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On the p-Laplacian and p-fluids

Part I

p-Laplace and basic properties

The Dirichlet problem for the Laplacian

Strong formulation

$$\begin{aligned} -\Delta u &= f \qquad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where f is given data.

Classical solution: Find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Variational approach: Classical solution minimizes

$$\mathcal{J}(w) := \int_\Omega \frac{1}{2}|\nabla w|^2\,dx - \int fw\,dx = \int_\Omega \frac{1}{2}|\nabla w|^2\,dx - \langle f, w\rangle.$$

on $X := \{w \in C^1(\overline{\Omega}) \,:\, w|_{\partial\Omega} = 0\}$.

Variational approach

First variation:

$$(\delta J)(w)(\xi) := \texttt{directional derivative} \\ = \frac{d}{dt} J(w + t\xi)\Big|_{t=0}.$$

For $\mathcal{J}(w) = \int_\Omega \frac{1}{2}|\nabla w|^2\,dx - \langle f, w\rangle$, we have

$$\begin{aligned}(\delta J)(w)(\xi) &= \frac{d}{dt}\left(\frac{1}{2}\langle \nabla(w+t\xi), \nabla(w+t\xi)\rangle - \langle f, w+t\xi\rangle\right)\Big|_{t=0} \\ &= \langle \nabla w, \nabla \xi\rangle - \langle f, \xi\rangle \\ &= \langle -\Delta w - f, \xi\rangle\end{aligned}$$

Minimizer $u \quad \Leftrightarrow \quad (\delta\mathcal{J})(u)(\xi) = 0$ for all $\xi \quad \Leftrightarrow \quad -\Delta u = f$.

p-Laplacian

Let $1 < p < \infty$.

Variational definition: Minimize $\mathcal{J}(w) := \int_\Omega \frac{1}{p}|\nabla w|^p \, dx - \langle f, w \rangle$

on Sobolev space $W_0^{1,p}(\Omega)$ ($W^{1,p}$-closure of $C_0^\infty(\Omega)$).

Euler-Langrange equation: $u \in W_0^{1,p}(\Omega)$ minimizer, then

$$0 \overset{!}{=} (\delta\mathcal{J})(u)(\xi) = \frac{d}{dt}\bigg(\int_\Omega \frac{1}{p}|\nabla(w + t\xi)|^p \, dx - \langle f, w + t\xi \rangle \bigg)\bigg|_{t=0}$$
$$= \big\langle |\nabla w|^{p-2}\nabla w, \nabla\xi \big\rangle - \langle f, w \rangle.$$

Thus $-\underbrace{\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big)}_{=:\Delta_p w} - f = 0 \quad$ in $(W_0^{1,p}(\Omega))^*$

The maximum principle (1/2)

Let h be harmonic, i.e. $-\Delta h = 0$.

$\Rightarrow$ for every ball B: $$u(x) = \fint_{B(x)} u(y)\,dy$$

Theorem (Strict maximum principle)

h cannot have strict maximum in interior!

Theorem (Maximum principle)

$$\min h(\partial\Omega) \le \min h(\Omega) \le \max h(\Omega) \le \max h(\partial\Omega)$$

In other words: $h(\Omega) \subset h(\partial\Omega)$.

The maximum principle (2/2)

Theorem

Let $u \in W_0^{1,2}(\Omega)$ with $-\Delta u \leq 0$. Then $u \leq 0$ on Ω.

Proof:

Define $u^+ := \max\{u, 0\} = \chi_{\{u \geq 0\}} u \in W_0^{1,2}(\Omega)$

Then $\nabla u^+ = \chi_{\{u \geq 0\}} \nabla u$.

$$\|\nabla u^+\|_2^2 = \langle \nabla u, \nabla u^+ \rangle = \langle u^+, f \rangle \leq 0.$$

Thus $u^+ = 0$, i.e. $u \leq 0$.

Convex hull property (1/4)

Vectorial: $u : \Omega \to \mathbb{R}^N$

Theorem (Convex hull property)

Let $-\Delta h = 0$. *Then* $h(\Omega) \subset \overline{\mathrm{conv\,hull}}\, h(\partial\Omega)$.

This generalizes the maximum principle!

Proof: Use linear functionals and the scalar maximum principle.

Theorem (Convex hull property – non-linear)

Let $-\Delta_p h = -\mathrm{div}(|\nabla h|^{p-2}\nabla h) = 0$. *Then* $h(\Omega) \subset \overline{\mathrm{conv\,hull}}\, h(\partial\Omega)$.

Proof: By projection, see below!

Convex hull property (2/4)

The set $K := \overline{\mathrm{conv\,hull}}\, h(\partial\Omega)$ is convex.

Use nearest point projection

$$\Pi_K x := \arg\min_{y \in K} |x - y|.$$

Then $|\Pi_K x - \Pi_K Y| \leq |x - y|$.

Define $(\Pi_K h)(x) := \Pi_K\big(h(x)\big)$ (point-wise projection)

Then $|\nabla \Pi_K h| \leq |\nabla h|$ (since difference quotients are reduced!)

Convex hull property (3/4)

Recall $|\nabla \Pi_K h| \leq |\nabla h|$.

Since $u(\partial\Omega) \subset K = \overline{\mathrm{conv\,hull}}\, h(\partial\Omega)$, we have $h = \Pi_K h$ on $\partial\Omega$.

$$\Rightarrow \qquad \mathcal{J}(\Pi h) = \int_\Omega \frac{1}{p} |\nabla \Pi_K h|^p \, dx \leq \int_\Omega \frac{1}{p} |\nabla h|^p \, dx \leq \mathcal{J}(h).$$

Uniqueness (with same boundary values) implies $h = \Pi_K h$.

$\Rightarrow$ No projection needed!

Thus $h(\Omega) \subset K = \overline{\mathrm{conv\,hull}}\, h(\partial\Omega)$.

$\Rightarrow$ Convex hull property!.

Convex hull property (4/4)

Theorem (Scalar case!)

Let $u \in W_0^{1,p}(\Omega)$ with $-\Delta_p u \leq 0$. Then $u \leq 0$ on Ω.

Proof:

Let $K := (-\infty, 0]$.

$$\mathcal{J}(w) = \int_\Omega \frac{1}{p} |\nabla w|^p \, dx - \int_\Omega f \, w \, dx \qquad \text{with } f \leq 0.$$

Then $\mathcal{J}(\Pi_K u) \leq \mathcal{J}(u)$ and $\Pi_K u = u$ on $\partial\Omega$.

Uniqueness implies $\Pi_K u = u$ on Ω.

Thus, $u(\Omega) \subset K = (-\infty, 0]$.

Part II

p-harmonic functions

p-harmonic functions

We say that h is p-harmonic if $\quad -\Delta_p h = -\mathrm{div}(|\nabla h|^{p-2}\nabla h) = 0.$

p-harmonic functions are local minimizers of

$$\mathcal{J}(w) = \int_\Omega \frac{1}{p}|\nabla h|^p\,dx,$$

i.e. $\mathcal{J}(u) \leq \mathcal{J}(u + t\,\xi)$ for all $\xi \in C_0^1(\Omega)$.

Define $A(Q) := |Q|^{p-2}Q$. Then

$$-\mathrm{div}(A(\nabla u)) = 0.$$

Monotonicity (1/3)

Consider $\langle A(\nabla u) - A(\nabla w), \nabla u - \nabla w\rangle$

For example used for uniqueness.

Pointwise estimate (with $[Q,P]_t := (1-t)Q + tP$)

$$
\begin{aligned}
\big(A(P) - A(Q)\big) \cdot (P-Q) &= \sum_j \big(A_j(P) - A_j(Q)\big)(P_j - Q_j) \\
&= \int_0^1 \frac{d}{dt} A_j([Q,P]_t)\, dt\, (P-Q)_j \\
&= \int_0^1 \underbrace{(\partial_k A_j)([Q,P]_t)}_{=|M|^{p-2}(\delta_{j,k} + (p-2)\frac{M_j M_k}{|M|^2})} \, dt\, (P-Q)_k (P-Q)_j
\end{aligned}
$$

Monotonicity (2/3)

Note that

$$|M|^{p-2}\left(\delta_{j,k} + (p-2)\frac{M_j M_k}{|M|^2}\right) \geq |M|^{p-2} \min\{p-1, 1\}\delta_{j,k}$$

Thus,

$$\begin{aligned}\big(A(P) - A(Q)\big) \cdot (P - Q) &\geq c \int_0^1 |[Q, P]_t|^{p-2}\, dt\, |P - Q|^2 \\ &\geq c\,(|Q| + |P|)^{p-2}\, |P - Q|^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\big(A(P) - A(Q)\big) \cdot (P - Q) &\sim (|Q| + |P|)^{p-2}\, |P - Q|^2, \\ |A(P) - A(Q)| &\sim (|Q| + |P|)^{p-2}\, |P - Q|.\end{aligned}$$

Monotonicity (3/3)

Recall $A(Q) = |Q|^{p-2}Q$.

Define $V(Q) = |Q|^{\frac{p-2}{2}}Q$.

Then $|V(Q)|^2 = A(Q) \cdot Q$ and $\dfrac{V(Q)}{|Q|} = \dfrac{A(Q)}{|Q|} = \dfrac{Q}{|Q|}$.

Then $|V(P) - V(Q)| \sim (|Q| + |P|)^{\frac{p-2}{2}} |P - Q|$.

Theorem

$$\big(A(P) - A(Q)\big) \cdot (P - Q) \sim (|Q| + |P|)^{p-2}|P - Q|^2 \sim |V(P) - V(Q)|^2,$$

$$|A(P) - A(Q)| \sim (|Q| + |P|)^{p-2}\, |P - Q|.$$

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Caccioppoli

Start with $\langle A(\nabla u), \nabla \xi \rangle = 0 \quad$ for $\xi \in W^{1,p}_0(\Omega)$.

Let $\xi \in C^\infty_0(2B)$ with $\chi_B \leq \xi \leq \chi_{2B}$ and $\|\nabla \eta\|_\infty \leq c\, r^{-1}$.

Choose $\xi = (u - \langle u \rangle_{2B})\eta^{p'}$. Then

$$\langle A(\nabla u), \eta \nabla u \rangle = \langle A(\nabla u), (u - \langle u \rangle_{2B}) \nabla(\eta^{p'}) \rangle.$$

$$\Rightarrow \qquad \int \eta^{p'} |\nabla u|^p \, dx \leq c \int \eta^{p'-1} |\nabla u|^{p-1} \frac{|u - \langle u \rangle_{2B}|}{r} \, dx$$

Young's inequality implies:

Lemma (Caccioppoli estimate)

$$\fint_B |\nabla u|^p \, dx \leq c \fint_{2B} \left| \frac{u - \langle u \rangle_{2B}}{r} \right|^p dx$$

Reverse Hölder's estimate

Lemma (Caccioppoli estimate)

$$\fint_B |\nabla u|^p \, dx \leq c \fint_{2B} \left| \frac{|u - \langle u \rangle_{2B}|}{r} \right|^p dx$$

Then Poincaré implies

Lemma (Reverse Hölder)

For some $\theta \in (0,1)$

$$\fint_B |\nabla u|^p \, dx \leq c \left(\fint_{2B} |\nabla u|^{\theta p} \, dx \right)^{\frac{1}{\theta}}$$

Gehring

Lemma (Gehring)

Assume that for all balls B and some $\theta \in (0,1)$

$$\fint_B |f|\,dx \leq c\bigg(\fint_{2B} |f|^\theta\,dx\bigg)^{\frac{1}{\theta}} + \fint_B |g|\,dx$$

Then there exists $s > 1$ such that

$$\bigg(\fint_B |f|^s\,dx\bigg)^{\frac{1}{s}} \leq c\fint_{2B} |f|\,dx + c\bigg(\fint_B |g|^s\,dx\bigg)^{\frac{1}{s}}$$

$$\Rightarrow \qquad \bigg(\fint_B |\nabla u|^{sp}\,dx\bigg)^{\frac{1}{s}} \leq c\fint_{2B} |\nabla u|^p\,dx$$

Higher order (1/2)

Difference quotient technique: $\tau_h f(x) := \frac{f(x+h) - f(x)}{|h|}$.

Test function $\xi = \tau_{-h}(\eta^{p'} \tau_h(u-a))$ with a linear.

For $p = 2$:

$$\begin{aligned} \langle \nabla u, \nabla \xi \rangle &= \langle \tau_h \nabla u, \nabla(\eta^2 \tau_h u) \rangle \\ &= \int \eta^2 |\tau_h \nabla u|^2 \, dx + \int \tau_h \nabla u \, \tau_h(u-a) \nabla(\eta^2) \, dx \end{aligned}$$

With $h \to 0$ we get $\int \eta^2 |\nabla^2 u|^2 \, dx \leq c \int \eta |\nabla^2 u| \frac{|\nabla(u-a)|}{r} \, dx.$

We get $\int_B |\nabla^2 u|^2 \, dx \leq c \int_B \left| \frac{\nabla(u-a)}{r} \right|^2 dx.$

Higher order (2/2)

Difference quotient technique: $\tau_h f(x) := \dfrac{f(x+h) - f(x)}{|h|}$.

Test function $\xi = \tau_{-h}(\eta^{p'} \tau_h(u-a))$ with a linear.

$p \neq 2$: Main part gives $\langle \tau_h A(\nabla u), \eta^{p'} \tau_h \nabla u \rangle \sim \int |\tau_h V(\nabla u)|^2 \, dx$.

Now, $h \to 0$ gives $\int \eta^{p'} |\nabla V(\nabla u)|^2 \, dx \leq \texttt{lower order term}$.

Attention: This is not $u \in W^{2,p}$.

Shifted N-functions

$$A(Q) = |Q|^{p-2}Q, \qquad V(Q) = |Q|^{\frac{p-2}{2}}Q, \qquad \varphi(t) = \tfrac{1}{p}t^p.$$

Shifted φ-functions: $\varphi_a(t) \approx (a+t)^{p-2}t^2$ [Diening, Ettwein '05]

$$\begin{aligned}(A(P) - A(Q)) \cdot (P - Q) \sim |F(P) - F(Q)|^2 &\sim \varphi_{|P|}(|P-Q|) \\ |A(P) - A(Q)| &\sim \varphi'_{|P|}(|P-Q|)\end{aligned}$$

Δ_2-condition: $\varphi_a(2t) \leq c\,\varphi_a(t)$.

Young's inequality: $\psi'(s)\,t \leq \delta\psi(s) + c_\delta\,\psi(t)$

Conjugate function: $\varphi^*(s) = \sup_{t\geq 0}\big(st - \varphi(t)\big)$.

Then $\varphi^*(t) = \frac{1}{p'}t^{p'}$ and $\varphi^{**} = \varphi$.

Higher order reverse Hölder (1/2)

For $\xi \in W_0^{1,p}(\Omega)$ and arbitrary constant Q

$$0 = \langle A(\nabla u), \nabla \xi \rangle = \langle A(\nabla u) - A(Q), \nabla \xi \rangle.$$

Let $\xi = \eta^{p'}(u - q)$ with q linear and $\nabla q = Q$. Then

$$\fint \eta^{p'}(A(\nabla u) - A(Q)) \cdot (\nabla u - Q)\, dx \leq c \int \eta^{p'-1} \varphi'_{|Q|}(|\nabla u - Q|) \left| \frac{u-q}{r} \right| dx.$$

With $(A(P) - A(Q)) \cdot (P - Q) \sim \varphi_{|Q|}(|P - Q|)$ and Young's inequality

$$\fint_B \varphi_{|Q|}(|\nabla u - Q|)\, dx \leq c \fint_{2B} \varphi_{|Q|}\left(\left| \frac{u-q}{r} \right| \right) dx.$$

Higher order reverse Hölder (2/2)

$$\fint_B \varphi_{|Q|}(|\nabla u - Q|)\,dx \le c \fint_{2B} \varphi_{|Q|}\bigg(\bigg|\frac{u-q}{r}\bigg|\bigg)\,dx.$$

Poincaré's inequality implies (for $\langle u - q\rangle_{2B} = 0$)

$$\fint_B \varphi_{|Q|}(|\nabla u - Q|)\,dx \le \bigg(\fint_{2B} \varphi^{\theta}_{|Q|}(|\nabla u - Q|)\,dx\bigg)^{\frac{1}{\theta}}$$

Thus, for all balls B

$$\fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2\,dx \le c\bigg(\fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^{2\theta}\,dx\bigg)^{\frac{1}{\theta}}$$

Subsolution property (1/2)

Formally $\xi = \partial_j(\eta \partial_j u)$

$$\begin{aligned} 0 &= \langle \partial_k A_k(\nabla u), \partial_j(\eta \partial_j u)\rangle \\ &= \langle \partial_j A_k(\nabla u), \partial_k(\eta \partial_j u)\rangle \\ &= \langle \partial_j A_k(\nabla u), \eta \partial_k \partial_j u)\rangle + \langle \partial_j A_k(\nabla u), (\partial_k \eta)\partial_j u)\rangle =: (I) + (II). \end{aligned}$$

Then $(I) \sim \int |\nabla u|^{p-2} |\nabla^2 u|^2 \eta \, dx \sim \int |\nabla V(\nabla u)|^2 \eta \, dx \geq 0.$

Moreover,

$$II = \int \underbrace{\bigg(\delta_{j,k} + (p-2)\frac{\partial_j u \partial_k u}{|\nabla u|^2}\bigg)}_{=:a(x)} \partial_k \Big(\frac{1}{p}|\nabla u|^p\Big) \partial_k \eta \, dx.$$

Subsolution property (2/2)

Recall: $a(x) = \delta_{j,k} + (p-2)\dfrac{\partial_j u \partial_k u}{|\nabla u|^2}$ (tensor)

Then $\lambda\,\mathrm{Id} \leq a(x) \leq \Lambda\,\mathrm{Id}$ and

$$-\mathrm{div}\Big(a(x)\nabla\Big(\frac{1}{p}|\nabla u|^p\Big)\Big) \leq 0.$$

L^∞- estimates: $\sup_B |\nabla u|^p \leq c \fint_{2B} |\nabla u|^p\,dx$

Harnack inequality:

$$\fint_{2B} |\nabla u|^p\,dx \leq c \inf_B |\nabla u|^p \leq c\Big(\sup_{2B} |\nabla u|^p - \sup_B |\nabla u|^p\Big)$$

Decay estimate

After a few more steps . . .

Theorem (decay estimate)

There exists $\alpha > 0$ *such that*

$$\fint_{B_r} |V(\nabla u) - \langle V(\nabla u)\rangle_{B_r}|^2\,dx \le c\left(\frac{r}{R}\right)^{\alpha} \fint_{B_R} |V(\nabla u) - \langle V(\nabla u)\rangle_{B_R}|^2\,dx$$

By characterization of $C^{0,\alpha}$ $\quad\Rightarrow\quad$ $V \in C^{0,\alpha}$

Since V^{-1} is Hölder continuous: $\nabla u \in C^{0,\beta}$.

This includes any $n, N \ge 1$.

In the plane (1/3)

Consider $-\Delta_p h = 0$ on $\mathbb{R}^2$ (scalar valued, i.e. $N = 1$).

[Bojarski, Iwaniec '83]: singular points $\nabla u(x) = 0$ are isolated.

Detailed study by: Iwaniec-Manfredi, Dobrowolski, Aronsson, Lindqvist

Aronsson: Hodograph transform

We will use a shorter but formal approach here!

In the plane (2/3)

Define
$$q := \nabla_x u$$
$$v(q) := q \cdot x - u(x).$$

Then
$$x = \nabla_q v$$
$$\nabla_q^2 v = (\nabla_x^2 u)^{-1}.$$

Thus $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ becomes

$$0 = |\nabla u|^{p-2}\Big(\Delta u + (p-2)\frac{\partial_k u \partial_j}{|\nabla u|^2}\partial_j \partial_k u\Big).$$

Hence, $0 = \nabla_x^2 u : \big(\mathrm{Id} + (p-2)\hat{q} \otimes \hat{q}\big) = (\nabla_q^2 v)^{-1} : \big(\mathrm{Id} + (p-2)\hat{q} \otimes \hat{q}\big)$

We get for $n = 2$: $\quad 0 = \nabla_q^2 v : \big(\mathrm{Id} + (p'-2)\hat{q} \otimes \hat{q}\big)$

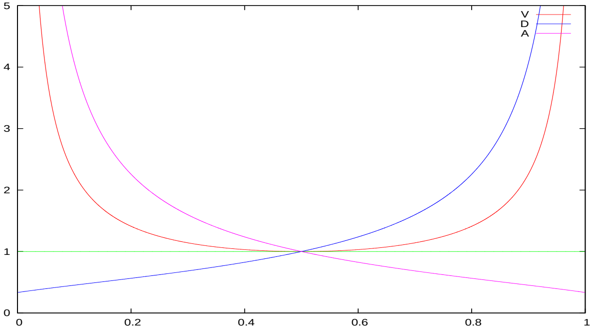
In the plane (3/3)

Recall: $0 = \nabla_q^2 v : (\mathrm{Id} + (p' - 2)\hat{q} \otimes \hat{q})$

Ansatz: $v(q) = |q|^\alpha q_1 q_2$ works with $0 = \alpha^2 + (p+2)\alpha + (4-2p)$.

We get $u \in C^{0,\gamma}$ with $\gamma = \frac{7p-6+\sqrt{p^2+12p-12}}{6p-6}$.

$D := \nabla u$
$A := A(\nabla u)$
$V := V(\nabla u)$



Part III

p-Stokes

Motion of fluid

Incompressible fluids with constant density

$$\begin{aligned} \partial_t v - \operatorname{div}(S) + [\nabla v]v + \nabla q &= f \\ \operatorname{div} v &= 0 \end{aligned}$$

$$\texttt{plus boundary conditions}$$

with

$$\begin{aligned} v &= \texttt{velocity} \\ q &= \texttt{pressure} \end{aligned}$$

Convective term $[\nabla v]v$ by change of coordinates!

Frame indifference (objectivity) gives: $A = A(\varepsilon(v))$

with $\varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$

Non-Newtonian fluids (or generalized Newtonian)

Properties of fluids are described by $A(\varepsilon(v))$.

Newtonian fluid: water, air

$$A(\varepsilon(v)) = 2\nu\varepsilon(v)$$

Then $\operatorname{div}(A(\varepsilon(v)) = \nu\Delta v + \nu\nabla\operatorname{div} v = \nu\Delta v$.

Power law fluid (generalized Newtonian): honey, ketchup, blood

$$A(\varepsilon(v)) = \begin{cases} (\gamma + |\varepsilon(v)|)^{p-2}\varepsilon(v), \\ (\gamma^2 + |\varepsilon(v)|^2)^{\frac{p-2}{2}}\varepsilon(v), \end{cases}$$

with $1 < p < \infty$ and $\gamma \geq 0$.

p-Stokes (1/2)

Consider time independent flow; no convection

$$\begin{aligned} -\operatorname{div}(A(\varepsilon(v))) + \nabla q &= f \\ \operatorname{div} v &= 0 \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Can be written as variational problem (for $A(\varepsilon(v)) = |\varepsilon(v)|^{p-2}\varepsilon(v)$)

$$\texttt{Energy:} \qquad \mathcal{J}(w) := \int \frac{1}{p}|\varepsilon(w)|^p\,dx - \int fw\,dx$$

Minimize $\mathcal{J}$ on $W^{1,p}_{0,\operatorname{div}}(\Omega) = \{v \in W^{1,p}_0 : \operatorname{div} v = 0\}$.

p-Stokes (2/2)

Minimize $\mathcal{J}(w) := \int \frac{1}{p}|\varepsilon(w)|^p\,dx - \int fw\,dx$ on $W^{1,p}_{0,\mathrm{div}}$.

Pressure free formulation: For all $\xi \in C^\infty_{0,\mathrm{div}}(\Omega)$

$$0 = \langle A(\varepsilon(v)), \nabla\xi\rangle - \langle f, \xi\rangle = \langle A(\varepsilon(v)), \varepsilon(\xi)\rangle - \langle f, \xi\rangle.$$

Reconstruction of pressure:

By "De Rahm" exists pressure $q \in \mathcal{D}'$ with

$$-\mathrm{div}(A(\varepsilon(v))) + \nabla q = f.$$

Later: Recover regularity of pressure q

Gradients ∇v vs. symmetric gradient $\varepsilon(v)$

Function spaces: $W^{1,p}_{0,\mathrm{div}}(\Omega)$

Energy controls: $\int |\varepsilon(v)|^p \, dx$, recall: $\varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$

$\Rightarrow$ Need control of ∇v by $\varepsilon(v)$

Pointwise: Not possible!

rigid-motions: $v(x) = Qx + b$ with Q anti-symmetric

However, $\partial_j \partial_k v_l = \partial_j \varepsilon_{kl}(v) + \partial_k \varepsilon_{lj}(v) + \partial_l \varepsilon_{jk}(v)$.

Thus $|\nabla \varepsilon(v)| \leq |\nabla^2 v| \leq 3\, |\nabla \varepsilon(v)|$.

Korn's inequality for $p = 2$

Case $p = 2$ and $v \in W^{1,2}_{0,\mathrm{div}}(\Omega)$:

$$\begin{aligned}
\|\varepsilon(v)\|_2^2 &= \int \varepsilon_{jk}(v)\varepsilon_{jk}(v)\,dx \\
&= \int \frac{1}{2}|\nabla v|^2\,dx + \int \frac{1}{2}\partial_j v_k \partial_k v_j\,dx \\
&= \int \frac{1}{2}|\nabla v|^2\,dx + \int \frac{1}{2}|\mathrm{div} v|^2\,dx \\
&= \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|\mathrm{div}(v)\|_2^2
\end{aligned}$$

Note: $\mathrm{div} v = \mathrm{tr}(\varepsilon(v))$

Thus, $\|\nabla v\|_2^2 \leq 2\|\varepsilon(v)\|_2^2.$

Negative norm theorem (1/2)

What about $W^{1,p}_0(\Omega)$? Idea: $\nabla^2 u \sim \nabla\varepsilon(v)$.

Define:

$$\langle f\rangle_\Omega := \fint_\Omega f\,dx$$

$$L^p_0(\Omega) := \{f \in L^p(\Omega) \,:\, \langle f\rangle_\Omega = 0\},$$

$$W^{-1,p}(\Omega) := \big(W^{1,p'}_0(\Omega)\big)^*.$$

Theorem (Negative norm theorem by Nečas)

Ω bounded, $\partial\Omega \in C^1$. Then for all $u \in L^p_0(\Omega)$

$$\|\nabla u\|_{W^{-1,p}(\Omega)} \sim \|u\|_{L^p(\Omega)}.$$

Negative norm theorem (2/2)

Theorem

Ω bounded, $\partial\Omega \in C^1$. Then for all $u \in L^p_0(\Omega)$

$$\|\nabla u\|_{W^{-1,p}(\Omega)} \sim \|u\|_{L^p(\Omega)}.$$

Easy part of the proof: $u \in L^p(\Omega)$, $H \in W_0^{1,p'}(\Omega)$

$$\langle \nabla u, H\rangle = -\langle u, \mathrm{div} H\rangle = -\langle u - \langle u\rangle_\Omega, \mathrm{div} H\rangle.$$

Thus $\quad \big|\langle \nabla u, H\rangle\big| \leq \|u - \langle u\rangle_\Omega\|_p \|H\|_{1,p}.$

In particular, $\quad \|\nabla u\|_{-1,p} \leq \|u - \langle u\rangle_\Omega\|_p.$

Difficult part: Later!

Korn's inequality

Theorem

Ω bounded, $\partial\Omega \in C^1$. Then

$$\|\nabla v - \langle \nabla v \rangle_\Omega\|_p \le c\,\|\varepsilon(v) - \langle \varepsilon(v)\rangle_\Omega\|_p, \qquad \text{for } v \in W^{1,p}(\Omega).$$

$$\|\nabla v\|_p \le c\,\|\varepsilon(v)\|_p \qquad \text{for } v \in W_0^{1,p}(\Omega).$$

Proof: Using $\partial_j \partial_k v_l = \partial_j \varepsilon_{kl}(v) + \partial_k \varepsilon_{lj}(v) + \partial_l \varepsilon_{jk}(v)$.

$$\|\nabla v - \langle \nabla v \rangle_\Omega\|_p \sim \|\nabla^2 v\|_{-1,p} \sim \|\nabla \varepsilon(v)\|_{-1,p} \sim \|\varepsilon(v) - \langle \varepsilon(v)\rangle_\Omega\|_p.$$

For $v \in W_0^{1,p}(\Omega)$ we have $\langle \nabla v \rangle_\Omega = \langle \varepsilon(v) \rangle_\Omega = 0$.

The pressure

We get v as $W^{1,p}_{0,\mathrm{div}}(\Omega)$-minimizer of

$$\mathcal{J}(w) := \int \frac{1}{p}|\varepsilon(w)|^p \, dx - \int fw \, dx$$

Pressure free formulation: For all $\xi \in W^{1,p}_{0,\mathrm{div}}(\Omega)$

$$\langle A(\varepsilon(v)), \nabla v \rangle = \langle f, \xi \rangle.$$

De Rahm gives distributional pressure q with

$$\langle A(\varepsilon(v)), \nabla v \rangle + \langle \nabla q, \xi \rangle = \langle f, \xi \rangle \qquad \text{for all } \xi \in C_0^\infty(\Omega).$$

Estimate for pressure:

$$\|q - \langle q \rangle\|_{p'} \sim \|\nabla q\|_{-1,p'} \leq \|A(\varepsilon(v))\|_{p'} + \|f\|_{-1,p'} \leq c(f).$$

Summary

Theorem

The p-Stokes system (with $f \in W^{-1,p'}(\Omega)$)

$$\begin{aligned} -\operatorname{div}(A(\varepsilon(v))) + \nabla q &= f \\ \operatorname{div} v &= 0 \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $v \in W^{1,p}_{0,\operatorname{div}}(\Omega)$ and $q \in L^{p'}_0(\Omega)$.

Uniqueness of v: Energy is strict convex

Uniqueness of q: Fixed the mean value of q

Part IV

Maximal function and covering theorems

Maximal function

For $f \in L^1_{\mathrm{loc}}$ define the (uncentered) maximal function

$$(Mf)(x) := \sup_{B \ni x} ⨍_B |f(y)| \, dy$$

(supremum over all balls B containing x)

For $0 \in B$ the mapping $f \mapsto ⨍_{x+B} |f| \, dy$ is continuous.

Thus, Mf is l.s.c. (lower semi continuous)

1. $Mf(x) \leq \liminf_{z \to x} Mf(z)$
2. $\{Mf > \lambda\}$ is open.

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Basic properties

Recall: $$(Mf)(x) := \sup_{B \ni x} \fint_B |f(y)| \, dy$$

M is sub-linear:
$$M(f+g) \leq Mf + Mg,$$
$$M(sf) \leq |s| Mf \qquad \text{for } s \in \mathbb{R}.$$

L^∞ estimate: $$\|Mf\|_\infty \leq \|f\|_\infty.$$

L^1 estimate: If $f \in C_0^\infty(\mathbb{R}^n)$ with $f \neq 0$, then Mf decays as $|x|^{-n}$.
Thus, $Mf \notin L^1$.

The L^1-case

Define $\|f\|_{\text{w-}L^1} := \sup\limits_{\lambda>0} \lambda \,|\{|f| > \lambda\}|$ (quasi-norm) Let

$\text{w-}L^1 := \{f \,:\, \|f\|_{\text{w-}L^1} < \infty\}$ (quasi-Banach space)

$$\lambda|\{|f| > \lambda\}| = \int \lambda\chi_{\{|f|>\lambda\}}\, dx \leq \int |f|\, dx = \|f\|_1.$$

Thus $L^1 \hookrightarrow \text{w-}L^1$.

Claim

$\|Mf\|_{\text{w-}L^1} \leq c\,\|f\|_1$.

Covering theorem (1/2)

For $\lambda > 0$ let $\quad \mathcal{O}_\lambda := \{Mf > \lambda\}$. (open set)

For all $x \in \mathcal{O}_\lambda$ exists B_x: $\quad \displaystyle ⨍_{Bx} |f|\, dy > \lambda.$

We have $B_x \subset \mathcal{O}_\lambda$ and $\quad \mathcal{O}_\lambda = \bigcup_{x \in \mathcal{O}_\lambda} B_x$.

Theorem (Basic covering theorem)

Let $\mathcal{O}$ be open, $\{B_x\}$ covering of balls with

1. $\sup_x |B_x| < \infty$
2. $\mathcal{O} \subset \bigcup_x B_x$.

Then there exists countable, pair wise disjoint $\{B_j\}$ with $\bigcup_x B_x \subset \bigcup_j 3B_j$.

Covering theorem (2/2)

Theorem (Basic covering theorem)

Let $\mathcal{O}$ be open, $\{B_x\}$ covering of balls with

1. $\sup_x |B_x| < \infty$
2. $\mathcal{O} \subset \bigcup_x B_x$.

Then there exists countable, pair wise disjoint $\{B_j\}$ with $\bigcup_x B_x \subset \bigcup_j 5B_j$.

Simplified proof for $\{B_x\}$ finite:

Start with $\mathcal{X} := \{B_x\}$ and $\mathcal{Y} := \emptyset$. Iteratively, do:

1. Find biggest B_x from $\mathcal{X}$ and move it from $\mathcal{X}$ to $\mathcal{Y}$.
2. Remove from $\mathcal{X}$ all B_x, which intersect some $B_j \in \mathcal{Y}$. $(\Rightarrow B_x \subset 3B_j)$
3. Start again.

The limit set $\mathcal{Y}$ is the desired family $\{B_j\}$.

w-L^1 estimates

Recall: $\fint_{B_x} |f|\,dy > \lambda$ and $\{B_x\}$ cover $\mathcal{O}_\lambda = \{Mf > \lambda\}$.

By covering theorem $\Rightarrow$ pair wise disjoint $\{B_j\}$ and $\{5B_j\}$ covers $\mathcal{O}_\lambda$.

$$\begin{aligned} \lambda \big|\{Mf > \lambda\}\big| &\leq \lambda \sum_j |5B_j| \leq \lambda\, 5^n \sum_j |B_j| \\ &\leq 5^n \sum_j \int_{B_j} |f|\,dy \leq 5^n \int_{\mathbb{R}^n} |f|\,dx = 5^n \|f\|_1. \end{aligned}$$

Theorem

$\|Mf\|_{w\text{-}L^1} \leq 5^n \|f\|_1$.

Marcinkiewicz

Theorem (Real interpolation)

Let be T sub-linear with $T : L^\infty \to L^\infty$ and $T : L^1 \to w\text{-}L^1$.
Then $T : L^p \to L^p$ for all $p > 1$.

$$\frac{1}{p}\int_{\mathbb{R}^n} |Mf|^p\,dx = \int_{\mathbb{R}^n}\int_0^\infty \chi_{\{Mf>\lambda\}}\,d\lambda\,dx = \int_0^\infty \lambda^{p-1}\big|\{Mf>\lambda\}\big|\,d\lambda =: (I).$$

Let $f_{0,\lambda} = f\chi_{\{|f|\leq\lambda/2\}}$ and $f_{0,\lambda} = f\chi_{\{|f|>\lambda/2\}}$. Then $f = f_{0,\lambda} + f_{1,\lambda}$.

Since $Mf_{0,\lambda} \leq \lambda/2$, we have $Mf_{1,\lambda} \geq Mf - Mf_{0,\lambda} > \lambda/2$.

$$(I) \leq \int_0^\infty \lambda^{p-1}\big|\{Mf_{1,\lambda} > \lambda/2\}\big|\,d\lambda \leq c\int_0^\infty \lambda^{p-2}\int_{\mathbb{R}^n} |f_{1,\lambda}|\,dx\,d\lambda$$

$$= c\int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} \lambda^{p-2}\,d\lambda\,dx = c\int_{\mathbb{R}^n} |f|^p\,dx.$$