# USING $\beta S$ IN COMBINATORICS AND DYNAMICAL SYSTEMS (SURVEY)

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**Introduction**

In this survey, we will investigate an object called $\beta S$, with the following features.

We start with an arbitrary semigroup $S$, with the discrete topology. $\beta S$ is then the Stone-Čech compactification of $S$, a compact Hausdorff space. The semigroup structure of $S$ extends in a natural way to $\beta S$, thus making $\beta S$ a compact right topological semigroup. In contrast to $S$, $\beta S$ has idempotents (i.e. elements $p$ satisfying $p \cdot p = p$), and these can be used to prove combinatorical results on $S$.

More precisely, a colouring of a set $X$ is simply a function $c : X \to I$; we think about $I$ as a set of colours and, for $x \in X$, call $c(x)$ the colour of $x$. $c$ induces the partition $X = \bigcup_{i \in I} A_i$ where $A_i = \{x \in X : c(x) = i\}$.

Now assume $X$ is large in some respect and the set $I$ is small, say finite. Is there a large monochromatic set, i.e. some $H \subseteq X$ and some $j \in I$, such that $c(x) = j$ holds for all $x \in H$? (Resp. such that $H \subseteq A_j$, for some $j$?) Ramsey theorems are results which give a positive answer, under appropriate conditions.

As an easy-to-understand example, call $A \subseteq \mathbb{N}$ a van der Waerden set if, for every $k \in \mathbb{N}$, there are $a, d \in \mathbb{N}$ such that $a, a + d, a + 2d, \ldots, a + kd \subseteq A$; the van der Waerden theorem states that for every partition $\mathbb{N} = A_1 \cup \cdots \cup A_r$ of $\mathbb{N}$ with finitely many colours, at least one $A_j$ is van der Waerden.

In the second part of the survey, we study dynamical systems over $S$, i.e. topological spaces $X$ on which $S$ operates in a continuous way; the algebraic-topological properties of $S$ resp $\beta S$ shed some light onto the dynamical structure of $X$. In particular, $\beta S$ is a dynamical system over $S$; algebraic properties of $\beta S$ can be translated into dynamical ones, and vice versa. Moreover, often theorems from topological dynamics imply combinatorial results.

The interplay of algebra, topology, and dynamics on $\beta S$ is what makes this object both useful and fascinating. This has been pointed out by A. Blass in [3] for the special and important case of the semigroup $(\mathbb{N}, +)$.

The elements of the set $\beta S$ are the **ultrafilters** on $S$; we recall resp. survey notions and results on ultrafilters in Sections 1 and define the topology on $\beta S$ in Section 2.

In Section 3, we introduce **semigroups** with the main result that compact right topological semigroups have idempotents. Section 4 introduces the (compact right topological) semigroup structure on $\beta S$.

Section 5 is the first example of a non-trivial application of existence of idempotents in $\beta S$ to **combinatorics**: for $S = A_1 \cup \cdots \cup A_r$ a partition of $S$, one $A_j$ is an IP set. More ecactly, there is a close connection between idempotents and IP-sets.

Section 6 defines a partial ordering on the set $E(S)$ of idempotents of a semigroup $S$ and explains how **minimal idempotents**, with respect to this partial order, explain some of the algebraic structure of $S$. In compact right topological semigroups, there are sufficiently many mininimal iemdpotents. This is applied in Section 7, where van der Waerden's theorem is proved, in fact a much more geneneral combinatorial result, the Hales-Jewett theorem.

In Sections 8 and 9, we study several types of **large subsets** of semigroupss, and how they are connected with ultrafilters on $S$.

Sections 10 and 11 introduce the notion of a **dynamical system** over $S$ and show how $\beta S$ can be considered as a dynamical system over $S$.

Sections 12 and 13 cover **recurrent points** in dynamical systems (points with strong return properties) and their existence, again connected with special ultrafilters on $S$. Section14 defines another very strong notion of recurrence.

Section 15 presents a technically simple application of idempotents in $\mathbb{N}$ to **diophantine approximation**, a topic of number theory.

I learnt most of the material in this survey from the monography [6] and the paper [3]; other parts are taken from [1], [2], [5]. However, I have tried to polish or simplify the presentation, wherever possible.

## 1. Basic properties of ultrafilters

The notion of a filter resp. an ultrafilter is a powerful tool both in set theory and in topology. The importance of ultrafilters in these notes lies in the fact that the set $\beta S$ of all ultrafilters on a fixed semigroup $S$ can be made into a compact right topological semigroup which will be our central object of study.

We present the basic notions on filters and ultrafilters and the principal tools required when dealing with them, in particular the existence theorems for filters resp. ultrafilters including a specified family of sets.

### A. Filters

A filter on a set $S$ is a collection of subsets of $S$ which may be viewed as being "large", with respect to some property, and compatible in the sense that any finitely many of them have non-empty intersection.

More exactly, let $S$ be a non-empty set. A *filter* on $S$ is a subset $p$ of the power set $\mathcal{P}(S)$ of $S$ satisfying:
– $S \in p$, and $\emptyset \notin p$
– if $A \in p$ and $A \subseteq B \subseteq S$, then $B \in p$
– if $A, B \in p$, then $A \cap B \in p$.

There is a simple criterion whether a family $\mathcal{A} \subseteq \mathcal{P}(S)$ ican be extended to a filter on $S$: this is possible iff $\mathcal{A}$ has the finite intersection property, i.e. iff every finite subfamily of $\mathcal{A}$ has non-empty intersection.

### B. Ultrafilters

An arbitrary filter $p$ on a set $S$ behaves nicely with respect to intersections: for $A, B \subseteq S$, $A \cap B$ is in $p$ if and only if both $A$ and $B$ are in $p$. We now consider filters which, in addition, behave nicely with respect to unions and complements:
– $p$ is an *ultrafilter* if for any $A \subseteq S$, either $A$ or $S \setminus A$ is in $p$.
– $p$ is a *prime filter* if for any $A, B \subseteq S$ satisfying $A \cup B \in p$, either $A \in p$ or $B \in p$.
– $p$ is a *maximal filter* if every filter including $p$ coincides with $p$.

A trivial example for an ultrafilter on $S$ is given by $\dot{s} = \{A \subseteq S : s \in A\}$, where $s \in S$.

1.1. **Theorem.** *For any filter on $S$, the properties of being an ultrafilter, prime, or maximal are equivalent.*

This equivalence shows, by Zorn's lemma, that sufficiently many ultrafilters exist.

1.2. **Theorem.** *Every filter is extendible to an ultrafilter.*

### C. Free and fixed filters

We denote the set of all ultrafilters on $S$ by $\beta S$. The canonical map $e : S \to \beta S$ is defined by $e(s) = \dot{s}$, for $s \in S$.

The function $e$ is one-one; thus $S$ is usually thought of as being a subset of $\beta S$, identifying every $s \in S$ with the ultrafilter $\dot{s}$ . For infinite $S$, this embedding is not a bijection, i.e. there are ultrafilters on $S$ which don't have the form $\dot{s}$, $s \in S$. The separating line between the trivial ultrafilters $e(s) = \dot{s}$ and the non-trivial ones is drawn by the notion of a fixed resp. free filter: the fixed ultrafilters are exactly the trivial ones of the form $e(s) = \dot{s}$, the free ultrafilters being the the non-trivial ones.

## 2. Applications of ultrafilters. The space $\beta S$

Ultrafilters, in particular free ones, are very useful in combinatorial arguments; we illustrate this by a proof of the infinite Ramsey theorem.

### A. Ramsey's theorem

For any set $X$ and $n \in \mathbb{N}$, we put

$$[X]^n = \{e \subseteq X : |e| = n\},$$

the set of all $n$-element subsets of $X$. Here $|e|$ denotes the cardinality of $e$.

2.1. **Theorem.** *(Ramsey) Let $X$ be an infinite set and $[X]^n = A_1 \cup \cdots \cup A_r$ a colouring of $[X]^n$ with finitely many colours. Then there exists an infinite monochromatic subset of $X$, i.e. some $H \subseteq X$ such that $[H]^n \subseteq A_j$, for some $j \in \{1, \ldots, r\}$.*

By topological reasoning, one can derive a finitary version of Ramsey's theorem from the infinite one.

### B. The $p$-limit of a sequence $(x_s)_{s\in S}$

As another application of ultrafilters, we define a notion of convergence for sequences $(x_s)_{s\in S}$ in a topological space, indexed by $S$, with respect to an arbitrary filter $p$ on $S$, and we see how it can describe topological facts: $x$ is a $p$-limit of $(x_s)_{s\in S}$ iff, for every neighbourhood $U$ of $x$, the set $\{s \in S : x_s \in U\}$ is in $p$.

Under appropriate assumptions, every sequence $(x_s)_{s\in S}$ has a unique $p$-limit.

2.2. **Theorem.** *For $X$ compact Hausdorff and $p$ an ultrafilter, there is a unique $p$-limit of $(x_s)_{s\in S}$, denoted by $p - \lim_{s\in S} x_s$ – the unique point in $\bigcap_{A\in p} \mathrm{cl}_X \{x_s : s \in A\}$.*

The following theorem demonstrates the usefulness of ultrafilters in topology – more exactly, of the set

$$\beta S = \{p : p \text{ is an ultrafilter on } S\}$$

of *all* ultrafilters on $S$: we can represent the closure of a set indexed by $S$ by using $\beta S$.

2.3. **Theorem.** *For a compact Hausdorff space $X$ and $(x_s)_{s\in S}$ a family of points in $X$, we have*

$$cl\{x_s : s \in S\} = \{p - \lim_{s\in S} x_s : p \in \beta S\}.$$

### C. $\beta S$ as a topological space

For a non-empty set $S$, we introduce the Stone topology on $\beta S$: for every $A \subseteq S$, the Stone set $\hat{A} \subseteq \beta S$ corresponding to $A$ is

$$\hat{A} = \{p \in \beta S : A \in p\}.$$

There is a unique topology on $\beta S$ having $\{\hat{A} : A \subseteq S\}$ as an open base, the Stone topology. We call $\beta S$, with the Stone topology, the Stone-Čech compactification of $S$.

We formulate the most important properties of the space $\beta S$.

2.4. **Theorem.** *The space $\beta S$, with the Stone topology, is Boolean, i.e. it is Hausdorff, compact, and has a base consisting of clopen sets.*
*More precisely, a subset of $\beta S$ is clopen iff it is the Stone set $\hat{A}$ of some $A \subseteq S$.*

### D. $\beta S$ as a compactification of $S$

The isolated points of $\beta S$ are exactly the fixed ultrafilters $\dot{s}$, for $s \in S$, hence the canonical map $e : S \to \beta S$ mapping $s \in S$ to the fixed ultrafilter $\dot{s}$ is a bijection from $S$ onto the set of isolated points of $\beta S$. Additionally, the set of isolated points is dense in $\beta S$.

2.5. **Theorem.** *Consider the set $S$ as a topological space with the discrete topology. Then the pair $(\beta S, e)$ is a compactification of $S$. I.e. $e$ is an embedding from $S$ into $\beta S$, and $e[S] \subseteq \beta S$ is dense.*

Identifying every point $s \in S$ with its image $\dot{s}$ under the map $e$, we shall view $S$ as a dense subspace of the compact space $\beta S$.

**E. The universal property of the compactification** $(\beta S, e)$

The compactification $(\beta S, e)$ of $S$ has a characteristic extension property.

2.6. **Theorem.** *Assume $X$ is a compact Hausdorff space and $f : S \to X$ is an arbitrary map. Then there is a unique continuous map $\tilde{f} : \beta S \to X$ such that $\tilde{f} \circ e = f$, the Stone-Čech extension of $f$.*
*Identifying $S$ with the dense subspace of $\beta S$ consisting of all isolated points, we see that $\tilde{f}$ is the unique continuous extension of $f$ to $\beta S$.*

## 3. Semigroups: basic facts and examples

We introduce the concept of semigroups, give a few examples and define the most important basic algebraic notions.

There are plenty of examples of semigroups without idempotent elements. Our main result is that every compact right topological semigroup does have idempotent elements, a fact which will be applied repeatedly in later chapters.

Finally we present a less obvious example of semigroups, the free ones, enjoying a particular universal property.

### A. Definition and examples

A semigroup is a pair $(S, \cdot)$ in which $S$ is a non-empty set and $\cdot$ is a binary associative operation on $S$. I.e. the equation $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds for all $x, y, z \in S$.

The most important example of a semigroup is the set $\omega = \{0, 1, 2, \dots\}$ of all natural numbers (including 0), under the usual addition of natural numbers.

### B. Special maps, elements, and subsets of semigroups

We define some algebraic notions from the theory of semigroups: homomorphisms, isomorphisms, subsemigroups, and state a few simple facts about them. We call an element $e$ of a semigroup idempotent if $e \cdot e = e$.

### C. Compact right topological semigroups and idempotents

We prove that semigroups from a very special class, the compact right topological ones, do have idempotents. This fact will first be used in Section 5 to prove a highly non-trivial combinatorial fact, Hindman's theorem.

A *compact right topological semigroup* is a triple $(S, \cdot, \mathcal{O})$ where $(S, \cdot)$ is a semigroup, $\mathcal{O}$ is a topology on $S$, the space $(S, \mathcal{O})$ is compact and Hausdorff, and for every $s \in S$, the right translation map $\rho_s$, defined by $\rho_s(x) = xs$, is continuous with respect to $\mathcal{O}$.

E.g, every finite semigroup, equipped with the discrete topology, is a compact topological semigroup.

3.1. **Theorem.** *Every compact right topological semigroup has an idempotent element.*

### D. Free semigroups

We define a very special example of semigroups, the free ones. They are quite important for the theory of semigroups because of their characteristic universal property stated in 3.3 and 3.4.

3.2. **Example.** Let $L$ be an arbitrary non-empty set, an *alphabet*; the elements of $L$ are the *letters* of $L$.
$L^*$ is the set of all ordered sequences in $L$ of finite length greater than 0, the *words* (of finite length) over the alphabet $L$. We consider $L$ to be a subset of $L^*$, identifying $b \in L$ with $(b) \in L*$.
The set $L^*$ is made into a semigroup, the *free semigroup over* $L$, under the operation of concatenation.

The semigroup $L^*$ has the following universal property.

3.3. **Theorem.** *For a map $f : L \to S$ from $L$ into an arbitrary semigroup $S$, there is a unique semigroup homomorphism $\bar{f} : L^* \to S$ extending $f$.*

Theorem 3.3 is the reason for calling the semigroup $L^*$ *free* over $L$: the elements of $L \subseteq L^*$ do not satisfy any equations, except those which hold in all semigroups.

The following corollary is the reason for an effect we will meet later: in many situations, combinatorial principles valid for free semigroups carry over to arbitrary semigroups.

3.4. **Corollary.** *Every semigroup is the homomorphic image of a free semigroup $L^*$, for sufficiently large $L$.*

## 4. $\beta S$ as a semigroup

For an arbitrary semigroup $(S, \cdot)$, we use the universal property of the Stone-Čech compactification $\beta S$ of $S$ to extend the multiplication given on $S$ to $\beta S$ in a manner which makes right and left translations on $\beta S$ continuous, to a considerable extent. It turns out that $\beta S$ is a right topological semigroup.

### A. The multiplication on $\beta S$

For $s \in S$, we define the function $q \mapsto s \cdot q$ (from $\beta S$ into itself) as the Stone-Čech extension of $t \mapsto s \cdot t$ (from $S$ into $\beta S$). Then for $q \in \beta S$,we define $p \mapsto p \cdot q$ (from $\beta S$ into itself) as the Stone-Čech extension of $s \mapsto s \cdot q$ (from $S$ into $\beta S$).

This defines $pq = p \cdot q$, for arbitrary $q \in \beta S$ and $p \in \beta S$, and ensures that the multiplication thus defined on $\beta S$ extends the multiplication on $S$.

4.1. **Theorem.** *$\beta S$ is a compact right topological semigroup.*

Recall that, for $f : S \to \beta S$, the Stone-Čech extension $\tilde{f}$ is given by $\tilde{f}(p) = p - \lim_{s \in S} f(s)$. This gives a description of $p \cdot q$: for $s \in S$ and $p, q \in \beta S$, $s \cdot q = q - \lim_{t \in S} st$ and $p \cdot q = p - \lim_{s \in S} sq = p - \lim_{s \in S}(q - \lim_{t \in S} st)$.

### B. Sets in $p \cdot q$

We explain which subsets of $S$ belong to the product ultrafilter $p \cdot q$. For $A \subseteq S$ and $s \in S$ and $q \in \beta S$, we put

$$s^{-1}A = \{t \in S : st \in A\}, \quad A^{-q} = \{s \in S : s^{-1}A \in q\}.$$

4.2. **Proposition.** *Assume that $s \in S$, $q \in \beta S$, and $A \subseteq S$. Then $A \in sq$ iff $s^{-1}A \in q$ iff $s \in A^{-q}$.*

The most concrete description of the sets in $pq$ is (d) below.

4.3. **Theorem.** *For $p, q \in \beta S$ and $A \subseteq S$, the following are equivalent:*
*(a)* $A \in pq$
*(b)* $A^{-q} \in p$
*(c)* $\{s \in S : \{t \in S : st \in A\} \in q\} \in p$
*(d) there are $V \in p$ and a family $(W_v)_{v \in V}$ of sets in $q$ such that $\bigcup_{v \in V} v \cdot W_v \subseteq A$.*

Recall that for $A, B \subseteq S$, $AB = A \cdot B$ is the set of all products $ab$ where $a \in A$ and $b \in B$. It turns out that for $A \in p$ and $B \in q$, $AB \in pq$, i.e. $\hat{A} \cdot \hat{B} \subseteq \widehat{AB}$.
Thus if $T$ is a subsemigroup of $S$, then $\hat{T}$ is a subsemigroup of $\hat{S} = \beta S$.

### C. Schur's theorem

Our first and most simple nontrivial application of idempotent elements in $\beta S$ is a theorem due to I. Schur, in the semigroup $(\mathbb{N}, +)$.

4.4. **Theorem.** *Assume $(S, \cdot)$ is a semigroup and $S = A_1 \cup \cdots \cup A_r$ is a colouring of $S$ with $r$ colours. Then there are $j \in \{1, \ldots, r\}$ and elements $v, w$ of $S$ such that $v, w, vw \in A_j$ – i.e. $v, w$ and $vw$ have the same colour $j$.*

E.g. for a colouring $\mathbb{N} = A_1 \cup \cdots \cup A_r$ of $\mathbb{N}$, there are $j \in \{1, \ldots, r\}$ and $v < w$ in $\mathbb{N}$ such that $v, w, v + w \in A_j$ – i.e. $v, w$ and $v + w$ have the same colour $j$.

### D. Non-commutativity in $\beta S$

Unless the associative law $x(yz) = (xy)z$, other algebraic laws hardly ever carry over from $S$ to $\beta S$. The most prominent example of this effect is the failure of the commutative law $xy = yx$ in $\beta S$; we give a criterion for $\beta S$ to be commutative. In particular $(\beta\omega, +)$ is not commutative.

## 5. Hindman's finite sums theorem

Using idempotents in $\beta S$, we prove a vast generalization of Schur's theorem: for every colouring of a semigroup $S$ with finitely many colours, at least one of the colours includes an IP-set, i.e. a (usually infinite) set with strong combinatorial structure. More precisely, we will see how idempotents of $\beta S$ and IP-sets are intimately connected.

### A. Statement of Hindman's theorem

For any set $X$, $\mathcal{P}_f(X)$ is the set of all non-empty finite subsets of $X$.
For $(S,\cdot)$ a semigroup, $\bar{x} = (x_i)_{i\in\omega}$ an infinite sequence in $S$, and $e\ in \mathcal{P}_f(\omega)$, $p_e$ is the product of the $x_i$, $i \in e$, multiplied from left to right in the order of their appearance in the sequence $\bar{x}$.
Given $\bar{x}$ as in (b), the subset $FP(\bar{x})$ of $S$ (the set of $f$inite $p$roducts over $\bar{x}$) is defined by

$$FP(\bar{x}) = \{p_e : e \in \mathcal{P}_f(\omega)\}.$$

$A \subseteq S$ is an IP-set if $FP(\bar{x}) \subseteq A$, for some infinite sequence $\bar{x}$ in $S$.
We can now state the abstract version of Hindman's theorem.

5.1. **Theorem.** *(Hindman) Assume $(S,\cdot)$ is a semigroup and $S = A_1 \cup \cdots \cup A_r$ is a colouring of $S$ with $r$ colours. Then at least one of the sets $A_j$, $j \in \{1,\ldots,r\}$, is an IP-set.*

For $(S,+)$ a commutative semigroup in additive notation, we will write $s_e = \Sigma_{i\in e} x_i$ and $FS(\bar{x}) = \{s_e : e \in \mathcal{P}_f(\omega)\}$. $A \subseteq S$ is an IS-set if $FS(\bar{x}) \subseteq A$, for some infinite sequence $\bar{x}$ in $S$.
The abstract version of Schur's theorem turns out to be an extreme special case of Hindman's theorem. The classical finite sums theorem due to Hindman is the following special case.

5.2. **Corollary.** *(Hindman's finite sums theorem) Assume $\omega = A_1 \cup \cdots \cup A_r$ is a colouring of $\omega$ with $r$ colours. Then at least one of the sets $A_j$ is an IS-set, in $(\omega,+)$.*
*Moreover the sequence $\bar{x} = (x_i)_{i\in\omega}$ proving this can be taken strictly increasing, i.e. such that $x_0 < x_1 < \ldots$.*

### B. Proof of Hindman's theorem

Hindman's theorem follows immediately from the implication from (a) to (b) in the following equivalence. We work in an arbitrary semigroup $(S,\cdot)$ and define the missing notions below.

5.3. **Theorem.** *(The main theorem on IP-sets) For any subset $A \subseteq S$, the following are equivalent.*
*(a) $A$ is contained in an idempotent ultrafilter on $S$ (i.e. the Stone set $\hat{A}$ contains an idempotent of $\beta S$)*
*(b) $A$ is an IP-set*
*(c) the power set $\mathcal{P}(A)$ of $A$ includes a multiplicative family*
*(d) $\hat{A}$ includes a closed subsemigroup of $\beta S$.*

5.4. **Corollary.** *If an IP-set $A \subseteq S$ is the union of finitely many subsets $A_1,\ldots,A_r$, then at least one $A_j$ is an IP-set.*

The implication from (d) to (a) holds by existence of idempotents in compact right topological semigroups, and those from (c) to (d) and from (b) to (c) work smoothly with the following notion and lemma.
A family $\mathcal{C}$ of subsets of a semigroup $(S,\cdot)$ is called *multiplicative* if it is nonempty, has the finite intersection property, and for every $C \in \mathcal{C}$ and every $c \in C$, there is some $D \in \mathcal{C}$ such that $c \cdot D \subseteq C$.
Multiplicative families give examples of closed subsemigroups of $\beta S$:

5.5. **Lemma.** *For a multiplicative family $\mathcal{C} \subseteq \mathcal{P}(S)$, the set $T = \bigcap_{C\in\mathcal{C}} \hat{C} = \{p \in \beta S : \mathcal{C} \subseteq p\}$ is a closed subsemigroup of $\beta S$.*

Thus in the main theorem, (c) implies (d). To see that (b) implies (c), let $\bar{x}$ certify that $A$ is an IP-set, i.e. such that $FP(\bar{x}) \subseteq A$. Then for every $n \in \omega$, $C_n = FP((x_i)_{n \leq i < \omega}) \subseteq A$, and $\mathcal{C} = \{C_n : n \in \omega\}$ is multiplicative.

The implication from (a) to (b) works by the following definition and lemma. Recall the notation $s^{-1}A = \{x \in S : sx \in A\}$ and $A^{-q} = \{s \in S : s^{-1}A \in q\} = \{s \in S : A \in sq\}$, for $A \subseteq S$, $s \in S$ and $q \in \beta S$. For $p \in \beta S$, we define $A^* \subseteq S$ by $A^* = A \cap A^{-p}$.

5.6. **Lemma.** *Assume $A^*$ is defined with respect to an idempotent ultrafilter $p$. Then the following hold.*
*(a) $A \in p$ iff $A^{-p} \in p$ iff $A^* \in p$.*
*(b) $A^{**} = A^*$.*
*(c) Assume that $A \in p$ and $L$ is a finite subset of $A^*$. Then there is some $W \in p$ such that $W \subseteq A^*$ and $L \cdot W \subseteq A^*$.*

### C. More on idempotent ultrafilters and IP-sets

We have proved that the sets contained in some idempotent ultrafilter on $S$ are exactly the IP-sets. Conversely, if $p$ is an ultrafilter on $S$ consisting of IP-sets, does it follow that $p$ is idempotent? Not quite, but a weaker statement holds.

5.7. **Corollary.** *Let $p$ be any ultrafilter on a discrete semigroup $S$. Then every set in $p$ is an IP-set iff $p \in \mathrm{cl}_{\beta S} E(\beta S)$.*

The set $\mathbb{N}$ of natural numbers is a semigroup both under addition and under multiplication, and we can apply the above theory both to $(\mathbb{N}, +)$ and to $(\mathbb{N}, \cdot)$. Here we have to distinguish the sets $E(\beta\mathbb{N}, +)$ and $E(\beta\mathbb{N}, \cdot)$, and similarly IP-subsets of $\mathbb{N}$ with respect to addition and to multiplication, i.e. IS-sets and IP-sets. There is a nontrivial result connecting IS- and IP-subsets of $\mathbb{N}$.

## 6. Semigroups: ideals and minimal idempotents

The abstract version of Hindman's Theorem is the paradigmatic example how idempotents of the semigroup $\beta S$ can be used to prove combinatorial results on a discrete semigroup $S$. For more refined results, however, just idempotent elements of $\beta S$ are not enough – we will have to use minimal idempotents. They are intimately connected with minimal left ideals in $\beta S$.

To ensure that such objects do exist, we make an extra assumption on our semigroup $S$: abundantness. It turns out that compact right topological semigroups are abundant; in particular, $\beta S$ is abundant for every semigroup $S$.

### A. (Minimal) Ideals

A subset $I$ of a semigroup $S$ is a left ideal of $S$ if it is non-empty and $SI \subseteq I$ holds, i.e. $i \in I$ and $s \in S$ implies that $si \in I$. It is a *two sided ideal* of $S$ if it is both a left ideal and a right ideal of $S$, i.e. we have $SIS \subseteq I$.

E. g. for $a \in S$, $Sa = \{sa : s \in S\}$ is a left ideal; if $a \in Sa$, then $Sa$ is clearly the smallest left ideal of $S$ containing $a$. Similarly, $SaS = \{sat : s, t \in S\}$ is a two sided ideal.

We call a left ideal $M$ of $S$ *minimal* if every left ideal $I$ of $S$ included in $M$ coincides with $M$. Minimal left ideals do not necessarily exist – e.g. the semigroup $(\mathbb{N}, +)$ has no minimal ideal. They are intimately connected to two sided ideals.

6.1. **Lemma.** *For a minimal left ideal $L$ of $S$, the following hold.*
*(a) For every $s \in S$, also $Ls$ is a minimal left ideal.*
*(b) $L$ is included in every two sided ideal of $S$.*

For any semigroup $S$, we put

$$K(S) = \bigcup \{L : L \text{ a minimal left ideal of } S\},$$

the union of all minimal left ideals of $S$. So $K(S) \neq \emptyset$ iff $S$ has at least one minimal left ideal. In this case, it is a left ideal, being the union of a non-empty family of left ideals. The following theorem explains the central role of the left ideal $K(S)$.

6.2. **Theorem.** *$K(S)$, if non-empty, is a two sided ideal of $S$ – in fact, the least one.*

### B. Abundant semigroups

Call a semigroup $S$ abundant if every left ideal of $S$ includes a minimal one and every (minimal) left ideal of $S$ contains an idempotent element. Abundance of $S$ guarantees that $K(S)$ is non-empty, and it holds for sufficiently many interesting semigroups:

6.3. **Theorem.** *Every compact right topological semigroup is abundant.*

### C. Minimal left ideals and idempotents in abundant semigroups

On the set $E(S)$ of idempotent elements of a semigroup $S$, we define the relation

$$e \leq f \quad \text{iff} \quad ef = fe = e,$$

a partial ordering on $E(S)$.

In abundant semigroups, minimal idempotents are connected to minimal left ideals.

6.4. **Theorem.** *Assume that $S$ is an abundant semigroup and that $e \in E(S)$. Then the following hold.*
*(a) If $L \subseteq Se$ is a minimal left ideal, then there is some idempotent $f \in L$ such that $f \leq e$.*
*(b) $e$ is a minimal idempotent iff $e \in L$ for some minimal left ideal $L$, i.e. iff $e \in K(S)$.*
*(c) $e$ is a minimal idempotent iff the left ideal $L = Se$ is minimal.*
*(d) There is some minimal idempotent $f$ such that $f \leq e$.*

**D. More on $K(\beta S)$**

The subset $K(S)$ of a semigroup $S$ was defined to be the union of all minimal left ideals of $S$; if non-empty, it is the least two sided ideal of $S$.

A subset $I$ of $S$ is called a right ideal of $S$ if $IS \subseteq I$. It is a non-trivial fact that, under suitable conditions, $K(S)$ is also the union of all minimal right ideals of $S$.

6.5. **Theorem.** *Assume that $S$ has a minimal left ideal containing an idempotent element. Then*

$$K(S) = \bigcup\{L : L \textit{ a minimal left ideal of } S\} = \bigcup\{R : R \textit{ a minimal right ideal of } S\}.$$

## 7. The Hales-Jewett theorem

We prove a deep combinatorial theorem by Hales and Jewett which, in its classical version, deals with colourings of free semigroups. In fact, a seemingly much more general theorem (the abstract Hales-Jewett theorem) holds. The proof heavily depends on the application of minimal idempotents in $\beta S$, for a semigroup $S$.

There are several attractive and more easily understandible consequences of the Hales-Jewett theorem, which we present below, e.g. van der Waerden's theorem for the additive semigroups $\mathbb{N}$ or $\mathbb{N}^k$ as its special cases.

### A. The abstract version

We call $W$ a nice semigroup of $(V,\cdot)$ if the remainder $R = V \setminus W$ of $V$ with respect to $W$ is a two sided ideal of $V$. (I.e. if $W$ is a proper subset of $V$, and the product of two elements $x, y$ of $V$ is in $W$ if and only if both $x$ and $y$ are in $W$.)
A retraction from $V$ to $W$ is a semigroup homomorphism $\sigma : V \to W$ such that the restriction of $\sigma$ to $W$ is the identity map on $W$.

7.1. **Theorem.** *(the Hales-Jewett theorem, abstract version) Assume that $W$ is a nice subsemigroup of $V$, and $\Sigma$ is a finite set of retractions from $V$ to $W$. Moreover assume that $W = B_1 \cup \cdots \cup B_r$ is a colouring of $W$ with finitely many colours. Then there is some $v \in R = V \setminus W$ such that $\{\sigma(v) : \sigma \in \Sigma\}$ is monochromatic, i.e. $\{\sigma(v) : \sigma \in \Sigma\} \subseteq B_j$, for some $j$.*

We state some consequences of the theorem. For a commutative semigroup $(S,+)$, $e \in S$ and $d \in \mathbb{N}$, we denote by $de$ the sum $e + \cdots + e$ in which $e$ is added $d$ times.

7.2. **Theorem.** *(Gallai's theorem for commutative semigroups) Assume $(S,+)$ is a commutative semigroup, $E \subseteq S$ is finite, and assume that $S = A_1 \cup \cdots \cup A_r$ is a colouring of $S$ with finitely many colours. Then there are $a \in S$ and $d \in \mathbb{N}$ such that $a + dE = \{a + de : e \in E\}$ is monochromatic.*

7.3. **Corollary.** *(van der Waerden's theorem) Assume that $\omega = A_1 \cup \cdots \cup A_r$ is a colouring of $\omega$ with finitely many colours and that $n \in \mathbb{N}$. Then there are $a \in \omega, d \in \mathbb{N}$ and some $j$ such that $\{a + di : 0 \leq i \leq n\} = \{a, a+d, a+2d, \ldots, a+dn\} \subseteq A_j$.*

A subset of $\omega$ of the form $\{a, a+d, a+2d, \ldots, a+dn\}$ is called an *arithmetic progression* (of length $n$). Van der Waerden's theorem says that at least one of the colours $A_j$ includes an arithmetic progression of length $n$. More generally, this holds in $k$ dimensions:

7.4. **Corollary.** *(the $k$-dimensional van der Waerden theorem) Assume that $k \in \mathbb{N}$, $\omega^k = A_1 \cup \cdots \cup A_r$ is a colouring of $\omega^k$ with finitely many colours, and that $n \in \mathbb{N}$. Then there are $a \in \omega^k, d \in \mathbb{N}$ and some $j$ such that for every $k$-tuple $e = (e_1, \ldots, e_k) \in \{0, 1, \ldots, n\}^k$, $a + de \in A_j$. I.e. some $A_j$ includes a homothetic copy of the $k$-dimensional cube $\{0, 1, \ldots, n\}^k$.*

### B. The classical version

This is a combinatorial result on free semigroups which turns out to be a special case of the abstract one. Recall from Section 3 the notion of the free semigroup $L^*$ on a set $L$: $L^*$ is the set of all finite words on the alphabet $L$, a semigroup under the operation of concatenation of words.

Assume the alphabet $L$ is split into two disjoint subsets $C$ and $X$. The elements of $C$ are called *constant letters*, the elements of $X$ are the *variable letters*. A word $w \in L^*$ is *constant* if $w \in C^*$, i.e. if $w$ contains only constant letters, and *variable* otherwise.
For $v \in L^*$, $x \in X$ and $a \in C$, $v(x/a)$ denotes the word arising from $v$ by replacing (substituting) the variable letter $x$ by the constant letter $a$, in all places where $x$ occurs in $v$.
A crucial point about substitutions is that, if $L = C \cup \{x\}$ and $a \in C$, the map $s_a : L^* \to C^*$ defined by $s_a(v) = v(x/a)$, is a semigroup homomorphism, the *substitution homomorphism* – in fact a retraction from $L^*$ to $C^*$.

7.5. **Theorem.** *(the classical Hales-Jewett theorem) Assume that* $L = C \cup \{x\}$ *(i.e.* $x$ *is the only variable letter) where* $C \neq \emptyset$, $C^* = B_1 \cup \cdots \cup B_r$ *is a colouring of* $C^*$ *with finitely many colours, and that* $F \subseteq C$ *is finite. Then there is a variable word* $v \in L^*$ *such that* $\{v(x/a) : a \in F\}$ *is monochromatic.*

Similarly, we can derive from the abstract Hales-Jewett theorem other generalizations of the classical one – we allow substitution of constant words $u \in C^*$ for $x$ , instead of just constant letters $a \in C$.

However, it has been observed by Hindman that the abstract version can be derived from the classical one.

## 8. Large sets: thick, central, IP-, and syndetic sets

In van der Waerden's theorem, we are given a finite colouring $\omega = A_1 \cup \dots \cup A_r$; the theorem states that (at least) one of the sets $A_i$ is large in the sense that it includes arbitrarily long arithmetic progressions. Can we say which of the sets $A_i$ has this largeness property? Respectively, in the abstract Hales-Jewett theorem, which of the sets $B_i$ is large in the sense that it includes $\{\sigma(v) : \sigma \in \Sigma\}$, for some $v \in V \setminus W$? In the next section, we will see that piecewise syndetic subsets of $W$ are large in the sense quoted above.

We will consider subsets of an arbitrary semigroup $S$ which can be considered as large, in different respects: thick sets, central sets, IP-sets, syndetic and piecewise syndetic sets. The connection between these notions is as follows:

- thick sets are central
- central sets are both IP-sets and piecewise syndetic.
- syndetic sets are piecewise syndetic.

The families of thick, central, etc. sets enjoy some of the following properties.

A family $\mathcal{A}$ of subsets of $S$ is called upward closed if $A \in \mathcal{A}$ and $A \subseteq B \subseteq S$ imply $B \in \mathcal{A}$. It is partition regular if, for any $A \in \mathcal{A}$ and $A = B \cup C$, $B \in \mathcal{A}$ or $C \in \mathcal{A}$ holds. A family which is both upwards closed and partition regular is a coideal.

We say that $M \subseteq \beta S$ provides an ultrafilter definition of $\mathcal{A} \subseteq \mathcal{P}(S)$ if $\mathcal{A} = \bigcup M$. I.e. $A \in \mathcal{A}$ iff $A \in p$ for some $p \in M$ iff $\hat{A} \cap M \neq \emptyset$.

It is not difficult to see that $\mathcal{A}$ is a coideal iff $\mathcal{A} = \bigcup M$, for some $M \subseteq \beta S$, i.e. iff $\mathcal{A}$ has an ultrafilter definition.

### A. Thick sets

We work in an arbitrary semigroup $(S, \cdot)$; recall that for $x \in S$ and $A \subseteq S$, $x^{-1}A = \{s \in S : xs \in A\}$. A subset $T$ of $S$ is *thick* if the familiy $\{x^{-1}\, T : x \in S\}$ of all backwards translations of $S$ has the finite intersection property.

The following characterizations of thick sets are straightforward.

8.1. **Theorem.** *For a subset $T$ of $S$, the following are equivalent.*
*(a) $T$ is thick*
*(b) for every finite subset $e$ of $S$, there is some $y \in S$ such that $ey \subseteq T$*
*(c) for some $p \in \beta S$, the (closed) left ideal $\beta S \cdot p$ of $\beta S$ is included in $\hat{T}$*
*(d) there is a minimal left ideal $L$ of $\beta S$ such that $L \subseteq \hat{T}$.*

8.2. **Example.** In the additive semigroup $(\omega, +)$, $A \subseteq \omega$ is thick iff it includes arbitrary long finite intervals.

### B. Syndetic sets

Syndetic sets have a simple description, by their very definition, and they are closely connected to thick sets. The notion of syndetic resp. piecewise syndetic sets arises in topological dynamics, a subject we will study later.

$A \subseteq S$ is called syndetic if there is a finite $e \subseteq S$ such

$$S = \bigcup_{t \in e} t^{-1}A,$$

i.e. if $S$ can be covered by finitely many backwards translations of $A$.

8.3. **Example.** In the semigroup $(\omega, +)$, there is a simple elementary description of syndetic subsets: $A$ is syndetic iff it has the *bounded gaps* property: there is some $k \in \omega$ such that each interval of length $k$ in $\omega$ meets $A$. (I.e. the distance between any $x \in \omega$ and the least $y \in A$ satisfying $x < y$ is at most $k$.)

There is a strong connection between thick and syndetic sets:

8.4. **Theorem.** *Assume $A, T \subseteq S$*
*(a) $A$ is syndetic iff it meets every thick subset of $S$ iff $S \setminus A$ is not thick.*
*(b) $T$ is thick iff it meets every syndetic subset of $S$ iff $S \setminus T$ is not syndetic.*

The families of thick resp. of syndetic sets are upwards closed but not partition regular.

**C. Central sets and IP-sets**

The notion of a central set has a short and quite abstract definition, but no simple elementary description. We give an elementary (but not really simple) description of central sets later.

$A \subseteq S$ is said to be central if $A \in p$ for some minimal idempotent $p$ of $\beta S$, i.e. iff the Stone set $\hat{A}$ meets the subset $E_{min}(\beta S)$ of $\beta S$.

8.5. **Proposition.** *Every thick set is central; every central set is an IP-set.*

The families of central resp. of IP-sets are both coideals: this holds for the family of central sets by the ultrafilter definition of centrality and it follows similarly for the family of IP-sets by the ultrafilter characterization of IP-sets in Section 5.

## 9. Large sets: piecewise syndetic sets

We study here another notion of largeness, for subsets of a semigroup. It is less easy to understand than those in the last section, but turns out to be strongly connected with the sets formerly handled.

### A. The notion of a piecewise syndetic set

A subset $A$ of a semigroup $S$ is called piecewise syndetic if there is a finite subset $e$ of $S$ such that $\bigcup_{t\in e} t^{-1}A$ is thick. In particular, every syndetic set is piecewise syndetic.

9.1. **Example.** The characterization of thick subsets of $(\omega, +)$ shows that $A$ is piecewise syndetic in $(\omega, +)$ iff there is some $k \geq 1$ such that there are arbitrarily long intervals in $\omega$ in which at least every $k$'th element is in $A$.

### B. Semigroups operating on sets

The following proofs become more transparent by using the notion of a semigroup operation and some relevant computational rules. In a topological setting, we will study semigroup operations in the next sections.

An operation (or action) of a semigroup $S$ on a set $X$ is a function $\mu : S \times X \to X$ such that, for $x \in X$ and $s, t \in S$, $\mu(1_s, x) = x$ and $\mu(s, \mu(t, x)) = \mu(st, x)$ hold. We abbreviate $\mu(s, x)$ by $s \cdot x$ or $sx$ and then obtain

$$s(tx) = (st)x.$$

E.g. if $S$ is a subsemigroup of $(T, \cdot)$, then $S$ operates on $T$ by left multiplication: $\mu(s, x) = s \cdot x$.

For $S$ operating on $X$, $U \subseteq X$, $s \in S$, and $p \in X$, we put

$$s^{-1}U = \{x \in X : sx \in U\}, \quad R(p, U) = \{s \in S : sp \in U\}.$$

$R(p, U)$ is called the return set of $p$ to $U$.

9.2. **Proposition.** *Assume $S$ operates on $X$, $U \subseteq X$, and $x \in X$.*
*(a) For $t \in S$, we have $t^{-1}R(x, U) = R(x, t^{-1}U)$.*
*(b) Let $e$ be an arbitrary subset of $S$; then $Sx \subseteq \bigcup_{t\in e} t^{-1}U$ is equivalent to $S = \bigcup_{t\in e} t^{-1}R(x, U)$.*
*(c) Let $t, y \in S$ and put $s = ty$. Then $t^{-1}R(x, y^{-1}U) = s^{-1}R(x, U)$.*
*(d) Let $e$ be a subset of $S$, $y \in S$, and define the subset $f$ of $S$ by $f = e \cdot y = \{ty : t \in e\}$. Then*

$$\bigcup_{t\in e} t^{-1}R(x, y^{-1}U) = \bigcup_{s\in f} s^{-1}R(x, U).$$

*In particular if $R(x, y^{-1}U)$ is syndetic, then so is $R(x, U)$.*

### C. The main results on piecewise syndetic sets

Using the technical remarks from 9.2, we obtain the following facts on piecewise syndetic sets.

9.3. **Proposition.** *If $A$ is piecewise syndetic, then $x^{-1}A$ is central, for some $x \in S$, and $\hat{A}$ meets $K(\beta S)$.*

9.4. **Theorem.** *$A \subseteq S$ is piecewise syndetic iff $A \in p$, for some $p \in K(\beta S)$.*
*In particular, every central set is piecewise syndetic.*

This gives an ultrafilter definition for the family of piecewise syndetic sets.

9.5. **Proposition.** *For $p \in K(\beta S)$ and $U \subseteq \beta S$ a neighbourhood of $p$, the return set $R(p, U)$ is syndetic.*

The following theorem connects the notions of syndecity, piecewise syndecity, and centrality.

9.6. **Theorem.** *The following are equivalent, for $A \subseteq S$:*
*(a) $A$ is piecewise syndetic*
*(b) $x^{-1}A$ is central, for some $x \in S$*
*(c) the set $\{x \in S : x^{-1}A \text{ is central }\}$ is syndetic.*

### D. Piecewise syndetic sets and the Hales-Jewett theorem

Piecewise syndetic sets give a refined version of the Hales-Jewett theorem.

9.7. **Theorem.** *Assume that $W$ is a nice subsemigroup of $V$, and $\Sigma$ is a finite set of retractions from $V$ to $W$. Let $B \subseteq W$ be piecewise syndetic. Then there is some $v \in R = V \setminus W$ such that $\{\sigma(v) : \sigma \in \Sigma\} \subseteq B$.*

Similarly, the consequences of the abstract Hales-Jewett theorem can be improved. E.g. van der Waerden's theorem obtains the form that every piecewise syndetic subset of the semigroup $(\omega, +)$ includes arbitrarily long arithmetic progressions.

## 10. Dynamical systems: basic notions

A dynamical system is a topological space with an action of a semigroup on it. We will see that the notion of a dynamical system $X$ over $S$ is a very intuitive one with many fascinating examples, both simple or more involved ones. The theory of dynamical systems over $S$ is a meeting point of topology and the algebra of semigroups: it is strongly connected with the arithmetic on $\beta S$ resp. with combinatorics on $S$.

This theory works more smoothly for monoids than for arbitrary semigroups: a monoid is a semigroup $S$ with an identity $1_S$, i.e. satisfying $1_S \cdot s = s \cdot 1_S = s$ for all $s \in S$.
In actions of a monoid $S$ on a set $X$, we require in addition that $1_S \cdot x = x$ holds for all $x \in X$.

### A. The notion of a dynamical system

A dynamical system over a monoid $S$ is a sequence $(X, S, \mu)$ where $X$ is a non-empty compact Hausdorff space and $\mu$ is an operation of $S$ on $X$ such that for every $s \in S$, the map $x \mapsto s \cdot x$ is continuous.

Dynamical systems over the monoid $(\omega, +)$ are called discrete. In this case, the operation of $(\omega, +)$ on $X$ is determined by the continuous function $t : X \to X$ given by $t(x) = 1 \cdot x = \mu(1, x)$, since $\mu(n, x) = n \cdot x = t^n(x)$. We usually identify $(X, \mu)$ with the pair $(X, t)$.

10.1. **Example.** Here $S$ is the set of non-negative real numbers, a monoid under addition; we interprete the elements of $S$ as points of time. Our space $X$ (non-compact in this example) is the plane $\mathbb{R}^2$; we represent the points of $X$ in the form $x = x_{r\phi} = (r\cos\phi, r\sin\phi)$ where $r \geq 0$ and $\phi \in \mathbb{R}$. Then $X$ becomes a dynamical system over $S$ by putting

$$t \cdot x_{r\phi} = x_{r'\phi'},$$

where $\phi' = \phi + t$ and $r' = re^{-t}$. This means that after time $t$, a point $x$ moves to a point $x'$ with its distance from the origin reduced by $e^{-t}$, and its argument increased by $t$.

### B. (Minimal) Subsystems

Minimal dynamical systems are easier to understand than arbitrary ones; and results on minimal systems can sometimes be used to obtain information about arbitrary ones. We prove an existence and a characterization theorem on these systems.

A subset $Y$ of a dynamical system $X$ over $S$ is invariant if, for $x \in Y$ and $s \in S$, also $sx \in Y$. Then also the closure of $Y$ in $X$ is invariant. We call $Y$ a subsystem of $X$ if it is non-empty, invariant and topologically closed.

A dynamical system is minimal if it has no proper subsystem. The existence theorem for these systems is a standard application of Zorn's lemma.

10.2. **Theorem.** *Every dynamical system has a minimal subsystem.*

10.3. **Notation.** For $X$ a dynamical system over $S$, we denote by $K(X)$ the union of all minimal subsystems of $X$.

For $x \in X$, the orbit of $x$ is $orb(x) = \{sx : s \in S\} = S \cdot x$, the least invariant subset of $X$ containing $x$. E.g. in a discrete dynamical system $(X, t)$, we have $orb(x) = \{t^n(x) : n \in \omega\}$.
The orbit closure of $x$ is $\bar{x} = cl_X orb(x)$, the least subsytem of $X$ containing $x$.
We call a subsystem of $X$ simply generated if it is the orbit closure $\bar{x}$ of some point $x \in X$.

10.4. **Proposition.** *For a dynamical system $X$ over $S$, the following are equivalemt:*
*(a) $X$ is minimal*
*(b) for every $x \in X$, the orbit closure of $x$ equals $X$*
*(c) for every $x \in X$, the orbit of $x$ is a dense subset of $X$*
*(d) for every non-empty open subset $U$ of $X$, there is a finite subset $e$ of $S$ such that $\bigcup_{s \in e} s^{-1}U = X$ (where $s^{-1}U = \{x \in X : sx \in U\}$).*

## 11. $\beta S$ as a dynamical system

For a monoid $(S, \cdot, 1_S)$, the multiplication on $\beta S$ defined in Section 4 gives rise to an operation of $S$ on $\beta S$. In this way, $\beta S$ is a dynamical system over $S$.

We describe its dynamics using the arithmetic of $\beta S$, the important feature being its universal property.

### A. The universal dynamical system $\beta S$

The subsystems of the dynamical system $\beta S$ are easily described:
For $p \in \beta S$, the orbit of $p$ in the dynamical system $\beta S$ is $orb(p) = \{sp : s \in S\} = S \cdot p$.
The orbit closure of $p$ is $\bar{p} = cl_{\beta S} Sp = \beta S \cdot p$.
The simply generated subsystems of $\beta S$ are just the simply generated left ideals of the semigroup $\beta S$. In particular, $\beta S$ is simply generated, being the orbit closure of $1_S \in \beta S$.
Similarly, the subsystems of $\beta S$ are exactly the closed left ideals of $\beta S$.
It follows that the minimal subsystems of $\beta S$ are exactly the minimal left ideals of the semigroup $\beta S$. Hence, the union of all minimal subsystems of the dynamical system $\beta S$ is the union of all minimal left ideals of the semigroup $\beta S$: the notation $K(\beta S)$ has the same meaning both in the dynamical and in the algebraic sense.

We will see that $\beta S$ is the most general simply generated dynamical system over $S$, in the following sense: for $X$ and $Y$ dynamical systems over $S$, a map $f : X \to Y$ is a homomorphism from $X$ to $Y$ if it is continuous and commutes with the operation of $S$ in the sense that, for $x \in X$ and $s \in S$, $f(sx) = sf(x)$.

11.1. **Theorem.** *Given a dynamical system $X$ over $S$ and $x \in X$, there is a unique $S$-homomorphism $f_x : \beta S \to X$ satisfying $f_x(1_s) = x$, namely $f_x(p) = p - lim_{s \in S} sx$. The image of $\beta S$ under $f_x$ is the orbit closure $\bar{x}$ of $x$.*

For $x \in X$ and $p \in \beta S$, we write

$$p \cdot x = px = p - lim_{s \in S} sx = f_x(p).$$

The map assigning $px$ to every $p \in \beta S$ is the homomorphism $f_x$ above, hence continuous.
For $x \in X$ and $p, q \in \beta S$, we have

$$p(qx) = (pq)x,$$

hence this defines an operation of $\beta S$ on $X$. However $X$ does *not* become a dynamical system over $\beta S$ (the map assigning $px$ to $x \in X$ is not necessarily continuous).

### B. $K(\beta S)$ and $K(X)$

For a dynamical system $X$ over $S$, we describe the connection between $K(X)$ (the union of all minimal subsystems of $X$) and $K(\beta S)$, using the homomorphisms $f_x$ above, resp. the notation $px$ for $f_x(p)$.

11.2. **Theorem.** *Let $X$ be a dynamical system over $S$. For $x \in X$, the following are equivalent:*
*(a) $x \in K(X)$*
*(b) there is an $S$-homomorphism $f : \beta S \to X$ such that $x \in f[K(\beta S)]$*
*(c) there are $p \in K(\beta S)$ and $y \in X$ such that $x = py$*
*(d) there is some $p \in K(\beta S)$ such that $x = px$.*
*Moreover in (d), we can pick $p \in L$, for any (minimal) left ideal $L$ of $\beta S$.*

11.3. **Proposition.** *Let $X$ be a dynamical system over $S$ and $x \in X$. Then $x \in K(X)$ iff there is a minimal idempotent $e$ of $\beta S$ satisfying $ex = x$.*
*Moreover if $L$ is a (minimal) left ideal of $\beta S$, we can pick $e \in L$.*

## 12. Recurrent and uniformly recurrent points

Working in a dynamical system $X$ over $S$, we define points of $X$ with special dynamical behaviour, characterize them using the arithmetic of $\beta S$ and show their existence.

### A. Recurrent points

For $x \in X$ and $U \subseteq X$, the return set of $x$ to $U$ is defined by $R(x, U) = \{s \in S : sx \in U\}$. Note that, if $x \in U$, then $1_S \in R(x, U)$.
A point $x$ is said to be recurrent if for every neighbourhood $U$ of $x$, $R(x, U) \neq \{1_S\}$.

It turns out that existence of recurrent points depends to a large extent upon whether $\beta S$ has an idempotent distinct from $1_S$.

12.1. **Example.** (a) If $p \in \beta S$ is an idempotent distinct from $1_S$, then for every $y \in X$, $py$ is recurrent.
(b) If $S$ is a finite group, then there are dynamical systems over $S$ without recurrent points. E.g. consider $X = S$ with the discrete topology, a dynamical system under left multiplication by $S$.

It is not difficult to prove that the following are equivalent, for a monoid $(S, \cdot, 1_S)$:
– $S$ is not a finite group
– there is a (minimal) left ideal of $\beta S$ not containing $1_S$
– $\beta S$ has a minimal idempotent distinct from $1_S$
– $\beta S$ has an idempotent distinct from $1_S$.
Call $S$ trivial if it is a finite group, non-trivial otherwise. So if $S$ is non-trivial, then every dynamical system over $S$ has recurrent points.

12.2. **Theorem.** *A point $x \in X$ is recurrent iff there is some $p \in \beta S$ such that $p \neq 1_S$ and $px = x$.*

Under appropriate conditions, an ultrafilter $p \neq 1_S$ satisfying $px = x$ can even taken to be idempotent. E.g. assume that in $S$, $s, t \neq 1_S$ implies $st \neq 1_S$. Then $T = \{p \in \beta S \setminus \{\, 1_S\} : px = x\}$ is a closed subsemigroup of $\beta S$, and every idempotent of $T$ is as required.

### B. Uniformly recurrent points

Recall from Section 8 the notion of a syndetic subset of $S$. A point $x$ in $X$ is said to be uniformly recurrent if for every neighbourhood $U$ of $x$, the return set $R(x, U)$ is syndetic.

12.3. **Theorem.** *$x \in X$ is uniformly recurrent iff $x \in K(X)$, i.e. $x \in M$ for some minimal subsystem $M$ of $X$. Hence every dynamical system has a uniformly recurrent point.*

For $S$ non-trivial, every uniformly recurrent point in a dynamical system over $S$ is recurrent, as follows from the characterization of trivial monoids.

### C. Dynamics in the shift system $({}^{\omega}c, t)$

We present a simple and intuitive example of a discrete dynamical system, the shift system. It allows for an elementary characterization of recurrent resp. uniformly recurrent points.

12.4. **Example.** We fix a finite non-empty set $c$ (the set of colours), e.g. $c = \{0, 1\}$, with the discrete topology, and consider the product space

$$X = {}^{\omega}c.$$

A point $x$ of $X$ is a sequence $x = (x_0, x_1, \dots)$, where $x_i \in c$, and may be considered as a colouring of $\omega$ by colours in $c$.
$X$ is a discrete dynamical system under the shift map $t : X \to X$ given by $t(x) = (x_1, x_2, \dots)$. For $n \in \omega$, we have $t^n(x) = (x_n, x_{n+1}, x_{n+2}, \dots)$.
A point $y$ of $X$ is recurrent iff every finite (initial) segment of $y$ appears once more in $y$ - in fact, infinitely often.

$y \in X$ is uniformly recurrent iff every finite (initial) segment $u$ of $y$ appears "syndetically often" in $y$. By our former characterization of syndeticity in $(\omega, +)$, this means that there is some $k$ such that every segment of $y$ of length $k$ has $u$ as a subsegment.

## 13. Proximality in dynamical systems

We define the notion of proximality of two points $x, y$ in a dynamical system $X$ over $S$. For the special case of a discrete dynamical system $(X, t)$ with underlying compact metric space $(X, d)$, proximality means that for every $\epsilon > 0$, there is some $n \in \omega$ satisfying $d(t^n x, t^n y) < \epsilon$. The main result here is the Auslander-Ellis theorem; we use it to give some non-trivial characterizations of central subsets of a monoid $S$.

### A. Proximality and the Auslander-Ellis theorem

We work in a compact Hausdorff space $X$. In the product space $X^2 = X \times X$, the diagonal of $X$ is the closed subspace $\Delta = \{(x, x) : x \in X\}$. A subset $V$ of $X^2$ is a neighbourhood of $\Delta$ if there is an open set $W \subseteq X^2$ such that $\Delta \subseteq W \subseteq V$. Intuitively, for "small" $V$, $(x, y) \in V$ means that $x$ is close to $y$.

More specifically, we can choose $W$ to have the form $W = \bigcup_{i \in I}(U_i \times U_i)$ where $(U_i)_{i \in I}$ is an open covering of $X$ (and $I$ is finite, by compactness of $X$). In this case, $(x, y) \in W$ means that $x, y$ are in the same $U_i$, for some $i \in I$.

Assume $X$ is a dynamical system over $S$. For $x, y \in X$ and $W \subseteq X^2$, we put $JR(x, y, W) = \{s \in S : (sx, sy) \in W\}$, the joint return set of $x, y$ to $W$.
$x$ is said to be proximal to $y$ if for every neighbourhood $V$ of $\Delta$, there is some $s \in S$ such that $(sx, sy) \in V$ - i.e. if $JR(x, y, V) \neq \emptyset$. For $V = \bigcup_{i \in I}(U_i \times U_i)$, this means that there are $s \in S$ and $i \in I$ such that $sx, sy \in U_i$.
The relation of being proximal is clearly reflexive and symmetric, but not transitive in general.

13.1. **Example.** Assume that $(X, d)$ is a compact metric space, $t : X \to X$ is continuous, and we consider the discrete dynamical system $(X, t)$. For $\epsilon > 0$, $W = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$ is a neighbourhood of $\Delta$, and $n \in JR(x, y, W)$ says that $d(t^n x, t^n y) < \epsilon$. Thus $x, y$ are proximal iff there is a sequence $(n_i)_{i \in \omega}$ of natural numbers such that $\lim t^{n_i} x = \lim t^{n_i} y$.

13.2. **Proposition.** *$x$ and $y$ are proximal, in a dynamical system $X$ over $S$, iff there is some $p \in \beta S$ such that $px = py$.*

The arithmetic of $\beta S$ proves that every point $x$ of a dynamical system $X$ over $S$ is proximal to a uniformly recurrent point $y$: pick a minimal idempotent $e$ in $\beta S$ and put $y = ex$; then $ey = eex = ex$. The Auslander-Ellis theorem gives a stronger statement.

13.3. **Theorem.** *(J. Auslander, R. Ellis) Assume that $x \in X$ and $M$ is subsystem of $X$ included in the orbit closure $\bar{x}$ of $x$. Then $x$ is proximal to a uniformly recurrent point $y$ in $M$.*

It has been observed by Furstenberg that Hindman's theorem for the semigroup $(\omega, +)$ (a purely mathematical assertion) can be derived directly from the Auslander-Ellis theorem, in fact from the weaker observation that in the shift system $(X = {}^{\omega}c, t)$, every point is proximal to a uniformly recurrent one (again a mathematical assertion).

### B. More about central sets

A subset $C$ of a monoid $S$ is called dynamically central if there exist objects $X, x, y, U$ satisfying

$$C = R(x, U)$$

where $X$ is a dynamical system over $S$ and $x, y \in X$; $y$ is uniformly recurrent and proximal to $x$; $U$ is a neighbourhood of $y$.

Using ideas from the proof of the Auslander-Ellis theorem, we prove that this notion is equivalent to the notion of centrality defined in Section 8, and we obtain a fairly elementary characterization of central sets.

Note that, given the definition of dynamical centrality, it is far from being obvious that supersets of dynamically central sets are dynamically central, and that, for $C = C_0 \cup C_1$

dynamically central, one of the $C_i$ must be dynamically central, i.e. that the family of dynamically central subsets of $S$ is partition regular. But this follows from the theorem:

13.4. **Theorem.** *(V. Bergelson, N. Hindman in a special case; H. Shi, H. Yang in the general case; S. Koppelberg for (c)) For $C \subseteq S$, the following are equivalent:*
*(a) $C$ is central*
*(b) $C$ is dynamically central*
*(c) in the shift space $X = {}^S2$ over $S$, there exists a uniformly recurrent point $y$ proximal to $\chi_C$ satisfying $y(1_S) = 1$.*

The formulation and the proof use a generalization of the shift system in 12.4:

13.5. **Example.** For an arbitrary monoid $(S, \cdot, 1_S)$, the shift space over $S$ is the product space $X = {}^S2$ where $2 = \{0, 1\}$ has the discrete topology. $X$ becomes a dynamical system over $S$ by putting, for $x \in X$ and $s \in S$,

$$(sx)(i) = x(is).$$

For $C \subseteq S$ and $U^* = \{z \in X : z(1_S) = 1\}$, the characteristic function $\chi_C$ of $C$ is a point in $X$ and $U^*$ is a clopen subset of $X$; it turns out that $R(\chi_C, U^*) = C$.

Part (c) of the theorem, together with the characterization of uniformly recurrent points in the shift system ${}^\omega\{0, 1\}$, gives a fairly elementary description of central subsets of $(\omega, +)$.

## 14. Multiple recurrent points

It is a result due to Birkhoff that every discrete dynamical system $(X,t)$ has a recurrent point $x$, which means that for every neighbourhood $U$ of $x$ there is $n \geq 1$ such that $t^n x \in U$. A much stronger result is due to Furstenberg and Weiss, the Multiple Birkhoff Recurrence theorem: for $X$ a minimal dynamical system over a commutative semigroup $S$, $x \in X$, $U$ a neighbourhood of $x$ and $e \subseteq S$ finite, there is $n \geq 1$ such that $\{a^n x : a \in e\} \subseteq U$.

In this section, we assume that $X$ is a dynamical system over a monoid $(S, \cdot, 1_S)$. We define the notion of a multiple recurrent point in Part A; in Part B, we prove the existence of such points under suitable conditions, a result due to Williams, Balcar, and Kalasek stronger than the Furstenberg-Weiss theorem. For the main results of the section, we have to assume that $S$ is commutative.

### A. Definition of multiple recurrent points and remarks

For $e \subseteq S$, call a point $x \in X$ $e$-recurrent if for every neighbourhood $U$ of $x$, there is some $n \geq 1$ such that $\{a^n x : a \in e\} \subseteq U$.
A point $x$ is multiple recurrent if it is $e$-recurrent for every finite $e \subseteq S$.

14.1. **Remark.** (a) Let $a \in S$, $e = \{a\}$ and define $t : X \to X$ by $t(x) = ax$. Then $x$ is $e$-recurrent iff it is a recurrent point of the discrete dynamical system $(X,t)$. Note that this implies $x \in aX = \{ax : x \in X\}$.
(b) For $a, b \in S$ and $e = \{a, b\}$, if $x$ is $e$-recurrent, then $x \in aX \cap bX$. Thus if $aX$ and $bX$ happen to be disjoint, then no $e$-recurrent point exists. This possibility is ruled out by commutativity of $S$.
(c) For $Y$ a subsystem of $X$ and $y \in Y$, clearly $y$ is $e$-recurrent in $X$ iff it is $e$-recurrent in $Y$. Looking for $e$-recurrent points or to multiple recurrent points, we can therefore concentrate to minimal subsystems of $X$.

### B. Existence of multiple recurrent points

We put up some machinery to prove an existence theorem for multiple recurrent points. For $U \subseteq X$, $e \subseteq S$, and $n \in \omega$, let

$$W(e,U,n) = \{y \in X : \{a^n y : a \in e\} \subseteq U\} = \bigcap_{a \in e} a^{-n} U.$$

(This set may be empty.) If $1_S \in e$, then $W(e,U,n) \subseteq U$.

For the rest of the section, we assume that $S$ is a commutative monoid.

14.2. **Theorem.** *(the topological van der Waerden theorem) Assume $U \subseteq X$ is open and meets $K(X)$ and that $e \subseteq S$ is finite. Then $W(e,U,n) \neq \emptyset$ for some $n \geq 1$.*
*Without loss of generality $1_S \in e$, so there is $y \in X$ such that $y \in \{a^n y : a \in e\} \subseteq U$ and $W(e,U,n) \subseteq U$.*

14.3. **Corollary.** *Assume $X = \bigcup_{i \in I} U_i$ is an open covering of $X$ and $e$ is finite. Then there are $i \in I$ and $n \geq 1$ such that $W(e,U_i,n) \neq \emptyset$.*

The proof of the topological van der Waerden theorem uses the abstract version of Gallai's theorem from Section 7. Conversely, Gallai's theorem can be derived from the topological van der Waerden theorem.

14.4. **Corollary.** *Assume that $X$ is a minimal dynamical system, $e \subseteq S$ is finite, and $B \subseteq X$ is open and non-empty. Then*

$$\mathcal{D}_e(B) = \{W(e,V,n) \neq \emptyset : n \geq 1, V \text{ open}, V \subseteq B \text{ or } V \subseteq X \setminus B\}$$

*is a $\pi$-base for $X$. I.e. for every non-empty open $U \subseteq X$ there is some $W \in \mathcal{D}_e(B)$ such that $W \subseteq U$.*

14.5. **Theorem.** *(S. Williams, B. Balcar, P. Kalasek) Assume $X$ is minimal and has a countable base and that $S$ is countable. Then the set $MR$ of multiple recurrent points of $X$ is non-empty – in fact dense in $X$.*
*This holds also under MA($\kappa$), for $\kappa < 2^\omega$, if $X$ satisfies the countable chain condition, $|S| \leq \kappa$, and $w(X) \leq \kappa$.*

## 15. Some diophantine approximation

### A. Introduction – the circle group $\mathbb{T}$ and a theorem by Kronecker

Working in the additive group $\mathbb{R}$ of real numbers, we have the set $\mathbb{Z}$ of integers as a subgroup, and we define the circle group $\mathbb{T}$ as the quotient $\mathbb{R}/\mathbb{Z}$. I.e. $x, y \in \mathbb{R}$ are identified modulo $\mathbb{Z}$ iff their difference $x - y$ is an integer. So $\mathbb{T}$ is a compact topological group; topologically we think about it as the unit interval $[0, 1]$ with the ends $0, 1$ identified. For $x, y \in \mathbb{T}$, let us denote by $\delta(x, y)$ the distance of $x$ and $y$ in $\mathbb{T}$, i.e. the length of the (shorter) arc on the circle $\mathbb{T}$ connecting $x$ and $y$.

For a real number $x$, we will for simplicity write $x$ for its equivalence class $\bar{x} = \{y \in \mathbb{R} : x - y \in \mathbb{Z}\}$. The following result can be proved in a purely elementary manner. ( In fact, Kronecker proved much stronger results.)

15.1. **Theorem.** *(L. Kronecker) Assume that $a \in \mathbf{T}$ and $\epsilon > 0$. Then there is some $n \geq 1$ such that $\delta(an, 0) < \epsilon$.*

More exactly, there are $m \in \mathbb{Z}$ and $n \geq 1$ such that (in $\mathbb{R}$) $|an - m| < \epsilon$, resp. $|a - m/n| < \epsilon/n$, holds – the rational number $m/n$ is an approximation of $a$.

It is the aim of this section to prove a vast generalization of Kronecker's theorem: the term $an$ (where $a \in \mathbb{R}$ and $n \in \omega$) can be replaced by a more general one: a polynomial $p(n)$ with constant term zero, i.e. satisfying $p(0) = 0$. Moreover the theorem holds for finitely many polynomials simultaneously.

### B. Furstenberg's theorem

The notion of an IP-set was considered in Sections 5 resp. 8. In an arbitrary semigroup $S$, we call $A \subseteq S$ an IP*-set if $A \cap E \neq \emptyset$ holds for every $IP$-set $E \subseteq S$. The following proposition is a routine consequence of Section 5.

15.2. **Proposition.** *$A$ is an IP*-set iff $A \in p$ holds for every idempotent $p \in \beta S$. Hence the family of IP*-subsets of $S$ is a filter on $S$.*

The main result of this section is due to Furstenberg.

15.3. **Theorem.** *(H. Furstenberg) Assume that $F$ is a finite set of polynomials with real coefficients and $f(0) = 0$ for each $f \in F$, and that $\epsilon > 0$. Then the set*

$$A_\epsilon = \{n \in \omega : \delta(f(n), 0) < \epsilon \text{ holds for each } f \in F\}$$

*is IP*, in $(\omega, +)$.*

E.g. putting $f(x) = ax^2$ where $a \in \mathbb{R}$, we obtain as a very special case an approximation theorem due due Hardy and Littlewood: for every $\epsilon > 0$, there are $m \in \mathbb{Z}$ and $n \geq 1$ such that $|a - m/n^2| < \epsilon/n^2$.

### C. Techniques for the proof

The proof of the theorem uses the arithmetic of $(\beta\omega, +)$.

15.4. **Lemma.** *Assume $(x_n)_{n\in\omega}$ is a sequence in a compact Hausdorff space and that $p, q \in \beta\omega$. Then*

$$(p + q) - lim_{r\in\omega}\, x_r = p - lim_{s\in\omega}\, q - lim_{t\in\omega}\, x_{s+t}.$$

*In particular if $p + p = p$, then $p - lim_{r\in\omega}\, x_r = p - lim_{s\in\omega}\, p - lim_{t\in\omega}\, x_{s+t}$.*

In the compact space $\mathbb{T}$, this gives the following results.

15.5. **Proposition.** *For $a \in \mathbb{R}$, $\epsilon > 0$ and $p = p + p$ in $\beta\omega$, we have $p - lim_{n\in\omega}\, na = 0$, in $\mathbb{T}$. Thus $A_\epsilon = \{n \in \omega : \delta(na, 0) < \epsilon\} \in p$.*

15.6. **Proposition.** *For $a \in \mathbb{R}$, $\epsilon > 0$, $k \geq 1$ and $p = p+p$ in $\beta\omega$, we have $p - lim_{n\in\omega}\, n^k a = 0$, in $\mathbb{T}$. Thus $A_\epsilon = \{n \in \omega : \delta(n^k a, 0) < \epsilon\} \in p$.*

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