Point processes and stable laws
Lecture notes

Youri Davydov
youri.davydov@univ-lille1.fr

Mateusz Topolewski*
woland@mat.uni.torun.pl

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1 Introduction

Non formal description:
point process = random cloud of points

Examples: Let $(X_k)$ be the iid random variables in $\mathbb{R}^d$.
1) empirical point process: $\mathcal{N}_n = \{X_1, X_2, \ldots, X_n\};$
2) $\mathcal{N}_{n,D} = \mathcal{N}_n \cap D$ for some fixed set $D \subset \mathbb{R}^d$;
3) $\mathcal{N} = \{X_1, \ldots, X_\nu\}$ where $\nu$ is random variable independent of sequence $(X_k)$ and $P(\nu \in \mathbb{N}) = 1$.

Simple point process:
Let $(\Omega, \mathcal{F}, P)$ be the probability space, $(E, \mathcal{E})$ — measurable space, called phase space. For us:

*Preparing \LaTeX version.
E is complete, separable metric space, 
\( E = \mathcal{B}_E \) is its Borel \( \sigma \)-algebra, 
\( K \) is a space of configurations. Formally 
\[ K = \{ \kappa = \{ x_i, i \in I \} \subset E, I \subset \mathbb{N} : \kappa \cap B(0, r) \text{ is finite for any } r > 0 \}. \]

Let \( K = \sigma(\pi_B, B \in \mathcal{E}) \) be \( \sigma \)-algebra generated by the family of projections \( \{ \pi_B, B \in \mathcal{E} \} \).

**Definition 1.1.** (Simple) point process (p.p.) is a measurable application from \((\Omega, \mathcal{F}, P)\) into \((K, K)\).

**Second approach:**

Point measure:
\[ \mu = \sum_{i \in I} \delta_{x_i}, \quad I \subset \mathbb{N}. \]

Simple point measure: \( \{ x_i \in E : i \in I \} \) is locally finite.
\[ M = \{ \mu : \mu \text{ is a simple point measure} \}, \quad \mathcal{M} = \sigma(\pi_B, B \in \mathcal{E}) \]

where
\[ \pi_B : \mathcal{M} \to \mathbb{R}, \quad \pi_B(\mu) = \mu(B), \quad B \in \mathcal{E}. \]

**Definition 1.2.** (Simple) point process (p.p.) is a measurable application from \((\Omega, \mathcal{F}, P)\) into \((M, \mathcal{M})\).

**Remark 1.3.**
- Application \( \varphi : K \to M \) such that
  \[ \varphi(\kappa) = \sum_{x \in \kappa} \delta_x \quad (= \mu) \]
  is bijection and
  \[ \pi_B(\kappa) = |\kappa \cap B| = \sum_{x \in \kappa} 1_B(x) = \mu(B). \]

- We can consider the family \( \{ N(B), b \in \mathcal{E} \} \) as a random process defined on \( \mathcal{E} \).

**Proposition 1.4.** (1) \( \iff \) (2) where:

1. \( \mathcal{N} : \Omega \to M \) is a p.p.
2. for all \( B \in \mathcal{E} \) the application \( \mathcal{N}(B) : \Omega \to \mathbb{N} \) is a random variable.

**Intensity measure:**

**Proposition 1.5.** Let \( \mathcal{N} \) be a p.p. The function \( m : \mathcal{E} \to \mathbb{R}_+ \),
\[ m(B) = \mathbb{E}[\mathcal{N}(B)] \]
is a measure. We call it intensity measure for \( \mathcal{N} \).

**Remark 1.6.**
1. It is possible that \( m(B) = \infty \) for very large subfamily of \( \mathcal{E} \).
2. \( m(\cdot) \) is a mean value of the process \( \{ \mathcal{N}(B), B \in \mathcal{E} \} \).
3) Assume that $X_1, \ldots, X_n$ are i.i.d. random variables with distribution $\mathcal{P}$. For $N_n = \sum_{i=1}^{n} \delta_{X_i}$, we have $E[N_n(B)] = n\mathcal{P}(B)$.

4) $E[N_n \cap D(B)] = n\mathcal{P}(B \cap D)$.

5) For $N = \sum_{i=1}^{\nu} \delta_{X_i}$, we have $E[N(B)] = E[\nu] \mathcal{P}(B)$.

Finite dimensional laws:

**Theorem 1.7.** Let $\mathcal{N}, \mathcal{N}_1$ be two p.p. in $(E, \mathcal{E})$. The following conditions are equivalent:

1) $\mathcal{N} \overset{d}{=} \mathcal{N}_1$;

2) for all $k \in \mathbb{N}$ and $B_1, \ldots, B_n \in \mathcal{E}$

$$
(N(B_1), \ldots, N(B_n)) \overset{d}{=} (N_1(B_1), \ldots, N_1(B_n));
$$

3) Equality (1) takes place for mutually disjoint sequences.

**2 Poisson point process (p.p.p.)**

**Definition 2.1.** Let $m$ be diffuse measure on $(E, \mathcal{E})$. The p.p. $\mathcal{N}$ is called Poissonian if

1) for any $B \in \mathcal{E}$ such that $0 < m(B) < \infty$

$$
\mathcal{N}(B) \sim \text{Pois}(m(B));
$$

(\mathcal{N}(B) has Poissonian distribution with parameter $m(B)$)

2) for all mutually disjoint sets $B_1, \ldots, B_n \in \mathcal{E}$ random variables $N(B_1), \ldots, N(B_n)$ are independent.

**Remark 2.2.** The measure $m$ is exactly intensity measure for $\mathcal{N}$.

**Theorem 2.3.** Let $m$ be a finite diffuse measure. Let $(X_k)$ be a sequence of i.i.d. random variables with common distribution $Q = \frac{1}{m(E)} m$. Let $\nu$ be a random variable independent of the sequence $(X_k)$ having Poissonian distribution with parameter $m(E)$. Then the p.p.

$$
\mathcal{N} = \sum_{i=1}^{\nu} \delta_{X_i}
$$

is p.p.p. with intensity measure $m$.

**Proof.** Let $B_1, \ldots, B_k \in \mathcal{E}$ be mutually disjoint sets such that $\bigcup_{i=1}^{k} B_i = E$. Moreover let $n_1 + \ldots + n_k = n$. Then

$$
P(\mathcal{N}(B_1) = n_1, \ldots, \mathcal{N}(B_k) = n_k) = P(\mathcal{N}(B_1) = n_1, \ldots, \mathcal{N}(B_k) = n_k, \nu = n)
$$

$$
= P(\nu = n) \frac{n!}{n_1! \ldots n_k!} \prod_{i=1}^{k} Q(B_i)^{n_i}
$$

$$
= e^{-m(E)} \frac{m(E)^n}{n!} \frac{n!}{n_1! \ldots n_k!} \prod_{i=1}^{k} \left( \frac{m(B_i)}{m(E)} \right)^{n_i}
$$

$$
= \prod_{i=1}^{k} e^{-m(B_i)} \frac{m(B_i)^{n_i}}{n_i!} = \prod_{i=1}^{k} P(\mathcal{N}(B_i) = n_i).
$$

$\square$

1) Let \( \mathcal{N} \) be p.p.p. with intensity measure \( m \) and \( f : E \rightarrow E_1 \) be a measurable injection such that for each ball \( B \subset E_1 \) the set \( f^{-1}(B) \) is bounded. Then \( \mathcal{N}_1 = \mathcal{N} f^{-1} \) is p.p.p. with intensity \( m_1 = m f^{-1} \).

**Example 3.1.** Homogeneous p.p.p. on \( \mathbb{R}^d \) is invariant relatively translations and rotations.

2) Let \( \mathcal{N}_1, \mathcal{N}_2 \) be two independent p.p.p. with diffuse intensity measures \( m_1, m_2 \) respectively. Then \( \mathcal{N}_1 + \mathcal{N}_2 \) is p.p.p. with intensity measure \( m_1 + m_2 \).

3) Let \( \mathcal{N}_1 \) be a p.p.p. with intensity measure \( m_1 \) defined on \( E_1 \). Let \( (Y_j) \) be a sequence of i.i.d. random variables with values in \( E_2 \) and with common distribution \( Q \). Then the p.p.p.

\[ \mathcal{N} = \{(x_i, Y_i), x_i \in \mathcal{N}_1\} \]

is p.p.p. on \( E_1 \times E_2 \) with intensity measure \( m = m_2 \times Q \). \( \mathcal{N} \) is called marked p.p.p. and variables \( (Y_j) \) are called marks.

**Theorem 3.2.** Existence of p.p.p. in general case follows from theorem 2.3

Homogeneous p.p.p. in \( \mathbb{R}^d \).

**Definition 3.3.** P.p.p. \( \mathcal{N} \) defined on \( E = \mathbb{R}^d \) is called homogeneous if \( m = \alpha \lambda^d \) (\( \lambda^d \) is Lebesgue measure on \( \mathbb{R}^d \)) and \( \alpha > 0 \) is called intensity of \( \mathcal{N} \).

**Case d = 1:** Let \( \pi \) be homogeneous p.p.p. with intensity 1. Consider the process

\[ X_t = \pi([0,t]), \quad t \geq 0. \]

Then

(i) \( X(0) = 0 \)

(ii) \( X(t) - X(s) = \pi((s,t]) \sim \text{Pois}(t - s) \) for \( t > s \)

(iii) \( X \) has independent increments

(iv) paths of \( X \) are right continuous almost surely

It means that \( X \) is a standard Poissonian process on \( \mathbb{R}_+ \).

**Theorem 3.4** ("Exponential description" of \( \pi \)). Let \( (\varepsilon_k) \) be a sequence of i.i.d. random variables having common exponential distribution with parameter 1. Let \( \Gamma_n = \varepsilon_1 + \ldots + \varepsilon_n \). Then

\[ \pi \overset{\mathcal{L}}{=} \sum_{k=1}^{\infty} \delta_{\Gamma_k}. \]

**Example 3.5.** \( \mathcal{N} \) is homogeneous p.p.p. on \( \mathbb{R}^d \) with intensity measure \( \lambda^d \) and \( (Y_j) \) a sequence of i.i.d. random variables which is independent of \( \mathcal{N} \). Let

\[ M = \{X_i + Y_i, \ X_i \in \mathcal{N}\}. \]

Then \( M \overset{\mathcal{L}}{=} \mathcal{N} \).
Example 3.6 (Process $\Pi_\alpha$). Let $(\gamma_k)$ be i.i.d. random variables, $\gamma_k \sim \mathcal{E}(1)$ and $\Gamma_k = \gamma_1 + \ldots + \gamma_k$ for $k = 1, 2, \ldots$. Let $(\varepsilon_k)$ be i.i.d. random variables in $\mathbb{R}^d$, $|\varepsilon_k| = 1$ a.s. Denote by $\sigma$ common law of $\varepsilon_k$. Let $\alpha > 0$.

Proposition 3.7. Point process

$$\Pi_\alpha = \sum_{k=1}^\infty \delta_{\varepsilon_k \Gamma_k^{-1/\alpha}}$$

is poissonian with intensity measure $m$ which in polar representation of $\mathbb{R}^d \setminus \{0\} = (\mathbb{R}^d \setminus \{0\}) \times S^{d-1}$ has the following structure:

$$m = \mu \times \sigma \quad \text{where} \quad \frac{d\mu}{dt} = \alpha t^{1-\alpha}.$$ 

Proof. Let $N = \sum_{k=1}^\infty \delta_{\Gamma_k}$ be homogeneous p.p.p. on $\mathbb{R}_+$. Then $\Pi = Nh^{-1}$ where $h : \mathbb{R}_+ \to (\mathbb{R}_+ \setminus \{0\})$, $h(\tau) = \tau^{-\frac{1}{\alpha}}$ is a p.p.p. and $\Pi = \sum_{k=1}^\infty \delta_{\Gamma_k^{-1/\alpha}} (\sim \mu)$. Now it is clear that $\Pi_\alpha$ is a marked p.p.p. constructed by $\Pi$ and marks $(\varepsilon_k)$.

4 Stability property

We consider only strict stability.

Definition 4.1. Let $X$ be a random variable with values in $\mathbb{R}^d$. $X$ is called strictly $\alpha$–stable if for all $a, b > 0$

$$a^\frac{1}{\alpha} X_1 + b^\frac{1}{\alpha} X_2 \sim (a + b)^\frac{1}{\alpha} X,$$ (2)

where $X_1, X_2$ are independent copies of $X$.

This class is very important: if the limit distribution for normed sums

$$\frac{Y_1 + \ldots + Y_n}{b_n}$$

exists, then it is strictly $\alpha$–stable for some $\alpha \in (0, 2]$ and in this case $b_n = nh(n)$, where $h$ is a slowly varying function.

Stability of p.p.:

Consider a p.p. $\mathcal{N}$ in $E = \mathbb{R}^d$. Let $D_a : \mathbb{R}^d \to \mathbb{R}^d$, $D_a x = ax$, for $a > 0$. For point measure $\mu$ we use the notation $D_a \circ \mu$ for $\mu D_a^{-1}$.

Definition 4.2. P.p. $\mathcal{N}$ is called $\alpha$–stable if for all $a, b > 0$

$$D_{a^\frac{1}{\alpha}} \mathcal{N}_1 \oplus D_{b^\frac{1}{\alpha}} \mathcal{N}_2 \sim D_{(a+b)^\frac{1}{\alpha}} \mathcal{N},$$ (3)

where $\mathcal{N}_1, \mathcal{N}_2$ are 2 independent copies of $\mathcal{N}$.

Proposition 4.3. The p.p.p. $\Pi_\alpha$ is strictly $\alpha$–stable.

Proof. Stability property for $\Pi_\alpha$ means that for every $a, b > 0$

$$D_{a^\frac{1}{\alpha}} \Pi_\alpha \oplus D_{b^\frac{1}{\alpha}} \Pi_\alpha \sim D_{(a+b)^\frac{1}{\alpha}} \Pi_\alpha,$$ (4)

where $\Pi_\alpha^{(1)}, \Pi_\alpha^{(2)}$ are 2 independent copies of $\Pi_\alpha$. To prove this equality it is sufficient to state that the intensity measures of the left part and the right part coincide. $\Box$
Let 

\[ f(\mu) = \int_E x\mu(dx) \]

be a "linear" functional on \( M \). Of course

\[ f(\mu_1 + \mu_2) = f(\mu_1) + f(\mu_2) \quad \text{and} \quad f(a \circ \mu) = af(\mu). \]

We have formally form [4] applying \( f \):

\[ a^{\frac{1}{\alpha}} Y^{(1)}_\alpha + b^{\frac{1}{\alpha}} Y^{(2)}_\alpha \overset{\mathcal{L}}{=} (a + b)^{\frac{1}{\alpha}} Y_\alpha, \]

where \( Y^{(1)}_\alpha, Y^{(2)}_\alpha \) are 2 independent copies of

\[ Y_\alpha = f(\Pi_\alpha) = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k. \]

When we can justify this passage?

**Answer:** It is sufficient to understand when the series \( \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k \) converges.

**Proposition 4.4.**

1) If \( \alpha \in (0, 1) \), then the convergence of \( \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k \) takes place almost surely.

2) For \( \alpha \in [1, 2) \) if \( \mathbb{E}[\varepsilon_1] = 1 \), then also we have this convergence.

**Proof.**

1) In this case a.s. the series converges absolutely because \( |\varepsilon_k| = 1 \) and \( \Gamma_k \sim k \) a.s.

2) Easily follows from Kolmogorov’s three-series theorem.

**Terminology:**

- The series \( \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k \) is called LaPage series.
- Its sum \( Y_\alpha \) is strictly \( \alpha \)-stable vector in \( \mathbb{R}^d \).
- \( \sigma \) is called spectral measure of \( Y_\alpha \).

**Comments:**

- Case \( \alpha = 2 \) corresponds to Gaussian law. There is no LePage representation in this case.
- For \( \alpha > 2 \) there is no stable laws in \( \mathbb{R}^d \).

**Theorem 4.5.** Let \( (\varepsilon_k) \) and \( (\Gamma_k) \) be 2 sequences as before. Let \( (\eta_k) \) be a sequence of i.i.d. real random variables, independent on \( (\Gamma_k) \) and \( (\varepsilon_k) \) and such that

\[ a^\alpha = \mathbb{E}[|\eta_k|^\alpha] < \infty. \]

Then the series

\[ Z_\alpha = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \eta_k \varepsilon_k \]

is almost surely convergent and in the case \( \eta_k \geq 0 \) almost surely

\[ Z_\alpha \overset{\mathcal{L}}{=} a Y_\alpha. \]
Remark 4.6. In general case: $Z_\alpha$ is strictly $\alpha$–stable and the last relation is fulfilled with $\tilde{Y}_\alpha$ having changed spectral measure $\tilde{\sigma}$:

$$\tilde{\sigma}(B) = \sigma(B)E[\eta^\alpha] + \sigma(-B)E[\eta^\alpha].$$

If $\sigma$ is symmetric, then $\tilde{\sigma} = \sigma$. If $\eta$ has a symmetric distribution, then

$$\tilde{\sigma}(B) = \sigma(B) + \sigma(-B).$$

Proof of theorem 4.5. It is sufficient to state the equality of intensity measures for two p.p.p.:

$$N_1 = \{\Gamma_k^{-\frac{1}{\alpha}}\eta_k\} \quad \text{and} \quad N_2 = \{a\Gamma_k^{-\frac{1}{\alpha}}\}.$$

Let compute the intensity measure $\mu_1$ for $N_1$:

$$\mu_1 = (m \times Q)i^{-1},$$

where $Q$ is common distribution of $\eta_k$ and

$$i : (0, \infty) \times \mathbb{R} \to \mathbb{R}, \quad i(x, t) = tx.$$

We have

$$(m \times Q)i^{-1}((\tau, \infty)) = (m \times Q)((x, t) : xt \geq \tau)) = \int_0^\infty \int_0^\infty 1_{\{xt \geq \tau\}} m(dx)Q(dt)$$

$$= \int_0^\infty m\left(\left\lfloor \frac{\tau}{t}, \infty\right\rfloor\right)Q(dt) = \int_0^\infty \left(\frac{\tau}{t}\right)^{-\alpha} Q(dt)$$

assuming $\eta_k \geq 0 a^\alpha m(\tau, \infty)).$

Hence $\mu_1 = a^\alpha m$. In particular, for $\eta_k \equiv a$ we obtain the same measure.

Remark 4.7. In symmetric case, choosing $\eta_k \sim \mathcal{N}(0,1)$ we see that the law of $Y_\alpha$ is a mixture of Gaussian distributions.

Stable Lévy processes:

Let $(U_k)$ be i.i.d. random variables with $[0,1]$–uniform distribution. Let

$$X(t) = \sum_{k=1}^\infty \Gamma_k^{-\frac{1}{\alpha}}\mathbb{1}_{[0,t]}(U_k)\xi_k, \quad t \in [0,1].$$

This process possesses the following properties:

1) It has independent, stationary increments.

2) $X(t) - X(s) \xrightarrow{L} (t-s)^\frac{\alpha}{2} X(1)$ (increments are strictly $\alpha$–stable)

3) Paths are right–continuous.

These properties follow directly from definition and theorem 4.5.
5 Convex cones

It is clear that if we want to formulate stability property, it is sufficient to have 2 operations: addition and multiplication on positive solutions. It means that the stability property could be considered in convex cones.

**Definition 5.1.** Topological space $\mathbb{K}$ is called convex cone if two operations (addition and scalar multiplication) satisfy the following conditions:

0) they are continuous;
1) for all $a \geq 0$ and $x, y \in \mathbb{K}$
   \[ a(x + y) = ax + ay; \]
2) for all $a, b \geq 0$ and $x \in \mathbb{K}$
   \[ a(bx) = (ab)x \]
3) there exists element 0 such that
   \[ x + 0 = x, \quad x \in \mathbb{K}, \quad \text{and} \quad c \cdot 0 = 0, \quad c \geq 0. \]

**Remark 5.2.** We do not require $(a + b)x = ax + bx$ (second distributive law).

**Examples:**

1) $\mathbb{B}$ — Banach space (or subcone of $\mathbb{B}$)

2) Let $T$ be a compact metric space. Take
   \[ C^\text{max}_+(T) := \{ f \geq 0 : f \text{ is continuous on } T \} \]
   with operations
   \[ (f + g)(t) = \max\{f(t), g(t)\}, \quad (cf)(t) = cf(t), \quad t \in T, c \geq 0. \]
   Remark that $f + f = f$ for all $f \in C^\text{max}_+(T)$!
   In particular, when $T$ is singleton, $\mathbb{K} = \mathbb{R}^\text{max}_+$, that is $\mathbb{K} = [0, \infty)$, $x + y = x \lor y$ and $cx$ is as usual.

3) $\mathbb{K} = \mathbb{K}(\mathbb{R}^d)$ space of all convex compact subsets of $\mathbb{R}^d$.
   $A \oplus B$ — Minkowski addition:
   \[ A \oplus B = \{ x + y : x \in A, \ y \in B \} \]
   $cA = \{ cx : x \in A \}$ is a dilatation.

   It is supposed that $\mathbb{K}$ is metrisable, $d$ is a metric on $\mathbb{K}$. Denote $|x| = d(0, x)$ and
   \[ S^1 = \{ x \in \mathbb{K} : |x| = 1 \}. \]
   Polar representation for $\mathbb{K}\{0\}$:
   \[ \mathbb{K} \simeq (0, \infty) \times S^1, \quad x \longleftrightarrow (|x|, \frac{x}{|x|}). \]
   Stability in $\mathbb{K}$:
   Definition of stability in $\mathbb{K}$ is the same as in $\mathbb{R}^d$. 

Theorem 5.3. If the series \( Y_\alpha = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k \) converges (in this case \( (\varepsilon_k) \) is i.i.d. sequence such that \( |\varepsilon_k| = 1 \) a.s.), then \( Y_\alpha \) will have a strictly \( \alpha \)-stable distribution.

Example 5.4. \( K = \mathbb{R}^{\max} \). Then \( \varepsilon_k \equiv 1 \) and
\[
\sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k = \max_k \{ \Gamma_k^{-\frac{1}{\alpha}} \} = \Gamma_1^{-\frac{1}{\alpha}}.
\]
It implies that for any \( \alpha > 0 \) there exists a strictly \( \alpha \)-max–stable law.


If we want to define the weak convergence of p.p., it is necessary before to introduce a convenient topology into space \( \mathbb{M} \). Usually one uses the so-called vague topology.

We consider \( \mathbb{M} \) as a subspace of \( \mathbb{M}_+(E) \) — space of all positive locally bounded measures on \( E \). We say that \( \mu_n \in \mathbb{M}_+(E) \) converges vaguely to \( \mu \) (denote \( \mu_n \overset{\text{vague}}{\longrightarrow} \mu \)) if
\[
\int_E f d\mu_n \longrightarrow \int_E f d\mu
\]
for each continuous function \( f \) having a bounded support. The topology induced by such type of convergence is called vague topology.

Laplace functional:
Laplace functional is an analogue of Laplace transform for random vectors and it is very useful in the study of weak convergence of p.p.

- Laplace transform for random variables:
  Let \( X \in \mathbb{R}_+ \) be a random variable. Laplace transform of \( X \) is the function
  \[
  L(t) = \mathbb{E} \left[ e^{-tX} \right], \quad t \in \mathbb{R}_+.
  \]
  It is known that Laplace transform defines completely the distribution of \( X \).

**Exercise:** Let \( X \sim \text{Pois}(\alpha) \) be a Poissonian random variable. Then \( L(t) = \exp (\alpha(e^{-t} - 1)) \).

- Laplace transform for random vectors:
  Let \( X \in \mathbb{R}^d_+ \) be a random vector. Then
  \[
  L(t) = \mathbb{E} \left[ e^{-\langle t, X \rangle} \right], \quad t \in \mathbb{R}^d_+.
  \]

**Proposition 6.1.** Let \( (X_n) \) be a sequence of random variables. Then weak convergence \( X_n \Rightarrow X \) is equivalent to convergence of Laplace transforms:
\[
L_n(t) \longrightarrow L(t), \quad t \in \mathbb{R}_+.
\]

**Proof.** "\( \Rightarrow \)" is evident.

"\( \Leftarrow \)" Suppose that \( L_n(t) \to L(t) \), where \( L \) is Laplace transform of \( X \). Let \( Y_n = e^{-X_n} \), then \( Y_n \in [0, 1] \) a.s. Hence sequence \( \{Y_n\} \) is tight. Let \( Y = e^{-X} \). If for \( \{n_k\} \subset \mathbb{N} \)
\[
Y_{n_k} \Rightarrow \tilde{Y},
\]

then $\mathbb{E} [Y_{n_k}^t] \to \mathbb{E} [\tilde{Y}^t]$ for $t \in \mathbb{R}_+$. Moreover

$$\mathbb{E} [Y_{n_k}^t] = L_{n_k}(t) \to L(t) = \mathbb{E} [Y^t].$$

Therefore we have $\tilde{Y} \overset{d}{=} Y$. Hence $Y_n \Rightarrow Y$ and it follows that $X_n \Rightarrow X$. \qed

**Definition 6.2.** Let $\mathcal{N}$ be a p.p. *Laplace functional* of $\mathcal{N}$ is operator

$$L_{\mathcal{N}}(f) = \mathbb{E} [e^{-\mathcal{N}(f)}],$$

where $\mathcal{N}(f) = \langle f, \mathcal{N} \rangle = \int_{E} f d\mathcal{N} = \sum_{x \in \text{supp} \mathcal{N}} f(x)$, $f$ being nonnegative function with bounded support.

By the transfer theorem

$$L_{\mathcal{N}}(f) = \int_{\mathcal{M}} \exp \left( - \int_{E} f(x) \kappa(dx) \right) \mathcal{P}_{\mathcal{N}}(d\kappa).$$

**Proposition 6.3.** $L_{\mathcal{N}}$ uniquely defines $\mathcal{P}_{\mathcal{N}}$.

**Proof.** Let $k \geq 1$ and $B_1, \ldots, B_k \in \mathcal{E}$, $\lambda_1, \ldots, \lambda_k \geq 0$. Let $f(x) = \sum_{i=1}^{k} \lambda_i \mathbb{1}_{B_i}(x)$. Then

$$\mathcal{N}(f) = \sum_{i=1}^{k} \lambda_i \mathcal{N}(B_i)$$

and $L_{\mathcal{N}}(f) = \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{k} \lambda_i \mathcal{N}(B_i) \right) \right]$ represents the joint Laplace transform of the random variable $(\mathcal{N}(B_1), \ldots, \mathcal{N}(B_k))$. It means that $L_{\mathcal{N}}$ defines all finite-dimensional distributions of $\mathcal{N}$, hence $\mathcal{P}_{\mathcal{N}}$ itself. \qed

**Proposition 6.4.** Let $\mathcal{N}$ be a p.p.p. with intensity measure $\mu$. Then

$$L_{\mathcal{N}}(f) = \exp \left( - \int_{E} (1 - e^{-f(x)}) \mu(dx) \right).$$

**Proof.** **Step 1:** $f(x) = c \mathbb{1}_{F}(x)$. Then $\mathcal{N}(f) = c \mathcal{N}(F)$ and

$$L_{\mathcal{N}}(f) = \mathbb{E} [e^{-\mathcal{N}(f)}] = \mathbb{E} [e^{-c\mathcal{N}(F)}] \quad \text{[as $\mathcal{N}(F) \sim \text{Pois}(\mu(F))$]}

= \exp \left( (e^{-c} - 1) \mu(F) \right) = \exp \left( - \int_{E} (1 - e^{-f(x)}) \mu(dx) \right).$$

**Step 2:** $f(x) = \sum_{i=1}^{k} c_i \mathbb{1}_{F_i}(x)$ where sets $(F_i)$ are disjoint and $c_i \geq 0$. Then $\{\mathcal{N}(F_i)\}$ are independent. Hence

$$L_{\mathcal{N}}(f) = \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{k} c_i \mathcal{N}(F_i) \right) \right] = \prod_{i=1}^{k} \mathbb{E} \left[ \exp \left( - c_i \mathcal{N}(F_i) \right) \right]$$

$$= \prod_{i=1}^{k} \exp \left( - \int_{E} (1 - e^{-c_i \mathbb{1}_{F_i}(x)}) \mu(dx) \right)

= \exp \left( - \int_{E} \sum_{i=1}^{k} (1 - e^{-c_i \mathbb{1}_{F_i}(x)}) \mu(dx) \right)

= \exp \left( - \int_{E} (1 - e^{-\sum_{i=1}^{k} c_i \mathbb{1}_{F_i}(x)}) \mu(dx) \right)

= \exp \left( - \int_{E} (1 - e^{-f(x)}) \mu(dx) \right).$$
We used here this obvious functional equality
\[ \sum_{i=1}^{k} (1 - e^{-c_i P_i(x)}) = 1 - e^{-\sum_{i=1}^{k} c_i P_i(x)}. \]

**Step 3:** Let \( f \geq 0 \) and \((f_n)\) be non-decreasing sequence of step-functions such that \( f_n \not\rightarrow f \). Then \( \mathcal{N}(f_n) \rightarrow \mathcal{N}(f) \) and
\[
L_{\mathcal{N}}(f) = \mathbb{E}[\exp(-\mathcal{N}(f))] \rightarrow \mathbb{E}[\exp(-\mathcal{N}(f))] = L_{\mathcal{N}}(f).
\]

\[ \square \]

**Proposition 6.5.** \( \mathcal{N}_n \Rightarrow \mathcal{N} \) iff \( L_{\mathcal{N}_n}(f) \rightarrow L_{\mathcal{N}}(f) \) for all \( f \in C^+_K(E) \).

**Sketch of proof.** "\( \Rightarrow \)": If \( \mathcal{N}_n \Rightarrow \mathcal{N} \) then by continuous mapping theorem
\[
\langle f, \mathcal{N}_n \rangle \Rightarrow \langle f, \mathcal{N} \rangle, \quad f \in C^+_K(E)
\]
and then \( \mathbb{E}[\exp(-\langle f, \mathcal{N}_n \rangle)] \rightarrow \mathbb{E}[\exp(-\langle f, \mathcal{N} \rangle)] \), that is \( L_{\mathcal{N}_n}(f) \rightarrow L_{\mathcal{N}}(f) \).

"\( \Leftarrow \)": Main point:

**Lemma.** \( (\mathcal{N}_n) \) is tight in \( \mathbb{M}_+(E) \) iff for all \( f \in C^+_K(E) \) the sequence \( (\mathcal{N}_n(f)) \) is tight in \( \mathbb{R} \).

Suppose \( L_{\mathcal{N}_n}(f) \rightarrow L_{\mathcal{N}}(f) \). It means that for any \( t \in \mathbb{R}^+ \)
\[
\mathbb{E}\left[\exp(-t\mathcal{N}_n(f))\right] \rightarrow \mathbb{E}\left[\exp(-t\mathcal{N}(f))\right],
\]
that is Laplace transform of random variable \( \mathcal{N}_n(f) \) converges to Laplace transform of \( \mathcal{N}(f) \). Hence
\[ \mathcal{N}_n(f) \Rightarrow \mathcal{N}(f) \]
and we see that \( (\mathcal{N}_n(f)) \) is tight. By lemma we get the tightness of \( (\mathcal{N}_n) \). Now to finish the proof it is sufficient to use standard arguments. \( \square \)

### 7 Convergence of empirical p.p.

**Theorem 7.1.** Let \((X_{k,j})_{j=1,2,...,n}\) be a triangular array of random variables independent and identically distributed in each row. Let \( \mathcal{N}_n = \sum_{j=1}^{n} \delta_{X_{n,j}} \). Let \( \mathcal{N} \) be p.p.p. with intensity measure \( \mu \). Then \( \mathcal{N}_n \Rightarrow \mathcal{N} \) iff
\[ nP(X_{n,1} \in \cdot) \overset{\text{vague}}{\longrightarrow} \mu. \]  \( (5) \)

**Proof.** It is sufficient to study the behaviour of Laplace functional. We have
\[
L_{\mathcal{N}_n}(f) = \mathbb{E}\left[\exp\left(-\sum_{j=1}^{n} f(X_{n,j})\right)\right] = (\mathbb{E}[\exp(-f(X_{n,1}))])^n
= \left(1 - \frac{\int_{E}(1 - e^{-f(x)})nP(X_{n,1} \in dx)}{n}\right)^n.
\]
We know that \( (1 - \frac{a_n}{n})^n \rightarrow e^{-a} \) iff \( a_n \rightarrow a \). Using this we see that \( L_{\mathcal{N}_n}(f) \rightarrow L_{\mathcal{N}}(f) \) iff \( (5) \) is fulfilled. \( \square \)
Corollary 7.2. Let \((X_k)\) be a sequence of i.i.d. random variables, \(X_{n,i} = \frac{X_i}{b_n}\), \(i = 1, 2, \ldots, n\). Then \(N_n = \sum_{i=1}^n \delta_{X_{n,i}}\) is empirical p.p. The condition (5) now looks like the following

\[
nP\left(\frac{X_i}{b_n} \in \cdot \right) \xrightarrow{\text{vague}} \mu.\tag{6}
\]

It is possible to show that (6) in case \(E = \mathbb{R}^d\) is equivalent to

1) \(b_n = n^{\frac{1}{2}} l(n)\), for some \(\alpha > 0\) and slowly varying function \(l : \mathbb{N} \to \mathbb{R}\).

2) \(\mu = m \times \sigma\) where \(m([\tau, \infty)) = \tau^{-\alpha}\) for \(\tau > 0\) and \(\sigma(S^{d-1}) = 1\).

3)

\[
nP\left(\frac{X_1}{|X_1|} \in A, |X_1| \geq \tau b_n\right) \to \sigma(A)m([\tau, \infty)) \quad \text{(RV)}
\]

for all \(\tau > 0\) and measurable \(A \subset S^{d-1}\) such that \(\sigma(\partial A) = 0\).

4) \(N = \Pi_\alpha\).

Relation (RV) is called condition of regular variation for the distribution of \(X_1\). We can define this condition in the same way for any convex cone \(\mathcal{K}\). Parameter \(\alpha\) is called tail index (or exposant), \(\sigma\) is called spectral measure.

Let \((X_i)\) be i.i.d. random variables such that \(X_i \overset{\mathcal{L}}{=} X\). Under (RV) the empirical p.p. \(N_n = \sum_{i=1}^n \delta_{X_{n,i}}\) converges weakly to \(\Pi_\alpha\).

8 Application to order statistics

Let \(\mathcal{K} = \mathbb{R}^d \setminus \{0\}\) and \((X_i)\) be i.i.d. random variables satisfying (RV). The application \(f : \mathbb{K} \to \mathbb{R}\) such that

\[
f(\kappa) = \max\{x \: x \in \kappa\}
\]

is continuous with respect to vague topology. We get immediately from \(N_n \Rightarrow \Pi_\alpha\) that

\[
\frac{M_n}{b_n} := \frac{1}{b_n} \max\{X_1, \ldots, X_n\} \Rightarrow \max\left\{\Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k \: k \in \mathbb{N}\right\},
\]

here \(\varepsilon_k \in \{-1, +1\}\), \(P(\varepsilon_k = \pm 1) = \sigma(\pm 1)\).

If \(X_k \geq 0\) then \(\sigma(-1) = 0\) and we see \(\frac{M_n}{b_n} \Rightarrow \Gamma_1^{\frac{-1}{\alpha}}\).

By the same way we deduce the results for all order statistics.

Now let \(\mathcal{K} = \mathbb{R}^d \setminus \{0\}\) and \((X_i)\) be i.i.d. random variables satisfying (RV). We can consider the convex hull of the sample \(\{X_1, \ldots, X_n\}\) (denoted by \(\text{conv}\{X_1, \ldots, X_n\}\)) as an analogue of the extremal segment \([\min_{k=1}^n X_k, \max_{k=1}^n X_k]\) in one-dimensional case. Remind that in \(\mathbb{R}^d \setminus \{0\}\) the topology corresponds to the metric

\[
d(x, y) = \frac{1}{|x|} - \frac{1}{|y|} + \frac{|x|}{|x|} - \frac{|y|}{|y|}.
\]

The application \(f\) from \(\mathbb{R}\) to the space of compact convex subsets of \(\mathbb{R}^d\) (notation \(\mathcal{K}(\mathbb{R}^d)\)) defined

\[
f(\kappa) = \text{conv}\{\kappa\}
\]
being continuous, we get from convergence $N_n \Rightarrow \Pi_\alpha$ that
\[
\frac{1}{b_n} \text{conv}\{X_1, \ldots, X_n\} \Rightarrow \{\Pi_\alpha\}.
\]

By the same way we can consider the behaviour of the second convex hull
\[
V_n^{(2)} = \text{conv}\{X_i : X_i \in V_n^0\},
\]
the third one, and so on.

The limiting convex sets are:
\[
V^{(1)} = \text{conv}\{\Pi_\alpha\}
\]
\[
V^{(2)} = \text{conv}\{x \in \Pi_\alpha : x \in [V^{(1)}]^0\}
\]
\[
\ldots
\]

The sequence of convex sets $\{V^{(k)}\}$ is decreasing. We say that it is constructed by piling procedure. Is it possible to deduce from the convergence $N_n \Rightarrow \Pi_\alpha$ the convergence of sums
\[
\frac{S_n}{b_n} \Rightarrow Y_\alpha?
\]

**Yes:** for $\alpha < 1$ it is possible to change the topology of vague convergence in $M$ replacing it by stronger in such a way that

1) the convergence $N_n \Rightarrow \Pi_\alpha$ still be true,
2) the application $f(\mu) = \int_E x\mu(dx)$ becomes continuous.

For details see [DaEg].

9 Application to random zonotopes

**Definition 9.1.** (i) Zonotope $= \sum_{i=1}^m \Delta_i$, where $\Delta_i$ are segments.

(ii) Zonoid $= \lim_n Z_n$, where $Z_n$ are zonotopes.

(iii) $Z = \sum_{i=1}^\infty Z_i$ is called countable zonotope.

Consider $K = \mathcal{K}(\mathbb{R}^d)$ equipped by Hausdorff distance. It is convex cone (with second distributivity law). Let $(X_k)$ be a sequence of i.i.d. random vectors in $\mathbb{R}^d$. We compare the behaviour of normalized sums $\frac{Z_n}{b_n}$ with the behaviour of the sums
\[
\frac{Z_n}{b_n} = \frac{1}{b_n} \sum_{i=1}^n [0, X_i].
\]

**Theorem 9.2 ([DPR]).**
\[
\frac{S_n}{b_n} \Rightarrow Y_\alpha \quad \text{iff} \quad \frac{Z_n}{b_n} \Rightarrow Z_\alpha,
\]
where $Z_\alpha = \sum_{k=1}^\infty \Gamma_k^{-\frac{1}{2}} [0, \varepsilon_k]$ and $Y_\alpha = \sum_{k=1}^\infty \Gamma_k^{-\frac{1}{2}} \varepsilon_k$.  

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It means that the limit of $\frac{Z_n}{b_n}$ is a countable zonotope which is strictly $\alpha$–stable in $\mathbb{R}$.

Comments to the proof:
We know that condition [RV] for $X_1$ gives the convergence $\frac{X_n}{b_n} \Rightarrow Y_\alpha$ and RV–condition for $[0, X_1]$ gives the convergence of $\frac{Z_n}{b_n}$ to $Z_\alpha$. But [RV] for $X_1$ is equivalent to [RV] for $[0, X_1]$! More exactly, if we have [RV] for $X_1$ with spectral measure $\sigma$, then [RV] for $[0, X_1]$ will be fulfilled with the measure $\tilde{\sigma}$ concentrated on the subset $W$ of $S^d$ composed by the segments $[0, x], x \in \mathbb{R}^d, |x| = 1$, and if

$$\varphi : S^{d-1} \to W, \quad \varphi(x) = [0, x],$$

then $\tilde{\sigma} = \sigma \varphi^{-1}$. We have now:

1) $\{\frac{X_n}{b_n} \Rightarrow Z_\alpha\} \Rightarrow \{\text{known fact}\} \Rightarrow \{\text{RV for } X_1\} \Rightarrow \{\text{RV for } [0, X_1]\} \Rightarrow \{\frac{X_n}{b_n} \Rightarrow Z_\alpha\}$.

2) $\{\frac{X_n}{b_n} \Rightarrow Z_\alpha\} \Rightarrow \{\frac{X_n}{b_n} \Rightarrow Y_\alpha\}$ we get this directly due to the fact, that the knowledge of $Z_n$ allows us to find all segments $[0, X_i], i = 1, \ldots, n$, and the application $Z \longmapsto S_n$ is continuous.

Info on the structure of the boundary of $Z_\alpha$. The set of extreme points of $Z_\alpha$ is a Cantor-type set and its Lebesgue measure is equal to 0.

10 Exercises

1. Let $\mathcal{N}$ be a point process. Consider intensity measure $\mu(B) = \mathbb{E}[\mathcal{N}(B)]$, $B \in \mathcal{E}$. Check that $\mu(B)$ is a measure.

2. Suppose $X_1, \ldots, X_n \in \mathbb{R}^d$ are i.i.d. random variables and define

$$\mathcal{N}(B) = \sum_{j=1}^{\nu} \mathbb{1}_{B}(X_j), \quad B \in \mathcal{B}^d, \quad (7)$$

where $\nu$ is a random variable independent of $(X_j)_{j=1,2,\ldots,n}$. Show that $\mathcal{N}$ is a point process.

3. Let $\mathcal{N}$ be a point process defined by (7). Show that $\mathbb{E}[\mathcal{N}(B)] = \mathbb{E}[\nu] \cdot \mathbb{P}(B)$ if $\nu$ is independent of $(X_j)_{j=1,2,\ldots,n}$.

4. Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are point processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with state spaces $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ respectively. Then independence of

$$\{\mathcal{N}_1(F_i), 1 \leq i \leq k\} \quad \text{and} \quad \{\mathcal{N}_2(G_j), 1 \leq j \leq l\}$$

for any $k, l, F_1, \ldots, F_k \in \mathcal{E}_1, G_1, \ldots, G_l \in \mathcal{E}_2$ implies $\mathcal{N}_1$ and $\mathcal{N}_2$ independence.

5. Suppose $\mathcal{N}$ is a point process with state space $\mathbb{R}^d$. If $\tau$ is a random vector in $\mathbb{R}^d$, show $\mathcal{N}(\cdot + \tau)$ is a point process. Note, the value of $\mathcal{N}(\cdot + \tau)$ on the set $F$ for realization $\omega$ is $\mathcal{N}(\omega; F + \tau(\omega))$, where $F + \tau = \{x + \tau, x \in F\}$.
6. Two point processes $N_1$ and $N_2$ are equal in distribution ($N_1 \overset{d}{=} N_2$) iff for each $f \geq 0$ bounded and measurable we have

$$N_1(f) \overset{d}{=} N_2(f)$$

as random variables. ($N(f) = \int_E f dN$).

7. Let $\mathcal{N}$ be a Poissonian point process (p.p.p.) with intensity measure $\mu(dt) = t^{-1}dt$ on $E = (0, \infty)$. Express $\mathcal{N}$ as a time changed homogeneous process.

8. Let $\mathcal{N}$ be a nonhomogeneous p.p.p. on $\mathbb{R}_+$ with mean measure $\mu(B) = \int_B \lambda(s)ds$, where $\lambda$ is a locally integrable. Express $\mathcal{N}$ as a function of a homogeneous p.p.p.

9. Let $(\varepsilon_i)_{i=1,2,\ldots}$ be i.i.d. exponential random variables on $\mathbb{R}_+$; $P(\varepsilon_i > x) = e^{-x}$, $x > 0$. Let $\Gamma_n = \sum_{i=1}^n \varepsilon_i$. Show that $\mathcal{N} = \sum_{n=1}^\infty \delta_{\Gamma_n}$ is a homogeneous p.p.p. on $\mathbb{R}_+$. Here $\delta_a$ is $\delta$-measure concentrated in $a$.

10. Suppose $\mathcal{N}$ be a homogeneous Poissonian point process on $\mathbb{R}_+$ with intensity measure $\lambda$, where $\lambda$ is Lebesgue measure. Suppose $\mathcal{N}$ is an integer number. Show that almost surely $\lim_{n \to \infty} \mathcal{N}([0,n])_n = 1$.

11. (Continuation) Suppose $T$ is a real number, show that

$$\lim_{T \to \infty} \frac{\mathcal{N}([0,T])}{T} = 1.$$ 

12. Suppose $\mathcal{N}'$ be a Poissonian point process on $\mathbb{R}_+$ such that its intensity measure $\mu$ is given by

$$\mu(dx) = \frac{1}{x}dx, \quad x \in \mathbb{R}_+.$$ 

Show that

$$\lim_{T \to \infty} \frac{\mathcal{N}'([0,T])}{b(T)}$$

exists almost surely if one chooses proper normalization $b(T)$. Find $b(T)$ and prove the result.

13. Suppose $\mathcal{N}$ be a Poissonian point process with intensity measure $\mu$. Denote the restriction of $\mathcal{N}$ on $A$ as $\mathcal{N} |_A$, where $\mathcal{N} |_A (B) = \mathcal{N}(A \cap B)$. Suppose $A_1, A_2, \ldots, A_m$ are mutually disjoint sets. Show that p.p.p. $\mathcal{N} |_{A_1}, \ldots, \mathcal{N} |_{A_m}$ are independent.

14. Suppose $A_1, A_2, \ldots, A_m$ are mutually disjoint sets, $A = \bigcup_{j=1}^m A_j$. Show that vector $(\mathcal{N}(A_1), \ldots, \mathcal{N}(A_m))$ has conditional multinomial distribution

$$P(\mathcal{N}(A_1) = k_1, \ldots, \mathcal{N}(A_m) = k_m \mid \mathcal{N}(A) = k_1 + \ldots + k_m)$$

$$= \frac{k!}{k_1! \cdots k_m!} \left[ \frac{\mu(A_1)}{\mu(A)} \right]^{k_1} \cdots \left[ \frac{\mu(A_m)}{\mu(A)} \right]^{k_m}, \quad k = k_1 + \ldots + k_m.$$
15. Show that $P(N \mid A \in \cdot \mid N(A) = k) = P(\sum_{i=1}^{k} \delta_{\xi_i} \in \cdot)$, where $(\xi_i)_{i=1,2,\ldots}$ are i.i.d. with common distribution $\frac{\mu_A(\cdot)}{\mu(A)}$.

16. Suppose $f \geq 0$. Show that

$$
\mathbb{E}[N \mid A(f)] = e^{-\mu(A)} \left( f(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{A} \ldots \int_{A} f\left( \sum_{i=1}^{k} \delta_{x_i} \right) \mu(dx_1) \ldots \mu(dx_k) \right).
$$

17. Suppose $\Pi_{\alpha,c}$, $(\alpha > 0, c > 0)$ is p.p.p. on $\mathbb{R}^d \backslash \{0\}$ (that is identified with $S^{d-1} \times \mathbb{R}^+$). Intensity measure $m_{\alpha,c}$ is given by

$$m_{\alpha,c} = \sigma \times \mu_{\alpha,c},$$

where $\sigma$ is a probability measure of $S^{d-1}$ and $\mu_{\alpha,c}$ is a measure on $\mathbb{R}^+$ defined by

$$\frac{d\mu_{\alpha,c}}{d\lambda}(r) = cr^{-\alpha-1}.$$

It is known that $\Pi_{\alpha,c}$ satisfies stability condition [3]. Show that if p.p.p $\mathcal{N}$ satisfies [3] then $\mathcal{N} \overset{c}{=} \Pi_{\alpha,c}$ for a certain $c > 0$.

18. If we have $\mathcal{N}_1, \mathcal{N}_2$ independent p.p.p. defined on the same space $E$, with two diffuse measures $\mu_1, \mu_2$ then $\mathcal{N}_1 + \mathcal{N}_2$ is p.p.p. with intensity measure $\mu_1 + \mu_2$.

19. Let $X_1, \ldots, X_n$ be independent random variables with absolute continuous distributions. Let $a_1, \ldots, a_n \in \mathbb{R}^n$ be the base of $\mathbb{R}^n$. Show that the random vector $Z = \sum_{i=1}^{n} a_i Z_i$ has a density.

20. Let $\mathcal{P}$ be a probability measure on Borel $\sigma$–algebra $\mathcal{E}$ of a complete separable linear space $E$. Let $(X_n)$ be i.i.d. random elements of $E$ with common distribution $\mathcal{P}$. Prove that a.s. $\text{span}(X_1, \ldots, X_n) = \text{span}(\text{supp } \mathcal{P})$. Recall that $\text{supp } \mathcal{P}$ is a support of $\mathcal{P}$, that is

- $\text{supp } \mathcal{P}$ is closed subset of $E$,
- $\mathcal{P}(\text{supp } \mathcal{P}) = 1$,
- for any open ball $B \subset E$ such that $\text{supp } \mathcal{P} \cap B \neq \emptyset$ we have $\mathcal{P}(B) > 0$.

21. Using later two exercises, show that the distribution of strictly $\alpha$–stable vector

$$Y_\alpha = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \varepsilon_k \in \mathbb{R}^d$$

is absolutely continuous iff $\text{span}(\text{supp } \sigma) = \mathbb{R}^d$, where $\sigma$ is common distribution of $\varepsilon_k$.

22. Suppose $x_n, x \in E$, $c_n \geq 0$, $c > 0$. Show that

$$c_n \delta_{x_n} \overset{\text{vague}}{\Rightarrow} c \delta_x$$

iff $c_n \to c$ and $x_n \to x$, as $n \to \infty$. 

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23. Let
\[ \lambda_n = \frac{1}{n} \sum_{i=1}^{\infty} \delta_{\frac{i}{n}} \]
be a discrete version of Lebesgue measure \( \lambda \) on \([0, \infty)\). Show that
\[ \lambda_n \overset{\text{vague}}{\to} \lambda. \]

24. Let \( M \in \mathcal{M} \) be measurable subset of \( \mathcal{M} \). Show that \( M \) is relatively compact in vague topology iff
\[ \sup_{\mu \in M} \mu(f) < \infty \]
for any \( f \in C_K^+ \).

References


