_1! \cdot \ldots \cdot k_m!} \left[\frac{\mu(A_1)}{\mu(A)}\right]^{k_1} \ldots \left[\frac{\mu(A_m)}{\mu(A)}\right]^{k_m}, \quad k = k_1 + \ldots + k_m. \end{aligned}$$

15. Show that $\mathrm{P}(\mathcal{N}\mid_A \in \cdot \mid \mathcal{N}(A)=k) = \mathrm{P}(\sum_{i=1}^k \delta_{\xi_i} \in \cdot)$, where $(\xi_i)_{i=1,2,\dots}$ are i.i.d. with common distribution

$$\frac{\mu\mid_A(\cdot)}{\mu(A)}.$$

16. Suppose $f \geq 0$. Show that

$$\mathbb{E}\left[\mathcal{N}\mid_A(f)\right] = e^{-\mu(A)}\left(f(0) + \sum_{k=1}^{\infty}\frac{1}{k!}\underbrace{\int_A \cdots \int_A}_{k} f\left(\sum_{i=1}^k \delta_{x_i}\right)\mu(dx_1)\dots\mu(dx_k)\right).$$

17. Suppose $\Pi_{\alpha,c}$, $(\alpha > 0, c > 0)$ is p.p.p. on $\mathbb{R}^d \backslash \{0\}$ (that is identified with $S^{d-1} \times \mathbb{R}_+$). Intensity measure $m_{\alpha,c}$ is given by

$$m_{\alpha,c} = \sigma \times \mu_{\alpha,c},$$

where σ is a probability measure of S^{d-1} and $\mu_{\alpha,c}$ is a measure on $\mathbb{R}_+$ defined by

$$\frac{d\mu_{\alpha,c}}{d\lambda}(r) = cr^{-1-\alpha}.$$

It is known that $\Pi_{\alpha,c}$ satisfies stability condition (3). Show that if p.p.p $\mathcal{N}$ satisfies (3) then $\mathcal{N} \overset{\mathcal{L}}{=} \Pi_{\alpha,c}$ for a certain $c > 0$.

18. If we have $\mathcal{N}_1, \mathcal{N}_2$ independent p.p.p. defined on the same space E, with two diffuse measures μ_1, μ_2 then $\mathcal{N}_1 + \mathcal{N}_2$ is p.p.p. with intensity measure $\mu_1 + \mu_2$.

19. Let $X_1, \dots, X_n$ be independent random variables with absolute continuous distributions. Let $a_1, \dots, a_n \in \mathbb{R}^n$ be the base of $\mathbb{R}^n$. Show that the random vector $Z = \sum_{i=1}^n a_i Z_i$ has a density.

20. Let $\mathcal{P}$ be a probability measure on Borel σ–algebra $\mathcal{E}$ of a complete separable linear space E. Let (X_n) be i.i.d. random elements of E with common distribution $\mathcal{P}$. Prove that a.s. $\mathrm{span}(X_1, \dots, X_n) = \mathrm{span}(\mathrm{supp}\,\mathcal{P})$. Recall that $\mathrm{supp}\,\mathcal{P}$ is a support of $\mathcal{P}$, that is

    - $\mathrm{supp}\,\mathcal{P}$ is closed subset of E,
    - $\mathcal{P}(\mathrm{supp}\,\mathcal{P}) = 1$,
    - for any open ball $B \subset E$ such that $\mathrm{supp}\,\mathcal{P} \cap B \neq \emptyset$ we have $\mathcal{P}(B) > 0$.

21. Using later two exercises, show that the distribution of strictly α–stable vector

$$Y_\alpha = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k \in \mathbb{R}^d$$

is absolutely continuous iff $\mathrm{span}(\mathrm{supp}\,\sigma) = \mathbb{R}^d$, where σ is common distribution of ε_k.

22. Suppose $x_n, x \in E$, $c_n \geq 0$, $c > 0$. Show that

$$c_n \delta_{x_n} \xrightarrow{\text{vague}} c\delta_x$$

iff $c_n \to c$ and $x_n \to x$, as $n \to \infty$.

23. Let
$$\lambda_n = \frac{1}{n}\sum_{i=1}^{\infty}\delta_{\frac{i}{n}}$$
be a discrete version of Lebesgue measure λ on $[0,\infty)$. Show that
$$\lambda_n \xrightarrow{\text{vague}} \lambda.$$

24. Let $M \in \mathcal{M}$ be measurable subset of $\mathbb{M}$. Show that M is relatively compact in vague topology iff
$$\sup_{\mu\in M}\mu(f) < \infty$$
for any $f \in \mathcal{C}_K^+$.

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