Homology of invariant group chains

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1 Bass conjecture

In this section we shall recall some facts and state open problems concerning the Bass conjecture [1]. Throughout this section R will denote a ring with identity.

1.1 Projective modules and K_0 groups

Recall the following theorem, which may also be treated as a definition of projective R-modules.

Theorem 1.1. *Let P be an R-module. Then the following are equivalent:*

1. *P is a projective R-module,*
2. *every short exact sequence $0 \to A \to B \to P \to 0$ splits (and thus $B \cong A \oplus P$),*
3. *P is a direct summand of a free R-module.*

Let G be a group and let $\overline{X}$ be a generating set for G. Let F be a free group with basis in one-to-one correspondence to $\overline{X}$. The kernel of the canonical map $F \to G$ is denoted by $R(G, \overline{X})$ and is called the *relation subgroup* associated with $\overline{X}$. If we abelianize the group $R = R(G, \overline{X})$, we obtain a $\mathbb{Z}G$-module $M(G, \overline{X}) = R/[R, R]$, where the G-action is given by conjugation. This module is called the *relation module* associated with $\overline{X}$.

Example 1.2. Let $G = \langle x, y \mid x^2 = y^3 \rangle$ be the trefoil group. There exists a generating set $\overline{X}$ for G such that $\mathbb{Z}G \oplus \mathbb{Z}G \cong M(G, \overline{X}) \oplus \mathbb{Z}G$. Thus $M(G, \overline{X})$ is projective, as a direct summand of $\mathbb{Z}G \oplus \mathbb{Z}G$. However, it is not free. We shall see later that $M(G, \overline{X})$ is a representative of the 0-class in the reduced K_0 group.

Example 1.3. Let $G = \langle x, y \mid xy^2x^{-1} = y^3 \rangle$ and let $z = y^4$. Then the relation module $M(G, \{x, z\})$ satisfies $M(G, \{x, z\}) \oplus \mathbb{Z}G \cong \mathbb{Z}G \oplus \mathbb{Z}G$.

Both groups given in the above examples have interesting applications in algebraic topology. For a simpler example of projective modules that are not free consider the following situation.

Example 1.4. $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$, thus $\mathbb{Z}_2$ and $\mathbb{Z}_3$ treated as modules over $\mathbb{Z}_6$ are projective.

Consider the following:

- $\mathcal{P} = \{$isomorphism classes $\{P\}$ of finitely generated projective R-modules$\}$,
- $F(\mathcal{P}) = \langle \mathcal{P} \rangle$, the free abelian group generated by all elements of $\mathcal{P}$,

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- $\mathcal{R}(\mathcal{P}) = \langle \{\{B\} - \{A\} - \{C\} \mid \text{there exists a split exact sequence } 0 \to A \to B \to C \to 0\} \rangle \leq F(\mathcal{P})$,
- $K_0(R) = F(\mathcal{P})/\mathcal{R}(\mathcal{P})$, the *Grothendieck group* of the ring R.

$K_0(R)$ is the abelian group generated by the quotient classes $[P]$ of $\{P\} \in F(\mathcal{P})$ for all finitely generated projective R-modules P, subject to the relations $[B] = [A] + [C]$ whenever $B \cong A \oplus C$. The 0 of this group is $[0]$.

Let A, B be finitely generated R-modules. Then A and B are said to be *stably isomorphic* if $A \oplus R^k \cong B \oplus R^k$ for some free R-module R^k, $k \in \mathbb{N}$. We say that A is *stably free* if $A \oplus R^n = R^m$ for some $m, n \in \mathbb{N}$.

Lemma 1.5. *If $[P] = [Q]$ in $K_0(R)$, then there exists a finitely generated projective R-module V such that $P \oplus V \cong Q \oplus V$.*

Lemma 1.6. *Let P, Q be finitely generated projective R-modules. Then:*

1. *$[P] = [Q]$ in $K_0(R)$ if and only if P and Q are stably isomorphic.*
2. *$[P]$ is in the cyclic subgroup of $K_0(R)$ generated by $[R]$ if and only if P is stably free.*
3. *$K_0(R)$ is the cyclic group generated by $[R]$ if and only if all finitely generated projective R-modules are stably free.*

Define the *Wall group of R* to be $\widetilde{K_0}(R) = K_0(R)/\langle [R] \rangle$. (This is also called *reduced K_0 group* and has important applications in topology.)

Note that in both of the Examples 1.2 and 1.3 the considered modules are stably free, and thus are in $[R]$, the 0 class in $\widetilde{K_0}(R)$.

1.2 Traces

Example 1.7. Let V_n be a finite dimensional vector space with $n = \dim V_n$. Consider the identity map $\mathrm{id} : V_n \to V_n$. The matrix of id in a fixed basis $(e_1, \ldots, e_n)$ of V_n is $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$. We may calculate the trace:

$$\mathrm{tr} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = n = \dim(V_n).$$

Using the properties

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$$

and

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

we easily prove that the trace of a map is independent of the choice of basis. So a way to calculate the dimension of a vector space is to calculate the trace of the identity map.

We want to generalize the above observation to finitely generated projective R-modules, beginning with free R-modules.

Let F be a finitely generated free R-module and let $\alpha : F \to F$ be any endomorphism. Fixing a basis of F we get a matrix $\alpha = [a_{ij}]_{i,j=1,\dots,n}$. If we defined $\operatorname{tr}(\alpha) = \sum_{i=1}^{n} a_{ii}$, then the identity $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ may not be satisfied (since R may be non-commutative). As a consequence, $\operatorname{tr}(\alpha) = \sum_{i=1}^{n} a_{ii}$ would depend on the choice of basis of F, and thus it is not a good definition for the trace.

To solve this problem we should define the trace in the following way:

$$\operatorname{tr}(\alpha) = \sum_{i=1}^{n} \overline{a_{ii}},$$

where $\overline{a_{ii}} \in R/[R,R]$ is the image of a_{ii} in the abelian group $R/[R,R]$ via the natural projection map $R \to R/[R,R]$. Now we may define:

$$\operatorname{Rank}_R F = \operatorname{tr}(\mathrm{id}_F) \in R/[R,R],$$

which is independent of the choice of basis.

Finally, let P be a finitely generated projective R-module. Let $\alpha : P \to P$ be an endomorphism. Since P is projective, $P \oplus Q \cong F$ for some R-module Q and some free R-module F. We have the following situation: $F \xrightarrow{\pi} P \xrightarrow{\alpha} P \overset{i}{\hookrightarrow} F$, where π is the projection map. Define $\operatorname{tr}(\alpha) = \operatorname{tr}(i \circ \alpha \circ \pi)$. It turns out that this definition does not depend on the choice of F, π, i. We may define:

$$\operatorname{rank}_R(P) = \operatorname{tr}(\mathrm{id}_P) = \operatorname{tr}(i \circ \pi).$$

Note that $(i \circ \pi) \circ (i \circ \pi) = i \circ (\pi \circ i) \circ \pi = i \circ \pi$, so $i \circ \pi$ is an idempotent endomorphism. The rank of a projective modules is thus the trace of some idempotent matrix. However, the matrix need not be diagonal.

1.3 Hatori-Stallings trace and the Bass conjecture

For topological reasons people are interested in group rings $\mathbb{Z}G$ (where the group G may be for example the fundamental group of some topological space, etc.). In this subsection $R = \mathbb{Z}G$ for some fixed group G.

Note that

$$R/[R,R] = \mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G] \cong \langle \{\{g^h\}, g \in G\} \rangle$$

is the free abelian group generated by the conjugacy classes $\{g^h\}$ in G of elements $g \in G$. This follows from the following observation:

$$st - ts = st - s^{-1}(st)s = r - s^{-1}rs$$

Define now the *Hatori-Stallings trace* of a finitely generated projective R-module P to be

$$\mathrm{HS}(P) = \mathrm{tr}_{\mathbb{Z}G}\,\mathrm{id}_P = \sum_{\{g^h\}} n_{g^h} g^h = n_e e + n_{g_1^h} g_1^h + \cdots + n_{g_k^h} g_k^h \in R/[R,R].$$

(There is a slight abuse of notation here; we should write $\mathrm{HS}(P) = \sum_{\{g^h\}} n_{\{g^h\}} \{g^h\}$.)

The following is open.

Conjecture 1.8 (Bass conjecture)**.** Given a group G and a finitely generated projective $\mathbb{Z}G$-module P, all the elements $n_{g_k^h} = 0$ for $g_k^h \neq e$.

There is also a weaker version of the above conjecture.

Conjecture 1.9 (weak Bass conjecture)**.** Given a group G and a finitely generated projective $\mathbb{Z}G$-module P, the sum $\sum_{\{g_k^h\}\neq e} n_{g_k^h} = 0$.

Actually, the Hatori-Stallings trace may be seen as a map $\mathrm{HS} : K_0(\mathbb{Z}G) \to \langle \{g^h\} \rangle$, since for $[P] = [Q]$ we have $P \oplus \mathbb{Z}G^k \cong Q \oplus \mathbb{Z}G^k$ and by additivity of trace we have $\mathrm{tr}_{\mathbb{Z}G}\,\mathrm{id}_P = \mathrm{tr}_{\mathbb{Z}G}\,\mathrm{id}_Q$. Note that by similar reasoning it is easy to calculate $\mathrm{HS}(P)$ for $[P] \in \langle [R] \rangle$.

Theorem 1.10. *The Bass conjecture is true for finite groups.*

The above theorem is generalized by the following.

Theorem 1.11 ([2])**.** *The Bass conjecture is true for amenable discrete groups.*

Recall that a discrete group G is *amenable* if it has a finitely additive left-invariant probability measure. Amenable groups have the following properties.

a) Every subgroup of an amenable group is amenable.
b) A group extension of an amenable group by an amenable group is amenable.
c) Every quotient of an amenable group is amenable.
d) Direct limit of amenable groups is amenable.
e) Finite direct product of amenable groups is amenable

Groups that are formed from abelian and finite groups using the operations of taking subgroups, quotients, direct unions and extensions are called *elementary amenable*.

Example 1.12. Here we list some examples of amenable groups.

1) Finite groups are amenable. (This is trivial, set $\mu(A) = \frac{|A|}{|G|}$ for $A \subseteq G$.)
2) $\mathbb{Z}$ is amenable. (Here the measure is more tricky to construct.)
3) Abelian groups are amenable. (Any abelian group is a direct limit of finitely generated abelian groups, so this follows from properties d), e) and examples 1), 2) above.)
4) If all finitely generated subgroups of a group are amenable, then the group is amenable. (This follows from d).)
5) If a group has a finite index amenable subgroup, then it is amenable. (First we note that a normal finite index amenable subgroup exists. Next we use b).)

6) Solvable groups are amenable.
7) Finitely generated groups of subexponential growth are amenable.

Example 1.13. Now we give some examples of groups that are not amenable.
1) If a countable discrete group contains a (non-abelian) free group on 2 generators as a subgroup, then it is non-amenable.
2) Free non-abelian groups are not amenable.
3) The free Burnside group

$$B(m,n) = \langle X_1, \dots, X_m \mid x^n = 1 \text{ for any word } x \text{ in the letters } X_1, \dots, X_m \rangle$$

is not amenable.
4) $SL_2(\mathbb{Z})$ is not amenable.
5) Braid groups are not amenable.

However, for all of these groups the Bass conjecture is true.

Remark 1.14. The Thompson group F is the group given by the following finite presentation:

$$\langle A, B \mid [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = 1 \rangle$$

The group F also has an infinite presentation that is perhaps more intuitive:

$$\langle x_0, x_1, x_2, \dots \mid x_k^{-1} x_n x_k = x_{n+1} \text{ for } k < n \rangle.$$

Because of their unusual properties, the Thompson group and some related groups introduced by Thompson have been widely studied. See [5] for a gentle introduction to this topic.

Open problem 1.15. Is the Thompson group amenable? (It is not elementary amenable.)

An interesting class of groups are the groups which have only two conjugacy classes of elements. The simplest one (and the only finite one) is $\mathbb{Z}_2$. Infinite groups with this property were constructed by Denis Osin [9]. Thus, $\mathrm{HS}(P) = n_e e + n_g g$ for P a finitely generated projective $\mathbb{Z}G$-module with G a group with two conjugacy classes, so the weak and the strong Bass conjectures coincide for such groups. However, the speaker does not know whether the Bass conjecture is true for these groups.

Theorem 1.16. *Let R_1, R_2 be rings. Then: $K_0(R_1 \times R_2) \cong K_0(R_1) \times K_0(R_2)$.*

Open problem 1.17. Suppose the Bass conjecture is true for groups G_1, G_2. Prove that it is true for $G_1 \times G_2$.

Open problem 1.18. Let $0 \to G_1 \to G \to G_2 \to 0$ be a short exact sequence of groups. Suppose the Bass conjecture is true for G_1 and G_2. Prove it for G. (Even the behaviour of $K_0(G)$ is not clear here, unless the sequence splits.)

Theorem 1.19 (Bass, cf. [3, Lemma 5.3])**.** *If the Bass conjecture is true for all finitely presented groups, then it is true for all groups.*

Proof. We shall use the well-known fact that $K_0(R)$ is naturally isomorphic to the Grothendieck group of the monoid of conjugacy classes of idempotent matrices in $GL_\infty(R)$.

Let G be a group and let A be a matrix over $\mathbb{Z}G$ such that $A^2 = A$. Consider the finitely generated group $G_0 \leq G$ such that the elements of the matrix A are all in $\mathbb{Z}G_0$. Let $G_0 = F/R$, where F is a finitely generated free group. We lift the matrix A to a matrix B over $\mathbb{Z}F$. There is a finite subset W of R such that the entries of the matrix $B^2 - B$ all lie in the ideal of $\mathbb{Z}F$ generated by the set $\{1 - r \mid r \in W\}$. Define $R_1 \leq R$ to be the normal closure of W in F. Let $G_1 = F/R_1$. The group G_1 is finitely presented. Let A_1 be the image of the matrix B; the entries of A_1 belong to $\mathbb{Z}G_1$. The matrix A_1 is idempotent. The map $G_1 \to G_0 \to G$ transforms A_1 into A, and thus we get a map $K_0(\mathbb{Z}G_1) \ni [A_1] \mapsto [A] \in K_0(\mathbb{Z}G)$. By commutativity of the following diagram, the result follows.

$$\begin{array}{ccc} K_0(G_1) & \longrightarrow & \mathbb{Z}G_1/[\mathbb{Z}G_1, \mathbb{Z}G_1] \\ \downarrow & & \downarrow \\ K_0(G) & \longrightarrow & \mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G] \end{array}$$

□

Open problem 1.20. Suppose that $G = \varinjlim G_\alpha$ and the Bass conjecture is true for all groups G_α. Prove that it is true for G.

2 Homology of invariant group chains

This section contains the main part of the lecture, dedicated to computing homology of invariant group chains, a topic first studied by K. Knudson [6].

Throughout this section G will be a fixed group.

2.1 Group homology

We begin with recalling some of the fundamental facts on group homology.

In order to calculate $H_\bullet(G, \mathbb{Z})$, with $\mathbb{Z}$ a trivial G-module, we use a projective resolution of $\mathbb{Z}$ (any two such resolutions are homotopy equivalent)

$$\cdots \to P_1 \to P_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

and take its tensor product with $\mathbb{Z}$ over $\mathbb{Z}G$. The sequence we obtain

$$P_G : \cdots \to P_2 \otimes_{\mathbb{Z}G} \mathbb{Z} \to P_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \to P_0 \otimes_{\mathbb{Z}G} \mathbb{Z}$$

need not be exact. The homology of the chain complex P_G is by definition the homology of G: $H_\bullet(G, \mathbb{Z}) = H_\bullet(P_G)$.

There is also a topological approach. We begin with an example.

Example 2.1. Let S^1 be the circle with a CW structure consisting of one vertex v and one edge e. The CW structure lifts via the map $p(t) = (\cos(2\pi t), \sin(2\pi t))$ to $\mathbb{R}$, the universal cover of S^1. There is an action of $\pi_1(S^1) \cong \mathbb{Z}$ on $\mathbb{R}$ given by $\mathbb{Z} \times \mathbb{R} \ni (n, x) \mapsto x + n \in \mathbb{R}$. The free abelian groups generated by the edges and by the vertices are now free $\mathbb{Z}[\pi_1(S^1)]$-modules (with an action induced by the above action of $\pi_1(S^1)$ on $\mathbb{R}$). Since $\mathbb{R}$ is contractible, the following cellular chain complex

$$\cdots \to 0 \to C_1(\mathbb{R}) \xrightarrow{\partial} C_0(\mathbb{R}) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

which may be written more explicitly as

$$\cdots \to 0 \to \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z} \to 0,$$

is acyclic, thus exact, and so it gives a projective resolution of the trivial $\mathbb{Z}$-module $\mathbb{Z}$, which may be used to calculate $H_n(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & , n = 0 \\ \mathbb{Z} & , n = 1 \\ 0 & , n > 1 \end{cases}$.

The general recipe, whose special case was shown in the above example, is as follows. To calculate $H_\bullet(G, \mathbb{Z})$, where G is a discrete group, consider a cellular realisation of the classifying space BG (which, in the case of discrete groups, is just the Eilenberg-Maclane space $K(G, 1)$) and take its universal cover EG with the lifted CW structure. The cellular chain complex of the universal cover EG gives a resolution of $\mathbb{Z}$.

In particular, we obtain $H_\bullet(G, \mathbb{Z}) \cong H_\bullet(BG, \mathbb{Z})$, where BG is the classifying space of G (and equals $K(G, 1)$ if G is a discrete group). This is the reason why people are interested in computing homology and cohomology of classifying spaces.

Example 2.2. Some examples of classifying spaces are:

- $G = \mathbb{Z}$, $BG = S^1$,
- $G = \mathbb{Z} \times \mathbb{Z}$, $BG = S^1 \times S^1$,
- $G = \mathbb{Z}_2$, $BG = \mathbb{R}P^\infty$.

2.2 The bar resolution

Both for theoretical and for computational aspects of the homology theory of groups, it is often convenient to have an explicit description of a resolution of $\mathbb{Z}$ over a given group.

(a) The homogeneous bar resolution

We first describe the non-normalized bar resolution. Let $\bar{B}_n$, $n \geq 0$, be the free abelian group on the set of all $(n+1)$-tuples $(y_0, y_1, \cdots, y_n)$ of elements of G. Define a left G-module structure in $\bar{B}_n$ by

$$y(y_0, y_1, \cdots, y_n) = (yy_0, yy_1, \cdots, yy_n), \qquad y \in G.$$

It is clear that $\bar{B}_n$ is a free G-module, a basis being given by the $(n+1)$-tuples $(1, y_1, \cdots, y_n)$.

We define the differential in the sequence

$$\bar{B} : \cdots \to \bar{B}_n \xrightarrow{\partial_n} \bar{B}_{n-1} \to \cdots \to \bar{B}_1 \xrightarrow{\partial_1} \bar{B}_0$$

by the simplicial boundary formula

$$\partial_n(y_0, y_1, \cdots, y_n) = \sum_{i=0}^{n} (-1)^i (y_0, y_1, \cdots, \hat{y}_i, \cdots, y_n),$$

where the symbol $\hat{y}_i$ indicates that y_i is to be omitted. We also define the augmentation $\epsilon : \bar{B}_0 \to \mathbb{Z}$ by

$$\epsilon(y) = 1.$$

Obviously ∂_n, ϵ are G-module homomorphisms. Moreover, an elementary calculation, very familiar to topologists, shows that

$$\partial_{n-1}\partial_n = 0, \ \ n \geq 2; \ \ \epsilon\partial_1 = 0.$$

We claim that $\bar{B}$ is a free G-resolution of $\mathbb{Z}$. This is also a translation into algebraic terms of a fact familiar to topologists, but we will give a proof. We regard

$$\cdots \to \bar{B}_n \xrightarrow{\partial_n} \bar{B}_{n-1} \to \cdots \to \bar{B}_1 \xrightarrow{\partial_1} \bar{B}_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

as a chain complex of abelian groups and, as such, it may readily be seen to admit a contracting homotopy $\bar{\triangle}$, given by

$$\bar{\triangle}_{-1}(1) = 1, \ \ \bar{\triangle}_n(y_0, y_1, \cdots, y_n) = (1, y_0, \cdots, y_n).$$

We leave to the reader to verify that $\bar{\triangle}$ is indeed a contracting homotopy, that is, that

$$\epsilon\bar{\triangle}_{-1} = \mathrm{id}_{\mathbb{Z}}, \ \ \partial_1\bar{\triangle}_0 + \bar{\triangle}_{-1}\epsilon = \mathrm{id}_{\bar{B}_0}, \ \ \partial_{n+1}\bar{\triangle}_n + \bar{\triangle}_{n-1}\partial_n = \mathrm{id}_{\bar{B}_n}, \ \ n \geq 1.$$

The complex $\bar{B}$ is called the (*non-normalized*) *standard* (or *bar*) *resolution in homogeneous form*.

Now let $D_n \subseteq \bar{B}_n$ be the subgroup generated by the $(n+1)$-tuples $(y_0, y_1, \cdots, y_n)$ such that $y_i = y_{i+1}$ for at least one value of i, $i = 0, 1, \cdots, n-1$. Such an $(n+1)$-tuple will be called degenerate and clearly D_n is the submodule of $\bar{B}_n$, generated by the degenerate $(n+1)$-tuples with $y_0 = 1$. We claim that $\partial D_n \subseteq D_{n-1}$. For, if $(y_0, y_1, \cdots, y_n)$ is degenerate, say $y_j = y_{j+1}$, then $\partial_n(y_0, y_1, \cdots, y_n)$ is a linear combination of degenerate n-tuples, together with the term

$$(-1)^j(y_0, y_1, \cdots, y_{j-1}, y, y_{j+2}, \cdots, y_n) +$$
$$(-1)^{j+1}(y_0, y_1, \cdots, y_{j-1}, y, y_{j+2}, \cdots, y_n), \ \ y = y_j = y_{j+1},$$

which is clearly zero. Thus the submodules D_n yield a subcomplex D, called the degenerate subcomplex of $\bar{B}$. (Of course, we could choose other definitions of degeneracy; for example, we could merely require that any two of $y_0, y_1, \cdots, y_n$ are the same.)

We remark that $D_0 = 0$. We also notice that the contracting homotopy $\bar{\triangle}$ has the property that $\bar{\triangle}_n D_n \subseteq D_{n+1}$, $n \geq 0$. Thus we see that, passing to the quotient complex $B = \bar{B}/D$, each G-module B_n is free (on the $(n+1)$-tuples $(y_0, \cdots, y_n)$ with $y_0 = 1$ and $y_i = y_{i+1}$ for no value of i, $i = 0, 1, \cdots, n-1$), and B is a G-free resolution of $\mathbb{Z}$, the contracting homotopy $\triangle$ being induced by $\bar{\triangle}$. The complex B is called the (*normalized*) *standard* (or *bar*) *resolution in homogeneous form*. It is customary in homological algebra to use the normalized form with precisely this definition of degeneracy.

(b) The inhomogeneous bar resolution

Let $\bar{B}'_n$, $n \geq 0$, be the free left G-module on the set of all n-tuples $[x_1 \mid x_2 \mid \cdots \mid x_n]$ of elements of G. We define the differential in the sequence

$$\bar{B}' : \cdots \to \bar{B}'_n \xrightarrow{\partial_n} \bar{B}'_{n-1} \to \cdots \to \bar{B}'_1 \xrightarrow{\partial_1} \bar{B}'_0$$

by the formula

$$\partial_n[x_1 \mid x_2 \mid \cdots \mid x_n] = x_1[x_2 \mid x_3 \mid \cdots \mid x_n] + \sum_{i=1}^{n-1}(-1)^i[x_1 \mid x_2 \mid \cdots \mid x_i x_{i+1} \mid \cdots \mid x_n] + (-1)^n[x_1 \mid x_2 \mid \cdots \mid x_{n-1}]$$

and the augmentation $\epsilon : \bar{B}'_0 \to \mathbb{Z}$ by

$$\epsilon[\,] = 1.$$

The reader is advised to give a direct proof that $\bar{B}'$ is a G-free resolution of $\mathbb{Z}$, using the hint that the contracting homotopy is given by

$$\bar{\triangle}_{-1}(1) = [\,], \;\; \bar{\triangle}_n(x[x_1 \mid \cdots \mid x_n]) = [x \mid x_1 \mid \cdots \mid x_n], \;\; n \geq 0$$

(recall that $\bar{\triangle}_n$ is a homomorphism of abelian groups). However, we avoid this direct proof by establishing an isomorphism between $\bar{B}'$ and $\bar{B}$, compatible with the augmentations.

Thus we define $\varphi_n : \bar{B}_n \to \bar{B}'_n$ by

$$\varphi_n(1, y_1, \cdots, y_n) = [y_1 \mid y_1^{-1}y_2 \mid \cdots \mid y_{n-1}^{-1}y_n]$$

and $\psi_n : \bar{B}'_n \to \bar{B}_n$ by

$$\psi_n[x_1 \mid x_2 \mid \cdots \mid x_n] = (1, x_1, x_1x_2, \cdots, x_1x_2\cdots x_n)$$

It is easy to see that φ_n, ψ_n are mutual inverses, and that they are compatible with the differentials and the augmentations.

Moreover, if $D'_n = \varphi_n D_n$, then D'_n is the submodule of $\bar{B}'_n$ generated by the n-tuples $[x_1 \mid x_2 \mid \cdots \mid x_n]$ with at least one x_i equal to 1. The modules D'_n constitute the degenerate subcomplex D' of $\bar{B}'$ and the quotient complex $B' = \bar{B}'/D'$ is a G-free resolution of $\mathbb{Z}$, isomorphic to B, and called the (*normalized*) *standard* (or *bar*) *resolution in inhomogeneous form*.

Definition 2.3. Define the *bar complex* $C_\bullet(G)$ by $C_\bullet(G) = B_\bullet(G) \otimes_G \mathbb{Z}$:

$$C_\bullet(G) := \cdots \longrightarrow C_2(G) \stackrel{\partial_2 \otimes 1_{\mathbb{Z}}}{\longrightarrow} C_1(G) \stackrel{\partial_1 \otimes 1_{\mathbb{Z}}}{\longrightarrow} C_0(G)$$

where the differential $\partial_n : B_n(G) \to B_{n-1}(G)$ is given by:

$$\partial_n([g_1 \mid \cdots \mid g_n]) = g_1[g_2 \mid \cdots \mid g_n] + \sum_{k=1}^{n-1} (-1)^k [g_1 \mid \cdots \mid g_k g_{k+1} \mid \cdots \mid g_n] + (-1)^n [g_1 \mid \cdots \mid g_{n-1}]$$

Definition 2.4. Let A be an abelian group. We define the chain complex $C_\bullet(G) \otimes A$, where G acts on A trivially and the differential is given by $\partial \otimes 1_{\mathbb{Z}} \otimes 1_A$. Denote its homology by $H_\bullet(G, A)$.

Remark 2.5. The following isomorphisms hold:

$$C_*(G) \otimes_{\mathbb{Z}} A = (B_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \otimes_{\mathbb{Z}} A = B_*(G) \otimes_{\mathbb{Z}[G]} (\mathbb{Z} \otimes_{\mathbb{Z}} A) = B_*(G) \otimes_{\mathbb{Z}[G]} A.$$

In homology we have:

$$H_\bullet(G, \mathbb{Z}) = H_\bullet(C_*(G))$$

and

$$H_\bullet(G, M) = H_\bullet(B_*(G) \otimes_{\mathbb{Z}[G]} M)$$

for any $\mathbb{Z}[G]$-module M. By the above isomorphisms, we can write

$$H_\bullet(G, A) = H_\bullet(C_*(G) \otimes_{\mathbb{Z}} A).$$

2.3 Homology of invariant group chains

Definition 2.6. Let Q be a finite group acting on the group G as a group of automorphisms. The action of Q on G induces an action on $C_\bullet(G) \otimes A$ by

$$q([g_1 \mid \cdots \mid g_i] \otimes a) = [q(g_1) \mid \cdots \mid q(g_i)] \otimes a.$$

Denote the subcomplex of invariant chains of $C_\bullet(G) \otimes A$ by $(C_\bullet(G) \otimes A)^Q$, and denote its homology by $H^Q_\bullet(G, A)$. Moreover, the homology of the subcomplex $C_\bullet(G)^Q \otimes A$ will be denoted by $H_\bullet(C_\bullet(G)^Q, A)$.

We note that if Q acts trivially on A, then we have the isomorphism $(C_\bullet(G) \otimes A)^Q \cong (C_\bullet(G))^Q \otimes A$, hence the homology groups of these complexes are isomorphic.

Remark 2.7. In the above situation we can also define the cohomology groups:

$$H^\bullet_Q = H^\bullet(\mathrm{Hom}(C_*(G)^Q \otimes A)).$$

Remark 2.8. Of course, we have the following inclusions of chain complexes:

$$B_*(G^Q) \hookrightarrow B_*(G)^Q \hookrightarrow B_*(G),$$

so

$$C_*(G^Q) \hookrightarrow C_*(G)^Q \hookrightarrow C_*(G),$$

and in turn

$$H_\bullet(G^Q, \mathbb{Z}) \xrightarrow{i_*} H^Q_\bullet(G, \mathbb{Z}) \xrightarrow{j_*} H_\bullet(G, \mathbb{Z}).$$

One should note that $\mathrm{im}\, j_* \subseteq H_\bullet(G, \mathbb{Z})^Q$, since a Q-invariant cycle in $C_\bullet(G, A)^Q$ gives rise to a Q-invariant homology class in $H_\bullet(G, A)$, and thus we really have

$$H_\bullet(G^Q, \mathbb{Z}) \to H^Q_\bullet(G, \mathbb{Z}) \to H_\bullet(G, \mathbb{Z})^Q.$$

We know that $BG = EG/G$, where we may consider the infinite join model of $EG = G * G * G \ldots$, which has a CW structure. (For example, for $G = \mathbb{Z}_2$ we get $E\mathbb{Z}_2 = S^\infty$ and $B\mathbb{Z}_2 = S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty$.)

The cellular chain complex of BG is isomorphic to $C_*(G)$. The Q-action on G gives a Q-action on BG, which turns BG into a Q-CW complex. We may thus study the cellular chain complex of the quotient CW complex BG/Q. Now,

$$C_*(BG/Q) = \frac{\mathbb{Z}\{[g_1 \mid \ldots \mid g_n] | g_i \in G\}}{\langle [q(g_1) \mid \ldots \mid q(q_n)] - [g_1 \mid \ldots \mid g_n] \rangle} = C_*(G)_Q.$$

Definition 2.9. Recall that given a discrete group H, a commutative ring with identity R and a left RH-module M, by M_H we denote the *R-module of H-coinvariants* $M_H = M/\langle m - Hm \rangle$, where $\langle m - Hm \rangle$ is the submodule of M generated by the set $\{m - hm \mid m \in M, h \in H\}$.

In this situation we may define the *norm map* $N : M_H \to M^H$, $N(\overline{m}) = \sum_{h \in H} hm$.

Note that N composed with the canonical projection $M \to M_H$ sends $\overline{m} \in M_H$ into $|H|\overline{m} \in M_H$, thus the order of H in the above definition annihilates $\mathrm{Ker}\, N$, i.e. we have the equality $|H| \,\mathrm{Ker}\, N = \{0\}$. Similarly, $|H| \,\mathrm{Coker}\, N = \{0\}$.

Applying the definition to our situation we get the map $N : C_*(BG/Q) = C_*(G)_Q \to C_*(G)^Q$ induced by

$$[g_1 \mid \ldots \mid g_n] \mapsto \sum_{q \in Q} [q(g_1) \mid \ldots \mid q(g_n)].$$

But $C_*(BG/Q)$ is a free chain complex, so $|Q| \,\mathrm{Ker}\, N = 0$ implies that $\mathrm{Ker}\, N$ is trivial. So, N is injective. (In general it does not have to be surjective.)

2.4 Results of Kevin Knudson

Here we present some results of [6]. Their generalizations shall be given in the next subsection.

Proposition 2.10 ([6, Proposition 2.1])**.** *Suppose $Q = \mathbb{Z}/p$, where p is prime. Then* $\operatorname{Coker} N_n = C_n(G^Q, A)$ *for all $n \geq 0$.*

Proof. If $n = 0$, then $N_0 : C_0(G)_Q \to C_0(G)^Q$ is just the multiplication by p map

$$(\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z})_Q = \mathbb{Z} \ni n \mapsto \sum_{q \in Q} qn = pn \in \mathbb{Z}.$$

Thus, $\operatorname{Coker} N_0 = \mathbb{Z}/p\mathbb{Z}$. But $C_0(G^Q, \mathbb{Z}/p) = C_0(G^Q) \otimes \mathbb{Z}/p = \mathbb{Z} \otimes \mathbb{Z}/p = \mathbb{Z}/p$.

Consider now the case $n \geq 1$. Since p is prime, the orbit of any element $g \in G$ has length 1 or p. The same is true for any $[g_1 \mid \dots \mid g_n]$. The basis elements of $C_n(G)^Q$ are elements of the form $\sum_{q \in Q}[q(g_1) \mid \dots \mid q(g_n)]$ together with the elements $[g_1 \mid \dots \mid g_n]$ with $g_i \in G^Q$.

If $[g_1 \mid \dots \mid g_n]$ is a generator of $C_n(BG/Q)$, then

$$N_n([g_1 \mid \dots \mid g_n]) = \sum_{q \in Q} [q(g_1) \mid \dots \mid q(g_n)].$$

If the orbit of $[g_1 \mid \dots \mid g_n]$ has p elements, then $N([g_1 \mid \dots \mid g_n])$ is the corresponding basis element of $C_n(G)^Q$.

On the other hand, if $g_i \in G^Q$ for all $1 \leq i \leq n$, then

$$N_n([g_1 \mid \dots \mid g_n]) = p[g_1 \mid \dots \mid g_n].$$

Thus,

$$\operatorname{Coker} N_n = \frac{\mathbb{Z}\{[g_1 \mid \dots \mid g_n] \mid g_i \in G^Q\}}{\langle p[g_1 \mid \dots \mid g_n] \mid g_i \in G^Q \rangle} = C_n(G^Q, \mathbb{Z}/p),$$

which finishes the proof. □

Corollary 2.11. $H_n(\operatorname{Coker} N_*) = H_n(C_*(G^Q, \mathbb{Z}/p))$.

Proof. This follows at once from Proposition 2.10, since the boundary maps of $\operatorname{Coker} N_*$ and $C_*(G^Q, \mathbb{Z}/p)$ clearly coincide. □

Corollary 2.12. *If $Q = \mathbb{Z}/p$, then there is a long exact sequence:*

$$\cdots \to H_n(BG/Q, \mathbb{Z}) \to H_n^Q(G, \mathbb{Z}) \to H_n(G^Q, \mathbb{Z}/p) \to H_{n-1}(BG/Q, \mathbb{Z}) \to \dots$$

Corollary 2.13. *If $Q = \mathbb{Z}/p$ and $G^Q = \{e\}$, then $H_*(BG/Q, \mathbb{Z}) = H_n^Q(G, \mathbb{Z})$ for all $n > 0$.*

Proposition 2.14 ([6, Proposition 3.3])**.** *If $|Q|$ is invertible in A, then the natural map*

$$H_\bullet^Q(G, A) \to H_\bullet(C_\bullet(G) \otimes A)^Q$$

is an isomorphism.

2.5 Generalization to $\mathbb{Z}/p^k$

In this subsection we give new results, which generalize those of the paper [6] presented in the previous subsection.

Lemma 2.15. *Let $Q = \mathbb{Z}/p^k = \langle t\rangle$, where p is a prime number. Then*

$$C_n(G)^Q = \bigoplus_{i=0}^{k} X_i,$$

where we use the notation:

$$X_0 = \mathbb{Z}\left\{[g_1 \mid \cdots \mid g_n] \mid g_j \in G^Q\right\}$$

and for $1 \le i \le k$:

$$X_i = \mathbb{Z}\left\{\sum_{s=0}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n] \mid \ \forall_j \, g_j \in G^{\mathbb{Z}/p^{k-i}} \text{ and } \exists_{i_0 \in \{1,\cdots,n\}} \, g_{i_0} \notin G^{\mathbb{Z}/p^{k-(i-1)}}\right\}.$$

Proof. We first prove that $X_0, \cdots, X_k$ are submodules of $C_n(G)^{\mathbb{Z}/p^k}$. This is clear for $i = 0$, since $C_*(G^Q) \hookrightarrow C_*(G)^Q$ as it was noted in Remark 2.8. Fix $1 \le i \le k$. It is enough to prove that each sum $\sum_{s=0}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n]$ is a fixed point of Q.

Since $\mathbb{Z}/p^k$ is cyclic, we only need to prove that

$$t\sum_{s=0}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n] = \sum_{s=0}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n].$$

We may write the sum as

$$t\sum_{s=0}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n] = \sum_{s=1}^{p^i} t^s[g_1 \mid \cdots \mid g_n] = \sum_{i=1}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n] + t^{p^i}[g_1 \mid \cdots \mid g_n].$$

But $\mathbb{Z}/p^{k-i} = \langle t^{p^i}\rangle$ for each $i = 1, \cdots, k-1$. By hypothesis $g_j \in \ G^{\langle t^{p^i}\rangle}$ for $1 \le j \le n$. Therefore we have

$$\sum_{i=1}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n] + t^{p^i}[g_1 \mid \cdots \mid g_n] = \sum_{i=0}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n],$$

and thus $\sum_{s=0}^{p^i-1} t^s[g_1 \mid \cdots \mid g_n]$ is a fixed point of Q.

Now we prove that each $x \in C_n(G)^{\mathbb{Z}/p^k}$ can be written as $x = \sum_{i=0}^{k} x_i$ where $x_i \in X_i$ for all $i = 0, \cdots, k$.

Let $x \in C_n(G)^{\mathbb{Z}/p^k}$. Then $x = \sum_{r \in R} n_r[g_1 \mid \cdots \mid g_n]_r$ and $t^s x = x$ for each $0 \leq s \leq p^k - 1$. Since p is prime, the orbit $O([g_1 \mid \cdots \mid g_n]_r)$ of any generator has order of the form p^i, where i can take values $i = 0, \cdots, k$. Then x can be expressed in the following way:

$$x = \sum_{i=0}^{k} \sum_{r \in R_i} n_r[g_1 \mid \cdots \mid g_n]_r,$$

where $R_i = \{[g_1 \mid \cdots \mid g_n]_r \mid \ |O([g_1 \mid \cdots \mid g_n]_r)| = p^i\}$, $i = 0, \cdots, k$.

Lemma 2.16. *If $[g_1 \mid \cdots \mid g_n]_r$ has orbit of order p^i, then $g_j \in G^{\mathbb{Z}/p^{k-i}}$ for all $j = 1, \cdots, n$, and there is an $i_0 \in \{1, \cdots, n\}$ such that $g_{i_0} \notin G^{\mathbb{Z}/p^{k-(i-1)}}$.*

Proof. Since

$$|O([g_1 \mid \cdots \mid g_n]_r)| = \frac{|Q|}{|Q_{[g_1|\cdots|g_n]_r}|},$$

where $Q_{[g_1|\cdots|g_n]_r}$ denotes the isotropy group of the element $[g_1 \mid \cdots \mid g_n]_r$, and moreover $|O([g_1 \mid \cdots \mid g_n]_r)| = p^i$ and $|Q| = p^k$, we conclude that $|Q_{[g_1|\cdots|g_n]_r}| = p^{k-i}$. Thus $Q_{[g_1|\cdots|g_n]_r} = \mathbb{Z}/p^{k-i}$, hence $g_j \in G^{\mathbb{Z}/p^{k-i}}$ for all $j = 1, \cdots, n$.

Now if there is no $i_0 \in \{1, \cdots, n\}$ such that $g_{i_0} \notin G^{\mathbb{Z}/p^{k-(i-1)}}$, then $g_j \in G^{\mathbb{Z}/p^{k-(i-1)}}$ for all $1 \leq j \leq n$. So $|Q_{[g_1|\cdots|g_n]_r}| > p^{k-i}$ and we obtain a contradiction. □

On the other hand, let $x_i = \sum_{r \in R_i} n_r[g_1 \mid \cdots \mid g_n]_r$ for each $0 \leq i \leq k$. Then $t^s x_i = x_i$ for $0 \leq s \leq p^k - 1$; so for each $r \in R_i$ there is $j(t^s) \in R_i$ such that $t^s n_{j(t^s)}[g_1 \mid \cdots \mid g_n]_{j(t^s)} = n_r[g_1 \mid \cdots \mid g_n]_r$ for some $0 \leq s \leq p^k - 1$. Then

$$x_i = \sum_{s=0}^{p^k-1} \sum_{i} t^s n_{j_i(t^s)}[g_1 \mid \cdots \mid g_n]_{j_i(t^s)}$$

Since the elements of each n-tuple are in $G^{\langle t^{p^i} \rangle}$, reordering the terms of the sum, we get the following sum:

$$x_i = \sum_{s=0}^{p^i-1} \sum_{l} t^s n_l[g_1 \mid \cdots \mid g_n]_l,$$

with $x_i \in X_i$ for all $i = 0, \cdots, k$.

By construction of the submodules X_i we know that:

$$X_i \cap (X_0 + \cdots + \widehat{X}_i + \cdots + X_k) = \{0\},$$

and thus $C_n(G)^Q = \bigoplus_{i=0}^{k} X_i$. □

Let $Q = \mathbb{Z}/p^k$, where p is a prime. We consider the norm map $\overline{N} : C_n(G)_Q \to C_n(G)^Q$ induced by the map

$$\overline{[g_1 \mid \cdots \mid g_n]} \overset{N}{\longmapsto} \sum_{s=0}^{p^k-1} t^s[g_1 \mid \cdots \mid g_n].$$

Lemma 2.17. *Let $Q = \mathbb{Z}/p^k$, where p is a prime, and let $\overline{N} : C_n(G)_Q \to C_n(G)^Q$ be the norm map. Then* $\operatorname{Im} \overline{N} = \bigoplus_{i=0}^{k} p^{k-i} X_i$.

Proof. If $\langle \{[g_1 \mid \cdots \mid g_n]_r\}_{r \in R} \rangle = C_n(G)$, then given $x \in C_n(G)_Q$ we have that $x = \sum_{r \in R} n_r \overline{[g_1 \mid \cdots \mid g_n]}$. So:

$$\overline{N}(\sum_{r \in R} n_r \overline{[g_1 \mid \cdots \mid g_n]}) = \sum_{r \in R} n_r (\sum_{s=0}^{p^k-1} t^s [g_1 \mid \cdots \mid g_n]_r).$$

Analogously as we did in the proof of Lemma 2.15, we separate the generators by their orbit orders:

$$x = \sum_{i=0}^{k} \sum_{r \in R_i} n_r \sum_{s=0}^{p^k-1} t^s [g_1 \mid \cdots \mid g_n]_r,$$

where $R_i = \{[g_1 \mid \cdots \mid g_n]_r \mid |O([g_1 \mid \cdots \mid g_n]_r)| = p^i\}$, $i = 0, \cdots, k$. We shall prove that $x_i \in p^{k-i} X_i$ for all i, where

$$x_i = \sum_{r \in R_i} n_r \sum_{s=0}^{p^k-1} t^s [g_1 \mid \cdots \mid g_n]_r.$$

For that we need to show that if $[g_1 \mid \cdots \mid g_n]_r$ has an orbit of order p^i, then :

$$\sum_{s=0}^{p^k-1} t^s [g_1 \mid \cdots \mid g_n]_r = p^{k-i} \sum_{s=0}^{p^i-1} t^s [g_1 \mid \cdots \mid g_n]_r.$$

We may write the sum as

$$\sum_{s=0}^{p^k-1} t^s [g_1 \mid \cdots \mid g_n]_r = \sum_{m=1}^{p^{k-i}} \left(\sum_{s=(m-1)p^i}^{mp^i-1} t^s [g_1 \mid \cdots \mid g_n]_r \right) = \sum_{m=1}^{p^{k-i}} \left(\sum_{s=0}^{p^i-1} t^{(m-1)p^i} t^s [g_1 \mid \cdots \mid g_n]_r \right).$$

But we know from Lemma 2.16 that $t^{p^i}[g_1 \mid \cdots \mid g_n]_r = [g_1 \mid \cdots \mid g_n]_r$. Now the claim follows:

$$\sum_{m=1}^{p^{k-i}} \left(\sum_{s=0}^{p^i-1} t^{(m-1)p^i} t^s [g_1 \mid \cdots \mid g_n]_r \right) = \sum_{m=1}^{p^{k-i}} \left(\sum_{s=0}^{p^i-1} t^s [g_1 \mid \cdots \mid g_n]_r \right) = p^{k-i} \sum_{s=0}^{p^i-1} t^s [g_1 \mid \cdots \mid g_n]_r.$$

Therefore $x_i = p^{k-i} \sum_{r \in R_i} \sum_{s=0} t^s n_r [g_1 \mid \cdots \mid g_n]_r$, so by Lemma 2.15 we have that $x_i \in p^{k-i} X_i$ for all i. Thus $\operatorname{Im} \overline{N} = \bigoplus_{i=0}^{k} p^{k-i} X_i$. □

Now we present a generalization of Proposition 2.10.

Corollary 2.18. $\operatorname{Coker} \overline{N} = \bigoplus_{i=0}^{k-1} X_i \otimes_{\mathbb{Z}} \mathbb{Z}/p^{k-i}$.

Proof. By Lemmas 2.15, 2.17 we have that:

$$\operatorname{Coker} \overline{N} = \frac{\bigoplus_{i=0}^{k} X_i}{\bigoplus_{i=0}^{k} p^{k-i} X_i} = \bigoplus_{i=0}^{k-1} X_i / p^{k-i} X_i \cong \bigoplus_{i=0}^{k-1} (X_i \otimes_{\mathbb{Z}} \mathbb{Z}/p^{k-i}).$$

□

Lemma 2.19. *If $Q = \mathbb{Z}/p^k$ and A is a flat G-module, then we obtain the following long exact sequence*

$$\cdots \to H_n(C_\bullet(G)_Q, A) \to H_n^Q(G, A) \to H_n(\operatorname{Coker} \overline{N}, A) \to H_{n-1}(C_\bullet(G)_Q, A) \to \cdots$$

Proof. Since the domain of N is free abelian, the map $\overline{N}$ is injective, and since A is flat, it follows that the following sequence of complexes is exact:

$$0 \to C_\bullet(G)_Q \otimes A \stackrel{\overline{N} \otimes \mathrm{id}}{\longrightarrow} C_\bullet(G)^Q \otimes A \stackrel{\pi \otimes \mathrm{id}}{\longrightarrow} \operatorname{Coker} \overline{N} \otimes A \to 0$$

Then using the connecting homomorphisms [10] we get the desired long exact sequence. □

Definition 2.20. Let $Q = \mathbb{Z}/p^k$. Define for the differential graded Q-module $C_\bullet(G) \otimes A$ a filtration $F^\bullet$ in the following way.

Let $F_s^i = C_i(G)^{Q_{k-s}} \otimes A$, $Q_{k-s} = \mathbb{Z}/p^{k-s}$, $0 \leq s \leq k$; that is, we have a family of subcomplexes

$$F_0^\bullet \subset F_1^\bullet \subset \cdots \subset F_{k-1}^\bullet \subset C_\bullet(G) \otimes A$$

such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
\longrightarrow & C_{n+1}(G)\otimes A & \longrightarrow & C_n(G)\otimes A & \longrightarrow & C_{n-1}(G)\otimes A & \longrightarrow \\
& \uparrow{\scriptstyle j^{n+1}_{k-1}} & & \uparrow{\scriptstyle j^{n}_{k-1}} & & \uparrow{\scriptstyle j^{n-1}_{k-1}} & \\
\longrightarrow & F^{n+1}_{k-1} & \longrightarrow & F^{n}_{k-1} & \longrightarrow & F^{n-1}_{k-1} & \longrightarrow \\
& \uparrow{\scriptstyle j^{n+1}_{k-2}} & & \uparrow{\scriptstyle j^{n}_{k-2}} & & \uparrow{\scriptstyle j^{n-1}_{k-2}} & \\
\longrightarrow & F^{n+1}_{k-2} & \longrightarrow & F^{n}_{k-2} & \longrightarrow & F^{n-1}_{k-2} & \longrightarrow \\
& \uparrow & & \uparrow & & \uparrow &
\end{array}
$$

where the homomorphisms j^i_s are given by inclusion.

Given the filtration $F^\bullet$ of the complex $C_\bullet(G)\otimes A$, the homomorphism j^i_s induces the homomorphisms $(j^i_s)_* : H_i(F^i_s) \to H_i(G,A)$. Since $F^i_{s-1} \subset F^i_s$, we have that $\operatorname{Im}(j^i_{s-1})_* \subset \operatorname{Im}(j^i_s)_*$. So $(j^\bullet)_*$ is a filtration of $H_\bullet(G,A)$.

But

$$F^i_s = C_i(G)^{Q_{k-s}} \otimes A \cong (C_i(G)\otimes A)^{Q_{k-s}}$$

for all $0 \le s \le k$. So $H_\bullet(F^\bullet_s) \cong H^{Q_{k-s}}_\bullet(G,A)$ for all $0 \le s \le k$.

Then, we have a family of subcomplexes

$$(j^\bullet_0)_*\left(H^{Q_k}_\bullet(G,A)\right) \subset (j^\bullet_1)_*\left(H^{Q_{k-1}}_\bullet(G,A)\right) \subset \cdots \subset (j^\bullet_{k-1})_*\left(H^{Q_1}_\bullet(G,A)\right) \subset H_\bullet(G,A)$$

such that the following diagram commutes:

$$
\begin{array}{ccccccc}
\longrightarrow & H_n(G,A) & \longrightarrow & H_{n-1}(G,A) & \longrightarrow \\
& \uparrow & & \uparrow & \\
\longrightarrow & (j^n_{k-1})_*(H^{Q_1}_n(G,A)) & \longrightarrow & (j^{n-1}_{k-1})_*(H^{Q_1}_{n-1}(G,A)) & \longrightarrow \\
& \uparrow & & \uparrow & \\
\longrightarrow & (j^n_{k-2})_*(H^{Q_2}_n(G,A)) & \longrightarrow & (j^{n-1}_{k-2})_*(H^{Q_2}_{n-1}(G,A)) & \longrightarrow \\
& \uparrow & & \uparrow &
\end{array}
$$

That is, $H_\bullet(G, A)$ is a differential graded Q-module with filtration $(j^\bullet)_*$, so by [8, Theorem 3.1] we have the following result.

Theorem 2.21. *There exists a spectral sequence (E^r, d^r) determined by $H_\bullet(G, A)$ that converges to the homology of the complex $H_\bullet(G, A)$ such that*

$$E^1_{s,q} \cong H_{s+q}\left(\frac{\left(j_s^{s+q}\right)_*\left(H_{s+q}^{Q_{k-s}}(G,A)\right)}{\left(j_{s-1}^{s+q}\right)_*\left(H_{s+q}^{Q_{k-s+1}}(G,A)\right)}\right) \quad s = 0, \cdots, k, \quad q \in \mathbb{Z}.$$

Proof. It remains to prove that our spectral sequence converges.

The differential is given by:

$$E^r_{s+r,q-r+1} \to E^r_{s,q} \xrightarrow{d^r} E^r_{s-r,q+r-1},$$

and $0 \le s \le k$, so we have 3 cases:

1. If $s = 0$, then $E^r_{r,q-r+1} \to E^r_{0,q} \xrightarrow{d^r} E^r_{-r,q+r-1}$.
   Since $E^r_{p,q} = 0$ if $p < 0$, we have that $E^r_{-r,q+r-1} = 0$, so
   $$E^{r+1}_{0,q} = \frac{E^r_{0,q}}{\operatorname{Im}\{E^r_{r,q-r+1} \to E^r_{0,q}\}}.$$
   But $E^r_{p,q} = 0$ if $p > k$, so $E^r_{r,q-r+1} = 0$ if $r \ge k+1$, hence $E^{k+1}_{0,q} = E^{k+2}_{0,q} = \cdots = E^\infty_{0,q}$.

2. If $s = k$, then $E^r_{k+r,q-r+1} \to E^r_{k,q} \xrightarrow{d^r} E^r_{k-r,q+r-1}$.
   $E^r_{p,k} = 0$ if $p > k$. Since $k, r > 0$, we have that $E^r_{k+r,q-r+1} = 0$. Hence $E^{r+1}_{k,q} = \operatorname{Ker} d^r$. But $E^r_{p,q} = 0$, if $p < 0$, so $E^r_{k-r,q+r-1} = 0$ if $r \ge k+1$. Thus $E^{k+1}_{k,q} = E^{k+2}_{k,q} = \cdots = E^\infty_{k,q}$.

3. Now we consider the case $0 < s < k$.
   If $r = k$, then $E^k_{s+k,q-k+1} \to E^k_{s,q} \xrightarrow{d^r} E^k_{s-k,q+k-1}$. But since $s < k$, we have that $E^k_{s-k,q+k-1} = 0$. On the other hand $E^r_{s+k,q-r+1} = 0$, thus $E^k_{s,q} = E^{k+1}_{s,q} = \cdots = E^\infty_{s,q}$.

The spectral sequence converges to the complex $H_\bullet(G, A)$ because the filtration is bounded. □

With Proposition 2.14 we have another way to look at Lemma 2.19 and Theorem 2.21:

Theorem 2.22. *If $Q = \mathbb{Z}/p^k$ and A is a flat G-module such that $|Q|$ is invertible in A, then the following sequence is exact.*

$$\cdots \to H_n(C_\bullet(G)_Q, A) \to H_n(G, A)^Q \to H_n(Coker\,\overline{N}, A) \to H_{n-1}(C_\bullet(G)_Q, A) \to \cdots$$

Theorem 2.23. *Let A be a flat G-module, let $Q = \mathbb{Z}/p^k$ be such that $|Q|$ is invertible A. Then there exists a spectral sequence (E^r, d^r) converging to the homology of the complex $H_\bullet(G, A)$, such that*

$$E^1_{s,q} \cong H_{s+q}\left(\frac{H_{s+q}(G,A)^{Q_{k-s}}}{H_{s+q}(G,A)^{Q_{k-s+1}}}\right).$$

Proof. The theorem follows from Proposition 2.14 and Theorem 2.21. □

2.6 Open problems

1. We could consider the situation when A is not a trivial G-module. How shall we then define $H^Q_\bullet(G, A)$ and what can be said about it?
2. Not much is known when Q is not $\mathbb{Z}/p^k$. For some calculations check [6].
3. Any applications of the homology of invariant group chains would be interesting.

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