ANALYSIS AND GEOMETRY ON CONFIGURATION SPACES: THEORY AND APPLICATIONS

INVITED LECTURES IN LUBLIN

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Abstract. Many fundamental phenomena in nature are described by usually nonlinear partial differential equations derived in a rather phenomenological way. Perhaps the most known ones are the Bolzmann equation, the Navier-Stokes equation, the reaction-diffusion equation, Fisher's equation, Vlasov's equation, etc. These equations describe a substance, e.g. gas or liquid, as a whole, without explicit considering its microscopic structure. They are fairly deterministic and operate with such macroscopic notions as pressure, fluid velocity, viscosity, and so on. On the other hand, the systems of large number of inter- acting microscopic agents (particles) are described by infinite chains of linear differential equations. For example, the systems of interacting gas molecules are described by the Bogoliubov hierarchy of linear differential equations, also called BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy. In this case, the motion of the particles and the inter-particle interactions are described explicitly and – due to the huge number of particle – in a probabilistic way. This means that the solutions give the time evolution of the probability distributions on the system's phase space. Since the very appearance of these methods, the problem of deriving the macroscopic description of interacting particle systems from their microscopic statistical mechanical description was considered as a challenging mathematical task. The proposed cycle of lectures presents a number of examples where such derivation can be performed, including also the description on the intermediate (mesoscopic) level. Along with the statistical mechanical models, there will be considered models used in plant ecology, genetics, oceanology, economic and social sciences. In these models, the particles can die, be born, diffuse, perform jumps, etc. The microscopic description is performed in terms of the Markov evolution of states of infinite systems of interacting particles located in the space Rd. Then the mesocopic description, which leads to nonlocal nonlinear differential equations, is obtained by means of a procedure called scaling. In this way, a number of known phenomenological equations are obtained and studied.

Contents

1. Gaussian White Noise Analysis

The main object to study is a Gaussian measure on a real Hilbert space $\mathcal{H}$.

1.1. **Finite dimensional case.** We first consider the simplest case where the underlying space $\mathcal{H}$ is one-dimensional, i.e. we deal with $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mathcal{B}(\mathbb{R})$ being the Borel σ-field of subsets of $\mathbb{R}$. In this case, the measure is

$$d\mu_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx, \qquad \mu_1 \in \mathcal{N}(0,1). \tag{1.1}$$

Its characteristic function is

$$\tilde{\mu}_1(\varphi) := \int_{\mathbb{R}} \exp(i\varphi x) d\mu_1(x) = \exp(-\varphi^2/2). \tag{1.2}$$

Let $\mathcal{M}_1(\mathbb{R})$ be the set of all probability measures on $\mathbb{R}$. Then we have a map $\mathcal{M}_1(\mathbb{R})\mu \mapsto \tilde{\mu} = k : \mathbb{R} \to \mathbb{C}$, the latter functions have the following properties:

(1) k is continuous on $\mathbb{R}$;

(2) $k(0) = 1$;

(3) k is positive definite.

According to the latter property, each k is such that

$$\forall \varphi_1, \ldots, \varphi_n \in \mathbb{R} \qquad \forall \xi_1, \ldots, \xi_n \in \mathbb{C}$$

$$\sum_{j,k=1}^{n} k(\varphi_j - \varphi_k)\xi_j \bar{\xi}_k \geq 0. \tag{1.3}$$

Let us prove (3). Given n, $\varphi_1, \ldots, \varphi_n \in \mathbb{R}$, and $\xi_1, \ldots, \xi_n \in \mathbb{C}$, we set

$$f(x) = \sum_{j=1}^{n} \xi_j e^{i\varphi_j x}, \quad x \in \mathbb{R}.$$

Then

$$|f(x)|^2 = f(x) \cdot \overline{f(x)} = \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k \exp\left[i(\varphi_j - \varphi_k)\right],$$

and hence

$$\sum_{j,k=1}^{n} k(\varphi_j - \varphi_k)\xi_j \bar{\xi}_k = \int_{\mathbb{R}} |f(x)|^2 \, d\mu(x) \geq 0,$$

which is (3). It turns out that the converse is also true.

Theorem 1.1 (Bochner theorem)**.** *A function $k : \mathbb{R} =\!\longrightarrow \mathbb{C}$ is the characteristic function of a measure $\mu \in \mathcal{M}_1(\mathbb{R})$ if and only if it possesses the properties (1), (2), and (3).*

Now let $\mathcal{H}$ be such that $\dim\mathcal{H} = N \in \mathbb{N}$. Then $\mathcal{H} \simeq \mathbb{R}^N$. For $\mu \in \mathcal{M}(\mathcal{H})$, we write

$$\tilde{\mu}(\varphi) = \int_{\mathcal{H}} \exp\left(i\langle \varphi, x\rangle\right) d\mu(x), \quad \varphi \in \mathbb{R}^N, \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{H}$. The Bochner theorem holds true also in this case. The standard Gaussian measure on $\mathcal{H} \simeq \mathbb{R}^N$ now is

$$d\mu_{\mathcal{H}}(x) = \bigotimes_{j=1}^{N} \frac{1}{\sqrt{2\pi}} \exp(-x_j^2/2) dx_j, \tag{1.5}$$

and hence

$$\tilde{\mu}_{\mathcal{H}}(\varphi) = \exp\left(-\langle \varphi, \varphi\rangle/2\right) = \exp\left(-\|\varphi\|_{\mathcal{H}}^2/2\right). \tag{1.6}$$

In the infinite dimensional case, the situation gets more complicated and, in general, neither (1.6) nor Theorem 1.1 holds.

1.2. **Infinite dimensional case.** Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal base of $\mathcal{H}$. Then the map

$$\mathcal{H} \ni x = \sum_{k\geq 1} x_k e_k \mapsto \xi = (x_k)_{k\in\mathbb{N}}$$

establishes the isomorphism

$$\mathcal{H} \simeq \ell^2 = \{\xi = (x_k)_{k\in\mathbb{N}},\ x_k \in \mathbb{R} \mid \sum_{k\geq 1} x_k^2 < \infty\}.$$

We also consider

$$\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots \ni \omega = (\omega_1, \dots, \omega_n, \dots)$$

and

$$\mathbb{R}_0^\infty \subset \mathbb{R}^\infty, \quad \mathbb{R}_0^\infty = \{\varphi = (\varphi_k)_0^\infty \mid \varphi_n = 0,\ n \geq n_0(\varphi)\}.$$

Then

$$\mathbb{R}^\infty \supset \ell^2 \supset \mathbb{R}_0^\infty.$$

The scalar product in ℓ^2 is

$$\langle h, g\rangle = \sum_{k\geq 1} h_k g_k. \tag{1.7}$$

For $\varphi \in \mathbb{R}^\infty_)$, the map $\omega \mapsto \langle \omega, \varphi\rangle$ can be extended to the whole $\mathbb{R}^\infty$ yielding

$$\langle \omega, \varphi\rangle = \sum_{k\geq 1} \omega_k \varphi_k.$$

The space $\mathbb{R}^\infty$ can be equipped with the *cylinder* σ-field $\Sigma(\mathbb{R}^\infty)$, being the smallest σ-field containing the set of *cylinder* sets

$$C = B \times \mathbb{R} \times \cdots \times \mathbb{R} \cdots$$

where B is a Borel subset of $\mathbb{R}^n$. For such cylinder sets, one can set

$$\mu_{\mathcal{H}}(C) = \left(1/\sqrt{2\pi}\right)^n \int_{\mathbb{R}^n} \mathbf{1}_B(x_1, \dots, x_n) \exp\left(-\sum_{j=1}^n x_j^2/2\right) dx_1 \cdots dx_n, \tag{1.8}$$

where $\mathbf{1}_B$ is the indicator function of the set B, i.e.

$$\mathbf{1}_B(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } (x_1, \dots, x_n) \in B; \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

It turns out that such a measure can be defined on $(\mathbb{R}^\infty, \Sigma(\mathbb{R}^\infty))$. For this measure, we, however, would have

$$\mu_{\mathcal{H}}(\mathbb{R}^\infty) = 1, \quad \mu_{\mathcal{H}}(\ell^2) = 0.$$

Definition 1.2. For $p = (p_k)_1^\infty$, $p_k > 0$, the weighted Hilbert space $\ell^2(p)$ is set to be

$$\ell^2(p) = \{\omega \in \mathbb{R}^\infty \mid \sum_{k\geq 1} p_k \omega_k^2 < \infty\}.$$

The scalar product herein is

$$\langle \omega, \tilde{\omega} \rangle = \sum_{k\geq 1} p_k \omega_k \tilde{\omega}_k.$$

The situation with the introduced above measure $\mu_\mathcal{H}$ is described by the following

Theorem 1.3 (Komogorov-Khinchine criterion)**.** *Suppose that all p_k, $k \geq 1$ are such that $p_k \leq 1$. Then*

$$\mathbb{R}^\infty \supset \ell^2(p) \supset \ell^2 \supset \mathbb{R}_0^\infty,$$

and

$$\mu_\mathcal{H}\left(\ell^2(p)\right) = \begin{cases} 0, & \text{if } \sum_{k\geq 1} p_k = +\infty; \\ 1, & \text{if } \sum_{k\geq 1} p_k < +\infty. \end{cases} \tag{1.10}$$

Proof. For $\varepsilon > 0$ and $N \in \mathbb{N}$, we set

$$f_{N,\varepsilon}(\omega) = \exp\left(-\varepsilon \sum_{k=1}^N p_k \omega_k^2\right). \tag{1.11}$$

Then by (1.8) we get

$$\begin{aligned} \int_{\mathbb{R}^\infty} f_{N,\varepsilon}(\omega) d\mu_\mathcal{H}(\omega) &= \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^N} \int_{\mathbb{R}^N} \exp\left(-\varepsilon \sum_{k=1}^N p_k \omega_k^2 - \frac{1}{2}\sum_{k=1}^N \omega_k^2\right) d\omega_1 \cdots d\omega_N \\ &= \prod_{k=1}^N \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \exp\left(-\frac{1}{2}(1+2\varepsilon p_k)\omega_k^2\right) d\omega_k \\ &= \prod_{k=1}^N \frac{1}{\sqrt{1+2\varepsilon p_k}}. \end{aligned} \tag{1.12}$$

For every $\varepsilon > 0$, the sequence $\{f_{N,\varepsilon}\}_{N\in\mathbb{N}}$ is clearly bounded, which by the dominated convergence theorem yields

$$\lim_{N\to\infty} \int_{\mathbb{R}^\infty} f_{N,\varepsilon}(\omega) d\mu_\mathcal{H}(\omega) = \int_{\mathbb{R}^\infty} f_\varepsilon(\omega) d\mu_\mathcal{H}(\omega), \tag{1.13}$$

where

$$f_\varepsilon(\omega) = \begin{cases} \exp\left(-\varepsilon \sum_{k=1}^\infty p_k \omega_k^2\right), & \omega \in \ell^2(p) \\ 0 & \omega \in \mathbb{R}^\infty \setminus \ell^2(p). \end{cases}$$

Now we consider the following possibilities:

(a) $\sum_{k=1}^\infty p_k = +\infty$, hence $\prod_{k=1}^\infty \sqrt{1+2\varepsilon p_k} = +\infty$;

(b) $\sum_{k=1}^\infty p_k < +\infty$, hence $\prod_{k=1}^\infty \sqrt{1+2\varepsilon p_k} < +\infty$.

In case (a), from (1.12) and (1.13) we have

$$\int_{\ell^2(p)} f_\varepsilon(\omega) d\mu_{\mathcal{H}}(\omega) = 0, \tag{1.14}$$

which holds for all $\varepsilon > 0$ and hence yields in the limit $\varepsilon \downarrow 0$

$$\mu_{\mathcal{H}}(\ell^2(p)) = 0$$

Similarly, in case (b) by the dominated convergence theorem we get

$$\begin{aligned} \mu_{\mathcal{H}}(\ell^2(p)) &= \lim_{\varepsilon\downarrow 0} \int_{\ell^2(p)} f_\varepsilon(\omega) d\mu_{\mathcal{H}}(\omega) \\ &= \lim_{\varepsilon\downarrow 0} \prod_{k=1}^\infty \frac{1}{\sqrt{1+2\varepsilon p_k}} = 1, \end{aligned}$$

which completes the proof, see (1.10). □

Every separable Hilbert space $\mathcal{H}$ is isomorphic to ℓ^2, which we shall always have in mind. Thus, we can write

$$\mathbb{R}^\infty \supset \ell^2(p) \supset \ell^2 \supset \mathbb{R}_0^\infty, \tag{1.15}$$

where p is such that $\sum_{k=1}^\infty p_k < +\infty$. One observes that by Theorem 1.3 $\mu_{\mathcal{H}}(\ell^2(p)) = 1$, however, $\mu_{\mathcal{H}}(\ell^2) = 0$. Now let us consider $\varphi \in \mathbb{R}_0^\infty$. Then for any $\omega \in \mathbb{R}^\infty$, we can define $\exp(i\langle\varphi,\omega\rangle) = \exp\left(i\sum_{k\geq 1}\varphi_k\omega_k\right)$ which leads to the following

$$\tilde{\mu}_{\mathcal{H}}(\varphi) = \int_{\mathbb{R}^\infty} \exp(i\langle\varphi,\omega\rangle)\, d\mu_{\mathcal{H}}(\omega). \tag{1.16}$$

Since $\varphi \in \mathbb{R}_0^\infty$, one finds $n \in \mathbb{N}$ such that $\varphi_k = 0$ for all $k > n$; hence, we have in (1.15)

$$\tilde{\mu}_{\mathcal{H}}(\varphi) = \exp(-\langle\varphi,\varphi\rangle/2) = \exp\left(-\|\varphi\|_{\mathcal{H}}^2/2\right). \tag{1.17}$$

Let us prove that, for a given $h \in \ell^2$, the map

$$\ell^2(p) \ni \omega \mapsto \langle\omega, h\rangle \tag{1.18}$$

is a measurable linear functional. For this $h = (h_1, h_2, \dots, h_n, \dots)$, we set $\varphi_n = (h_1, \dots, h_n, 0, \dots,)$. Clearly, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in ℓ^2 to h. Set $l_{\varphi_n}(\omega) = \langle \omega, \varphi_n\rangle$, $\omega \in \ell^2(p)$. For $\lambda \in \mathcal{C}$ and $\varphi \in \mathbb{R}_0^\infty$, we consider

$$f_\varphi(\lambda) = \int_{\ell^2(p)} \exp\left(\lambda\langle\omega,\varphi\rangle\right) d\mu_{\mathcal{H}}(\omega) = \exp\left(\frac{\lambda}{2}\|\varphi\|_{\mathcal{H}}^2\right). \tag{1.19}$$

Then we have

$$\begin{aligned} f'_\varphi(0) &= \int_{\ell^2(p)} \langle\omega,\varphi\rangle d\mu_{\mathcal{H}}(\omega) = 0, \\ f''_\varphi(0) &= \int_{\ell^2(p)} \langle\omega,\varphi\rangle^2 d\mu_{\mathcal{H}}(\omega) = \|\varphi\|_{\mathcal{H}}^2. \end{aligned} \tag{1.20}$$

In the same way, for $\psi \in \mathbb{R}_0^\infty$, we obtain

$$\int_{\ell^2(p)} \langle\omega,\varphi\rangle\langle\omega,\psi\rangle d\mu_{\mathcal{H}}(\omega) = \langle\varphi,\psi\rangle. \tag{1.21}$$

Then by (1.20), for any $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\ell^2(p)} \left|l_{\varphi_n}(\omega) - l_{\varphi_m}(\omega)\right|^2 d\mu_{\mathcal{H}}(\omega) &= \int_{\ell^2(p)} \left|\langle\omega,\varphi_n - \varphi_m\rangle\right|^2 d\mu_{\mathcal{H}}(\omega) \\ &= \|\varphi_n - \varphi_m\|_{\mathcal{H}}^2, \end{aligned}$$

which yields that $\{l_{\varphi_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\ell^2(p), \mu_{\mathcal{H}})$. Therefore, there exists a unique $l \in L^2(\ell^2(p), \mu_{\mathcal{H}})$ such that $l_{\varphi_n} \to l$. This, and the convergence $\varphi_n \to h$, imply that

$$\langle\varphi_n, \omega\rangle \to \langle h, \omega\rangle \qquad \mu_{\mathcal{H}} - \text{a.s.}$$

In a more general setting, we have that, for a separable Hilbert space $\mathcal{H}$, there exists another Hilbert space $\mathcal{H}_-$ such that $\mathcal{H}_- \supset \mathcal{H}$, and a Gaussian measure $\mu_{\mathcal{H}}$ on $\mathcal{H}_-$, with the property

$$\tilde{\mu}_{\mathcal{H}}(\varphi) = \exp\left(-\frac{1}{2}\|\varphi\|_{\mathcal{H}}^2\right), \qquad \mu_{\mathcal{H}}(\mathcal{H}_-) = 1.$$

Furthermore, for every $h \in \mathcal{H}$, the map $\mathcal{H}_- \ni \omega \mapsto \langle\omega, h\rangle_{\mathcal{H}}$ defines a measurable map on $\mathcal{H}_-$. Then this $\mu_{\mathcal{H}}$ is called the *canonical Gaussian measure* corresponding to $\mathcal{H}$.

1.3. **Gaussian White Noise measure.** Let us consider the case where the Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$. By $C_0^\infty(\mathbb{R})$ we denote the space of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are infinitely differentiable and have compact support. The latter means that each φ vanishes outside an interval $[a, b] \subset \mathbb{R}$, specific for this function. Note that $C_0^\infty(\mathbb{R})$ corresponds to $\mathbb{R}_0^\infty$ in the previous subsection. For such φ and for the canonical Gaussian measure corresponding to $\mathcal{H}$, we thus have

$$\tilde{\mu}_{\mathcal{H}}(\varphi) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\varphi(t)|^2 dt\right). \tag{1.22}$$

Let $\{\psi_n\}_{n\in\mathbb{N}_0}$ be a basis of $L^2(\mathbb{R})$. One can take Hermite functions

$$\psi_n(t) = (2^n n!)^{-1/2}(-1)^n \pi^{-1/4} e^{t^2/2} D^n e^{-t^2}, \quad D = \frac{d}{dt}. \tag{1.23}$$

For $\varphi, \phi \in C_0^\infty(\mathbb{R})$ and a sequence $\{p_n\}_{n\in\mathbb{N}_0}$, we set

$$\langle \varphi, \phi \rangle_{\mathcal{H}_-} = \sum_{n=0}^{\infty} p_n \langle \varphi, \psi_n \rangle_{\mathcal{H}} \langle \psi_n, \phi \rangle_{\mathcal{H}}. \tag{1.24}$$

It is clear that the left-hand side of (1.24) is independent of the particular choice of the basis of $\mathcal{H}$. Suppose now that $\sum_{n=0}^\infty p_n < +\infty$. Then the completion of $C_0^\infty(\mathbb{R})$ in the norm $\|\cdot\|_{\mathcal{H}_-}$, defined by (1.24), is a separable Hilbert space, which we denote by $\mathcal{H}_-$. The canonical Gaussian measure on $\mathcal{H}_-$, that is the one for which we have (1.22) is called the *Gaussian White Noise measure*, WNM for short. For this measure, and for $\omega \in \mathcal{H}_-$, we have

$$\int_{\mathcal{H}_-} \omega(t)\omega(s) d\mu_{\mathcal{H}}(\omega) = \delta(t-s), \tag{1.25}$$

where δ is the *Dirac* δ-function. It is a distribution, which is defined as

$$\int_{\mathbb{R}} \varphi(t)\delta(t-s)dt = \varphi(s), \qquad \varphi \in C_0^\infty(\mathbb{R}). \tag{1.26}$$

Indeed, by (1.21), for $\varphi, \phi \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathcal{H}_-} \langle \omega, \varphi \rangle_{\mathcal{H}} \langle \omega, \phi \rangle_{\mathcal{H}} d\mu_{\mathcal{H}}(\omega) = \int_{\mathbb{R}} \varphi(t)\phi(t)dt.$$

By (1.26), the latter can be rewritten

$$\int_{\mathcal{H}_-} \langle \omega, \varphi \rangle_{\mathcal{H}} \langle \omega, \phi \rangle_{\mathcal{H}} d\mu_{\mathcal{H}}(\omega) = \int_{\mathbb{R}}\int_{\mathbb{R}} \delta(t-s)\varphi(t)\phi(s)dtds. \tag{1.27}$$

On the other hand,

$$\int_{\mathcal{H}_-} \langle \omega, \varphi \rangle_{\mathcal{H}} \langle \omega, \phi \rangle_{\mathcal{H}} d\mu_{\mathcal{H}}(\omega)$$
$$= \int_{\mathbb{R}} \varphi(t)\phi(s) \left(\int_{\mathcal{H}_-} \omega(t)\omega(s) d\mu_{\mathcal{H}}(\omega), \right) dtds$$

which together with (1.27) yields (1.25).

As above, for any $h \in \mathcal{H}$, the map $\mathcal{H}_- \ni \omega \mapsto \langle \omega, h \rangle_{\mathcal{H}}$ defines a measurable linear functional (which of course in not continuous). For $t \in [0, +\infty)$, let us consider $h_t = \mathbb{I}_{[0,t)}$, where $\mathbb{I}_{[0,t)}(s) = 1$ if $s \in [0, t)$ and $\mathbb{I}_{[0,t)}(s) = 0$ otherwise. Then the map

$$\mathcal{H}_- \ni \omega \mapsto B_t(\omega) = \langle \omega, h_t \rangle_{\mathcal{H}} = \int_0^t \omega(s) ds \tag{1.28}$$

can be employed to define the map $[0, +\infty) \ni t \mapsto B_t(\omega)$, ω is fixed in $\mathcal{H}_-$. It has the following properties:

(i) $B_t : \mathcal{H}_- \to \mathbb{R}$ is a Gaussian random variable;

(ii) $\mathbb{E} B_t = \int_{\mathcal{H}_-} B_t(\omega) d\mu_{\mathcal{H}}(\omega) = 0$;

(iii) for every $t, s \in [0, +\infty)$, we have that

$$\begin{aligned} \mathbb{E} B_t B_s &= \int_{\mathcal{H}_-} B_t(\omega) B_s(\omega) d\mu_{\mathcal{H}}(\omega) \\ &= \int_{\mathbb{R}} \mathbb{I}_{[0,t)}(\tau) \mathbb{I}_{[0,s)}(\tau) d\tau \\ &= \min\{t; s\} = (t \wedge s). \end{aligned} \tag{1.29}$$

B_t is called the *Wiener process* or the *Brownian motion.*

1.4. **Chaos decomposition for the White Noise measure.** Let us return to the one dimensional case, i.e. now we set again $\mathcal{H} = \mathbb{R}$. In this case,

$$d\mu_{\mathcal{H}}(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\omega^2\right) d\omega \tag{1.30}$$

is a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. For a fixed $\varphi \in \mathbb{R}$, the Gaussian exponential is defined to be

$$\begin{aligned} \mathbb{R} \ni \omega \mapsto e_G(\varphi, \omega) &= \frac{\exp(\varphi\omega)}{\mathbb{E}_{\mu_{\mathcal{H}}} \exp(\varphi\omega)} \\ &= \frac{\exp(\varphi\omega)}{\exp(-\varphi^2/2)} = \exp\left(\varphi\omega - \frac{\varphi^2}{2}\right). \end{aligned} \tag{1.31}$$

How we fix $\omega \in \mathbb{R}$ and expand

$$\exp\left(\varphi\omega - \frac{\varphi^2}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(\omega) \varphi^n. \tag{1.32}$$

It is clear that each H_n is a polynomial such that $\deg H_n = n$. For example,

$$H_1(\omega) = \omega, \quad H_2(\omega) = \omega^2 - 1, \quad H_3(\omega) = \omega^3 - 3\omega.$$

One can show that, for all $n \in \mathbb{N}_0$,

$$H_n(\omega) = \left(n!\sqrt{\pi}\right)^{1/2} \psi_n(\omega/\sqrt{2}) e^{-\omega^2/4}, \tag{1.33}$$

where ψ_n is the same as in (1.23). The latter yields

$$\int_{\mathbb{R}} H_n(\omega) H_m(\omega) d\mu_{\mathcal{H}}(\omega) = n! \delta_{nm}, \tag{1.34}$$

which means that $\{H_n\}_{n \in \mathbb{N}_0}$ is an orthogonal basis of $L^2(\mathbb{R}, \mu_{\mathcal{H}})$. Thus,

$$F = \sum_{n=0}^{\infty} f^{(n)} H_n, \quad F \in L^2(\mathbb{R}, \mu_{\mathcal{H}}), \tag{1.35}$$

and

$$\|F\|_{L^2(\mathbb{R},\mu_{\mathcal{H}})} = \sum_{n=0}^{\infty} n! |f^{(n)}|^2 \tag{1.36}$$

The orthogonality (1.34) can also be proven directly from the definition (1.31). Indeed,

$$\begin{aligned} &\int_{\mathbb{R}} e_G(\phi, \omega) e_G(\psi, \omega) d\mu_{\mathcal{H}}(\omega) \\ &= \int_{\mathbb{R}} \exp\left((\phi + \psi)\omega - \frac{\phi^2}{2} - \frac{\psi^2}{2} \right) d\mu_{\mathcal{H}}(\omega) \\ &= \exp\left(\frac{1}{2}(\phi + \psi)^2 - \frac{\phi^2}{2} - \frac{\psi^2}{2} \right) = \sum_{n=0}^{\infty} \frac{(\phi\psi)^n}{n!} \\ &\sum_{n,m=0}^{\infty} \frac{\phi^n}{n!} \cdot \frac{\psi^m}{m!} \int_{\mathbb{R}} H_n(\omega) H_m(\omega) d\mu_{\mathcal{H}}(\omega), \end{aligned} \tag{1.37}$$

which holds for any $\phi, \psi \in \mathbb{R}$ and hence implies (1.34).

Now let us pass to the infinite dimensional case. Set $\mathcal{H} = L^2(\mathbb{R})$. For $\varphi \in C_0^\infty(\mathbb{R})$, we define

$$\begin{aligned} e_G(\varphi, \omega) &= \frac{\exp(\langle \varphi, \omega \rangle)}{\mathbb{E}_\mu \exp(\langle \varphi, \cdot \rangle)} \\ &= \exp\left(\langle \varphi, \omega \rangle - \frac{1}{2} \|\varphi\|_{\mathcal{H}}^2 \right), \end{aligned} \tag{1.38}$$

which can clearly be extended to all $\varphi, \omega \in \mathcal{H}$. Consider

$$\exp(\langle \varphi, \omega \rangle) = \sum_{n=0}^{\infty} \frac{1}{n!} B_n(\omega, \varphi). \tag{1.39}$$

Here, for a fixed $\omega \in \mathcal{H}$, $B_n(\cdot, \varphi)$ is a n-homogeneous continuous polynomial in $\varphi \in \mathcal{H}$. Indeed,

$$B_n(\omega, \varphi) = \langle \varphi, \omega \rangle^n = \left(\int_{\mathbb{R}} \varphi(t)\omega(t)dt \right)^n$$
$$= \int_{\mathbb{R}^n} \omega(t_1) \cdots \omega(t_n)\varphi(t_1) \cdots \varphi(t_n) dt_1 \cdots dt_n,$$

and by the Schwarz inequality

$$|B_n(\omega, \varphi)| \le \|\varphi\|_{\mathcal{H}}^n \cdot \|\omega\|_{\mathcal{H}}^n. \tag{1.40}$$

Now, for $\omega \in C_0^\infty(\mathbb{R})$, we set, c.f. (1.32),

$$\begin{aligned} e_G(\varphi, \omega) & \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} H_n(\omega)(t_1, \ldots, t_n)\varphi(t_1) \cdots \varphi(t_n) dt_1 \cdots dt_n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \big\langle H_n(\omega), \varphi^{\otimes n} \big\rangle, \end{aligned} \tag{1.41}$$

where as in (1.39) we have that

$$\mathcal{H} \ni \varphi \mapsto \big\langle H_n(\omega), \varphi^{\otimes n} \big\rangle \tag{1.42}$$

is a n-homogeneous continuous polynomial in $\varphi \in \mathcal{H}$. As in the one-dimensional case, H_n are the Hermite polynomials. The map (1.42) can be extended in ω to $\omega \in \mathcal{H}_-$, where the latter space is the same as in (1.24). Clearly, now this map is only measurable, for which we have

$$\begin{aligned} \int_{\mathcal{H}_-} \big\langle H_n(\omega), \phi^{\otimes n} \big\rangle \cdot \big\langle H_n(\omega), \psi^{\otimes n} \big\rangle d\mu_{\mathcal{H}}(\omega) & \\ = n!\delta_{nm} \big\langle \phi^{\otimes n}, \psi^{\otimes m} \big\rangle = n!\delta_{nm} \langle \phi, \psi \rangle^n. & \end{aligned} \tag{1.43}$$

Recall that

$$\big(L^2(\mathbb{R})\big)^{\otimes n} = L^2(\mathbb{R}^n).$$

Let

$$L^2_{\text{sym}}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n), \tag{1.44}$$

be the subspace consisting of the symmetric vectors, i.e. $\varphi^{(n)} \in L^2_{\text{sym}}(\mathbb{R}^n)$ means that

$$\varphi^{(n)}(t_{\sigma(1)}, \dots, t_{\sigma(n)}) = \varphi^{(n)}(t_1, \dots, t_n) \tag{1.45}$$

for all permutations $\sigma \in \Sigma_n$. Thus, for all $\omega \in \mathcal{H}_-$ we can define a measurable map

$$L^2_{\text{sym}}(\mathbb{R}^n) \ni \varphi^{(n)} \mapsto \big\langle H_n(\omega), \varphi^{(n)} \big\rangle, \tag{1.46}$$

such that

$$\begin{aligned} \int_{\mathcal{H}_-} \big\langle H_n(\omega), \phi^{(n)} \big\rangle \cdot \big\langle H_n(\omega), \psi^{(n)} \big\rangle d\mu_{\mathcal{H}}(\omega) \\ = n!\delta_{nm} \big\langle \phi^{(n)}, \psi^{(n)} \big\rangle_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{1.47}$$

For a given $n \in \mathbb{N}$, we call

$$\mathcal{F}_n = L^2_{\text{sym}}(\mathbb{R}^n)$$

the *n-particle space.* We also set $\mathcal{F}_0 = \mathbb{R}$ and consider the space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n. \tag{1.48}$$

This is the *Fock space* over the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. It consists of the vectors

$$\vec{f} = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, \dots), \qquad f^{(n)} \in \mathcal{F}_n.$$

By construction, the space (1.48) is also called *symmetric* or *Bose* Fock space. One also writes

$$\mathcal{F}(L^2(\mathbb{R})) = \text{Exp}(L^2(\mathbb{R})).$$

This construction leads us to the following

Theorem 1.4 (Chaos decomposition)**.** *For every $F \in L^2(\mathcal{H}_-, \mu_{\mathcal{H}})$, $\mathcal{H} = L^2(\mathbb{R})$, there exists a unique $\vec{f} \in \mathcal{F}(L^2(\mathbb{R}))$ such that*

$$F = \sum_{n=0}^{\infty} \big\langle H_n(\cdot), f^{(n)} \big\rangle, \tag{1.49}$$

and

$$\|F\|_{L^2(\mathcal{H}_-, \mu_{\mathcal{H}})} = \|\vec{f}\|_{\mathcal{F}(L^2(\mathbb{R}))}. \tag{1.50}$$

Therefore, the representation (1.49) establishes an isomorphism

$$L^2(\mathcal{H}_-, \mu_{\mathcal{H}}) \simeq \mathcal{F}(L^2(\mathbb{R})). \tag{1.51}$$

One observes that (1.50) means

$$\int_{\mathcal{H}_-} |F(\omega)|^2 d\mu_{\mathcal{H}}(\omega) = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} |f^{(n)}(t_1, \ldots, t_n)| dt_1 \cdots dt_n.$$

1.5. **Differential operators.** For a function $F : \mathcal{H}_- \to \mathbb{R}$, we are going to define the derivative ∇F, and also hight-order derivatives. For $h \in \mathcal{H}$, we set

$$(\nabla_h F)(\omega) = \lim_{t \to 0} [F(\omega + th) - F(\omega)] / t. \tag{1.52}$$

This is the so called the *derivative of F in direction h*, or just the *directional derivative.* If is is linear in h and continuous, i.e.

$$|(\nabla_h F)(\omega)| \leq C(\omega)\|h\|,$$

for some $C(\omega) > 0$, then, by the Riesz lemma, we can define $\nabla F(\omega) \in \mathcal{H}$, which is called the *gradient* of F.

Let us now recall the notion of the Laplace operator on the Euclidean space $\mathbb{R}^d$. For $f \in C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, we have

$$\triangle f(x) = \sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_k^2}(x). \tag{1.53}$$

Note that $L^2(\mathbb{R}^d)$ means the space of square-integrable functions with respect to the Lebesgue measure m defined on the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$. For any $f \in C_0^\infty(\mathbb{R}^d)$, one can define the *energy form*

$$\begin{aligned} \mathcal{E}(f,f) &= -\int_{\mathbb{R}^d} f(x) \triangle f(x) dm(x) \\ &= \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dm(x) \geq 0. \end{aligned} \tag{1.54}$$

The name *energy* comes from the fact that the operator $H = -(\hbar^2/2m)\triangle$ is the energy operator of a free quantum particle. Likewise we define

$$\mathcal{E}(F,F) = \int_{\mathcal{H}_-} \|\nabla F(\omega)\|_{\mathcal{H}}^2 d\mu_{\mathcal{H}}(\omega), \tag{1.55}$$

which is also called the *Dirichlet form.*

Recall that the directional derivative was defined in (1.52). For $h \in \mathcal{H}$, the adjoint directional derivative ∇_h^* is defined as follows. For $f, g \in$

$L^2(\mathcal{H}_-, \mu_{\mathcal{H}})$, we set

$$\int_{\mathcal{H}_-} (\nabla_h f)(\omega) g(\omega) d\mu_{\mathcal{H}}(\omega) = \int_{\mathcal{H}_-} f(\omega)(\nabla_h^* g)(\omega) d\mu_{\mathcal{H}}(\omega). \tag{1.56}$$

Lemma 1.5. *For every $h \in \mathcal{H}$, we have that*

$$\nabla_h^* = -\nabla_h + \langle h, \cdot \rangle. \tag{1.57}$$

Remark 1.6. Note that the map

$$\mathcal{H}_- \ni \omega \mapsto \langle h, \omega \rangle,$$

which appears in (1.57), is measurable. By (1.56), to prove (1.57) we have to show that

$$\int_{\mathcal{H}_-} (\nabla_h f)(\omega) g(\omega) d\mu_{\mathcal{H}}(\omega) = -\int_{\mathcal{H}_-} f(\omega)(\nabla_h g)(\omega) d\mu_{\mathcal{H}}(\omega) + \int_{\mathcal{H}_-} f(\omega) g(\omega) \langle h, \omega \rangle d\mu_{\mathcal{H}}(\omega). \tag{1.58}$$

Proof of Lemma 1.5: Let us show that the shifted measure $d\mu_{\mathcal{H}}(\cdot + h)$, where h is the same as in (1.57), is absolutely continuous with respect to $d\mu_{\mathcal{H}}$. In the one-dimensional case, we have

$$d\mu_{\mathcal{H}}(\omega + h) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\omega + h)^2\right) d\omega,$$

and hence

$$\frac{d\mu_{\mathcal{H}}(\omega + h)}{d\mu_{\mathcal{H}}(\omega)} = \exp\left(-h\omega - \frac{1}{2}h^2\right)$$

Let us show that a similar formula holds also in the considered case. Namely, we show that

$$\frac{d\mu_{\mathcal{H}}(\omega + h)}{d\mu_{\mathcal{H}}(\omega)} = \exp\left(-\langle h, \omega \rangle - \frac{1}{2}\|h\|_{\mathcal{H}}^2\right) := R(h, \omega). \tag{1.59}$$

By (1.16) and (1.17), we immediately see that, for any $h \in \mathcal{H}$,

$$\int_{\mathcal{H}_-} R(h, \omega) d\mu(\omega) = 1,$$

that is

$$d\nu_h(\omega) := R(h,\omega)d\mu_{\mathcal{H}}(\omega) \tag{1.60}$$

is a probability measure on $\mathcal{H}_-$. Thus, to prove (1.59) we have to show that, for all $h \in \mathcal{H}$,

$$d\nu_h(\omega) = d\mu_{\mathcal{H}}(\cdot + h). \tag{1.61}$$

Definition 1.7. For a measurable space $(S, \mathcal{S})$, a family $\mathfrak{F}$ of $\mathcal{S}/\mathcal{B}(\mathbb{R})$-measurable functions $F : S \to \mathbb{R}$ is called a *uniqueness class* if for any two probability measures on $(S, \mathcal{S})$, the equality

$$\int_S F d\mu = \int_S F d\nu,$$

which holds for all $F \in \mathfrak{F}$, implies that $\mu = \nu$.

Suppose that a set $\mathfrak{F}$ of functions $F : \mathcal{H}_- \to \mathbb{R}$ has the properties:

(a) for $F, G \in \mathfrak{F}$, their point-wise product $F \cdot G$ is also in $\mathfrak{F}$;

(b) for any distinct $\omega, \omega' \in \mathcal{H}_-$, there exists $F \in \mathfrak{F}$ such that $F(\omega) \neq F(\omega')$;

(c) $\mathfrak{F}$ contains a constant function.

Then $\mathfrak{F}$ is a uniqueness class. Let us take

$$\mathfrak{F} = \{F(\omega) = \exp\left(\langle \phi, \omega \rangle\right) : \phi \in C_0^\infty(\mathbb{R})\}. \tag{1.62}$$

Clearly, it has all the properties just mentioned. At the same time, direct calculations yield

$$\int_{\mathcal{H}_-} F(\omega) R(h,\omega) d\mu_{\mathcal{H}}(\omega) = \exp\left(-\langle \phi, h \rangle + \frac{1}{2}\|h\|^2_{\mathcal{H}}\right)$$

$$\begin{aligned}\int_{\mathcal{H}_-} F(\omega) d\mu_{\mathcal{H}}(\omega + h) &= \int_{\mathcal{H}_-} F(\omega - h) d\mu_{\mathcal{H}}(\omega) \\ &= \exp\left(-\langle \phi, h \rangle + \frac{1}{2}\|h\|^2_{\mathcal{H}}\right),\end{aligned}$$

which yields (1.61). Now we can prove (1.58). In view of (1.52), we have

$$\int_{\mathcal{H}_-} (\nabla_h f)(\omega) g(\omega) d\mu_{\mathcal{H}}(\omega) =$$

$$= \frac{d}{dt}\left[\int_{\mathcal{H}_-} f(\omega + th) g(\omega) d\mu_{\mathcal{H}}(\omega)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\int_{\mathcal{H}_-} f(\omega) g(\omega - th) d\mu_{\mathcal{H}}(\omega - th)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\int_{\mathcal{H}_-} f(\omega) g(\omega - th) R(-th, \omega) d\mu_{\mathcal{H}}(\omega)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\exp\left(-\frac{t^2}{2}\|h\|_{\mathcal{H}}^2\right) \int_{\mathcal{H}_-} f(\omega) g(\omega - th) \exp\left(t\langle h, \omega\rangle\right) d\mu_{\mathcal{H}}(\omega)\right]_{t=0}$$

$$= \int_{\mathcal{H}_-} f(\omega)\left[-(\nabla_h g)(\omega) + g(\omega)\langle h, \omega\rangle\right] d\mu_{\mathcal{H}}(\omega),$$

which readily yields (1.58). □

One observes that the map $f(\omega) \mapsto (\nabla f)(\omega)$ yields an element of $\mathcal{H} \simeq \mathcal{L}(\mathcal{H} \to \mathbb{R})$. Thus, the second derivative

$$(\nabla\nabla f)(\omega) := f''(\omega)$$

yields an element of $\mathcal{L}(\mathcal{H} \to \mathcal{H})$. Here, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, by $\mathcal{L}(\mathcal{H} \to \mathcal{H}')$ we denote the set of all continuous linear mappings. Then we set

$$(\triangle f)(\omega) := \mathrm{trace} f''(\omega). \tag{1.63}$$

Theorem 1.8. *The Dirichlet form introduced in (1.55) can be written in the form*

$$\mathcal{E}(F, F) = \int_{\mathcal{H}_-} F(\omega) \left(L_{\mathcal{H}} F\right)(\omega) d\mu_{\mathcal{H}}(\omega), \tag{1.64}$$

$$\left(L_{\mathcal{H}} F\right)(\omega) := -(\triangle F)(\omega) + \langle (\nabla F)(\omega), \omega\rangle. \tag{1.65}$$

Note that the operator $L_{\mathcal{H}}$ defined in (1.65) is called the *Ornstein-Uhlenbeck operator*. Having in mind the isomorphism (1.51) and the decomposition (1.48) we can define the action of $L_{\mathcal{H}}$ on the functions $f : \mathcal{F} \to \mathbb{R}$. Simple calculations show that

$$L_{\mathcal{H}}|_{\mathcal{F}_n} = n\mathbb{I}, \tag{1.66}$$

which is the multiplication operator by n.

2. Poisson Measures and Configuration Spaces

Now we develop a version of the theory presented above but based on the Poisson measure rather than on the Gaussian one.

2.1. **The Poisson measure.** First we consider the one-dimensional case. Let $\sigma > 0$ be fixed. It will serve as the intensity parameter of the Poisson measure which we introduce now. Set

$$m_k = \frac{\sigma^k}{k!} e^{-\sigma}, \qquad k \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}. \tag{2.1}$$

Let $\mathcal{M}^1(\mathbb{R})$ stand for the set of all probability measures on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then the Poisson measure $\pi_\sigma \in \mathcal{M}^1(\mathbb{R})$ is an atomic measure with atoms $\{k\}$, $k \in \mathbb{Z}_+$, such that

$$\pi_\sigma(\{k\}) = m_k, \quad k \in \mathbb{Z}_+. \tag{2.2}$$

Equivalently

$$\pi_\sigma = \sum_{k=0}^{\infty} m_k \delta_k, \tag{2.3}$$

where δ_x is the atomic measure which has a single atom at $x \in \mathbb{R}$ with mass one. That is, for $B \in \mathcal{B}(\mathbb{R})$,

$$\delta_x(B) = \mathbf{1}_B(x) = \begin{cases} 1, & x \in B, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The measure (2.3) can be used to describe a population of random size living in a zero-dimensional space. Its Fourier transform is

$$\begin{aligned} \tilde{\pi}_\sigma(\varphi) &= \int_{\mathbb{R}} \exp(i\varphi x)\, \pi_\sigma(dx) = \sum_{k=0}^{\infty} \exp(i\varphi k - \sigma) \frac{\sigma^k}{k!} \\ &= e^{-\sigma} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sigma e^{-\varphi}\right)^k = \exp\left(\sigma\left(e^{i\varphi} - 1\right)\right). \end{aligned} \tag{2.5}$$

Suppose now that the population which we describe lives in a locally compact space X, which, for simplicity, we shall always assume to be $\mathbb{R}^d$ with some $d \geq 2$. It can also be any Riemannian manifold.

Thus, we consider a measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and let $\mathcal{M}(\mathbb{R}^d)$ stand for the set of all measures thereon. The space X is equipped with the intensity measure $\sigma \in \mathcal{M}(\mathbb{R}^d)$, which is supposed to be non-atomic (e.i. $\sigma(\{x\}) = 0$ for all $x \in X$, and infinite but σ-finite, i.e.

$$\sigma(X) = +\infty, \quad X = \bigcup_{n=1}^{\infty} X_n, \quad \sigma(X_n) < \infty \text{ for all } n \in \mathbb{N}.$$

In particular, we assume that $\sigma(K) < \infty$ for any compact $K \subset X$. It is also natural to assume that σ has a density, ρ, which is locally integrable with respect to Lebesgue's measure. The simplest case is where ρ is constant. It this case, we write $\sigma(dx) = z dx$. Now we can introduce the Poisson measure by its Fourier transform

$$\tilde{\pi}_\sigma(\varphi) = \exp\left(\int_X \left(e^{i\varphi(x)} - 1\right) \sigma(dx)\right), \quad \varphi \in C_0^\infty(X). \tag{2.6}$$

The measure π_σ itself lives on a space, which we introduce right below.

2.2. **Configuration spaces.** Let X be as above. The space of locally finite configurations on X is

$$\Gamma(X) = \{\gamma \subset X : \forall \text{ compact } K \subset X \ \ |\gamma \cap K| < \infty\}. \tag{2.7}$$

Here $|\cdot|$ stands for cardinality. From this definition it follows immediately that γ cannot have coinciding points as well as accumulation points. Thus,

$$\Gamma(\mathbb{R}^d) \neq (\mathbb{R}^d)^\infty.$$

For $\Lambda \subset X$, we denote

$$\gamma_\Lambda = \gamma \cap \Lambda, \qquad \Gamma(\Lambda) = \{\gamma \in \Gamma(X) : \gamma \subset \Lambda\}.$$

Now we set a topology on the space $\Gamma(X)$. To this end we associate it with a subset of $\mathcal{M}(X)$ by means of the representation

$$\gamma = \sum_{x \in \gamma} \delta_x, \tag{2.8}$$

where δ_x is the Dirac measure as in (2.4). Then for a compact $K \subset X$,

$$\gamma(K) = \left(\sum_{x \in \gamma} \delta_x\right)(K) = \sum_{x \in \gamma} \delta_x(K) = |\gamma \cap K| < \infty.$$

Since we shall use the notion of the induced topology in the sequel, we discuss it now in more detail. Let Y be any nonempty set and $\Xi = \{\xi_i\}_{i \in I}$ be a family of maps $\xi_i : Y \to \mathbb{R}$; the index set I is arbitrary. The topology on Y *induced by the family* Ξ is the weakest topology in which all the maps $\xi_i : Y \to \mathbb{R}$ are continuous. This is exactly the family

$$\bigcup_{i\in I}\left\{\xi_i^{-1}(A) : A \text{ open subset of } \mathbb{R}\right\}.$$

Let $C_0(X)$ be the set of all continuous functions $f : X \to \mathbb{R}$ which have compact support (i.e. vanish outside compact sets). For $f \in C_0(X)$, we set

$$\langle f, \gamma\rangle = \int_X f(x)\gamma(dx) = \sum_{x\in\gamma} f(x). \tag{2.9}$$

Note that the latter sum is finite since it runs over $\gamma \cap \mathrm{supp} f$. Hence, each $f \in C_0(X)$ defines the map

$$\Gamma(X) \ni \gamma \mapsto \langle f, \gamma\rangle \in \mathbb{R}.$$

Definition 2.1. The vague topology on $\Gamma(X)$ is the topology induced thereon by the family

$$\Xi = \{\langle f, \cdot\rangle : f \in C_0(X)\}.$$

A net $\{\gamma_\alpha\}_{\alpha\in I}$ converges in this topology to a certain γ if

$$\forall\ f \in C_0(X) \quad \int_X f d\gamma_\alpha \to \int_X f d\gamma.$$

A very important fact about the vague topology is that it can be metrized, and the corresponding metric space will be complete and separable. Such spaces are called *Polish spaces*. The vague topology naturally induces the measurability on on $\Gamma(X)$. Recall that by $\mathcal{B}(X)$ we denote the σ-algebra of Borel subsets of $X = \mathbb{R}^d$. Let $\mathcal{B}_c(X) \subset \mathcal{B}(X)$ be the set of Borel subsets with compact closure. Each such a set is bounded in X. For $n \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(X)$, we set

$$\Gamma^{(n)}(\Lambda) = \{\gamma \in \Gamma(\Lambda) : |\gamma| = n\}, \quad \Gamma^{(0)}(\Lambda) = \{\emptyset\}. \tag{2.10}$$

Further, for $\Lambda \in \mathcal{B}_c(X)$, by Λ^n we denote the corresponding cartesian product consisting of tuples $(x_1, \dots, x_n)$, $x_i \in \Lambda$. Let also $\widetilde{\Lambda^n}$ be the off-diagonal part, i.e,

$$\widetilde{\Lambda^n} = \{(x_1, \dots, x_n) \in \Lambda^n : x_i \neq x_j \ \text{ for } i \neq j\}.$$

We say that two elements of $\widetilde{\Lambda^n}$ are equivalent if they coincide up to a permutation of their numbers, that is, $(x'_1, \dots, x'_n) \sim (x_1, \dots, x_n)$ if $(x'_1, \dots, x'_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for some permutation $\sigma \in \Sigma_n$. Then the factor set $\widetilde{\Lambda^n}/\Sigma_n$ can be identified with $\Gamma^{(n)}(\Lambda)$

$$\widetilde{\Lambda^n}/\Sigma_n \simeq \Gamma^{(n)}(\Lambda) \tag{2.11}$$

by the relation

$$\Gamma^{(n)}(\Lambda) \ni \gamma = \{x_1, \dots, x_n\} \simeq \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in \Sigma_n\} \in \widetilde{\Lambda^n}/\Sigma_n,$$

which naturally induces a metric on $\Gamma^{(n)}(\Lambda)$ by means of the metric of X restricted to Λ. Thereafter, we can present $\Gamma(\Lambda)$ as the disjoint union of $\Gamma^{(n)}(\Lambda)$, $n \in \mathbb{Z}_+$, i.e.,

$$\Gamma(\Lambda) = \coprod_{n=0}^{\infty} \Gamma^{(n)}(\Lambda), \tag{2.12}$$

and equip $\Gamma(\Lambda)$ with the topology of the disjoint union. This means that the open subsets of $\Gamma(\Lambda)$ are exactly the disjoint unions of open subsets of $\Gamma^{(n)}(\Lambda)$. The latter topology naturally defines the Borel σ-algebra of subsets of $\Gamma(\Lambda)$, which we denote by $\mathcal{B}(\Gamma(\Lambda))$. Now we take $\Lambda_1 \subset \Lambda_2 \in \mathcal{B}_c(X)$. Then the map

$$\Gamma(\Lambda_2) \ni \gamma \mapsto p_{\Lambda_2\Lambda_1}(\gamma) = \gamma_{\Lambda_1} \in \Gamma(\Lambda_1)$$

is the projection of $\Gamma(\Lambda_2)$ onto $\Gamma(\Lambda_1)$. One can also define

$$\Gamma(X) \ni \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda \in \Gamma(\Lambda), \quad \Lambda \in \mathcal{B}_c(X). \tag{2.13}$$

By $\mathcal{B}(\Gamma(X))$ we denote the σ-field of subsets of $\Gamma(X)$ induced by the family

$$\Xi = \{p_\Lambda : \Lambda \in \mathcal{B}_c(X)\}. \tag{2.14}$$

This defines $\Gamma(X)$ as the projective limit of $\Gamma(\Lambda)$, i.e.,

$$\Gamma(X) = \mathrm{prlim}_\Lambda \Gamma(\Lambda). \tag{2.15}$$

Our next aim is to define probability measures on $\Gamma(X)$ as projective limits of measures on $\Gamma(\Lambda)$, $\Lambda \in \mathcal{B}_c(X)$. We are going to do this by means of the celebrated Kolmogorov extension theorem. Note that the latter is usually employed to define a stochastic process by means of its finite-dimensional distributions.

In the sequel, by $\mathcal{M}(\Gamma(X))$ (respectively, $\mathcal{M}^1(\Gamma(X))$) we denote the set of all (respectively, all probability) measures on the measurable space $(\Gamma(X), \mathcal{B}(\Gamma(X)))$. For $\mu \in \mathcal{M}^1(\Gamma(X))$ and $\Lambda \in \mathcal{B}_c(X)$, we set

$$\mu^\Lambda = \mu \circ p_\Lambda^{-1} := p_\Lambda^* \mu \in \mathcal{M}^1(\Gamma(\Lambda)), \tag{2.16}$$

where p_Λ is the same as in (2.13). Then μ^Λ is called the *projection* of μ onto Λ. Let now $\Lambda_1 \subset \Lambda_2 \in \mathcal{B}_c(X)$. Then we readily have that

$$\mu^{\Lambda_1} = \mu^{\Lambda_2} \circ p_{\Lambda_2\Lambda_1}^{-1}. \tag{2.17}$$

The property (2.17) is called the *consistency* of the family $\{\mu^\Lambda : \Lambda \in \mathcal{B}_c(X)\}$. It comes from the consistency of the projections p_Λ and $p_{\Lambda_2\Lambda_1}$, which can be illustrated

$$\Gamma(X) \xrightarrow{p_{\Lambda_2}} \Gamma(\Lambda_2) \xrightarrow{p_{\Lambda_2\Lambda_1}} \Gamma(\Lambda_1)$$

$$\Gamma(X) \xrightarrow{p_{\Lambda_1}} \Gamma(\Lambda_1)$$

The consistency (2.17) can also be expressed in terms of integrals. Let $F : \Gamma(\Lambda) \to \mathbb{R}$ be bounded and measurable. For such F and $\mu \in \mathcal{M}^1(\Gamma(X))$, we have

$$\begin{aligned} \int_{\Gamma(\Lambda)} F(\eta) d\mu^\Lambda(\eta) &= \int_{\Gamma(X)} F(p_\Lambda \gamma) d\mu(\gamma) \\ &= \int_{\Gamma(\Lambda)} F(\eta) d\left(p_\Lambda^* \mu\right)(\eta) \\ &= \int_{\Gamma(\Lambda)} F(\eta) d\mu(p_\Lambda^{-1}\eta). \end{aligned} \tag{2.18}$$

Theorem 2.2 (Kolmogorov extension theorem)**.** *Suppose that there exists a consistent family of probability measures $\{\mu^\Lambda : \Lambda \in \mathcal{B}_c(X)\}$. Then there exists a unique probability measure $\mu \in \mathcal{M}^1(\Gamma(X))$ such that, for every $\Lambda \in \mathcal{B}_c(X)$, $\mu^\Lambda = p_\Lambda^* \mu$.*

2.3. **Lebesgue-Poisson and Poisson measures on configuration spaces.** An element $\sigma \in \mathcal{M}(X)$ is called a *Radon measure* if it is finite on compact subsets of X. As an element of our theory is the Radon measure σ prescribed to the manifold X, such that $\sigma(X) = +\infty$. Throughout these lectures we assume that X is the Euclidean space $\mathbb{R}^d$, $d \geq 2$, equipped with the measure

$$\sigma(dx) = \rho(x) m(dx), \tag{2.19}$$

where m is Lebesgue's measure on X. The density $\rho \geq 0$ is supposed to be locally integrable

$$\int_K \rho(x) m(dx) < \infty \qquad \text{for any compact } K \subset X.$$

The simplest choice is

$$\rho(x) \equiv z > 0. \tag{2.20}$$

In the latter case, z is called it activity.

Now we introduce the space of finite configurations in X. Namely, for $n \in \mathbb{N}$, we set

$$\Gamma^{(n)}(X) = \{\gamma \in \Gamma(X) : |\gamma| = n\}, \quad \Gamma^{(0)}(X) := \{\emptyset\}. \tag{2.21}$$

Then the space of finite configurations in X is

$$\Gamma_0(X) = \coprod_{n=0}^{\infty} \Gamma^{(n)}(X). \tag{2.22}$$

Note that $\Gamma_0(X)$ is a proper subset of $\Gamma(X)$. For $n \in \mathbb{N}$, let

$$\sigma^{\otimes n} = \sigma \times \cdots \times \sigma$$

be the product measure on the cartesian product $X^n = X \times \cdots \times X$. As above, we set

$$\widetilde{X^n} = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}.$$

It can readily be show that

$$\sigma^{\otimes n}\left(X^n \setminus \widetilde{X^n}\right) = 0.$$

As in (2.11) we identify the factor $\widetilde{X^n}/\Sigma_n$ with $\Gamma^{(n)}(X)$. Let

$$\mathrm{sym}_X^n(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\} \in \Gamma^{(n)}(X). \tag{2.23}$$

Then this map acts

$$\mathrm{sym}_X^n : \widetilde{X^n} \longrightarrow \Gamma^{(n)}(X).$$

Now we set

$$\sigma^{(n)} = \sigma^{\otimes n} \circ (\mathrm{sym}_X^n)^{-1}, \tag{2.24}$$

and also

$$\Gamma^{(0)}(X) = \{\emptyset\}, \qquad \sigma^{(0)}(\emptyset) = 1. \tag{2.25}$$

For a function $G : \Gamma^{(n)}(X) \to \mathbb{R}$, one finds a symmetric function $G^{(n)} : X^n \to \mathbb{R}$ such that

$$G(\{x_1, \ldots, x_n\}) = G^{(n)}(x_1, \ldots, x_n). \tag{2.26}$$

For appropriate such functions, we then have

$$\begin{aligned} &\int_{\Gamma^{(n)}(X)} G(\{x_1, \ldots, x_n\}) d\sigma^{(n)} \\ &\qquad = \int_{X^n} G^{(n)}(x_1, \ldots, x_n) d\sigma(x_1) \cdots d\sigma(x_n). \end{aligned} \tag{2.27}$$

The Lebesgue-Poisson measure λ_σ on $\Gamma_0(X)$ is set to be

$$\lambda_\sigma = \sum_{n=0}^{\infty} \frac{1}{n!}\sigma^{(n)}. \tag{2.28}$$

We now fix $\Lambda \in \mathcal{B}_c(X)$ and consider the restriction of σ to $(\Lambda, \mathcal{B}(\Lambda))$, which we denote by σ_Λ. By means of the latter measure we define $\sigma_\Lambda^{(n)}$ and thereby

$$\lambda_\sigma^\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!}\sigma_\Lambda^{(n)}. \tag{2.29}$$

The latter is a measure on

$$\Gamma(\Lambda) = \Gamma_0(\Lambda) := \coprod_{n=0}^{\infty} \Gamma^{(n)}(\Lambda). \tag{2.30}$$

In particular, we have that

$$\begin{aligned} \lambda_\sigma^\Lambda\left(\Gamma(\Lambda)\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\sigma_\Lambda^{(n)}\left(\Gamma(\Lambda)\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\sigma_\Lambda\left(\Lambda\right)\right]^n = \exp\left(\sigma(\Lambda)\right). \end{aligned} \tag{2.31}$$

Thereafter, we can introduce, c.f. (2.1), (2.2),

$$\pi_\sigma^\Lambda = e^{-\sigma(\Lambda)}\lambda_\sigma^\Lambda, \tag{2.32}$$

which is a probability measure on $\Gamma(\Lambda)$. Since we have such measures for every $\Lambda \in \mathcal{B}_c(X)$, we can check whether the family $\{\pi_\sigma^\Lambda : \Lambda \in \mathcal{B}_c(X)\}$ is consistent. For $\Lambda \in \mathcal{B}_c(X)$, consider

$$\Lambda = \Lambda_1 \cup \Lambda_2, \qquad \Lambda_1 \cap \Lambda_2 = \emptyset.$$

Then each γ_Λ can be decomposed $\gamma_\Lambda = \gamma_{\Lambda_1} \cup \gamma_{\Lambda_2}$, which also defines the decomposition $\Gamma(\Lambda) = \Gamma(\Lambda_1) \times \Gamma(\Lambda_2)$. It can be checked that

$$\lambda_\sigma^\Lambda = \lambda_\sigma^{\Lambda_1} \times \lambda_\sigma^{\Lambda_2}.$$

The latter decomposition yields

$$\begin{aligned} \pi_\sigma^\Lambda &= \exp\left[-\sigma(\Lambda_1 \cup \Lambda_2)\right]\lambda_\sigma^{\Lambda_1} \times \lambda_\sigma^{\Lambda_2} \\ &= \left(e^{-\sigma(\Lambda_1)}\lambda_\sigma^{\Lambda_1}\right) \times \left(e^{-\sigma(\Lambda_2)}\lambda_\sigma^{\Lambda_2}\right) = \pi_\sigma^{\Lambda_1} \times \pi_\sigma^{\Lambda_2}. \end{aligned} \tag{2.33}$$

The consistency in question comes from the latter decomposition in the following way. For $\Lambda_1 \subset \Lambda$ and an appropriate function $F : \Gamma(\Lambda_1)$, we have

$$\begin{aligned}\int_{\Gamma(\Lambda)} F(\gamma_{\Lambda_1}) d\pi_\sigma^\Lambda(\gamma_\Lambda) &= \int_{\Gamma(\Lambda_1)\times\Gamma(\Lambda_2)} F(\gamma_{\Lambda_1}) d\pi_\sigma^{\Lambda_1}(\gamma_{\Lambda_1}) \cdot d\pi_\sigma^{\Lambda_2}(\gamma_{\Lambda_2}) \\ &= \int_{\Gamma(\Lambda_1)} F(\gamma_{\Lambda_1}) d\pi_\sigma^{\Lambda_1}(\gamma_{\Lambda_1}).\end{aligned}$$

Hence, we have the consistency

$$p^*_{\Lambda\Lambda_1}\pi_\sigma^\Lambda = \pi_\sigma^{\Lambda_1}, \tag{2.34}$$

which holds for any $\Lambda_1 \subset \Lambda \in \mathcal{B}_c(X)$. Here $p^*_{\Lambda\Lambda_1}$ is the same as in (2.17). In view of (2.34), we can apply to the family $\{\pi_\sigma^\Lambda : \Lambda \in \mathcal{B}_c(X)\}$ Theorem 2.2 and obtain that there exists a unique probability measure on $(\Gamma(X), \mathcal{B}(\Gamma(X)))$ consistent with this family, which we denote by π_σ and call the *Poisson measure* with intensity measure σ.

Now we introduce the Laplace transform of the Poisson measure π_σ. For $f \in C_0(X)$, we set

$$\hat{\pi}_\sigma(f) = \int_{\Gamma(X)} \exp\left(\langle f, \gamma\rangle\right) d\pi_\sigma(\gamma), \tag{2.35}$$

where the pairing $\langle f, \gamma\rangle$ was defined in (2.9). Since f has compact support, one finds $\Lambda \in \mathcal{B}_c(X)$ such that f vanishes outside Λ. Having this in mind we obtain in (2.35)

$$\begin{aligned}\hat{\pi}_\sigma(f) &= \int_{\Gamma(X)} \exp\left(\langle f, \gamma_\Lambda\rangle\right) d\pi_\sigma(\gamma) = \int_{\Gamma(\Lambda)} \exp\left(\langle f, \eta\rangle\right) d\pi_\sigma^\Lambda(\eta) \\ &= e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \exp\left(\sum_{k=1}^n f(x_k)\right) d\sigma(x_1)\cdot d\sigma(x_n) \\ &= e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \left(\int_\Lambda e^{f(x)} d\sigma(x)\right)^n \\ &= \exp\left(-\sigma(\Lambda) + \int_\Lambda e^{f(x)} d\sigma(x)\right) \\ &= \exp\left(\int_\Lambda \left[e^{f(x)} - 1\right] d\sigma(x)\right).\end{aligned}$$

This representation can be extended to all measurable $f : X \to \mathbb{R}$ such that $e^f - 1 \in L^1(X, \sigma)$. For such functions we thus have

$$\hat{\pi}_\sigma(f) = \exp\left(\int_X \left[e^{f(x)} - 1\right] d\sigma(x)\right). \tag{2.36}$$

Suppose now that the manifold X has been decomposed into disjoint union

$$X = X_1 \sqcup X_2. \tag{2.37}$$

This implies the decomposition of any $\gamma \in \Gamma(X)$ into the pair $(\gamma_{X_1}, \gamma_{X_2})$ and hence the decomposition

$$\Gamma(X) = \Gamma(X_1) \times \Gamma(X_2). \tag{2.38}$$

Proposition 2.3. *The decompositions (2.37), (2.38) yield the following one*

$$\pi_\sigma^X = \pi_\sigma^{X_1} \times \pi_\sigma^{X_2}, \tag{2.39}$$

where π_σ^X is the Poisson measure for the manifold X - the same as in (2.35).

Proof. By (2.36), the decomposition (2.37) leads to

$$\begin{aligned} \hat{\pi}_\sigma^X(f) &= \exp\left(\int_{X_1} \left[e^{f(x)} - 1\right] d\sigma(x)\right) \cdot \exp\left(\int_{X_1} \left[e^{f(x)} - 1\right] d\sigma(x)\right) \\ &= \hat{\pi}_\sigma^{X_1}(f) \cdot \hat{\pi}_\sigma^{X_2}(f), \end{aligned}$$

which readily yields (2.39). □

The result just obtained can be generalized to any finite decomposition

$$X = X_1 \sqcup X_2 \sqcup X_3 \cdots \sqcup X_n,$$

which yields the decomposition

$$\pi_\sigma^X = \pi_\sigma^{X_1} \times \pi_\sigma^{X_2} \times \cdots \times \pi_\sigma^{X_n}, \tag{2.40}$$

that is, the measure π_σ^X is *infinitely divisible.*

There exists one more fact about the Poisson measure π_σ. For $B \in \mathcal{B}_c(X)$, we define the random variable N_B on $(\Gamma(X), \mathcal{B}(\Gamma(X)))$ by setting

$$\Gamma(X) \ni \gamma \mapsto N_B(\gamma) = |\gamma_B|, \tag{2.41}$$

that is, N_B is a counting measure.

Proposition 2.4. *For any $B_1, \ldots, B_n \in \mathcal{B}_c(X)$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$, the corresponding random variables are jointly independent Poisson variables with intensities $\sigma(B_i)$, $i = 1, \ldots, n$.*

Proof. First one observes that $N_B(\gamma) = \langle \mathbf{1}_B, \gamma\rangle$. Thus, by (2.36) we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sum_{i=1}^n \lambda_i N_{B_i}\right)\right] &= \int_{\Gamma(X)} \exp\left(\sum_{i=1}^n \lambda_i \langle \mathbf{1}_{B_i}, \gamma\rangle\right) d\pi_\sigma(\gamma) \\
&= \int_{\Gamma(X)} \exp\left(\left\langle \sum_{i=1}^n \lambda_i \mathbf{1}_{B_i}, \gamma\right\rangle\right) d\pi_\sigma(\gamma) \\
&= \exp\left\{\int_X \left(\exp\left(\sum_{i=1}^n \lambda_i \mathbf{1}_{B_i}(x)\right) - 1\right) d\sigma(x)\right\} \\
&= \exp\left(\sum_{i=1}^n \left(e^{\lambda_i} - 1\right)\sigma(B_i)\right) \\
&= \prod_{i=1}^n \exp\left[\left(e_i^\lambda - 1\right)\sigma(B_i)\right],
\end{aligned}$$

which readily yields the proof. □

The result just proven can be summarized in the following

Definition 2.5. A measure $\mu \in \mathcal{M}^1(\Gamma(X))$ is called a Poisson random field on X with intensity $\sigma \in \mathcal{M}(\Gamma(X))$ if for every family $B_1, \dots, B_n \in \mathcal{B}_c(X)$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$, the family $N_{B_1}, \dots, N_{B_n}$ is an independent family of Poisson random variables with intensities $\sigma(B_i)$, $i = 1, \dots, n$.

Remark 2.6. Probability measures on configuration spaces appear in various fields of science. In probability theory, elements of $\mathcal{M}^1(\Gamma(X))$ are called point processes or point random fields, e.g. the Poisson random field introduced above. In mathematical physics, people study giant systems of particles, such as classical gases or fluids in $\mathbb{R}^d$, $d = 2, 3$. Then γ is interpreted as a microscopic state of the gas, whereas $\mu \in \mathcal{M}^1(\Gamma(X))$ is its macroscopic state. In particular, π_σ with $\sigma(dx) = zm(dx)$, $z > 0$, is the macroscopic state of a homogeneous free gas with density z. Recently, elements of $\mathcal{M}^1(\Gamma(X))$ found applications in biology, ecology, sociology, to describe behavior of large complex systems (population, society, etc).

Let us now discuss the possibility to take into account possible multiplicity of the points in configurations. For such configurations, the representation (2.8) gets the form

$$\eta = \sum_{x\in\gamma} n_x \delta_x, \qquad n_x \in \mathbb{N}. \tag{2.42}$$

Herein the configuration itself is η, whereas $\gamma \in \Gamma(X)$ is its support. The set of all configurations (2.42) is denoted by $\ddot{\Gamma}(X)$. Clearly, $\Gamma(X) \subset \ddot{\Gamma}(X)$ and for all $B \in \mathcal{B}_c(X)$, we have that

$$\eta(B) = \sum_{x\in\gamma} n_x.$$

The set $\ddot{\Gamma}(X)$ can also be considered as a subset of $\mathcal{M}(X)$ consisting of all integer-valued Radon measures.

Another possible extension of the description presented above is to consider the so called *marked configurations*. Now the configuration is the set of pairs (x, m_x), where the mark m_x takes values in the space of marks M. As above, the set of positions x form the support of the configuration. Elements of $\ddot{\Gamma}(X)$ can be an example of such configurations. Another example is the space of configurations of a gas particles, each of which is characterized by position $x \in \mathbb{R}^3$ and velocity $v_x \in \mathbb{R}^3$.

Finally, let us turn to the so called *Mecke characterization* of the Poisson measure.

Proposition 2.7. *The Poisson measure π_σ obeys the Mecke formula*

$$\int_{\Gamma(X)} \sum_{x\in\gamma} F(x,\gamma)\,d\pi_\sigma(\gamma) = \int_X \int_{\Gamma(X)} F(x, \gamma\cup x)\,d\sigma(x)\,d\pi_\sigma(\gamma), \tag{2.43}$$

which holds for any integrable function $F : X \times \Gamma(X) \to \mathbb{R}$. And vice versa, any probability measure on $\Gamma(X)$ which satisfies the Mecke formula is the Poisson measure with intensity measure σ.

Proof. It is enough to prove (2.43) for the function

$$F(x,\gamma) = \varphi(x)\exp(\langle\psi,\gamma\rangle), \qquad \varphi,\psi \in C_0(X),$$

for which we have

$$\begin{aligned}(2.44)\quad &\int_{\Gamma(X)} \sum_{x\in\gamma} F(x,\gamma)d\pi_\sigma(\gamma)\\ &= \int_{\Gamma(X)} \sum_{x\in\gamma} \varphi(x)\exp(\langle\psi,\gamma\rangle)d\pi_\sigma(\gamma)\\ &= \int_{\Gamma(X)} \langle\varphi,\gamma\rangle \exp(\langle\psi,\gamma\rangle)d\pi_\sigma(\gamma)\\ &= \frac{d}{dt}\left[\int_{\Gamma(X)} \exp\left(\langle\psi+t\varphi,\gamma\rangle\right)d\pi_\sigma(\gamma)\right]_{t=0}\\ &= \frac{d}{dt}\left[\exp\left(\int_X \left[e^{\psi(x)+t\varphi(x)}-1\right]d\sigma(x)\right)\right]_{t=0}\\ &= \exp\left(\int_X \left[e^{\psi(x)}-1\right]d\sigma(x)\right)\cdot\int_X \varphi(x)e^{\psi(x)}d\sigma(x).\end{aligned}$$

At the same time, the right-hand side of (2.43) is

$$\begin{aligned}&\int_X\int_{\Gamma(X)} \varphi(x)\exp\left(\langle\psi,\gamma\cup x\rangle\right)d\sigma(x)d\pi_\sigma(\gamma)\\ &\qquad = \left(\int_X \varphi(x)e^{\psi(x)}d\sigma(x)\right)\cdot\exp\left(\int_X\left[e^{\psi(x)}-1\right]d\sigma(x)\right),\end{aligned}$$

which coincides with the last line in (2.44) and hence yields its proof.

□

2.4. **The diffeomorphism group and the Poisson measure.** In the sequel, for a measurable space, say, $(S,\mathcal{S})$, by $L^0(S,\mathcal{S})$ we denote the set of all $\mathcal{S}/\mathcal{B}(\mathbb{R})$-measurable functions $F:S\to\mathbb{R}$. Let F be in $L^0(\Gamma(X),\mathcal{B}(\Gamma(X)))$ and be a measurable map $T:\Gamma(X)\to\Gamma(X)$. Then $F\circ T$ is also in $L^0(\Gamma(X),\mathcal{B}(\Gamma(X)))$. For such T we define the adjoint map $T^*:\mathcal{M}^1(\Gamma(X)\to\mathcal{M}^1(\Gamma(X)$ in the following way.

$$(2.45)\qquad \int_{\Gamma(X)} F(\gamma)d(T^*\mu)(\gamma) = \int_{\Gamma(X)}(F\circ T)(\gamma)d\mu(\gamma),$$

holding for all $F\in L^0(\Gamma(X),\mathcal{B}(\Gamma(X)))$. Since $L^0(\Gamma(X),\mathcal{B}(\Gamma(X)))$ is a uniqueness class, see Definition 1.7, the measure $T^*\mu$, and hence T^*, are well-defined.

Definition 2.8. A map $T : \Gamma(X) \to \Gamma(X)$ is said to be admissible for a given $\mu \in \mathcal{M}^1(\Gamma(X)$ if $T^*\mu$ is absolutely continuous with respect to μ. That is, there exists a Radon-Nikodym derivative

$$R(T,\gamma) = \frac{d(T^*\mu)}{d\mu}(\gamma). \tag{2.46}$$

Recall that, in the case of Gaussian measures, we considered the shift transformation $\mathcal{H}_- \ni \omega \mapsto T_h\omega := \omega + h$, $h \in \mathcal{H}$. This map has its adjoint, which is admissible for $\mu_{\mathcal{H}}$, see (1.59). The infinitesimal transformation corresponding to the shift is the gradient. Our next aim is to develop the corresponding objects also for Poisson measures on configuration spaces.

A bijection $\Phi : X \to X$ is called a *diffeomorphism* if both Φ and Φ^{-1} are continuously differentiable. That is both $\nabla\Phi$ and $\nabla\Phi^{-1}$ are continuous maps from X to the space of all linear operators $A : X \to X$. It is clear that all diffeomorphisms $\Phi : X \to X$ constitute a group under convolution, which we denote by $\mathrm{Diff}(X)$. A subgroup of this group constitute the (local) diffeomorphisms which have compact support. By definition, each such Φ acts as the identity map on $X \setminus K$ for some compact K, specific for this Φ. By $\mathrm{Diff}_0(X)$ we denote the group of all local diffeomorphisms.

Consider $\gamma \in \Gamma(X)$ and $\Phi \in \mathrm{Diff}_0(X)$. Then we set

$$\Phi(\gamma) = \{\Phi(x) : x \in \gamma\}. \tag{2.47}$$

Since Φ is a monomorphism, $\Phi(x_1) \neq \Phi(x_2)$ for two distinct $x_1, x_2 \in \gamma$. Therefore $\Phi(\gamma) \in \Gamma(X)$ and hence Φ can be defined as a map $\Phi : \Gamma(X) \to \Gamma(X)$. Then by (1.11) we define also Φ^*. Of course, we can define the adjoint map to Φ as to a map $\Phi : X \to X$. In both cases we use the same notation as it is always clear from the context which one is meant.

Theorem 2.9. *For every* $\Phi \in \mathrm{Diff}_0(X)$*, it follows that*

$$\Phi^*\pi_\sigma = \pi_{\Phi^*\sigma}. \tag{2.48}$$

Proof. Since the family of functions $\{\langle f, \cdot\rangle : f \in C_0(X)\}$ is a uniqueness class, see Definition 1.7, it is enough to prove that the Laplace transforms of both measures in (2.48) coincide. Thus, we have

$$
\begin{aligned}
\int_{\Gamma(X)} \exp\left(\langle f,\gamma\rangle\right) d(\Phi^*\pi_\sigma)(\gamma) &= \int_{\Gamma(X)} \exp\left(\langle f,\Phi(\gamma)\rangle\right) d\pi_\sigma(\gamma)\\
&= \int_{\Gamma(X)} \exp\left(\sum_{x\in\gamma} f\left(\Phi(x)\right)\right) d\pi_\sigma(\gamma)\\
&= \int_{\Gamma(X)} \exp\left(\left\langle f\circ\Phi,\gamma\right\rangle\right) d\pi_\sigma(\gamma)\\
&= \exp\left(\int_X \left[\exp(f\circ\Phi(x))-1\right] d\sigma(x)\right)\\
&= \exp\left(\int_X \left[\exp(f(x))-1\right] d(\Phi^*\sigma)(x)\right)\\
&= \int_{\Gamma(X)} \exp\left(\langle f,\gamma\rangle\right) d\pi_{\Phi^*\sigma}(\gamma),
\end{aligned}
$$

which completes the proof. □

Let us now consider the action of Φ^* on the measure σ, see Definition 2.8. By definition,

$$
\begin{aligned}
\frac{d\left(\Phi^*\sigma\right)(x)}{d\sigma(x)} &= \frac{\rho(\Phi^{-1}(x))}{\rho(x)}\cdot\frac{dm(\Phi^{-1}(x)}{dm(x)}\\
&= \frac{\rho(\Phi^{-1}(x))}{\rho(x)}\cdot J_m(\Phi)(x)\\
&:= p^\sigma_\Phi(x).
\end{aligned}
\tag{2.49}
$$

Here

$$J_m(\Phi)(x) := \det\left[\nabla\Phi^{-1}(x)\right]$$

is the Jacobian of the diffeomorphism Φ.

Theorem 2.10 (Skorohod theorem)**.** *For the Poisson measure π_σ, every $\Phi\in\mathrm{Diff}_0(X)$ is admissible and*

$$
\begin{aligned}
\frac{d\left(\Phi^*\pi_\sigma\right)}{d\pi_\sigma}(\gamma) &= \prod_{x\in\gamma} p^\sigma_\Phi(x)\exp\left(\int_X \left[1-p^\sigma_\Phi(x)\right] d\sigma(x)\right)\\
&:= R(\Phi,\gamma).
\end{aligned}
\tag{2.50}
$$

Proof. For any $f \in C_0(X)$, we have

$$
\begin{aligned}
&\int_{\Gamma(X)} \exp\left(\langle f,\gamma\rangle\right) R(\Phi,\gamma) d\pi_\sigma(\gamma) \\
&\quad = \exp\left(\int_X \left[1 - p_\Phi^\sigma(x)\right] d\sigma(x)\right) \\
&\quad \times \int_{\Gamma(X)} \prod_{x\in\gamma} p_\Phi^\sigma(x) \exp\left(\langle f,\gamma\rangle\right) d\pi_\sigma(\gamma) \\
&\quad = \exp\left(\int_X \left[1 - p_\Phi^\sigma(x)\right] d\sigma(x)\right) \\
&\quad \times \int_{\Gamma(X)} \exp\left(\left\langle f + \ln p_\Phi^\sigma, \gamma\right\rangle\right) d\pi_\sigma(\gamma) \\
&\quad = \exp\left(\int_X \left[1 - p_\Phi^\sigma(x)\right] d\sigma(x)\right) \\
&\quad \times \exp\left(\int_X \left[e^{f(x)} p_\Phi^\sigma(x) - 1\right] d\sigma(x)\right) \\
&\quad = \exp\left(\int_X \left[e^{f(x)} - 1\right] p_\Phi^\sigma(x) d\sigma(x)\right) \\
&\quad = \exp\left(\int_X \left[e^{f(x)} - 1\right] d(\Phi^*\sigma)(x)\right) \\
&\quad = \int_{\Gamma(X)} \exp\left(\langle f,\gamma\rangle\right) d\pi_{\Phi^*\sigma}(\gamma),
\end{aligned}
$$

which completes the proof as the family of functions $\{\langle f, \cdot\rangle : f \in C_0(X)\}$ is a uniqueness class. □

2.5. **Differential geometry of configuration spaces.** Usually, differentiation is defined in linear spaces. For nonlinear metric spaces, e.g. Riemannian manifolds, the notion of the derivative is introduced by means of certain auxiliary objects. We are going to follow this way in the case of configuration spaces.

To get started let us consider first the linear case where the space is $X = \mathbb{R}^d$. Take $v : X \to X$, such that $\|v\| = 1$. We call is a *vector field.*

For an appropriate $f : X \to \mathbb{R}$, the derivative at x in direction $v(x)$ is

$$(\nabla_v f)(x) = \frac{d}{dt}\left[f(x + tv(x))\right]_{t=0}.$$

If this derivative is linear in v, it can be written in the form

$$(\nabla_v f)(x) = \langle \nabla f(x), v(x)\rangle_{T_xX} \tag{2.51}$$

where the scalar product $\langle \cdot, \cdot\rangle_{T_xX}$ is taken in the space *tangent* to X at point x. In the linear case, the latter is just the copy of the space X itself, that is, $T_xX \simeq X = \mathbb{R}$ for all $x \in X$. Then

$$TX = \bigcup_{x\in X} T_xX \tag{2.52}$$

is said to be the *tangent bundle*. Note that the gradient $\nabla f(x)$, if exists, is in T_xX. Suppose now that the vector field v is itself infinitely differentiable in X. The set of all such vector fields will be denoted by $\mathrm{Vec}(X)$; it is called the C^∞*-section* of the tangent bundle. Since both vectors in (2.51) are in the same space, we can write

$$(\nabla_v f)(x) = \sum_{i=1}^{d} v^i(x)\frac{\partial f}{\partial x^i}(x), \quad \mathbb{R}^d \ni x = (x^1, \ldots, x^d). \tag{2.53}$$

We let $\mathrm{Vec}_0(X) \subset \mathrm{Vec}(X)$ consist of v with compact support, i.e. $v(x) = 0$ for $x \in X \setminus K$ for some compact K. For $v \in \mathrm{Vec}_0(X)$, let us consider the Cauchy problem

$$\begin{cases} \frac{du_t}{dt} = v(u_t), & u : \mathbb{R} \to \mathbb{R}^d \\ u_0 = x, & x \text{ is fixed in } \mathbb{R}^d. \end{cases} \tag{2.54}$$

The solution of (2.54) is called a *flow*. It defines the map

$$\mathbb{R}^d \ni x \mapsto u_t := \Phi_t^v(x).$$

It is clear that $\Phi_t^v \circ \Phi_s^v = \Phi_{t+s}^v$ and $\Phi_0^v(x) = x$. Hence, $\{\Phi_t^v : t \in \mathbb{R}\}$ is a one-parameter group. Since $v \in \mathrm{Vec}_0(X)$, each Φ_t^v is in $\mathrm{Diff}_0(X)$. Thus, we can define

$$(\nabla_v f)(x) = \frac{d}{dt}\left[f\left(\Phi_t^v(x)\right)\right]_{t=0}, \tag{2.55}$$

which is called the *Lie derivative* of f along v.

Having done this job on X we can transport the notions just developed to the space of configurations, as it was done in the previous subsection. This procedure is called *lifting*. For $v \in \mathrm{Vec}_0(X)$, by means of $\Phi_t^v \in \mathrm{Diff}_0(X)$ we define

$$\Phi_t^v(\gamma) = \{\Phi_t^v(x) : x \in \gamma\}, \quad \gamma \in \Gamma(X). \tag{2.56}$$

As in (2.48), we have that $\Phi_t^v(\gamma)$ is in $\Gamma(X)$ and hence the latter Φ_t^v maps $\Gamma(X)$ into itself.

Definition 2.11. For a function $F : \Gamma(X) \to \mathbb{R}$, the Γ-derivative at $\gamma \in \Gamma(X)$ along $v \in \mathrm{Vec}_0(X)$ is the Lie derivative

$$\left(\nabla_v^\Gamma F\right)(\gamma) = \frac{d}{dt}\left[F\left(\Phi_t^v(\gamma)\right)\right]_{t=0}. \tag{2.57}$$

Note that in (2.57) the vector field v is "constant", i.e. is independent of γ. An important example of such a derivative is provided by the choice

$$F(\gamma) = \langle f, \gamma\rangle, \qquad f \in C_0^\infty(X).$$

In this case, we have

$$\begin{aligned}\left(\nabla_v^\Gamma F\right)(\gamma) &= \frac{d}{dt}\left[\langle f, \Phi_t^v(\gamma)\rangle\right]_{t=0} = \frac{d}{dt}\left[\sum_{x\in\gamma} f\left(\Phi_t^v(x)\right)\right]_{t=0} \\ &= \sum_{x\in\gamma}\left(\nabla_v f\right)(x) = \langle \nabla_v f, \gamma\rangle.\end{aligned} \tag{2.58}$$

Given $N \in \mathbb{N}$, by $\mathcal{D}$ we denote the set of all infinitely differentiable functions $g : \mathbb{R}^N \to \mathbb{R}$ with compact support, i.e., $\mathcal{D} = C_0^\infty(\mathbb{R}^N)$. *Smooth cylinder functions* on $\Gamma(X)$ are those $F : \Gamma(X) \to \mathbb{R}$ which have the representation

$$\begin{aligned}&F(\gamma) = g(\langle \phi_1, \gamma\rangle, \ldots, \langle \phi_N, \gamma\rangle), \\ &g \in \mathcal{D}, \quad \phi_1, \ldots, \phi_N \in C_0^\infty(X).\end{aligned} \tag{2.59}$$

The set of such F will be denoted by $\mathcal{F}C_0^\infty(\mathcal{D}, \Gamma(X))$. A generalization of (2.58) is given in the following

Proposition 2.12. *For every $F \in \mathcal{F}C_0^\infty(\mathcal{D}, \Gamma(X))$, we have that*

$$\left(\nabla_v^\Gamma F\right)(\gamma) = \sum_{j=1}^N \frac{\partial g}{\partial s_j}\left(\langle \nabla_v \phi_1, \gamma\rangle, \ldots, \langle \nabla_v \phi_N, \gamma\rangle\right). \tag{2.60}$$

Proof. First we observe that, for any $\phi \in C_0^\infty(X)$,

$$\langle \phi, \Phi_t^v(\gamma)\rangle = \sum_{x\in\gamma} \phi\left(\Phi_t^v(x)\right) = \sum_{x\in\gamma} \phi \circ \Phi_t^v\,(x). \tag{2.61}$$

Hence,

$$F\left(\Phi_t^v(\gamma)\right) = g(\langle \phi_1 \circ \Phi_t^v, \gamma\rangle, \ldots, \langle \phi_N \circ \Phi_t^v, \gamma\rangle),$$

which yields together with (2.58) that

$$\frac{d}{dt}\left[F\left(\Phi_t^v(\gamma)\right)\right]_{t=0} = \sum_{j=1}^{N} \frac{\partial g}{\partial s_j}\left(\langle \nabla_v \phi_1, \gamma\rangle, \ldots, \langle \nabla_v \phi_N, \gamma\rangle\right),$$

and thus completes the proof. □

Let us now address the question what could be the tangent space to $\Gamma(X)$ at a given γ. In other words, in which linear space $T_\gamma\Gamma(X)$ we can put the gradient $\nabla^\Gamma F$, c.f (2.51)?

Definition 2.13. The tangent space $T_\gamma\Gamma(X)$ is defined as the Hilbert space of vector fields $V_\gamma : X \to TX$ with the scalar product

$$\begin{aligned} \langle V_\gamma, W_\gamma\rangle_{T_\gamma\Gamma(X)} &= \int_X \langle V_\gamma(x), W_\gamma(x)\rangle_{T_xX}\gamma(dx) \\ &= \sum_{x\in\gamma} \langle V_\gamma(x), W_\gamma(x)\rangle_{T_xX}. \end{aligned} \tag{2.62}$$

One observes that in the above definition T_xX is a copy of $X = \mathbb{R}^d$ for every x. Thus, we have that

$$T_\gamma\Gamma(X) = L^2(X \to TX, \gamma), \tag{2.63}$$

and the corresponding tangent bundle is

$$T\Gamma(X) = \bigcup_{\gamma\in\Gamma(X)} T_\gamma\Gamma(X). \tag{2.64}$$

Each $v \in \mathrm{Vec}_0(X)$ defines the 'constant'[1] vector field V_γ^v on $\Gamma(X)$ by the relation

$$\langle V_\gamma^v, V_\gamma^v\rangle_{T_\gamma\Gamma(X)} = \int_X \langle v(x), v(x)\rangle_{T_xX}\gamma(dx). \tag{2.65}$$

Definition 2.14. For $F : \Gamma(X) \to \mathbb{R}$, the Γ-gradient is defined as the map

$$\Gamma(X) \ni \gamma \mapsto \left(\nabla^\Gamma F\right)(\gamma) \in T_\gamma\Gamma(X) = L^2(X \to TX, \gamma)$$

such that for $v \in \mathrm{Vec} + 0(X)$,

$$\left(\nabla_v^\Gamma F\right)(\gamma) = \langle \left(\nabla^\Gamma F\right)(\gamma), v\rangle_{T\gamma\Gamma(X)}. \tag{2.66}$$

[1] which is independent of γ.

2.5.1. *Integration by parts and divergence.* First we consider some motivating example. Take any $\phi, \psi \in C_0^\infty(X)$. For such ϕ, we have

$$(\nabla_v \phi)(x) = \sum_{i=1}^{d} v^i(x) \frac{\partial \phi}{\partial x^i}(x). \tag{2.67}$$

The integration-by-parts formula (with respect to Lebesgue's measure m) is

$$\begin{aligned} \int_{\mathbb{R}^d} (\nabla_v \phi)(x)\psi(x) dm(x) &= -\int_{\mathbb{R}^d} \phi(x)(\nabla_v \psi)(x) dm(x) \\ &- \int_{\mathbb{R}^d} \phi(x)\psi(x) \mathrm{div} v(x) dm(x), \end{aligned} \tag{2.68}$$

which is known as Stoke's formula. For short, we call it $(\mathrm{IbP})_m$. Here

$$\mathrm{div} v(x) = \sum_{i=1}^{d} \frac{\partial v^i}{\partial x^i}(x) \tag{2.69}$$

is the *divergence* of the vector field v at point x. At the same time, (2.68) can be considered as the definition of the adjoint gradient in $L^2(\mathbb{R}^d, m)$. That is, if we set

$$\int_{\mathbb{R}^d} (\nabla_v \phi)(x)\psi(x) dm(x) = \int_{\mathbb{R}^d} \phi(x)(\nabla_v^* \psi)(x) dm(x), \tag{2.70}$$

then

$$\nabla_v^* = -\nabla_v - \mathrm{div} v(\cdot), \tag{2.71}$$

where the latter is the multiplication operator. Clearly, for $\sigma = \rho m$ with constant density ρ, the $(\mathrm{IbP})_\sigma$ has the form (2.68). What can be said with this regard if ρ is nonconstant? Suppose that $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ and that ρ has continuous gradient. Then we can define

$$\beta^\sigma(x) = \frac{\nabla \rho(x)}{\rho(x)} \in \mathbb{R}^d \simeq T_x X. \tag{2.72}$$

By analogy, β^σ is called the logarithmic derivative of σ. For $v \in \mathrm{Vec}_0(X)$, we define

$$\beta_v^\sigma(x) = \langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div} v(x). \tag{2.73}$$

Theorem 2.15. *Let $\sigma(dx) = \rho(x)m(dx)$ be such that ρ is everywhere positive and has continuous gradient. Then the integration-by-parts formula* $(\mathrm{IbP})_\sigma$ *has the following form: for any $\phi, \psi \in C_0^\infty(X)$,*

$$\int_X \nabla_v \phi \cdot \psi d\sigma = -\int_X \phi \cdot \nabla_v \psi d\sigma - \int_X \phi \cdot \psi \cdot \beta_v^\sigma d\sigma. \tag{2.74}$$

Proof. In view of (2.67), by the usual integration-by-parts formula we have

$$\begin{aligned}\int_X \nabla_v \phi \cdot \psi d\sigma &= \sum_{i=1}^d \int_X \frac{\partial \phi(x))}{\partial x^i} \left[v^i(x)\psi(x)\rho(x) \right] dm(x) \\ &= - \int_X \phi(x) \left(\sum_{i=1}^d \frac{\partial v^i(x)}{\partial x^i} \right) \psi(x)\rho(x) dm(x) \\ &\quad - \int_X \phi(x) \left(\sum_{i=1}^d v^i(x) \frac{\partial \psi(x)}{\partial x^i} \right) \rho(x) dm(x) \\ &\quad - \int_X \phi(x)\psi(x) \frac{1}{\rho(x)} \left(\sum_{i=1}^d v^i(x) \frac{\partial \rho(x)}{\partial x^i} \right) \rho(x) dm(x) \\ &= \mathrm{RHS}(2.74).\end{aligned}$$

□

Definition 2.16. Given $v \in \mathrm{Vec}_0(X)$, the logarithmic derivative of the Poisson measure π_σ along this v is defined to be the map

$$\Gamma(X) \ni \gamma \mapsto B_v^{\pi_\sigma}(\gamma) \tag{2.75}$$

$$\begin{aligned}B_v^{\pi_\sigma}(\gamma) &:= \int_X \left[\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div} v(x) \right] d\gamma(x) \\ &= \sum_{x \in \gamma} \left[\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div} v(x) \right]\end{aligned}$$

Theorem 2.17 ($(\mathrm{IbP})_{\pi_\sigma}$)**.** *For every $F, G \in \mathcal{F}C_0^\infty(\mathcal{D}, \Gamma(X))$ and for all $v \in \mathrm{Vec}_0(X)$, it follows that*

$$\begin{aligned}\int_{\Gamma(X)} \nabla_v^\Gamma F \cdot G d\pi_\sigma &= - \int_{\Gamma(X)} F \cdot \nabla_v^\Gamma G d\pi_\sigma \\ &\quad - \int_{\Gamma(X)} F \cdot G \cdot B_v^{\pi_\sigma} d\pi_\sigma,\end{aligned} \tag{2.76}$$

thet is, in $L^2(\Gamma(X), \pi_\sigma)$, the adjoint gradient takes the form, c.f. (2.71)

$$\left(\nabla_v^\Gamma \right)^* = -\nabla_v^\Gamma - B_v^{\pi_\sigma}(\cdot). \tag{2.77}$$

Proof. In view of (2.57) and (2.56), we have

$$\int_{\Gamma(X)} F\left(\Phi_t^v(\gamma)\right) G(\gamma) d\pi_\sigma(\gamma) \tag{2.78}$$
$$= \int_{\Gamma(X)} F(\gamma) G\left(\Phi_{-t}^v(\gamma)\right) d\pi_{(\Phi^v)_t^*\sigma}(\gamma)$$
$$= \int_{\Gamma(X)} F(\gamma) G\left(\Phi_{-t}^v(\gamma)\right) R\left(\Phi_t^v, \gamma\right) d\pi_\sigma(\gamma),$$

see (2.48) and (2.50). Now we take the t-derivative of both sides of (2.78) at $t = 0$. Then the left-hand side turns into the left-hand side of (2.76). Furthermore,

$$\frac{d}{dt}\left[G\left(\Phi_{-t}^v(\gamma)\right)\right]_{t=0} = -\nabla_v^\Gamma G(\gamma), \tag{2.79}$$

which being plugged into (2.78) gives the first term on the right-hand side of (2.76). The second term can be obtained analogously by means of (2.48) - (2.50). □

2.6. **Representations of the Lie algebra** $\mathrm{Vec}_0(X)$**.** .

There exists a one-to-one correspondence between $\mathrm{Vec}_0(X)$ and $\mathrm{Diff}_0(X)$ established by the Cauchy problem

$$\begin{cases} \frac{d}{dt}\Phi_t^v(x) = v\left(\Phi_t^v(x)\right), \\ \Phi_0^v(x) = x. \end{cases}$$

Suppose that we have a real Hilbert space $\mathcal{H}$, and let $\mathfrak{U}(\mathcal{H})$ be the group of all unitary operators $V : \mathcal{H} \to \mathcal{H}$. For a group G, let the map $G \ni g \mapsto V_g \in \mathfrak{U}(\mathcal{H})$ be such that, for all $g_1, g_2 \in G$,

$$g_1 \cdot g_2 \mapsto V_{g_1} \cdot V_{g_2}. \tag{2.80}$$

Then the image of G in $\mathfrak{U}(\mathcal{H})$ is called the *unitary representation* of G.

Let $\mathcal{H} = L^2(\Gamma(X), \pi_\sigma)$. For $\Phi \in \mathrm{Diff}_0(X)$ and $F \in L^2(\Gamma(X), \pi_\sigma)$, we set

$$\left(V(\Phi)F\right)(\gamma) = F\left(\Phi(\gamma)\right)\sqrt{\frac{d\pi_\sigma(\Phi(\gamma))}{d\pi_\sigma(\gamma)}}. \tag{2.81}$$

One can verify that

$$V(\Phi_1) \cdot V(\Phi_2) F = V(\Phi_1 \cdot \Phi_1) F. \tag{2.82}$$

On the other hand,

$$
\begin{aligned}
\langle V(\Phi)F, V(\Phi)G\rangle_{\mathcal{H}} &= \int_{\Gamma(X)} F(\Phi(\gamma))G(\Phi(\gamma))\frac{d\pi_\sigma(\Phi(\gamma))}{d\pi_\sigma(\gamma)}d\pi_\sigma(\gamma) \\
&= \int_{\Gamma(X)} F(\Phi(\gamma))G(\Phi(\gamma))d\pi_\sigma(\Phi(\gamma)) \\
&= \int_{\Gamma(X)} F(\gamma)G(\gamma)d\pi_\sigma(\gamma) \\
&= \langle F, G\rangle_{\mathcal{H}},
\end{aligned}
$$

that is, each $V(\Phi)$, $\phi \in \mathrm{Diff}_0(X)$, is a unitary operator in $\mathcal{H}$. In view of (2.82), the image of $\mathrm{Diff}_0(X)$ in $\mathfrak{U}(\mathcal{H})$ is a unitary representation of the former group. Following this way, we obtain the unitary representation of $\mathrm{Vec}_0(X)$ in $\mathfrak{U}(\mathcal{H})$ defined by the map

$$
\mathrm{Vec}_0(X) \ni v \mapsto \Phi_t^v \mapsto V(\Phi_t^v) \in \mathfrak{U}(\mathcal{H}). \tag{2.83}
$$

By the Stone theorem, each $V(\Phi_t^v)$ has the form

$$
V(\Phi_t^v) = \exp\left(itJ_v\right), \tag{2.84}
$$

where J_v is a self-adjoint operator in $\mathcal{H}$. It turn out that

$$
J_v = \frac{1}{i}\nabla_v^\Gamma + \frac{1}{2i}B_v^{\pi_\sigma}(\cdot), \quad i = \sqrt{-1}, \tag{2.85}
$$

c.f. (2.77).

2.7. **Brownian motion on configuration spaces.** Here we present an analytic approach to the Markov dynamics.

Let X be a topological space (e.g. $X = \mathbb{R}^d$). BY $C_0^\infty(X)$ we denote the space of all infinitely differentiable functions $f : X \to \mathbb{R}$ with compact support. Let L be a linear operator $f \mapsto Lf$ with domain $\mathcal{D}(L) \subset C_0^\infty(X)$. For example,

$$
(Lf)(x) = \sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_k^2}(x) \in C_0^\infty(X). \tag{2.86}
$$

Consider the Cauchy problem

$$
\begin{cases} \frac{\partial u_t}{\partial t}(x) = (Lu_t)(x), \\ u_0(x) = \varphi(x), \end{cases} \tag{2.87}
$$

where $t \geq 0$ and $\varphi \in C_{\mathrm{b}}(X)$ - the (Banach) space of all bounded continuous functions $f : X \to \mathbb{R}$. Under certain conditions imposed on

L the solution of (2.87) can be presented in the form

$$u_t = T_t\varphi, \tag{2.88}$$

such that $T_0 = I$, $T_tT_s = T_{t+s}$, I being the identity operator $Iu = u$ for all u. By the latter conditions, $\{T_t\}_{t\geq 0}$ is a semigroup of linear operators on the space $C_{\rm b}(X)$.

Definition 2.18. We say that $\{T_t\}_{t\geq 0}$ is a Markov semigroup on $C_{\rm b}(X)$ if the following conditions are satisfied:

(1) for every $\varphi \in C_{\rm b}(X)$ such that $f \geq 0$, and all $t \geq 0$, it follows that $u_t = T_t\varphi \geq 0$ (positivity preservation);
(2) for all $t \geq 0$, it follows that $T1 = 1$, where 1 is the constant function taking value 1 (conservativity).

Example (heat equation): $L = \Delta$

$$\begin{cases} \frac{\partial u_t}{\partial t}(x) = (\Delta u)(x) = \sum_{k=1}^d \frac{\partial^2 u_t}{\partial x_k^2}(x), \\ u_0(x) = \varphi(x), \end{cases} \tag{2.89}$$

For $t > 0$, its solution has the form, $t > 0$,

$$u_t(x) = (T_t\varphi)(x) = \int_{\mathbb{R}^d} \varphi(y)P_t(xdy), \tag{2.90}$$

$$P_t(xdy) = \left(2\sqrt{\pi t}\right)^{-d} \exp\left(-\frac{|x-y|^2}{4t}\right) dy.$$

One can easily verify that $P_t(x, \mathbb{R}^d) = 1$ for all x, and

$$|u_t(x)| \leq \sup_{x\in\mathbb{R}^d} |\varphi(x)|P_t(x, \mathbb{R}^d) = \sup_{x\in\mathbb{R}^d} |\varphi(x)|,$$

that is the map $\varphi \mapsto u_t$ has the property

$$\|u_t\|_{C_{\rm b}(X)} = \|T_t\varphi\|_{C_{\rm b}(X)} \leq \|\varphi\|_{C_{\rm b}(X)}, \tag{2.91}$$

i.e., is a *contraction*. Furthermore, for $\varphi \geq 0$, by (2.90) we see that $u_t \geq 0$ and $T_t1 = 1$. That is $\{T_t\}_{t\geq 0}$ defined by (2.90) is a positivity preserving conservative semigroup of contractions. Such semigroups are called *stochastic*.

Let $\{T_t\}_{t\geq 0}$ be as above and $(\Omega, \Sigma(\Omega), P)$ be the probability space. For a given $\mu_0 \in \mathcal{M}^1(X)$ and $N \in \mathbb{N}$, we define random variables $\xi_1, \ldots, \xi_N$. $\xi_i : \Omega \to X = \mathbb{R}^d$ such that, for $A_1, \ldots, A_N \in \mathcal{B}(X)$ and $0 < t_1 < \cdots < t_N$,

$$\begin{aligned} &P\left(\xi_{t_1} \in A_1, \ldots, \xi_{t_N} \in A_N\right) \\ &\quad = \int_X \left(T_{t_1}\mathbf{1}_{A_1}T_{t_2-t_1}\mathbf{1}_{A_2}\cdots T_{t_n-t_{N-1}}\mathbf{1}_{A_N}\right)(x)d\mu_0(x). \end{aligned} \tag{2.92}$$

By Kolmogorov's consistency theorem, the above probabilities determine a stochastic process on $(\Omega, \Sigma(\Omega), P)$, which is the Markov process associated with the generator $L = \Delta$. It is the Brownian motion on $X = \mathbb{R}^d$. In particular, (2.92) determine the "one-time" distributions $\mu_t \in \mathcal{M}^1(X)$, given by

$$\mu_t(A) = \int_X (T_t \mathbf{1}_A)(x) d\mu_0(x) := (T_t^* \mu_0)(A). \tag{2.93}$$

This yields the adjoint semigroup $\{T_t^*\}_{t\geq 0}$, acting in $\mathcal{M}^1(X)$. It has the following interpretation: μ_t is the state of the underlying system at time t; the map $\mu_0 \mapsto \mu_t = T_t^* \mu_0$ is the adjoint evolution os states.

Suppose now that an operator L is given. Then we have the following problems.

(1) Does L determines a stochastic semigroup $\{T_t\}_{t\geq 0}$?

(2) If yes, how to get the adjoint semigroup $\{T_t^*\}_{t\geq 0}$?

Let us now give a geometric interpretation of the Brownian motion. For $\phi, \psi \in C_0^\infty(X)$, let us consider the following bilinear form

$$\mathcal{E}(\phi, \psi) = \int_X \langle \nabla \phi, \nabla \psi \rangle dm(x). \tag{2.94}$$

It determines the quadratic form

$$\mathcal{E}(\phi, \phi) = \int_X \|\nabla \phi\|^2_{\mathbb{R}^d} dm(x) = \int_X \|\nabla \phi\|^2_{T_x X} dm(x), \tag{2.95}$$

which is called the *energy form* or *Dirichlet form*. Applying in (2.94) the usual integration-by-parts formula, we get

$$\begin{aligned} \mathcal{E}(\phi, \psi) &= \int_X \langle \nabla \phi, \nabla \psi \rangle dm(x) \\ &= \sum_{j=1}^d \int_X \left(\frac{\partial \phi}{\partial x^j}(x) \cdot \frac{\partial \psi}{\partial x^j}(x) \right) dm(x) \\ &= -\sum_{j=1}^d \int_X \phi(x) \left(\frac{\partial \psi}{\partial x^j}(x) \right) dm(x) \\ &= -\int_X \phi(x) (\Delta \psi)(x) dm(x). \end{aligned} \tag{2.96}$$

Then we have the following sequence of implications

Geometry $\Rightarrow$ Dirichlet form $\Rightarrow$ Laplacian $\Rightarrow$ Brownian motion

Now let us consider the corresponding objects on the configuration spaces. Here we have

$$\Gamma(X), \quad T_\gamma\Gamma(X), \quad \nabla^\Gamma, \quad \pi_\sigma.$$

We also have the $(\mathrm{IbP})_{\pi_\sigma}$, see Theorem 2.17. The energy form in this case is

$$\mathcal{E}_{\pi_\sigma}(F,F) = \int_{\Gamma(X)} \|\nabla^\Gamma F(\gamma)\|^2_{T_\gamma\Gamma(X)} d\pi_\sigma(\gamma). \tag{2.97}$$

3. Combinatorial harmonic analysis on configuration spaces

3.1. **Space of finite configurations.** Now we again consider the configuration space $\Gamma(X)$. By $\mathcal{O}(X)$ we denote the totality of open subsets of X, whereas $\mathcal{O}_c(X)$ with stand for the totality of open subsets having compact closures. Recall also that $\mathcal{B}(X)$ and $\mathcal{B}_c(X)$ denote the families of Borel sets and Borel sets with compact closures, respectively. The space of n-particle configurations is

$$\Gamma^{(n)}(X) = \{\eta \in \Gamma(X) : |\eta| = n\}, \quad n \in \mathbb{N}. \tag{3.1}$$

Let $\widetilde{X^n}$ be the off-diagonal part of the Cartesian product

$$X^n = X \times \cdots \times X \ni (x_1, \ldots, x_n).$$

That is

$$\widetilde{X^n} = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}.$$

We say that two elements of $\widetilde{X^n}$ are equivalent if they coincide up to a permutation of their numbers, that is, $(x'_1, \ldots, x'_n) \sim (x_1, \ldots, x_n)$ if $(x'_1, \ldots, x'_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for some permutation $\sigma \in \Sigma_n$. Then the factor set $\widetilde{X^n}/\Sigma_n$ can be identified with $\Gamma^{(n)}(X)$

$$\widetilde{X^n}/\Sigma_n \simeq \Gamma^{(n)}(X) \tag{3.2}$$

by the relation

$$\Gamma^{(n)}(X) \ni \gamma = \{x_1, \ldots, x_n\} \simeq \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \sigma \in \Sigma_n\} \in \widetilde{X^n}/\Sigma_n,$$

which naturally induces a metric on $\Gamma^{(n)}(X)$ by means of the metric of X. Thereafter, we can present $\Gamma(X)$ as the disjoint union of $\Gamma^{(n)}(X)$, $n \in \mathbb{Z}_+$, i.e.,

$$\Gamma_0(X) = \coprod_{n=0}^{\infty} \Gamma^{(n)}(X), \tag{3.3}$$

which is called the *space of finite configurations*. This spaces is given the topology which cames from the above representation, i.e. $A \subset$

$\Gamma_0(X)$ is open if its intersection with each $\Gamma^{(n)}(X)$ is open in this space. Then $K \subset \Gamma_0(X)$ is compact if and only if there exists $N \in \mathbb{N}$ such that: (a) $K \cap \Gamma^{(n)}(X) = \emptyset$ for all $n > N$; (b) $K \cap \Gamma^{(n)}(X)$ is compact in $\Gamma^{(n)}(X)$ for all $n \leq N$. Naturally, the mentioned above topology of $\Gamma_0(X)$ determines also the corresponding Borel σ-algebra which we denote by $\mathcal{B}(\Gamma_0(X))$.

Definition 3.1. A subset $B \in \mathcal{B}(\Gamma_0(X))$ is called bounded if there exists $N \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(X)$ such that

$$B \subset \coprod_{n=0}^{N} \Gamma^{(n)}(\Lambda).$$

Recall the the space of all configurations $\Gamma(X)$ is given the vague topology defined by the pairing

$$\Gamma(X) \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x), \quad f \in C_0(X).$$

In the vague topology, $\Gamma(X)$ is a Polish space.

3.2. **Functions on configuration spaces.**

3.2.1. *Functions on* $\Gamma_0(X)$. By $\mathcal{B}_b(\Gamma_0(X))$ we define the set of all bounded sets $B \in \mathcal{B}(\Gamma_0(X))$, see Definition 3.1. Note that $\mathcal{B}_b(\Gamma_0(X))$ is a ring: it contains $\emptyset$ and is closed with respect to unions and intersections. Suppose that we have a map $\varrho : \mathcal{B}_b(\Gamma_0(X)) \to \mathbb{R}_+$, which has the following properties.

(1) $\varrho(B) < \infty$ for all $B \in \mathcal{B}_b(\Gamma_0(X))$.

(2) For any sequence $\{B_k\}_{k \in \mathbb{N}}$ such that $B_j \cap B_k = \emptyset$, $j \neq k$ and

$$\bigcup_{k=1}^{\infty} B_k \in \mathcal{B}_b(\Gamma_0(X)),$$

it follows that

$$\varrho\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \varrho(B_k).$$

Then such a map ϱ is called a *pre-measure* on $\mathcal{B}_b(\Gamma_0(X))$. In this case, there exists a unique extension of this pre-measure to a measure ϱ on the σ-algebra $\mathcal{B}(\Gamma_0(X))$.

In the sequel, by $L^0(\Gamma_0(X), \mathcal{B}(\Gamma_0(X)))$ we denote the set of all measurable functions $F : \Gamma_0(X) \to \mathbb{R}$. For short, we also use the notation $L^0(\Gamma_0)$. By $B(\Gamma_0)$ (respectively, $L^0_{\mathrm{ls}}(\Gamma_0)$) we denote the set of all bounded (respectively, with compact support) functions $F \in L^0(\Gamma_0)$. Note that $F \in L^0_{\text{ls}}(\Gamma_0)$ if and only if there exists $\Lambda \in \mathcal{B}_c(X)$ such that F is identically zero on $\Gamma_0(X) \setminus \Gamma(\Lambda)$. Next, we introduce the set of functions $F : \Gamma_0(X) \to \mathbb{R}$ which have *bounded support.* The latter consist of those such functions for each of which there exist $\Lambda \in \mathcal{B}_c(X)$ and $N \in \mathbb{N}$ such that F is identically zero on

$$\Gamma_0(X) \setminus \coprod_{n=0}^{N} \Gamma^{(n)}(\Lambda).$$

The set of the latter functions is denoted by $L^0_{\text{bs}}(\Gamma_0)$. Finally, we introduce

$$B_{\text{ls}}(\Gamma_0) = B(\Gamma_0) \cap L_{\text{ls}}(\Gamma_0), \qquad B_{\text{bs}}(\Gamma_0) = B(\Gamma_0) \cap L_{\text{bs}}(\Gamma_0). \tag{3.4}$$

Remark 3.2. In view of the representation (3.3), we have that every function $G : \Gamma_0(X) \to \mathbb{R}$ has the following structure: its restriction to any $\Gamma^{(n)}(X)$, which we denote by $G^{(n)}$, can be written in the form

$$G^{(n)}(\{x_1, \dots, x_n\}) = \hat{G}^{(n)}(x_1, \dots, x_n), \tag{3.5}$$

where $\hat{G}^{(n)} : X^n \to \mathbb{R}$ is a symmetric function. Thus, each $G : \Gamma_0(X) \to \mathbb{R}$ can be viewed as a sequence of symmetric functions $(\hat{G}^{(0)}, \hat{G}^{(1)}, \hat{G}^{(2)}, \dots, \hat{G}^{(n)}, \dots)$.

3.2.2. *Functions on* $\Gamma(X)$. First we define the so called *cylinder sets.* For $\Lambda \in \mathcal{B}_c(X)$, we consider

$$\{\gamma \in \Gamma(X) : \exists \Lambda' \subset \Lambda \text{ such that } \gamma_{\Lambda'} \in A \in \mathcal{B}(\Lambda')\}. \tag{3.6}$$

The σ-algebra $\mathcal{B}_\Lambda(\Gamma)$ generated by such cylinder sets with support in $\mathcal{B}(\Lambda)$ is clearly isomorphic to the Borel σ-algebra $\mathcal{B}(\Lambda)$. Then we set

$$\mathcal{B}_{\text{cyl}}(\Gamma) = \bigcup_{\Lambda \in \mathcal{B}_c(X)} \mathcal{B}_\Lambda(\Gamma). \tag{3.7}$$

This is the *algebra* (not σ-algebra) of all cylinder subsets of $\Gamma(X)$.

Definition 3.3. The set of functions $F : \Gamma(X) \to \mathbb{R}$, denoted by $L^0(\Gamma, \mathcal{B}_{\text{cyl}})$, consists of all those $F : \Gamma(X) \to \mathbb{R}$ for each of which there exists $\Lambda \in \mathcal{B}_c(X)$ and a function $F_\Lambda : \Gamma(\Lambda) \to \mathbb{R}$, such that

$$F(\gamma_\Lambda = F_\Lambda(\gamma_\Lambda), \quad \text{and} \quad F_\Lambda \in L^0(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))).$$

We consider the following functions $F : \Gamma(X) \to \mathbb{R}$:

(1) continuous functions $C(\Gamma)$, and continuous cylinder functions $\mathcal{F}C(\Gamma)$;

(2) measurable cylinder functions $\mathcal{F}L^0(\Gamma, \mathcal{B}(\Gamma))$;

(3) polynomially bounded measurable cylinder functions $\mathcal{F}L^0_{\mathrm{pb}}(\Gamma, \mathcal{B}(\Gamma))$.

The latter set consist of those $F : \Gamma(X) \to \mathbb{R}$ for each of which there exists $\Lambda \in \mathcal{B}_c(X)$ and a polynomial P in $|\gamma_\Lambda|$ such that, for all $\gamma \in \Gamma(X)$,

$$|F(\gamma)| \leq P(|\gamma_\Lambda|). \tag{3.8}$$

3.3. **Combinatorial Fourier transform.** The combinatorial Fourier is also called the K-transform.

3.3.1. *Definition and main statement.*

Definition 3.4. For $G \in L^0_{\mathrm{ls}}(\Gamma_0)$, we define

$$(KG)\gamma) = \sum_{\xi \Subset \gamma} G(\xi) \quad \gamma \in \Gamma(X), \tag{3.9}$$

where the summation is performed over all *finite* sub-configurations $\xi \Subset \gamma$.

In view of their applications in the theory of complex systems, the functions $F : \Gamma(X) \to \mathbb{R}$ are called *observables*, whereas the functions $G : \Gamma_0(X) \to \mathbb{R}$ – *quasi-observables*.

Why does the K transform (3.9) is well-defined? Since $G \in L^0_{\mathrm{ls}}(\Gamma_0)$, there exists $\Lambda \in \mathcal{B}_c(X)$ such that G is identically zero on $\Gamma_0(X) \setminus \Gamma(\Lambda)$. Then the sum in (3.9) is finite.

Theorem 3.5. *The K-transform (3.9) has the following properties.*

(i) *Let $G \in L^0_{\mathrm{ls}}(\Gamma_0)$. Then its K-transform KG is in $\mathcal{F}L^0(\Gamma)$. Since G is identically zero on $\Gamma_0(X) \setminus \Gamma(\Lambda)$ for some $\Lambda \in \mathcal{B}_c(X)$, for this Λ, $KG \in L^0(\Gamma, \mathcal{B}_\Lambda(\Gamma))$.*

(ii) *K maps $B_{\mathrm{bs}}(\Gamma_0)$ into $\mathcal{F}L^0_{\mathrm{pb}}(\Gamma)$.*

(iii) *K maps $L^0_{\mathrm{ls}}(\Gamma_0)$ into $\mathcal{F}L^0(\Gamma)$ and is invertible. Furthermore,*

$$\left(K^{-1}F\right)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0(X). \tag{3.10}$$

(iv) *K is linear and positivity preserving.*

(v) *For every $G \in C_{\mathrm{ls}}(\Gamma_0)$, KG is in $\mathcal{F}C(\Gamma)$.*

Proof. (i) Since G is identically zero on $\Gamma_0(X) \setminus \Gamma(\Lambda)$, we have that

$$(KG)(\gamma) = \sum_{\xi \subset \gamma_\Lambda} G(\xi) = (KG)(\gamma_\Lambda),$$

hence, KG is in $\mathcal{F}L^0(\Gamma)$.

(ii) Since G is identically zero on

$$\Gamma_0(X) \setminus \coprod_{n=0}^{N} \Gamma^{(n)}(\Lambda),$$

for some $N \in \mathbb{N}$, we have that

$$\begin{aligned} |(KG)(\gamma)| &= |(KG)(\gamma_\Lambda)| \le \sum_{\xi \subset \gamma_\Lambda} |G(\xi)| \\ &\le \sup_{\Gamma_0} |G| \sum_{k=0}^{N} \binom{|\gamma_\Lambda|}{k} = (1 + |\gamma_\Lambda|)^N \sup_{\Gamma_0} |G|, \end{aligned}$$

hence $KG \in \mathcal{F}L^0_{\mathrm{pb}}(\Gamma)$.

(iii) First we observe that K^{-1} in (3.10) is well-defined. Then

$$\begin{aligned} \left(K^{-1}\left(KG\right)\right)(\eta) &= \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \left(\sum_{\zeta \subset \xi} G(\zeta) \right) \\ &= \sum_{\zeta \subset \eta} G(\zeta) \sum_{\xi \subset (\eta \setminus \zeta)} (-1)^{|\xi|} = G(\eta). \end{aligned}$$

At the same time, for $F \in \mathcal{F}L^0(\Gamma)$, we have that $F(\gamma) = F(\gamma_\Lambda)$ for some Λ, and hence,

$$K(K^{-1}F) = F.$$

(iv) Clearly, $KG \ge 0$ whenever $G \ge 0$. (v) is obvious. □

3.3.2. *Examples.*

(1) Let

$$G(\eta) = \begin{cases} f(x) & \text{if } \eta = \{x\}, \quad x \in X; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta) = \sum_{x \in \gamma} f(x) = \langle f, g \rangle.$$

Hence, K : Functions$(\Gamma_0) \to$ Functions(Γ), i.e. it is *lifting.*

(2) Let

$$G(\eta) = \begin{cases} V(x,y) & \text{if } \ \eta = \{x,y\}, \quad x,y \in X; \\ 0 & \text{otherwise.} \end{cases}$$

Here $V \in C_0(X \times X)$. Then

$$(KG)(\gamma) = \sum_{\{x,y\} \subset \gamma} V(x,y) := E^V(\gamma),$$

which is called *energy functional* corresponding to V.

From these examples we see that K maps quasi-observables into observables.

Let us define the following map

$$\mathbb{K} : \mathcal{B}_b(\Gamma_0) \times \Gamma \to \mathbb{R}_+,$$

which acts according to the rule: for $A \in \mathcal{B}_b(\Gamma_0)$ and $\gamma \in \Gamma$,

$$\mathbb{K}(A,\gamma) = (K\mathbf{1}_A)(\gamma) = \sum_{\xi \Subset \gamma} \mathbf{1}_A(\xi). \tag{3.11}$$

Clearly, $\mathbb{K}$ is additive in the sense that

$$\mathbb{K}(A_1 \cup A_2, \gamma) = \mathbb{K}(A_1,\gamma) + \mathbb{K}(A_2,\gamma), \quad \text{if } \ A_1 \cap A_2 = \emptyset. \tag{3.12}$$

Furthermore, for every $A \in \mathcal{B}_b(\Gamma_0)$, $\mathbb{K}(A,\cdot)$ is measurable and, for every $\gamma \in \Gamma$, $\mathbb{K}(\cdot,\gamma)$ is a pre-measure on the ring $\mathcal{B}_b(\Gamma_0)$. If

$$A = \coprod_{k=1}^{\infty} A_k, \quad \text{all } A_k \in \mathcal{B}_b(\Gamma_0) \ \text{ and } \ A \in \mathcal{B}_b(\Gamma_0),$$

then

$$\mathbb{K}(A,\gamma) = \sum_{k=1}^{\infty} \mathbb{K}(A_k,\gamma). \tag{3.13}$$

Theorem 3.6. *The above introduced* $\mathbb{K}$ *can be extended to a kernel* $\mathbb{K} : \mathcal{B}(\Gamma_0) \times \Gamma \to \mathbb{R}_+$. *For any* $G \in L^0_{\text{ls}}(\Gamma_0)$, *it follows that*

$$\int_{\Gamma_0} G(\eta)\mathbb{K}(d\eta,\gamma) = \sum_{\eta \Subset \gamma} G(\eta) = (KG)(\gamma). \tag{3.14}$$

Proof. Since $\mathbb{K} : \mathcal{B}(\Gamma_0) \times \Gamma \to \mathbb{R}_+$ is a pre-measure, it can uniquely be extended to a measure $\mathbb{K}(d\eta,\gamma)$ on $\mathcal{B}(\Gamma_0)$. Take $G = \mathbf{1}_A$, $A \in \mathcal{B}_b(\Gamma_0)$. Then

$$\int_{\Gamma_0} \mathbf{1}_A(\eta)\mathbb{K}(d\eta,\gamma) = \mathbb{K}(A,\gamma) = \sum_{\eta \Subset \gamma} \mathbf{1}_A(\eta) = (\mathbb{K}\mathbf{1}_A)(\gamma).$$

Now for a given $N \in \mathbb{N}$ and real $\alpha_1, \dots \alpha_N$, we take a step function

$$G(\eta) = \sum_{k=1}^{N} \alpha_k \mathbf{1}_{A_k}(\eta), \quad A_1, \dots A_N \in \mathcal{B}_b(\Gamma_0), \tag{3.15}$$

for which we obtain by repeating the above calculations

$$\int_{\Gamma_0} \mathbf{1}_A(\eta) \mathbb{K}(d\eta, \gamma) = \sum_{k=1}^{N} \alpha_k \mathbb{K}(A_k, \gamma) = (KG)(\gamma). \tag{3.16}$$

□

Definition 3.7. For $G_1, G_2 \in L^0(\Gamma_0)$, we define the combinatorial convolution

$$(G_1 \star G_2)(\eta) = \sum_{(\xi_1, \xi_2, \xi_3) \in \mathcal{P}_3(\eta)} G_1(\xi_1 \cup \xi_2) G_2(\xi_2 \cup \xi_3), \tag{3.17}$$

where $\mathcal{P}_3(\eta)$ is the family of all partitions of $\eta \in \Gamma_0$ into the sum of ξ_1, ξ_2, ξ_3, such that $\xi_j \cap \xi_k \neq \emptyset$.

Clearly, $G_1 \star G_2 \in L^0(\Gamma_0)$. Another representation of this convolution is

$$(G_1 \star G_2)(\eta) = \sum_{\xi \subset \eta,\ \zeta \subset \eta:\ \xi \cup \zeta = \eta} G_1(\xi) G_2(\zeta). \tag{3.18}$$

Proposition 3.8. *For any $G_1, G_2 \in L^0_{\mathrm{ls}}(\Gamma_0)$, it follows that*

$$K(G_1 \star G_2) = KG_1 \cdot KG_2. \tag{3.19}$$

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