Ergodic Ramsey Theory

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Introduction

These notes accompany the lectures given at the Jagiellonian University in Kraków in September 2013. Ergodic Ramsey Theory is now too advanced an area of mathematics to be presented comprehensively during such a short course. The goal here is to give some flavor only as this material is intended to be as elementary as possible. For more details the reader is referred to surveys [1, 2, 3, 4, 5, 6] and to a now classic book of Furstenberg [7]. All surveys are available at `https://people.math.osu.edu/bergelson.1/`

1 Notes

We start with the basic and arguably the first result of ergodic theory. As usual in this area it goes back to Poincaré.

The following lemma is nowadays called the Poincaré Recurrence Theorem.

Poincaré Recurrence Theorem. *Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $T\colon X \to X$ be a μ-preserving transformation. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$.*

Proof. Consider $A, T^{-1}A, T^{-2}A, \dots, T^{-n}A, \dots$ Since T preserves μ we have $\mu(T^{-n}A) = \mu(A)$ for each n. Therefore there exist $i < j$ such that $0 < \mu(T^{-i}A \cap T^{-j}A)$ as otherwise $\{T^{-n}A\}$ would be an infinite family of essentially disjoint (disjoint $\mod \mu$) sets of equal positive measure. Take $n = j - i > 0$ and note that

$$0 < \mu(T^{-i}A \cap T^{-j}A) = \mu(T^{-i}(A \cap T^{-(j-i)}A)) = \mu(A \cap T^{-n}A),$$

where we use invariance of μ one more time. $\square$

The above lemma gives us much more than it seems. It is really surprising that such a simple result has so many interesting corollaries.

Corollary 1. *Let $(X, \mathcal{B}, \mu, T)$ be a probability preserving system. Take any $A \in \mathcal{B}$ with $\mu(A) > 0$. Then for μ-almost every $x \in A$ there exists $n = n(x) > 0$ such that $T^n x \in A$.*

Proof. Assume that this is not true. Then there is a set $A \in \mathcal{B}$ such that the set A_0 of those $x \in A$ which do not return to A has positive measure. By the Poincaré Recurrence Theorem there is an $n > 0$ such that $\mu(A_0 \cap T^{-n}(A_0)) > 0$. But then for any $x \in A_0 \cap T^{-n}(A_0)$ we have $x \in A_0$ and $T^n x \in A_0$ contradicting the definition of A_0. $\square$

Exercise 1. Prove that the set A_0 defined above is indeed measurable.

Remark 1. In the proof of the Poincaré Recurrence Theorem we used two general principles:

1. The Pigeon Hole Principle.

2. Stationarity: $\mu(T^{-n}A) = \mu(A)$ for all $n \geq 0$.

There are some interesting questions regarding the Poincaré Recurrence Theorem. But to state some of them we will need the following definitions.

Definition 1. Let $(X, \mathcal{B}, \mu, T)$ be a probability preserving system. Take any $A \in \mathcal{B}$ with $\mu(A) > 0$. Define the *set of returns* by

$$R_A = \{n : \mu(A \cap T^{-n}A) > 0\},$$

and, given $\varepsilon > 0$, the *set of large returns*

$$R_{A,\varepsilon} = \{n : \mu(A \cap T^{-n}A) > \mu^2(A) - \varepsilon\}.$$

Now it is natural to ask: *What do we know about R_A? $R_{A,\varepsilon}$? How big are those sets?* It turns out that at least the following is true and there is interesting mathematics lurking behind.

Theorem 1. *Let $(X, \mathcal{B}, \mu, T)$ be a probability preserving system. For any $A \in \mathcal{B}$ with $\mu(A) > 0$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $n^2 \in R_{A,\varepsilon}$.*

Theorem 1 is a special case of a result due to Bergelson, Furstenberg and McCutcheon and we postpone the proof for now. The result is quite surprising: the set of squares

$$S = \{n^2 : n \in \mathbb{N}\}$$

is a sparse set. Why there must be a square in any $R_{A,\varepsilon}$? What other sets have similar property? It is worthwhile to have a name for this property.

Definition 2. A set $S \subset \mathbb{N}$ is *a set of recurrence* if for any probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in S$ such that $\mu(A \cap T^{-n}A) > 0$, or equivalently $S \cap R_A \neq \emptyset$.

It follows from Theorem 1 that the squares is a set of recurrence. What are other sets of recurrence? It turns out that some sets are and some are not good for recurrence.

Example 1. The following sets

$$\{p+1 : p \text{ prime}\},\ \{p-1 : p \text{ prime}\},\ \{n^2-1 : n \in \mathbb{N}\},\ \{[n^\alpha] : \alpha > 0\},$$
$$\{[p^\alpha] : \alpha > 0, \alpha \notin \mathbb{N}, p \text{ prime}\}$$

are sets of recurrence. The set $\{n^2+1 : n \in \mathbb{N}\}$ is not a set of recurrence, neither is the set of primes.

Exercise 2. Prove that $\{2^n : n \in \mathbb{N}\}$ is not a set of recurrence.

Upper density is one of many notions of largeness one may consider for subsets of $\mathbb{N}$ (or $\mathbb{Z}$, or even more general groups).

Definition 3. The *upper density* $\bar{\mathrm{d}}(E)$ of a set $E \subset \mathbb{N}$ is defined as

$$\bar{\mathrm{d}}(E) = \limsup_{N\to\infty} \frac{|E \cap \{1, \dots, N\}|}{N}.$$

With the above notation we may state the Sárközy theorem.

Theorem 2 (Sárközy)**.** *If $E \subset \mathbb{N}$ and $\bar{\mathrm{d}}(E) > 0$, then there exist $x, y \in E$ such that $x - y = n^2$ for some $n \in \mathbb{N}$.*

One can show that the fact that the squares is the set of recurrence is equivalent to the Sárközy theorem.

Definition 4. Given a family $\mathcal{S}$ of subsets of $\mathbb{N}$ (or $\mathbb{Z}$) we say that $E \subset \mathbb{N}$ is an $\mathcal{S}^*$-set if for any $A \in \mathcal{S}$ we have $A \cap E \neq \emptyset$.

Definition 5. A set $S \subset \mathbb{N}$ (or $\mathbb{Z}$) is *syndetic* (or *relatively dense*) if it has bounded gaps. That is, S is syndetic if $S \cap [a, b]$ is nonempty for any long enough interval $[a, b]$ in $\mathbb{N}$ (or $\mathbb{Z}$).

Exercise 3. Prove that if $E \subset \mathbb{N}$ is syndetic, then $\bar{\mathrm{d}}(E) > 0$.

Exercise 4. Prove that $E \subset \mathbb{N}$ is syndetic if and only if there exists a finite set $F \subset \mathbb{N}$ such that $E - F = \{n - m \in \mathbb{N} : n \in E,\, m \in F\} = \mathbb{N}$.

Definition 6. A set $E \subset \mathbb{N}$ is *thick* if E contains arbitrarily long intervals.

Exercise 5. Let $\mathcal{S}$ be the family of all syndetic subsets of $\mathbb{N}$. Prove that $E \subset \mathbb{N}$ is an $\mathcal{S}^*$ if and only if E contains arbitrarily long intervals.

Syndetic sets may be considered big, but the notion of a syndetic set is not the best notion of largeness. One of the desired properties that any notion of largeness should have is partition regularity. A notion of largeness is *partition regular* if whenever one divides a large set into finite many disjoint subsets then one of these sets is also large.

Exercise 6. Prove that the property of being syndetic is not partition regular. Hint: one can represent $\mathbb{N}$ as a disjoint union of two non-syndetic sets.

We can easily modify the proof of the Poincaré Recurrence Theorem to obtain the following:

Corollary 2. *Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $T\colon X \to X$ be a μ-preserving transformation. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ for any strictly increasing integer sequence $\{n_i\}_{i=1}^{\infty}$ there exists $n \in \{n_j - n_i : j > i\}$ such that $\mu(A \cap T^{-n}A) > 0$.*

Exercise 7. Mimic the proof of Poincaré Recurrence Theorem and prove this Corollary.

Definition 7. We say that $E \subset \mathbb{N}$ is a Δ-set if E is the set of differences of an infinite subset of $\mathbb{N}$.

Now we may consider the family of Δ^* sets, that is, the collection of all sets whose intersection with a set differences of any infinite set is nontrivial. We may also restate our corollary of the Poincaré Recurrence Theorem as follows:

Corollary 3. *Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $T\colon X \to X$ be a μ-preserving transformation. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ the set R_A is a Δ^*-set. Equivalently, the set differences of any infinite set is a set of recurrence.*

It follows from the above fact that for any probability preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$ the set R_A is syndetic.

Exercise 8. Prove it. The proof is outlined in the exercises below.

Exercise 9. Prove that if $E \subset \mathbb{N}$ is thick, then there is a strictly increasing integer sequence $\{a_n\}_{n=1}^{\infty}$ such that $\{a_j - a_i : j > i\} \subset E$.

Exercise 10. Prove that every Δ^*-set is syndetic.

Exercise 11. Prove that not every syndetic set is a Δ^*-set. Hint: consider odd numbers.

Now we know that any set of returns is always a Δ^*-set (and hence syndetic), so we note another important property of Δ^*-sets.

Exercise 12. If the sets S_1 and S_2 are Δ^*-sets, then $S_1 \cap S_2$ is also a Δ^*-set.

Definition 8. A family $\mathcal{S}$ of subsets of $\mathbb{N}$ has the *filter property* if $A, B \in \mathcal{S}$ imply $A \cap B \in \mathcal{S}$.

One of possible solutions uses the following fact, which is a version of Ramsey's theorem. For $E \subset \mathbb{N}$ let $E^{(2)}$ denote the set of all two element subsets of E.

Theorem 3. *Let $\mathbb{N}^{(2)} = \bigcup_{j=1}^{r} C_j$ be any finite coloring of $\mathbb{N}^{(2)}$. Then there is a infinite set $S \subset \mathbb{N}$ such that $S^{(2)}$ is contained in one of C_j's.*

Exercise 13. Prove the above statement. Hint: it may be helpful to think of $\mathbb{N}^{(2)}$ as the set of edges of an infinite graph, whose set of vertices is $\mathbb{N}$.

An easy consequence of the above theorem is partition regularity of difference sets.

Corollary 4. *Let D be a set of differences of some strictly increasing sequence of integers. If $D = \bigcup_{j=1}^{r} C_j$ is any finite partition of D, then one of C_j contains a set of differences.*

Exercise 14. Prove the above statement.

Note that the partition regularity of difference sets is equivalent to the filter property of Δ^*-sets. Even more is true.

Exercise 15. Let $\mathcal{F}$ be a collection of subsets of $\mathbb{N}$ which is hereditary upwards, that is, if $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$. Assume that $\mathcal{F}$ is proper, that is, $\emptyset \notin \mathcal{F}$. Prove: the family $\mathcal{F}$ is partition regular if and only if its dual family $\mathcal{F}^*$ has the filter property.

Exercise 16. Let $(X, \mathcal{B}_X, \mu, T)$ and $(Y, \mathcal{B}_Y, \nu, S)$ be probability measure preserving systems. For any $A \in \mathcal{B}_X$ and $B \in \mathcal{B}_Y$ with $\mu(A) > 0$ and $\nu(B) > 0$ the set $R_A \cap R_B$ is a Δ^*-set.

It is natural to ask if a similar result holds for sets $R_{A,\varepsilon}$? The following result is usually called the Khintchine's recurrence theorem.

Theorem 4. *Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving system. For any $A \in \mathcal{B}$ with $\mu(A) > 0$ the set $R_{A,\varepsilon}$ is syndetic.*

Khintchine's recurrence theorem immediately follows from the following corollary of the classical von Neumann theorem.

Theorem 5. *For any probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ one has*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A) \geq \mu(A)^2.$$

For a proof we refer to any ergodic theory textbook. Here we will use another approach.

Lemma 1. *Let $(X, \mathcal{B}, \mu)$ be a probability space. Assume that for each $n \in \mathbb{N}$ we have $A_n \in \mathcal{B}$ with $\mu(A_n) \geq a > 0$. Then for every $\varepsilon > 0$ there exist $j > i$ such that $\mu(A_i \cap A_j) > a^2 - \varepsilon$.*

Proof. Let $\mathbb{I}_j$ be a characteristic function of A_j. Then for each $n \in \mathbb{N}$ we have (using the Cauchy-Schwartz inequality):

$$n^2a^2 \leq \left(\sum_{j=1}^{n} \mu(A_j)\right)^2 = \left(\int_X \sum_{j=1}^{n} \mathbb{I}_j \mathrm{d}\mu\right)^2 \leq \int_X \left(\sum_{j=1}^{n} \mathbb{I}_j\right)^2 \mathrm{d}\mu =$$

$$\sum_{j=1}^{n} \mu(A_j) + 2 \sum_{1\leq i<j\leq n} \mu(A_i \cap A_j).$$

Now, if for some $\varepsilon > 0$ we have $\mu(A_i \cap A_j) \leq a^2 - \varepsilon$ for all $i < j$, then we arrive at contradiction with the above inequality $\square$

Apply our lemma to $A_n = \mu(T^{-i_n}A)$. We get that the set of large returns $R_{A,\varepsilon}$ is a Δ^*-set.

Let D be a difference set. Choose any $i > j > k$ and set $x = n_i - n_j$, $y = n_j - n_k$ and $z = n_j - n_k$. Clearly, $x, y, z \in D$ and $x + y = z$. That is, there is a solution to the equation $x + y = z$ in one of the cells of our partition.

Theorem 6 (Schur (1916))**.** *For any finite coloring* $\mathbb{N} = \bigcup_{j=1}^{r} C_j$ *one of* C_j*'s contains* $x, y, x + y$ *for some* $x, y \in \mathbb{N}$.

In other words, the equation $x + y = z$ is partition regular.

Remark 2. For every $t \in \mathbb{Z}$ we have $\bar{\mathrm{d}}(E - t) = \bar{\mathrm{d}}(E)$. (Prove it!)

Remark 3. For every $\varepsilon > 0$ and $E \subset \mathbb{N}$ with $\bar{\mathrm{d}}(E) > 0$ there is $n \in \mathbb{N}$ such that $\bar{\mathrm{d}}(E \cap (E - n)) > \bar{\mathrm{d}}(E)^2 - \varepsilon$.

Exercise 17. What is the density version of Schur's theorem?

Schur was motivated by the Fermat's Last Theorem (proved by Wiles in 1995). Schur's theorem was only used as a lemma for the main theorem.

Theorem 7. *Let* $n > 1$. *There exists a prime* q *such that for all primes* $p > q$ *the congruence* $x^n + y^n = z^n \pmod{p}$ *has a solution in the integers with* $xyz \neq 0 \pmod{p}$.

Exercise 18. Derive from theorem 6 the following "finitistic" version: *For any* $r \in \mathbb{N}$ *there exists* $N_0 = N_0(r)$, *such that for any* $N > N_0$ *and any* r*-coloring* $\{1, 2, \ldots, N\} = \bigcup_{j=1}^{r} C_j$, *one of* C_j*'s contains* $x, y, x + y$.

Exercise 19. Use finitistic version of Schur's theorem to prove the theorem about Fermat equation mod p stated above. Hint: Let p be a prime and let $Z_p^* = \{1, 2, ..., p - 1\}$ be the group (under multiplication) of nonzero residues modulo p. Let $S = \{x^n \pmod{p} : x \in Z_p^*\}$. Notice that S is a subgroup of Z_p^*. Hence, we can write Z_p^* as a union of cosets of S (S induces a partition of Z_p^*). If p is large enough, one may use finitistic version of Schur's theorem.

Folkman's theorem generalizes Schur's theorem by stating that for any partition of the positive integers into a finite number of parts there exist arbitrarily large sets of integers all of whose nonempty sums belong to the same cell of partition. The theorem had been discovered and proved independently by several mathematicians, before it was named "Folkman's theorem", as a memorial to Jon Folkman, by Graham, Rothschild, and Spencer. Even further generalizations are known to be true. One is Rado's theorem which concerns a similar problem: the theorem characterizes the integer matrices A with the property that for every finite partition of the positive integers the system of linear equations $Ax = 0$ can be guaranteed to have a solution in which every coordinate of the solution vector x belongs to the same cell of the partition. Folkman's theorem is equivalent to the partition regularity of the system of equations

$$x_T = \sum_{i \in T} x_{\{i\}},$$

where T ranges over the nonempty subsets of the set $\{1, 2, \ldots, m\}$.

One may replace addition by multiplication in Schur's (or even Folkman's) theorem: if the natural numbers are finitely partitioned, then for any k there exists a finite set S with k elements such that all products of nonempty subsets of S belong to a single partition set. Indeed, if one restricts the partition of $\mathbb{N}$ to powers of two, then this result follows immediately from the additive version of Folkman's theorem. However, it is open whether for every finite partition there always exist arbitrarily large finite sets such that all sums and all products of nonempty subsets belong to a single partition cell. It is not even known whether there must necessarily exist a set of the form $\{x, y, x + y, xy\}$ contained in some partition cell.

Folkman's (and Schur's) theorem is even further generalized by Hindman's theorem.

Theorem 8. *For any finite coloring* $\mathbb{N} = \bigcup_{j=1}^{r} C_j$ *one of* C_j*'s contains a finite sums set of the form*

$$\mathrm{FS}(n_i)_{i=1}^{\infty} = \{n_{i_1} + n_{i_2} + \ldots + n_{i_k} : 0 < i_1 < i_2 < \ldots < i_k,\ k \in \mathbb{N}\}$$

for some infinite and strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$.

Recall that any set containing a FS-set of some infinite sequence is called an IP-set. Hence Hindman's theorem says *For any finite coloring* $\mathbb{N} = \bigcup_{j=1}^{r} C_j$ *one of* C_j*'s is an* IP*-set.*

Exercise 20. Prove that the class of IP sets is partition regular.

Exercise 21. Prove that every thick set is also an IP-set.

Definition 9. We say that a set $E \subset \mathbb{N}$ (or $\mathbb{Z}$) is AP-rich if it contains arbitrary long arithmetic progressions.

The following theorem was conjectured by Schur and proved by van der Waerden.

Theorem 9 (van der Waerden (1926)). *For any finite coloring* $\mathbb{N} = \bigcup_{j=1}^{r} C_j$ *one of* C_j*'s is AP-rich.*

A *density version* of van der Waerden is the following:

Theorem 10 (Szemerédi (1975)). *If* $E \subset \mathbb{N}$ *and* $\bar{\mathrm{d}}(E) > 0$*, then* E *is AP-rich.*

Theorem 11 (Furstenberg's Correspondence Principle). *For every* $E \subset \mathbb{N}$ *with* $\bar{\mathrm{d}}(E) > 0$ *there exist a probability measure preserving system* $(X, \mathcal{B}, \mu, T)$ *and* $A \in \mathcal{B}$ *with* $\mu(A) = \bar{\mathrm{d}}(E)$ *such that for any finite subset* $\{n_1, \ldots, n_k\} \subset \mathbb{N}$ *one has*

$$\bar{\mathrm{d}}(E \cap (E - n_1) \cap \ldots \cap (E - n_k)) \geq \mu(A \cap T^{-n_1}A \cap \ldots \cap T^{-n_k}A).$$

The set $\{n_1, \ldots, n_k\} \subset \mathbb{N}$ should be thought of as a configuration and then we can say that for any configuration in $\mathbb{N}$ we can find many shifted copies of it in E. It is a general principle that if a set is large, then it should be rich (it should contain many configurations).

Exercise 22. Show that there exists a finite coloring of $\mathbb{N}$ such that no cell contains $\{n, 2n\}$ for some n.

Exercise 23. Show that Szemerédi's theorem is equivalent to the following claim: *If* $E \subset \mathbb{N}$ *and* $\bar{\mathrm{d}}(E) > 0$*, then* E *contains an affine image of any finite set.* In other words: for any finite set $F \subset \mathbb{N}$ there exist $a, b \in \mathbb{N}$ such that

$$a + bF = \{a + bx : x \in F\} \subset E.$$

Exercise 24. Show that Szemerédi's theorem is equivalent to the following claim: *For every* $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *there exists an* $L \in \mathbb{N}$ *such that if* $N \geq L$ *and* $E \subset \{1, \ldots, N\}$ *with* $|E| \geq N\varepsilon$*, then* E *contains a* k*-term arithmetic progression.*

Take $E \subset \mathbb{Z}$ and identify E with its characteristic function $\omega = \mathbb{I}_E \in \Omega = \{0,1\}^{\mathbb{Z}}$. Now let $\Lambda = \overline{\{T^n\omega : n \in \mathbb{Z}\}}$, where $T(\omega)_n = \omega_{n-1}$ is the shift map. Then Λ is compact and shift-invariant.

Theorem 12 (Krylov-Bogoliouboff)**.** *If X is compact and $T\colon X \to X$ is continuous, then there exists a T invariant Borel probability measure on X.*

There are many density notions of largeness in $\mathbb{Z}$. A general way to construct one of them is as follows: Take a sequence $\{I_n\}$ of finite sets $I_n = \{a_n, a_n+1, \ldots, b_n\}$, where $b_n - a_n \to \infty$ as $n \to \infty$. Set

$$\bar{\mathrm{d}}_{\{I_n\}} = \limsup_{n\to\infty} \frac{|E \cap I_n|}{|I_n|}$$

Using the above notion of largeness we may give yet another form of the Szemerédi theorem.

Theorem 13. *If $E \subset \mathbb{N}$ and $\bar{\mathrm{d}}_{\{I_n\}}(E) > 0$ for some sequence $\{I_n\}$ of finite sets $I_n = \{a_n, a_n+1, \ldots, b_n\}$ with $b_n - a_n \to \infty$ as $n \to \infty$, then E is AP-rich.*

Exercise 25. Prove the equivalence of Theorems 10 and 13 (that is, show that any one of them may be easily proved by assuming the other is true).

Exercise 26. Extend this circle of ideas to $\mathbb{Z}^d$.

"Experimental" fact: *For any finite coloring of $\mathbb{N}$ one of the cells is a super-good set — it has a lot of structure.* Here is an example of this phenomenon.

Theorem 14. *For any partition $\mathbb{N} = \bigcup_{j=1}^{r} C_j$ one of C_j's is simultaneously AP-rich and GP-rich.*

Proof. Write numbers in $\mathbb{N}$ as $2^{k-1}(2n-1)$ and use a partition of $\mathbb{N}$ to induce a coloring of pairs $(k,n) \in \mathbb{N}^2$. Then use a version of two-dimensional van der Waerden. □

The set $\{2^n : n \in \mathbb{N}\}$ is clearly GP-rich, but not AP-rich.

Let S be a set of all square-free integers. One can prove that, in some sense, more than half integers are square-free.

Exercise 27. Prove that $\bar{\mathrm{d}}(S) = 6/\pi^2$.

But the set S of square-free integers is AP-rich but not GP-rich. This raises the following question, which the reader may try to answer.

Exercise 28. What is the density version of the Theorem 14?

Exercise 29. Prove directly that the set S of square-free integers is AP-rich.

To deal with geometric progressions we need to study the structure of multiplicative semigroup $\mathbb{N}$ ($\mathbb{N}$ with respect to multiplication). In $(\mathbb{N}, \cdot)$ we introduce the Følner sets

$$F_N = \{p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_N^{i_N} : 0 \leq i_j \leq N,\, j = 1, \ldots, N\},$$

where p_n is the n-th prime. Even more generally, we may consider

$$F_N = \{a_N p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_N^{i_N} : 0 \leq i_j \leq N,\, j = 1, \ldots, N\},$$

for arbitrary sequence of positive integers $\{a_N\}$.

Theorem 15. *If $E \subset \mathbb{N}$ and*

$$\bar{\mathrm{d}}^{\times}_{\{F_N\}}(E) = \limsup_{N\to\infty} \frac{|E \cap F_N|}{|F_N|} > 0$$

for some sequence $\{F_N\}$ defined above, then E is GP-rich. Actually, E is also AP-rich!

Sets $E \subset \mathbb{N}$ with $\bar{\mathrm{d}}^{\times}_{\{F_N\}}(E) > 0$ may be called multiplicatively large. Even more is true, for example ever such E contains configurations of the form

$$\{a(b + ic)^j, 0 \leq i, j \leq N\},$$

but even this only scratches the surface.

There is whole polynomial Szemerédi theory. But for this we refer the reader to the surveys cited below.

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