CHAPTER 1

Classical Calderón–Zygmund decomposition, and real interpolation

We start with the Riesz rising sun lemma, and then pass to its many-dimensional substitute, i.e., the Calderón–Zygmund lemma. We discuss some applications of the latter, the John–Nirenberg and Campanato–Morrey inequalities being among them. To relate the matter with real interpolation, we observe that the so-called "good" part of the Calderón–Zygmund decomposition is a near-minimizer for the couple (L^1, L^∞). Next, in a rather elementary setting, we prove that certain near-minimizers are stable under the action of *long-range regular operators.*

1. Riesz rising sun lemma and the Calderón–Zygmund procedure

1.1. Riesz rising sun lemma. In 1932, F. Riesz (see [**R**]) suggested a simple proof of the Hardy–Littlewood maximal theorem based on the following lemma, the geometric interpretation of which has led to its name.

LEMMA 1.1. (Rising sun lemma) *Given a function $f \in L^1(\mathbb{R})$ and a number $t > 0$, there exist at most countably many closed intervals $Q_i = [a_i, b_i]$ $(i \in I)$ with disjoint interiors such that*

$$\frac{1}{b_i - a_i} \int_{a_i}^{b_i} |f(s)|\, ds = t, \qquad i \in I, \tag{1.1}$$

and

$$\left\| f \chi_{\mathbb{R} \setminus \bigcup Q_i} \right\|_{L^\infty} \leq t. \tag{1.2}$$

PROOF. Consider the function

$$F(x) = \int_0^x |f(s)|\, ds - tx. \tag{1.3}$$

Since $f \in L^1$, it is clear that $F(x)$ is continuous and satisfies

$$\lim_{x \to -\infty} F(x) = \infty, \qquad \lim_{x \to \infty} F(x) = -\infty. \tag{1.4}$$

Consider the graph of $F(x)$ and perceive it as a mountain chain illuminated by rising sun (the rays arrive from the right and are parallel to the x-axis); see Figure 1. Let Ω denote the projection to the x-axis of the shadowed pieces of the graph.

Now, $x \in \Omega$ means that there exist y with

$$y > x \text{ and } F(y) > F(x).$$

Since F is continuous, relations (1.4) imply that Ω is open and, consequently, it splits into at most countably many open intervals (a_i, b_i). At the ends of each

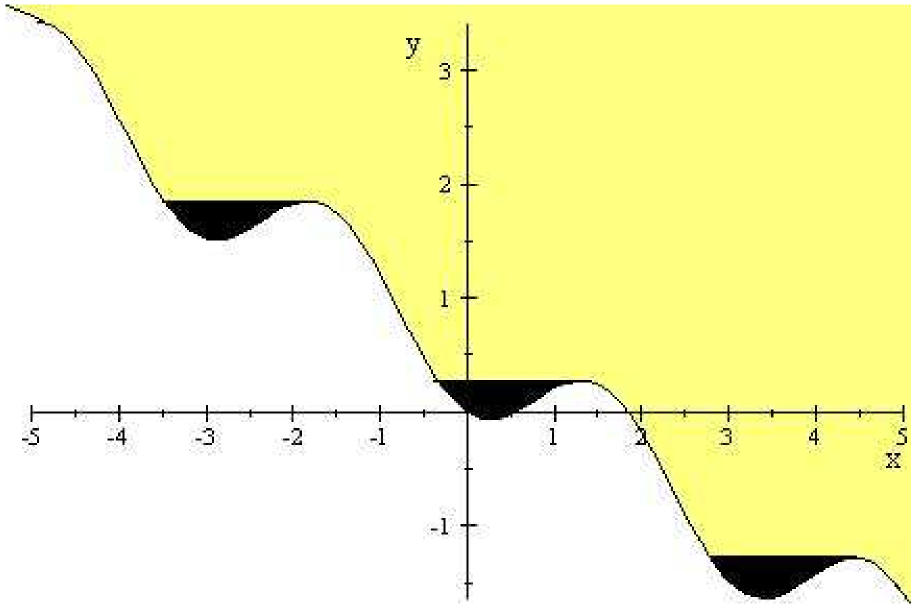

FIGURE 1. The graph of $F(x)$ illuminated by light rays parallel to the x-axis and arriving from the right.

interval, $F(x)$ takes equal values, that is, $F(a_i) = F(b_i)$. Thus,

$$\int_0^{a_i} |f(s)|\, ds - ta_i = \int_0^{b_i} |f(s)|\, ds - tb_i,$$

which implies (1.1).

We prove (1.2). If x lies outside Ω, then the point $(x, F(x))$ of the graph is not shadowed, which means that $F(x) \geq F(x + h)$ for every $h > 0$, implying

$$\int_0^{x} |f(s)|\, ds - tx \geq \int_0^{x+h} |f(s)|\, ds - t(x + h).$$

Consequently,

$$\frac{1}{h}\int_x^{x+h} |f(s)|\, ds \leq t$$

for $h > 0$; by the Lebesgue theorem (Theorem 0.19), we have $|f(x)| \leq t$ a.e. outside Ω, which proves (1.2). □

1.2. Calderón–Zygmund lemma. In 1952, for the purposes of the theory of singular integrals, A. Calderón and A. Zygmund (see [**CZ**]) proved the following statement, which can be viewed as a multidimensional analog of the rising sun lemma. We refer the reader to “Definitions, notation, and some standard facts” after the Introduction for the notion of dyadic cubes, to be used in the proof.

LEMMA 1.2. (Calderón–Zygmund) *Given a finction $f \in L^1(\mathbb{R}^n)$ and a number $t > 0$, there exists an at most countable family Q_i $(i \in I)$ of dyadic cubes with* *mutually disjoint interiors and such that*

$$t \le \frac{1}{|Q_i|} \int_{Q_i} |f(s)|\, ds < 2^n t, \qquad i \in I, \tag{1.5}$$

$$\left\| f \chi_{\mathbb{R}^n \setminus \bigcup Q_i} \right\|_{L^\infty} \le t. \tag{1.6}$$

Note that (1.6) is a precise analog of (1.2), but instead of equality in (1.1) we only have equivalence in (1.5).

PROOF. In the set D of dyadic cubes, consider the subset $D_{f,t}$ of all cubes satisfying the left inequality in (1.5):

$$D_{f,t} = \left\{ Q \in D \ : \ \frac{1}{|Q|} \int_Q |f(s)|\, ds \ge t \right\}. \tag{1.7}$$

For the role of $\{Q_i\}_{i \in I}$, we can take the family of maximal cubes in $D_{f,t}$ (a cube $Q \in D_{f,t}$ is *maximal* if it it is not included in any other cube of the family $D_{f,t}$). If Q_i is a maximal cube in $D_{f,t}$, take the dyadic cube $\tilde{Q}_i$ containing Q_i and of volume $2^n |Q_i|$. Then $\tilde{Q}_i$ is not in $D_{f,t}$, whence

$$\frac{1}{|Q_i|} \int_{Q_i} |f(s)|\, ds \le \frac{2^n}{\left|\tilde{Q}_i\right|} \int_{\tilde{Q}_i} |f(s)|\, ds < 2^n t.$$

Thus, Q_i satisfies (1.5). To prove (1.6), first we observe that, since $f \in L^1$, for every cube $Q \in D_{f,t}$ we have

$$|Q| \le \frac{1}{t} \int_Q |f(s)|\, ds \le \frac{1}{t} \|f\|_{L^1}. \tag{1.8}$$

This volume restriction implies that an arbitrary cube $Q \in D_{f,t}$ is included in some maximal cube $Q_i \in D_{f,t}$. We see that if $x \in \mathbb{R}^n \setminus \bigcup Q_i$, then any dyadic cube Q containing x is not in $D_{f,t}$, whence

$$\frac{1}{|Q|} \int_Q |f(s)|\, ds < t.$$

Now, the Lebesgue theorem (see Theorem 0.19) shows that $|f(x)| \le t$ for a.e. $x \in \mathbb{R}^n \setminus \bigcup Q_i$, which is (1.6). □

We mention a "more constructive" form of the same procedure in the original exposition by Calderón and Zygmund. They started with splitting $\mathbb{R}^n$ into congruent dyadic cubes of volume greater than $\frac{1}{t}\|f\|_{L^1}$. This ensures the inequality $\frac{1}{|Q|}\int_Q |f(s)|\, ds < t$ for each of them; also, it is easily seen that any member of $D_{f,t}$ is included in one of these cubes (see (1.8)). At the next step, any such cube underwent splitting in 2^n congruent dyadic cubes, among which those belonging to $D_{f,t}$ (i.e., satisfying $\frac{1}{|Q|}\int_Q |f(s)|\, ds \ge t$) were selected. The procedure was repeated consecutively for the remaining cubes (no cubes selected at some stage underwent further splitting). Clearly, this leads to the same family $\{Q_i\}_{i \in I}$ of cubes as above. An experienced reader will readily recognize here a stopping time argument from the theory of martingales.

It should be noted that in place of the *standard* dyadic grid D we could work with any grid D_1 obtained from D by dilation and shift. A useful application of this is as follows: if Q is an arbitrary cube, we can choose D_1 in such a way that it contains Q.

1.3. Calderón–Zygmund decomposition. Amazingly, the simple geometric lemmas discussed above lead to deep results in Fourier analysis. A bridge to many of them is splitting a function $f \in L^1(\mathbb{R}^n)$ in a "good" and a "bad" part.

As before, suppose that we are given a function $f \in L^1(\mathbb{R}^n)$ and a number $t > 0$. Taking the family $\{Q_i\}_{i\in I}$ from the Calderón–Zygmund lemma, we introduce the "good" part of f by the formula

$$f_t = \sum_{i\in I} \left(\frac{1}{|Q_i|} \int_{Q_i} f(s)ds \right) \chi_{Q_i} + f\,\chi_{\mathbb{R}^n \setminus \bigcup Q_i}. \tag{1.9}$$

The "bad" part is then given by

$$f - f_t = \sum_{i\in I} \left(f - \frac{1}{|Q_i|} \int_{Q_i} f(s)ds \right) \chi_{Q_i}. \tag{1.10}$$

The Calderón–Zygmund decomposition of f (at the level t) is defined to be

$$f = f_t + (f - f_t)\,. \tag{1.11}$$

With this decomposition, we relate two linear operators $P_{f,t}$ and $I - P_{f,t}$, given by

$$P_{f,t}(h) = \sum_{i\in I} \left(\frac{1}{|Q_i|} \int_{Q_i} h(s)ds \right) \chi_{Q_i} + h\chi_{\mathbb{R}^n \setminus \bigcup Q_i} \tag{1.12}$$

and

$$(I - P_{f,t})(h) = \sum_{i\in I} \left(h - \frac{1}{|Q_i|} \int_{Q_i} h(s)ds \right) \chi_{Q_i}. \tag{1.13}$$

They (and especially their analogs in subtler settings) will be of much use in what follows. Clearly,

$$P_{f,t}(f) = f_t,\ \ (I - P_{f,t})f = f - f_t. \tag{1.14}$$

The Calderón–Zygmund decomposition possesses a series of simple but important properties. We collect them in the lemma below. In order to state it, we denote

$$B_{L^\infty}(t) = \{g \in L^\infty :\ \|g\|_{L^\infty} \le t\} \tag{1.15}$$

(this is the ball of radius t and centered at zero in L^∞) and introduce the distance functional

$$\operatorname{dist}_{L^1}(f, B_{L^\infty}(t)) = \inf_{g\in B_{L^\infty}(t)} \|f - g\|_{L^1}\,. \tag{1.16}$$

Also, we refer the reader to "Definitions, notation, and some standard facts" after the Introduction for the definition of the space BMO.

LEMMA 1.3. *The Calderón–Zygmund decomposition has the following properties.*

(CZ1) (A capacity estimate.) *The volumes of the cubes Q_i, $i \in I$, obey the inequality*

$$\sum_{i\in I} |Q_i| \le \frac{2}{t} \operatorname{dist}_{L^1}(f, B_{L^\infty}(\frac{t}{2})). \tag{1.17}$$

(CZ2) (Splitting the "bad" function into blocks.) *Put*

$$b_i = \left(f - \frac{1}{|Q_i|}\int_{Q_i} f(s)ds\right)\chi_{Q_i} \qquad i \in I. \tag{1.18}$$

Then

$$f - f_t = \sum_{i\in I} b_i \tag{1.19}$$

and all the blocks b_i, $i \in I$ *satisfy*

$$\operatorname{supp} b_i \subset Q_i, \qquad \int_{\mathbb{R}^n} b_i(s)ds = 0, \qquad \|b_i\|_{L^1} < 2^{n+1}t\,|Q_i|\,. \tag{1.20}$$

(CZ3) (Approximation property.) *We have*

$$\|f_t\|_{L^\infty} \le 2^n t; \tag{1.21}$$

$$\|f - f_t\|_{L^1} \le 2^{n+2}\operatorname{dist}_{L^1}(f, B_{L^\infty}(\frac{t}{2})). \tag{1.22}$$

(CZ4) (Operator estimates.) *We have*

$$\|P_{f,t}\|_{L^1\to L^1} \le 1, \qquad \|I - P_{f,t}\|_{\mathrm{BMO}\to L^1} \le \frac{2}{t}\operatorname{dist}_{L^1}(f, B_{L^\infty}(\frac{t}{2})). \tag{1.23}$$

Proof. (CZ1) Let $g \in B_{L^\infty}(\frac{t}{2})$. The left inequality in (1.5) shows that

$$\begin{aligned}\|f-g\|_{L^1} &\ge \sum_{i\in I}\int_{Q_i}|f-g| \ge \sum_{i\in I}\int_{Q_i}|f| - \sum_{i\in I}\int_{Q_i}|g| \\ &\ge t\sum_{i\in I}|Q_i| - \frac{t}{2}\sum_{i\in I}|Q_i| = \frac{t}{2}\sum_{i\in I}|Q_i|\,.\end{aligned}$$

Since $g \in B_{L^\infty}(\frac{t}{2})$ is arbitrary, we obtain (1.17).

(CZ2) This is an immediate consequence of (1.10) and the right inequality in (1.5).

(CZ3) Inequality (1.21) follows from the definition (1.9) and the Calderón–Zygmund lemma. By (1.19) and (1.20), we see that

$$\|f - f_t\|_{L^1} < 2^{n+1}t\sum_{i\in I}|Q_i|\,.$$

It remains to apply (1.17).

(CZ4) The inequality $\|P_{f,t}(h)\|_{L^1} \le \|h\|_{L^1}$ is a consequence of the definition (1.12). If $h \in \mathrm{BMO}$, then (1.13) implies

$$\begin{aligned}\|(I - P_{f,t})h\|_{L^1} &= \sum_{i\in I}\int_{Q_i}\left|h(y) - \frac{1}{|Q_i|}\int_{Q_i}h(s)\,ds\right|dy \\ &\le \|h\|_{\mathrm{BMO}}\sum_{i\in I}|Q_i|\,.\end{aligned}$$

It remains to refer to (1.17). □

1.4. A weak type inequality for linear operators. We want to illustrate the use of the Calderón–Zygmund decomposition. The arguments that follow are a slight formalization of a procedure in [**CZ**]. We need a definition.

DEFINITION 1.4. Suppose $1 < p \le \infty$, and let $T : L^p \to L^p$ be a bounded linear operator. This operator is said to be *long-range L^1-regular* (or often simply long-range regular) if for every function $f \in L^p \cap L^1$ supported on a cube Q and having zero average (i.e., $\int f(s)ds = 0$) we have

$$\int_{\mathbb{R}^n \setminus 2Q} |Tf(s)|\, ds \le \mathrm{lr}(T)\, \|f\|_{L^1} \tag{1.24}$$

with a constant $\mathrm{lr}(T) > 0$ independent of f and Q.

Surely, T is long-range regular if it is bounded on L^1. However, we shall see that some very important operators not bounded on L^1 are long-range regular. At the moment we show (under the guidance of [**CZ**]) that the long-range regular operators send L^1 to $L^{1,\infty}$.

THEOREM 1.5. *Let $1 < p \le \infty$, and let T be a linear operator that maps the space $L^p = L^p(\mathbb{R}^n)$ into itself boundedly. If T is long-range L^1-regular, then for all $f \in L^p \cap L^1$ we have*

$$\|Tf\|_{L^{1,\infty}} \le C\, \|f\|_{L^1} \tag{1.25}$$

with a constant $C > 0$ independent of f.

We remind the reader that the space $L^{1,\infty}$ (or weak-L^1) is determined by the quasinorm $\|f\|_{1,\infty} = \sup_{t>0} t|\{|f| > t\}|$.

REMARK 1.6. An operator T satisfying (1.25) (equivalently, satisfying $|\{|Tf| > t\}| \le ct^{-1}\|f\|_{L^1}$, $t > 0$) is said to be *of weak type* $(1, 1)$. This definition applies also to sublinear[1] operators that may fail to be linear.

REMARK 1.7. The proof below yields $C \le d(2^{2n(1-1/p)}\|T\|_{L^p \to L^p} + \mathrm{lr}(T))$, where d is a numerical constant. We leave apart the question about the optimality of this estimate.

REMARK 1.8. The theorem shows that T extends by continuity up to a bounded operator from L^1 to $L^{1,\infty}$. For this extension, (1.24) remains true if, instead of $f \in L^p \cap L^1$, we demand merely that $f \in L^1$.

PROOF. We must prove that

$$|\{x : |(Tf)(x)| > t\}| \le Ct^{-1}\, \|f\|_{L^1}$$

for all $t > 0$. Fixing t, we consider the Calderón–Zygmund decomposition $f = f_{wt} + (f - f_{wt})$ (see (1.11)), for some positive constant w. Next, we take two more positive numbers u and v such that $u + v = 1$. Since

$$\{x : |(Tf)(x)| > t\} \subset \{x : |(Tf_{wt})(x)| > ut\} \cup \{x : |T(f - f_{wt})(x)| > vt\}\,, \tag{1.26}$$

it suffices to estimate the measures of the sets on the right separately.[2]

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[1] An operator T is *sublinear* if $|T(\alpha f + \beta g)| \le |\alpha||Tf| + |\beta||Tg|$

[2] If we are not interested in an estimate for C (see Remark 1.7), we can simply take, say, $w = 1$ and $u = v = 1/2$.

First, assume that $p < \infty$ (otherwise the proof simplifies; see below). The definition (1.9) of the "good" function f_{wt} implies

$$\|f_{wt}\|_{L^1} \le \|f\|_{L^1};$$

consequently, $f_{wt} \in L^1 \cap L^\infty \subset L^p$. Since T is bounded on L^p and $\|f_{wt}\|_{L^\infty} \le 2^n wt$ (cf. (1.21)), we see that

$$\begin{aligned} |\{x : |(Tf_{wt})(x)| > ut\}| &\le \frac{1}{(ut)^p} \int_{\mathbb{R}^n} |Tf_{wt}(s)|^p \, ds \\ &\le \frac{1}{(ut)^p} \|T\|^p_{L^p \to L^p} \int_{\mathbb{R}^n} |f_{wt}(s)|^p \, ds \le \frac{1}{(ut)^p} \|T\|^p_{L^p\to L^p} (2^n wt)^{p-1} \|f_{wt}\|_{L^1} \\ &\le \|T\|^p_{L^p\to L^p} (2^n w)^{p-1} \frac{1}{u^p} \frac{1}{t} \|f\|_{L^1}. \end{aligned} \tag{1.27}$$

To estimate the second summand in (1.26), we observe that

$$|\{x : |T(f - f_{wt})(x)| > vt\}| \le |\cup 2Q_i| + |\{x \in \mathbb{R}^n \setminus \cup 2Q_i : |T(f - f_{wt})(x)| > vt\}|.$$

We remind the reader that the "bad" function $f - f_{wt}$ is the sum of the blocks $b_i = (f - \frac{1}{|Q_i|}\int_{Q_i} f(s)ds)\chi_{Q_i}$ with properties (1.20). Now, long-range regularity implies

$$\begin{aligned} |\{x \in \mathbb{R}^n \setminus \cup 2Q_i : |T(f - f_{wt})(x)| > vt\}| &\le \frac{1}{vt} \int_{\mathbb{R}^n \setminus \cup 2Q_i} |T(f - f_{wt})(s)| \, ds \\ \le \frac{1}{vt} \sum_{i\in I} \int_{\mathbb{R}^n \setminus 2Q_i} |(Tb_i)(s)| \, ds &\le \operatorname{lr}(T) \frac{1}{vt} \sum_{i \in I} \int_{\mathbb{R}^n} |b_i(s)| \, ds \le \frac{2\operatorname{lr}(T)}{vt} \int_{\mathbb{R}^n} |f(s)| \, ds. \end{aligned}$$

Therefore, the inequality

$$\Big|\bigcup 2Q_i\Big| \le 2^n \sum_{i\in I} |Q_i| \tag{1.28}$$

combined with (1.5) yields

$$\begin{aligned} |\{x : |T(f - f_{wt})(x)| > vt\}| &\le 2^n \sum_{i \in I} |Q_i| + \frac{2\operatorname{lr}(T)}{vt} \int_{\mathbb{R}^n} |f(s)| \, ds \\ &\le \left(\frac{2^n}{w} + \frac{2\operatorname{lr}(T)}{v}\right) \frac{1}{t} \int_{\mathbb{R}^n} |f(s)| \, ds. \end{aligned} \tag{1.29}$$

Since $\sum_{i\in I} |Q_i| \le t^{-1} \int_{\mathbb{R}^n} |f(s)| ds$ by (1.5), inequalities (1.27) and (1.29) show that, indeed, the measure of the set where $|Tf(x)| > t$ does not exceed $Ct^{-1}\|f\|_{L^1}$, as required. The constant C can easily be written out; the expression for it will involve the norm of T on L^p, the quantity $\operatorname{lr}(T)$, and three positive parameters u, v, and w ($u + v = 1$). Minimization over these parameters yields Remark 1.7.

For $p = \infty$, we simply put $u = v = 1/2$ and $w = 1/(4 \cdot 2^n \|T\|_{L^\infty \to L^\infty})$. Then the first summand in (1.26) is 0, and the above estimates imply inequality (1.25) with $C = d(2^{2n}\|T\|_{L^\infty\to L^\infty} + \operatorname{lr}(T))$. □

It should be mentioned that long-range regularity is typical of singular integral operators. The basics of their theory will be presented in the sequel. For the moment we mention that the Hilbert transformation, or the projection Q onto the gradient vector fields (discussed in the Introduction), or the partial sum operators related to wavelet expansions are all long-range regular.

1.5. Hardy–Littlewood maximal operator. We remind the reader that the Hardy–Littlewood maximal operator is defined on the functions $f \in L^1_{\text{loc}}$ by the formula

$$Mf(x) = \sup_{Q\,:\,x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy. \tag{1.30}$$

Though nonlinear, this operator is subadditive, i.e.,

$$M(\lambda f) = |\lambda|\, Mf, \qquad M(f+g) \le Mf + Mg. \tag{1.31}$$

Also, it is bounded on L^∞, that is

$$\|Mf\|_{L^\infty} \le \|f\|_{L^\infty}.$$

Since also $Mf \ge |f|$ by the definition (1.30) and the Lebesgue theorem (Theorem 0.19), we see that in fact

$$\|Mf\|_{L^\infty} = \|f\|_{L^\infty}. \tag{1.32}$$

The operator M is of weak type (1, 1), i.e., satisfies the inequality

$$\|Mf\|_{L^{1,\infty}} \le c\,\|f\|_{L^1} \tag{1.33}$$

with a constant $c > 0$ depending only on the dimension n. This fact is very well known, so, rather, the emphasis in the statement below is on an intimate relationship between Lebesgue sets of Mf and the cubes involved in the Calderón–Zygmund decompositions of f. (This statement does imply (1.33), though.) The techniques employed will be similar to but somewhat different from those in the proof of Theorem 1.5.[3]

THEOREM 1.9. *Suppose that $f \in L^1$ and $s > 0$. Put $t = \frac{s}{10^{n+1}}$ and denote by Q_i, $i \in I$, the cubes occurring in the Calderón–Zygmund decomposition of f at the level t. Then*

$$\{x : Mf(x) > s\} \subset \bigcup_{i\in I} 2Q_i. \tag{1.34}$$

PROOF. By the subadditivity of M, we can write (as before):

$$\{x : Mf(x) > s\} \subset \left\{x : Mf_t(x) > \frac{s}{2}\right\} \cup \left\{x : M(f-f_t)(x) > \frac{s}{2}\right\}. \tag{1.35}$$

The choice of t shows that $\|f_t\|_{L^\infty} \le 2^n t \le \frac{s}{2}$ (see (1.21)), whence $|Mf_t| \le s/2$ everywhere. Therefore, the first set on the right in (1.35) is empty. We claim that the second set is included in $\bigcup_{i\in I} 2Q_i$.

To establish the claim, take $x \notin \bigcup 2Q_i$. We again use the formula $f - f_t = \sum_{i\in I} b_i$, where the blocks b_i possess the properties (1.20). Therefore, for every cube Q containing x we have

$$\frac{1}{|Q|}\int_Q |f - f_t| \le \frac{1}{|Q|}\sum_{i:Q_i\cap Q\ne\varnothing}\int_{Q_i}|b_i| \le \frac{2^{n+1}t}{|Q|}\sum_{i:Q_i\cap Q\ne\varnothing}|Q_i|. \tag{1.36}$$

[3]Subsequently, it will be explained that (1.33) can be obtained by direct application of Theorem 1.5 to a certain linear and long-range regular operator bounded on L^∞. Surely, this is also well known.

Since x is in Q but outside $\bigcup_{i\in I} 2Q_i$, we see that any Q_i intersecting Q is included in $5Q$. Since the interiors of the Q_i are disjoint, it follows that

$$\sum_{i:Q_i\cap Q\neq\varnothing} |Q_i| \leq |5Q| = 5^n\,|Q|\,.$$

So, by (1.36), for $x \notin \cup 2Q_i$ we obtain

$$\begin{aligned} M(f-f_t)(x) &= \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f-f_t| \leq \sup_{Q\ni x} \frac{2^{n+1}t}{|Q|} \sum_{i:Q_i\cap Q\neq\varnothing} |Q_i| \\ &\leq 2^{n+1}5^n t \leq \frac{s}{2}, \end{aligned}$$

proving the claim. □

As has already been mentioned, the weak type inequality (1.33) is a consequence of the above theorem. Moreover, it implies the following (formally) stronger *Kolmogorov inequality.*

Theorem 1.10. *For every $s > 0$, we have*

$$|\{x:\, Mf(x) > s\}| \leq \frac{c}{s}\operatorname{dist}_{L^1}(f, B_{L^\infty}(\frac{s}{c}))$$

with a constant c depending only on the dimension n.

Proof. Indeed, we put $t = \frac{s}{10^{n+1}}$ as in the preceding theorem and combine it with Lemma 1.3 to obtain

$$\begin{aligned} |\{x:\, Mf\,(x) > s\}| &\leq \Big|\bigcup_{i\in I} 2Q_i\Big| \leq 2^n\cdot\frac{2}{t}\operatorname{dist}_{L^1}\Big(f, B_{L^\infty}\Big(\frac{t}{2}\Big)\Big) \\ &= 20^{n+1}\frac{1}{s}\operatorname{dist}_{L^1}\Big(f, B_{L^\infty}\Big(\frac{s}{20^{n+1}}\Big)\Big). \end{aligned}$$

□

We include a well-known calculation based on the Kolmogorov inequality (i.e., on Theorem 1.10) and showing that the operator M is bounded on L^p for $1 < p < \infty$. Since $\operatorname{dist}_{L^1}(f, B_{L^\infty}(s/c)) \leq \int_{|f|\geq s/c} |f(x)|dx$, we can write

$$\begin{aligned} \|Mf\|_{L^p}^p &= p\int_0^\infty s^{p-1}|\{Mf > s\}|\,ds \leq C\int_0^\infty s^{p-2}\int_{|f|>s/c} |f(x)|\,dx\,ds \\ &= C\int_{\mathbb{R}^n} |f(x)| \int_0^{c|f(x)|} s^{p-2}\,ds\,dx = C_1\int_{\mathbb{R}^n} |f(x)|^p\,dx. \end{aligned}$$

2. Norms on BMO and Lipschitz spaces

Here we present two results the proofs of which are spectacular applications of the Calderón–Zygmund procedure. The arguments are similar in nature, but differ at a point, so we give them separately. Both results are about the Morrey–Campanato spaces $\dot{C}_p^{s,k}$ defined in (0.14), (0.15). (The definitions will be reproduced and discussed below.) Now our interest lies in the case where the smoothness parameter s is nonnegative; then, for historical reasons, it is natural to call the spaces in question simply *Campanato spaces.* The first result says that, in the case of zero smoothness, the space does not depend on p for $1 \leq p < \infty$.

PROPOSITION 1.11. BMO $= \dot{C}_p^{0,1}$ *for every* $1 \le p < \infty$.

The second result says that for $s > 0$ the equivalence of norms on $\dot{C}_p^{s,k}$ extends up to $p = \infty$ inclusive.

PROPOSITION 1.12. *If* $s > 0$, *the space* $\dot{C}_p^{s,k}$ *does not depend on* p, $1 \le p \le \infty$. *In other words, the functional*

$$f \mapsto \sup_Q \inf_{q \in \mathcal{P}_k} |Q|^{|s|/n} \operatorname{ess\,sup}_Q |f - q| \tag{1.37}$$

(*see* (0.15)) *is an equivalent seminorm on* $\dot{C}_p^{s,k}$ *for every finite* $p \ge 1$.

2.1. John–Nirenberg inequality. We remind the reader that BMO is the space of locally integrable functions f on $\mathbb{R}^n$ for which the following seminorm is finite:

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q|,$$

where $f_Q = |Q|^{-1} \int_Q f$ and the supremum is taken over all cubes Q in $\mathbb{R}^n$ (as usual, with edges parallel to coordinate axes). This seminorm vanishes precisely on the constant functions; so, BMO is sooner a space of classes modulo constants than of functions.

In fact, up to renorming, we deal here with the (Morrey–)Campanato space $\dot{C}_1^{0,1}$. We recall the formula for the seminorm on the latter space:

$$\|f\|_{\dot{C}_1^{0,1}} = \sup_Q \inf_c \frac{1}{|Q|} \int_Q |f(x) - c| dx,$$

where the inner infimum is taken over all constants c (polynomials of degree zero). Comparison of the formulas shows that in the first case we subtract in the integrand the average $f_Q = |Q|^{-1} \int_Q f$, and in the second case we subtract (in fact) a constant of best approximation (the infimum over c is attained), so that, clearly, the second seminorm does not exceed the first. As to the reverse estimate, for every constant c we have $|f_Q - c| \le |Q|^{-1} \int_Q |f - c|$, so that we make an admissible error when subtracting f_Q in place of the best constant.

It turns out that, in fact, the functions in BMO obey a condition much stronger than local integrability and much stronger than it was claimed in Proposition 1.11.

THEOREM 1.13. (John–Nirenberg inequality) *Suppose* $f \in$ BMO. *Then for every cube* Q *and every* $t > 0$ *we have*

$$|\{x \in Q : |f(x) - f_Q| > t\}| \le C|Q| \exp\left(\frac{-ct}{\|f\|_{\mathrm{BMO}}}\right), \tag{1.38}$$

where the constants C *and* c *depend only on the dimension* n.

PROOF. Without loss of generality, we assume that $\|f\|_{\mathrm{BMO}} \le 1$. We fix a cube Q and put

$$\varphi(t) = \sup_{\|g\|_{\mathrm{BMO}} \le 1} \frac{|\{x \in Q : |g(x) - g_Q| > t\}|}{\int_Q |g - g_Q|}.$$

Multiplication of the independent variable by a constant generates an isometry of BMO, and the same is true for a shift in $\mathbb{R}^n$. So, the function φ does not depend on Q (in particular, we may and do assume that Q is a dyadic cube). Clearly, φ is monotone nonincreasing and $\varphi(t) \le t^{-1}$. We claim that

$$\varphi(d + 2^n A) \le \varphi(d)/A$$

whenever $d > 0$ and $A > 1$.

To show this, we consider the function $G = |g - g_Q|\chi_Q$, where the BMO-norm of g is at most one, and apply the Calderón–Zygmund lemma at the level A to it. Since $G_Q \le 1 \le A$ and we have agreed that Q is a dyadic cube, all resulting cubes Q_i will be included in Q. Now, we have

$$|g_Q - g_{Q_i}| \le \frac{1}{|Q_i|}\int_{Q_i} |g - g_Q| \le 2^n A, \tag{1.39}$$

whence it follows that, for every i, if $x \in Q_i$ and $|g(x) - g_Q| > d + 2^n A$, then $|g(x) - g_{Q_i}| > d$. Since $|g - g_Q| \le A$ outside the union of Q_i on Q, we obtain

$$\begin{aligned} \{x \in Q:\ |g(x) - g_Q| > d + 2^n A\} &\subset \bigcup_i \{x \in Q_i:\ |g(x) - g_Q| > d + 2^n A\} \\ &\subset \bigcup_i \{x \in Q_i:\ |g(x) - g_{Q_i}| > d\}. \end{aligned} \tag{1.40}$$

Recalling that $\int_{Q_i} |g - g_{Q_i}| \le |Q_i|$ (because the BMO-norm of g is at most 1), finally we can write

$$\begin{aligned} |\{x \in Q:\ |g - g_Q| > d + 2^n A\}| &\le \sum_i |\{x \in Q_i:\ |g - g_{Q_i}| > d\}| \\ &\le \varphi(d) \sum_i \int_{Q_i} |g - g_{Q_i}| \le \varphi(d) \sum_i |Q_i| \\ &\le \frac{\varphi(d)}{A} \sum_i \int_{Q_i} |g - g_Q| \le \frac{\varphi(d)}{A} \int_Q |g - g_Q|, \end{aligned}$$

which is our claim.

The rest is easy. We put $A = e$, $d = 2^n e$, and recall that $\varphi(d) \le d^{-1}$. Adding consecutively $2^n A$ to the argument, we arrive at the inequality $\varphi(2^n ek) \le \frac{1}{2^n e^k}$, $k \in \mathbb{N}$. Since φ is monotone nonincreasing, we obtain $\varphi(t) \le C\exp(-ct)$ if $t \ge 2^n e$. Finally, if f is a function with BMO-norm not exceeding 1, then $|Q| \ge \int_Q |f - f_Q|$ for every cube Q, and we arrive at

$$\frac{|\{x \in Q:\ |f - f_Q| > t\}|}{|Q|} \le \varphi(t) \le C\exp(-ct)$$

if $t \ge 2^n e$. But the fraction on the left does not exceed 1 for all t. $\square$

The John–Nirenberg inequality allows us to conclude that, whenever $f \in \mathrm{BMO}$ is of norm at most 1, for every $p \in (1, \infty)$ and every cube Q we have

$$\frac{1}{|Q|}\int_Q |f - f_Q|^p \le p\int_0^\infty t^{p-1}\varphi(t)dt \le C(p, n)^p,$$

where $C(p, n)$ is a constant depending only on p and n. Thus,

$$f \mapsto \sup_Q \left(\frac{1}{|Q|}\int |f - f_Q|^p\right)^{1/p}$$

is an equivalent seminorm on BMO for every $p \ge 1$. This proves Proposition 1.11.