IMPAN Lectures on the Muskat problem

The purpose of these lectures is to show recent mathematical results concerning the classical Muskat problem. The Muskat problem models the evolution of incompressible fluids of different nature in porous media. The starting point is to motivate the problem and to establish several basic properties from its modeling equations. The final goal is to show well-posed scenarios in order to prove formation of singularities in finite time and existence of solutions for all time.

1. Setting the problem: modeling and equations

   In this talk we introduce the basic equations that model the problem: incompressibility, conservation of mass and Darcy's law. We provide several models from fluid mechanics to motivate Muskat, leading to very important questions as well-posedness, global-in-time existence or finite time blow-up. Finally we show how to understand it from a weak formulation, considering the dynamics of several incompressible fluids.

2. Contour equations

   Next, we introduced Muskat as a contour dynamics problem. For two incompressible fluids of different densities and viscosities, we find the contour evolution equation which describes the dynamics. It is given by a system with several important features, in particular it is nonlinear, nonlocal and given by unbounded operators. We explain how the contour dynamics equation is equivalent to satisfy the weak formulation.

3. Features and main results

   In this lecture we provide some basic properties for the contour equation, as finite energy preservation, stationary states, mean conservation, maximum principles and scaling invariance of solutions. We also show the main results on the problem, to get a general overview.

4. Ill-posedness: Rayleigh-Taylor instabilities

   In this talk we focus on an important feature of the problem. The Muskat problem shows instabilities which are understand from the nonlinear contour equation by means of the Rayleigh-Taylor condition. It yields that if the gradient of the pressure jump in the normal direction has a negative sign then the system is ill-posed. On the other hand, if it has a positive sign, then the system is proved to be well-posed. It is related with setting a denser fluid above a lighter fluid, or expanding a lower-viscosity fluid into a higher viscosity fluid and viceversa.

5. Well-posedness

   Next, we show the main ingredient to proved local-existence. It is based on energy estimates, dealing with nonlinear singular integral operators which appear in the contour evolution system. A crucial part of the argument is dealing with the Rayleigh-Taylor sign condition, due to the fact that this is the main ingredient to obtain a stable scenario. We also point out the importance of the arc-chord condition, which set the contour as a one-to-one regular curve.

6. Global existence for small initial data

   In the talk we show some recent results of global existence for small initial data. They provide global existence of classical solutions and instant analyticity of the contour. These two ingredients show by a time reversal argument ill-posedness in the unstable case.

7. Global existence for medium size initial data

   In this lecture we focus on the size and the criticality in the norm of the initial data in order to obtain global-in-time results. In particular, we show that solutions with initial slope less than one are global in time understanding the contour equation in a weak sense. In particular, this size is independent of the physical constants of the problem such us viscosity, gravity, density and permeability.

8. Rayleigh-Taylor breakdown

   In this talk we provide a finite time breakdown for the problem. Starting in a stable regime, we prove that the Muskat problem enter into unstable regime. In particular we show that smooth graphs with the denser fluid below turn to interface which are no longer graphs in finite time. This turning effect is dramatic as it changes the character of the equation. In particular the significance of a turnover is that the Rayleigh-Taylor condition breaks down.

9. Regularity breakdown

Next we continue with a new feature of the Muskat solutions given in the last talk. We prove that some of these smooth initial interfaces in the stable regime turn to the unstable regime and later blow-up. Therefore Muskat develops finite time singularities with loss of regularity starting from well-posed scenarios. This is the first case of singularity formation in contour dynamics of incompressible fluids in an initially well-posed scenario. The pattern of these initial data is far from trivial; some simulations have shown that there exist initial data with steep slopes for which a regularizing effect appears.

10. Splash singularities

Finally we show results for Muskat concerning a new type of singularity. We call it "splash singularity". In this scenario contours evolving intersect at a single point while the free boundary remains smooth. We show as Muskat with two fluids can not develop splash singularities. Meanwhile, for the one-phase case (with a dry region) there exist initial data whose Muskat solutions blow-up in finite time in a splash singularity. This is the first example of a singularity in a stable scenario for the Muskat problem.