SOME EXERCISES FOR THE MINICOURSE: BANACH REPRESENTATIONS OF DYNAMICAL SYSTEMS

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1. Exercises

Exercise 1.1. Show that the one point compactification $A(\lambda)$ of a discrete space with cardinality $\lambda \geq \omega$ is uniformly Eberlein.

Exercise 1.2. Let G be a countable discrete group. Show that there exists a topological group embedding $G \to \mathrm{Iso}_{\,lin}(l_2)$.

Exercise 1.3. Let G be the Polish symmetric group $S_{\mathbb{N}}$ (of all permutations of $\mathbb{N}$) with the pointwise topology. Show that there exists a topological group embedding $S_{\mathbb{N}} \to \mathrm{Iso}_{\,lin}(l_2)$.

Exercise 1.4. Let V be a reflexive space and $B \subset V, A \subset V^*$ are bounded subsets. Show that the function

$$A \times B \to \mathbb{R}, (x, f) \mapsto < x, f >= f(x)$$

has DLP.

Exercise 1.5. Show that the original norm of the Banach space c_0 does not satisfy DLP.
Hint: Define $u_n := e_n$ and $v_m := \sum_{i=1}^{m} e_i$.

Exercise 1.6. Show that the Banach space $L_{2k}[0, 1]$ has DLP for every $k \in \mathbb{N}$.

Exercise 1.7. Give an example of a bounded countable family of continuous functions $F \subset C[0, 1]$ such that F does not satisfy DLP (double limit property) on $[0, 1]$.

Exercise 1.8. Let a topological group G admit a left-invariant metric with DLP. Show that G is reflexively representable.

Call a continuous representation $h : G \to \mathrm{Iso}_{\,lin}(V)$ on a Banach space V *adjoint continuous* if the adjoint representation $h^* : G \to \mathrm{Iso}_{\,lin}(V^*)$ is also continuous. It is a well known phenomenon in Functional Analysis that continuous representations on *general* Banach spaces need not be adjoint continuous (even for compact groups).

Exercise 1.9. The regular representation of the circle group $\mathbb{T}$ on $V := C(\mathbb{T})$ is continuous but not adjoint continuous.

Hint: ("Point measures are responsible for this") Indeed, the continuity of the adjoint representation $\mathbb{T} \to \mathrm{Iso}\,(C(\mathbb{T})^*)$ is equivalent to the norm continuity of all orbit maps $\tilde{v} : \mathbb{T} \to C(\mathbb{T})^*, t \mapsto tv$ for every functional $v \in C(\mathbb{T})^*$. Now observe that the map $\mathbb{T} \to C(\mathbb{T})^*, t \mapsto t\delta_{x_0}$ is discontinuous for every point measure δ_{x_0}, where $\delta_{x_0}(f) := f(x_0)$.

Exercise 1.10. Let L be the left uniform structure of the topological group $\mathrm{Iso}\,(l_2)$. Show that the uniform space $(\mathrm{Iso}\,(l_2), L)$ is uniformly embedded into the uniform space l_2.

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Exercise 1.11. Let $\pi : S_{\mathbb{N}} \times l_1 \to l_1$ be the natural (linear isometric) action of the Polish symmetric group $S_{\mathbb{N}}$ on the Banach space l_1 (permutations of coordinates). Show that the dual action

$$S_{\mathbb{N}} \times l_\infty \to l_\infty, \ (gf)(x) := f(g^{-1}(x))$$

on the dual Banach space $l_1^* = l_\infty$ is not continuous. So, the natural representation of $S_{\mathbb{N}}$ on l_1 is (continuous but) not adjoint continuous.

Exercise 1.12.

(1) For a topological space X consider the semigroup $(X^X, \circ)$ of all selfmaps $f : X \to X$ with respect to pointwise (=product) topology. Show that X^X is a right topological semigroup.
(2) $C(X, X)$ is a semitopological subsemigroup of X^X.
 Is it true that $C([0,1],[0,1])$ is a topological semigroup ?
(3) Prove that the left translation $l_f : X^X \to X^X$ is continuous if and only if $f \in C(X, X)$. Derive that if X is T_1, then the right topological semigroup X^X is semitopological iff X is discrete.

Exercise 1.13. Let $(G, \cdot, \tau)$ be a locally compact non-compact Hausdorff topological group. Denote by $S := G \cup \{\infty\}$ the 1-point compactification of G.

Show that $(S, \cdot, \tau_\infty)$ is a semitopological but not topological semigroup.

Exercise 1.14. Let G be a Hausdorff topological group and $H \leq G$ be its topological subgroup. If H is locally compact then H is closed in G.

Exercise 1.15. If S is a compact Hausdorff topological semigroup and if G is a subgroup of S then $cl(G)$ is a (compact) topological group.

Hint: e_G is an idempotent of S and also a neutral element of $T := cl(G)$.

Exercise 1.16. Let S be the interval $[0,1]$ with the multiplication

$$st = \begin{cases} t, & \text{if } 0 \leq t < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that: S is a compact right topological semigroup with $\Lambda(S) = \emptyset$. The subset $T := [0, \frac{1}{2})$ is a subsemigroup of S and $cl(T) = [0, \frac{1}{2}]$ is not a subsemigroup of S.

Example 1.17. Let $S := \mathbb{Z} \cup \{-\infty, \infty\}$ be the two-point compactification of $\mathbb{Z}$. Extend the usual addition by:

$$n + t = t + n = s + t = t \quad n \in \mathbb{Z}, \ s, t \in \{-\infty, \infty\}$$

Show: $(S, +)$ is a noncommutative compact right topological semigroup having dense topological centre $\Lambda(S) = \mathbb{Z}$. S is not semitopological.

Exercise 1.18. Show that the right topological semigroup S of the previous exercise is topologically isomorphic to the *enveloping semigroup* of the invertible cascade $(\mathbb{Z}, [0,1])$ generated by the homeomorphism $\sigma : [0,1] \to [0,1], \sigma(x) = x^2$.

Exercise 1.19. Prove that:

(1) for every metric space (M, d) the semigroup $S := \Theta(M, d)$ of all *non-expanding maps* [1] $f : X \to X$ (that is, $d(f(x), f(y)) \leq d(x, y)$) is a topological monoid with respect to the topology of pointwise convergence;
(2) the group $\text{Iso}\,(M)$ of all onto isometries is a topological group;
(3) the evaluation map $S \times M \to M$ is a continuous monoidal action.

Exercise 1.20. Let $S \times X \to X$ be contractive action of S on (X, d). Show that the following conditions are equivalent:

(i) The action is continuous.

[1] in another terminology: *Lipschitz 1 maps*

(ii) The action is separately continuous.
(iii) The natural homomorphism $h : S \to \Theta(X, d)$ of monoids is continuous.

Exercise 1.21. Prove that $\Theta(V)$ and $L(V, V)$ are semitopological monoids with respect to the weak operator topology for every Banach space V.

Exercise 1.22. For every Banach space $(V, ||\cdot||)$ show that:

(1) The semigroup $\Theta(V)_s$ (with SOT) is a topological monoid.
(2) The subspace $\mathrm{Iso}\,(V)_s$ of all linear onto isometries is a topological group.

Exercise 1.23. Let Unif(Y, Y) be the set of all uniform self-maps of a uniform space (Y, μ). Denote by $\mu_{\sup}$ the uniformity of uniform convergence on Unif(Y, Y). Show that

(1) under the corresponding topology $top(\mu_{\sup})$ on Unif(Y, Y) and the usual composition we get a topological monoid;
(2) If G is any subgroup of the monoid Unif(Y, Y) then G is a topological group;
(3) For every subsemigroup $S \subset$ Unif(Y, Y) the induced action $S \times Y \to Y$ is continuous;

Exercise 1.24. Let Y be a compact space. Show that:

(1) The semigroup $C(Y, Y)$ endowed with the compact open topology is a topological monoid;
(2) The subset $H(Y)$ in $C(Y, Y)$ of all homeomorphisms $Y \to Y$ is a topological group;
(3) For every subsemigroup $S \subset C(Y, Y)$ the induced action $S \times Y \to Y$ is continuous;
(4) Furthermore, it satisfies the following remarkable minimality property. If τ is an arbitrary topology on S such that $(S, \tau) \times Y \to Y$ is continuous then $\tau_{co} \subset \tau$.

Exercise 1.25. Let X be a compact space and $F \subset C(X)$ be a bounded subset. Show that F has DLP on X iff F has DLP on B^*, where $B^* = B_{C(X)^*}$.

Exercise 1.26. Show that if V is Asplund then $\mathrm{Iso}\,(V)_w$ (in WOT) is a topological group.

Exercise 1.27. If $\nu_1 : X \to Y_1$ and $\nu_2 : X \to Y_2$ are two compactifications, then ν_2 *dominates* ν_1, that is, $\nu_1 = q \circ \nu_2$ for some (uniquely defined) continuous map $q : Y_2 \to Y_1$ iff $\mathcal{A}_{\nu_1} \subset \mathcal{A}_{\nu_2}$. Show that if in addition, X, Y_1 and Y_2 are S_d-systems (i.e., all the s-translations on X, Y_1 and Y_2 are continuous) and if ν_1 and ν_2 are S-maps, then q is also an S-map. Furthermore, if the action on Y_1 is (separately) continuous then the action on Y_2 is (respectively, separately) continuous. If ν_1 and ν_2 are homomorphisms of semigroups then q is also a homomorphism.

Exercise 1.28. (Greatest ambit) Let G be a topological group and $\beta_G : G \to \beta G$ be the compactification induced by the algebra RUC(G). Show that it is the universal semigroup G-compactification of G with jointly continuous G-action. (The universality means that for every semigroup G-compactification $\nu : G \to P$ with continuous action $G \times P \to P$ there exists a unique continuous G-homomorphism $q : \beta G \to P$ such that $\beta_G \circ q = \nu$.

Exercise 1.29. (Enveloping semigroups)

(1) Let X be a compact S-space with the enveloping semigroup $E(X)$ and L a subset of $C(X)$ such that L separates points of X. Then the Ellis compactification $j : S \to E(X)$ is equivalent to the compactification of S which corresponds to the subalgebra $\mathcal{A}_L := \langle m(L, X) \rangle$, the smallest norm closed S-invariant unital subalgebra of $C(S)$ which contains the family
$$\{m(f, x) : S \to \mathbb{R},\ s \mapsto f(sx)\}_{f \in L,\ x \in X}.$$
(2) Let $q : X_1 \to X_2$ be a continuous onto S-map between compact S-spaces. There exists a (unique) continuous onto semigroup homomorphism $Q : E(X_1) \to E(X_2)$ with $j_{X_1} \circ Q = j_{X_2}$.
(3) Let Y be a closed S-subspace of a compact S-system X. The map $r_X : E(X) \to E(Y),\ p \mapsto p|_Y$ is the unique continuous onto semigroup homomorphism such that $r_X \circ j_X = j_Y$.
(4) Let $\alpha : S \to P$ be a right topological compactification of a semigroup S. Then the enveloping semigroup $E(S, P)$ of the semitopological system (S, P) is naturally isomorphic to P.
(5) If X is metrizable then $E(X)$ is separable. Moreover, $j(S) \subset E(X)$ is separable.

Proof. (1) The proof is straightforward using the Stone-Weierstrass theorem.

(2) By Remark 1.27 it suffices to show that the compactification $j_{X_1} : S \to E(X_1)$ dominates the compactification $j_{X_2} : S \to E(X_2)$. Equivalently we have to verify the inclusion of the corresponding algebras. Let $q(x) = y, f_0 \in C(X_2)$ and $f = f_0 \circ q$. Observe that $m(f_0, y) = m(f, x)$ and use (1).

(3) Is similar to (2).

(4) Since $E(S, P) \to P, a \mapsto a(e)$ is a natural homomorphism, $j_P : S \to E(S, P)$ dominates the compactification $S \to P$. So it is enough to show that, conversely, $\alpha : S \to P$ dominates $j_P : S \to E(S, P)$. By (1) the family of functions

$$\{m(f, x) : S \to \mathbb{R}\}_{f \in C(P),\ x \in P}$$

generates the Ellis compactification $j_P : S \to E(S, P)$. Now observe that each $m(f, x) : S \to \mathbb{R}$ can be extended naturally to the function $P \to \mathbb{R},\ p \mapsto f(px)$ which is *continuous*.

(5) Since X is a metrizable compactum, $C(X, X)$ is separable and metrizable in the compact open topology. Then $j(S) \subset C(X, X)$ is separable (and metrizable) in the same topology. Hence, the dense subset $j(S) \subset E(X)$ is separable in the pointwise topology. This implies that $E(X)$ is separable. □

Exercise 1.30. Let K be a compact space which is norm-fragmented in $C(K)^*$. Show that K is scattered.

Exercise 1.31. If X is (locally) fragmented by $f : X \to Y$, where (X, τ) is a Baire space and (Y, ρ) is a pseudometric space then f is continuous at the points of a dense G_δ subset of X.

Exercise 1.32. Let K be a RN compactum. Show that K has a dense subset $Y \subset X$ such that y has a countable local bases in X for every $y \in Y$.

Exercise 1.33. When X is compact and (Y, ρ) metrizable uniform space then $f : X \to Y$ is fragmented iff f has a *point of continuity property* (i.e., for every closed nonempty $A \subset X$ the restriction $f_{|A} : A \to Y$ has a continuity point).

Exercise 1.34. Let (X, τ) be a separable metrizable space and (Y, ρ) a pseudometric space. Suppose that $f : X \to Y$ is a fragmented onto map. Then Y is separable. Hint: use the idea of the Cantor-Bendixon theorem.

Exercise 1.35. Show that $F = \{f_i : X \to \mathbb{R}\}_{i \in I}$ is a fragmented family iff the induced map $X \to (\mathbb{R}^F, \xi_U)$ is fragmented, where ξ_U is the uniformity of uniform convergence on $\mathbb{R}^F$.

Exercise 1.36. Give an example of a bounded family F of continuous functions $[0, 1] \to \mathbb{R}$ such that F is eventually fragmented but not fragmented.

2. Some Solutions

Definition. *Let (Y, τ) be a topological space and X be a set. Denote by Y^X the set of all maps $f : X \to Y$ endowed with the product topology of Y^X. This topology has the standard base α which consists of all the sets:*

$$O(x_1, \cdots, x_n; U_1, \cdots, U_n) := \{f \in Y^X :\ f(x_i) \in U_i\}$$

where, $F := \{x_1, \cdots, x_n\}$ is a finite subset of X and U_i are nonempty open subsets in Y. Other names of this topology are: pointwise *topology,* point-open *topology. Sometimes we use a short notation $(x_1, \cdots, x_n; U_1, \cdots, U_n)$ instead of $O(x_1, \cdots, x_n; U_1, \cdots, U_n)$.*

Exercise 2.1.

(1) For every topological space X consider the semigroup $(X^X, \circ)$ of all selfmaps $f : X \to X$ with respect to pointwise (=product) topology. Show that X^X is a right topological semigroup.
(2) $C(X, X)$ is a semitopological subsemigroup of X^X.
Is it true that $C([0, 1], [0, 1])$ is a topological semigroup ?
(3) Prove that the left translation $l_f : X^X \to X^X$ is continuous if and only if $f \in C(X, X)$. Derive that if X is T_1, then the right topological semigroup X^X is semitopological iff X is discrete.

Proof. First a general

Remark 2. *The product topology on X^X can be described by nets as the pointwise topology. A* net *(generalized sequence) f_i in X^X converges to $f \in X^X$ iff the net $f_i(x)$ in X converges to $f(x)$ for each $x \in X$. This explains the term: "pointwise topology".*

(1)

First proof: (using the nets)

We have to show that $r_h : X^X \to X^X$ is continuous for every given $h \in X^X$. It is equivalent to show

$$\lim f_i = f \Rightarrow \lim f_i h = fu$$

for every net f_i in X^X. $\lim f_i = f$ means (see Remark 2) that $\lim f_i(x) = f(x)$ for every $x \in X$. Then substituting $h(x)$ we have $\lim f_i(h(x)) = f(h(x))$. This exactly means that $\lim f_i h = fh$ (again, Remark 2).

Second proof: (using the nbds)

First we recall the following general topological

Fact 2. *For the continuity of a map it is enough to show that the preimage of any basic nbd is an nbd. Moreover, in fact, it is enough even to check the same for a subbase.* [2]

Consider the following family

$$\gamma := \{(x; U) : x \in X, U \in \tau\}, \qquad (x; U) := \{f \in X^X : \; f(x) \in U\}$$

Then γ is a subbase of the standard base (of the pointwise topology on X^X)

$$(x_1, \cdots, x_n; U_1, \cdots, U_n) := \{f \in Y^X : \; f(x_i) \in U_i\}.$$

Now we can prove (1) using Fact 2. Let $h \in X^X$. For every given $(x; U)$ consider the open set $(h(x); U)$. Then for every $f \in (h(x); U)$ we have $fh \in (x; U)$.

(2) $C(X, X)$ is evidently a subsemigroup of X^X so it is enough to show that for $h \in C(X, X)$ the corresponding left translation $l_h : X^X \to X^X$ is continuous (i.e., $C(X, X) \subset \Lambda(X^X)$).

First proof: Let $h \in C(X, X)$. If $\lim f_i = f$ in X^X then $\lim f_i(x) = f(x)$ in X for every $x \in X$. Then by the continuity of h we have $\lim h(f_i(x)) = h(f(x))$. This means that $\lim h(f_i) = h(f)$. Now use Remark 2.

Second proof: Let $h \in C(X, X)$. For every standard subbase nbd $(a; U) \in \gamma$ consider the open set $(a; h^{-1}(U))$ (the continuity of h guarantees that $h^{-1}(U)$ is open in X). Then $f \in (a; h^{-1}(U))$ implies that $hf \in (a; U)$. By Fact 2 we obtain that $l_h : X^X \to X^X$ is continuous.

$C([0, 1], [0, 1])$ **is not a topological semigroup.** We have to show that the multiplication m (the composition !) is not continuous. In fact, we will show much more that m is not continuous at any point $(h_0, f_0) \in C[0, 1] \times C[0, 1]$. Let $a := h_0(f_0)(1)$. Consider an open nbd $(a; (-\frac{1}{2}, \frac{1}{2}))$ of $h_0 \circ f_0$ in the space $C([0, 1], [0, 1])$. Then for every basic nbds

$$h_0 \in (x_1, \cdots, x_n; U_1, \cdots, U_n) \qquad f_0 \in (y_1, \cdots, y_m; V_1, \cdots, V_m)$$

there exists a pair f, h such that

$$h \in (x_1, \cdots, x_n; U_1, \cdots, U_n) \qquad f \in (y_1, \cdots, y_m; V_1, \cdots, V_m)$$

but $h \circ f \notin (a; (-\frac{1}{2}, \frac{1}{2}))$. Indeed using a freedom [3] in the building of continuous functions (and the fact that each of the nbds U_i and V_k are infinite sets) one may choose $f \in (y_1, \cdots, y_m; V_1, \cdots, V_m)$ s.t. $f(1) \notin \{x_1, \cdots, x_n\}$. Now we can choose $h \in (x_1, \cdots, x_n; U_1, \cdots, U_n)$ s.t. $h(f(1)) \notin (-\frac{1}{2}, \frac{1}{2})$.

(3) (First part)

First proof: Let $l_h : X^X \to X^X$ be continuous. We have to show that $h \in C(X, X)$. It is equivalent to show that h preserves the convergence of nets in X in the following sense:

$$\lim x_i = x \;\Rightarrow\; \lim h(x_i) = h(x)$$

[2] Recall that a family γ of open subsets is said to be a *subbase* if the finite intersections (that is, the family $\gamma^{\cap fin}$) is a topological base.

[3] Namely the fact that every map $F \to [0, 1]$ on a finite subset $F \subset [0.1]$ can be extended to a continuous map $[0, 1] \to [0, 1]$

For every $y \in X$ consider the constant function $c_y : X \to X, c_y(t) = t$. Then $\lim c_{x_i} = c_x$ in X^X. The continuity of $l_h : X^X \to X^X$ means that it preserves the convergence in X^X. So, in particular, we have $\lim l_h(c_{x_i}) = l_h(c_x)$. But this means that $\lim h(x_i) = h(x)$, as desired.

Second proof: (Gal Lavi and Noam Lifshitz)

We have to show that $h : X \to X$ is continuous at every given $a \in X$. Let $U \in N(h(a))$ in X. Consider the open nbd $(a; U)$ in X^X. Consider the constant function $c_a : X \to X, x \mapsto a$. Then $(h \circ c_a)(x) = h(a)$ for every $x \in X$. In particular, $hc_a \in (a; U)$. By our assumption the left transition $l_h : X^X \to X^X$ is continuous. Therefore, there exists a basic nbd

$$W := (x_1, \cdots, x_n; V_1, \cdots, V_n)$$

of c_a in X^X s.t. $hW \subset (a; U)$. Each V_i is a nbd of a (because, $c_a(x_i) = a$). Then also $V := \cap_i V_i \in N(a)$. Now observe that $c_v \in W$ for every $v \in V$. On the other hand, $hW \subset (a; U)$ leads us to $hc_v = f(v) \in U$ for every $v \in V$. Hence, $h(V) \subset U$. This proves the continuity of h at a.

(3) (Second part)

If X is discrete then of course $X^X = C(X, X)$ which is semitopological by (2).

If X^X is semitopological then by the first part of (3) we know that $X^X = C(X, X)$. Let $X \in T_1$. We have to show that X is discrete. Since $X \in T_1$, every singleton $\{a\}$ is closed in X. Choose one of them. For every nonempty $A \subset X$ consider a function $f_A : X \to X$ s.t. $f_A^{-1}(a) = A$. Since f is continuous we get that A is closed. So, every subset of X is closed, hence X is discrete.

One may show that in general if $X^X = C(X, X)$ then either X is discrete or X has the trivial topology. So, the assumption $X \in T_1$ can be replaced by the assumption that the topology on X is not trivial.

□

Definition. *Let X be a topological space. A* compactification *of X is a continuous map $f : X \to Y$ where Y is a compact Hausdorff space and $f(X)$ is dense in Y. We say:* proper compactification *when, in addition, f is required to be a topological embedding.*

One of the standard examples of a proper compactification is the so-called *1-point compactification* $\nu : X \hookrightarrow X_\infty := X \cup \{\infty\}$ defined for every locally compact non-compact Hausdorff space (X, τ). Recall the topology

$$\tau_\infty := \tau \cup \{X_\infty \setminus K : \quad K \text{ is compact in } X\}.$$

Exercise 2.2. Let $(G, \cdot, \tau)$ be a locally compact non-compact Hausdorff topological group. Denote by $S := G \cup \{\infty\}$ the 1-point compactification of G.

Show that $(S, \cdot, \tau_\infty)$ is a semitopological but not topological semigroup.

Proof. First we show that S **is semitopological**. Let $a \in S$. We have to show that $l_a : S \to S$ and $r_a : S \to S$ are continuous. We consider only the case of l_a. The second case is similar. So, we have to check that $l_a : S \to S$ is continuous at every $y \in S$. For $a = \infty$ the transition l_a is a constant map. WRG assume that $a \neq \infty$, hence $a \in G$. We have two cases for $y \in S$:

(a) If $y \neq \infty$ then for every open nbd $U \in N(y), U \subset G$ just take the open nbd $V := a^{-1}U \in N(a^{-1}y)$. Then $l_a(V) = U$.

(b) Let $y = \infty$ and $U \in N(\infty)$ is an open nbd. Then by the definition of the 1-point compactification topology, $U = S \setminus K$, where K is compact in G. Then $a^{-1}K$ is also compact in G. So, $V := S \setminus a^{-1}K \in N(\infty)$ and $l_a(V) = U$.

Now we show that S **is not topological**. That is, the multiplication is not continuous. Indeed, we show that the multiplication $m : S \times S \to S$ is not continuous at the point (∞, ∞).

First proof:

Choose the open nbd $U := S \setminus \{e\}$ of ∞. It is enough to show that for every nbd $V \in N(\infty)$ we have $e \in VV$ (this will mean that VV is not a subset of U). Observe that every $V \in N(\infty)$ contains an open nbd $S \setminus K$, where K is compact and **symmetric** (indeed, WRG replace K by $K \cup K^{-1}$). Now observe that for every $x \in S \setminus K$ we have $x^{-1} \in S \setminus K$ but $xx^{-1} = e \notin U$.

Second proof:

Assuming the contrary let $m : S \times S \to S$ be continuous. Then

$$A := m^{-1}(\{e\}) = \{(x, x^{-1}) \in S \times S : \ x \in G\}$$

is a closed subset of the product $S \times S$. Since, S is compact then A is compact, too. Consider the projection $\pi_1 : S \times S \to S, (a, b) \mapsto a$. Then $\pi_1(A)$ is a compact subset of S. But $\pi_1(A) = G$. So, we obtain that G is compact, a contradiction (because G is assumed to be noncompact).

□

As we know a locally compact Hausdorff group G admits an embedding into a compact Hausdorff group iff G is compact. Exercise 2.2 shows that such G at least admits a proper *semigroup compactification* $\nu : G \hookrightarrow S$ such that S is a compact semitopological monoid.

Exercise 2.3. Let G be a Hausdorff topological group and $H \leq G$ be its topological subgroup. If H is locally compact then H is closed in G.

Proof. It is equivalent to prove in the case of $cl(H) = G$. So we have to show that H is closed in $cl(H)$. It suffices to show that H is open in $G = Cl(H)$ (because every open subgroup is closed).

Since H is LC one may choose a compact nbd K of e in H.

$$\exists U \in N_G(e) \cap \tau : \ U \cap H \subset K$$

$$U = U \cap G = U \cap cl(H) \subset cl(U \cap H) \subset cl(K) = K$$

(remark1: for every open $O \subset X$ and $A \subset X$ we have $O \cap cl(A) \subset cl(O \cap A)$)
(remark2: every compact subset is closed in a Hausdorff space)

So, $U \subset K$. Therefore, $U \subset H$. Hence, $int_G(H) \neq \emptyset$. It follows that that the subgroup H is open in $cl(H)$. Hence, also closed. So, $H = cl(H)$. □

It is impossible to embed a locally compact noncompact group into any Hausdorff compact group. In particular, there is no finite-dimensional topologically faithful representation by linear isometries of a locally compact noncompact groups (like $\mathbb{Z}$, $\mathbb{R}$) on finite-dimensional Euclidean spaces.

Exercise 2.4. If S is a compact Hausdorff topological semigroup and if G is a subgroup of S then $cl(G)$ is a (compact) topological group.

Hint: e_G is an idempotent of S and also a neutral element of $T := cl(G)$.

Proof. The simplest way here is to use the technique of the nets (generalized sequences).

1. $T = cl(G)$ **is a topological subsemigroup of** S**.**

Indeed, let $x, y \in T := cl(G)$. Then there exist nets $\{x_i\}_{i \in \Gamma}$ and $\{y_i\}_{i \in \Gamma}$ such that $\lim x_i = x, \lim y_i = y$ and $x_i, y_i \in G$. [4] Then by the continuity of the multiplication we have $\lim(x_i y_i) = (\lim x_i)(\lim y_i) = xy$. Since $x_i y_i \in G$ we obtain that $xy \in cl(G)$.

2. e_G **is a neutral element in** $T = cl(G)$**. So,** T **is a topological monoid.**

Indeed, if $\lim x_i = x \in cl(G)$ and $x_i \in G$ then $\lim(x_i e_G) = \lim x_i = x$. On the other hand, by the continuity of the multiplication we have $\lim(x_i e_G) = (\lim x_i) e_G = x e_G$. So, $x e_G = x$. Similarly, $e_G x = x$.

3. T **is a group.**

Let $t \in T$ and g_i be a net in G converging to t. By compactness we may assume that some subnet of g_i^{-1} converges to some $s \in T$. For simplicity (WRG) we assume that g_i^{-1} itself converges to some $s \in T$. Since S is topological we have $g_i g_i^{-1}$ converges to ts. By the Hausdorff axiom we necessarily have $e = ts$. Similarly, $e = st$.

4. T **is a (compact) topological group.**

Now (after 1-3) it suffices to show that the inversion $j : T \to T, t \mapsto t^{-1}$ is continuous. Let $\lim t_i = t$ in T. We have to show that the limit $\lim t_i^{-1}$ exists in T and it equals to t^{-1}. Consider the net t_i^{-1} in T. Since T is compact, there exists a converging subnet $t_{i_j}^{-1}$. Let $\lim t_{i_j}^{-1} = y \in$

[4]Note that for every two converging nets $\nu_1 : (\Gamma_1, \leq_1) \to X, \ \nu_2 : (\Gamma_2, \leq_2) \to X$ one may assume WRG that they have the same domain ($\Gamma := \Gamma_1 \times \Gamma_2$ for example)

T. A subnet of a converging net converges (to the same limit). So, $\lim t_{i_j} = t$. Then by the continuity of the multiplication in T we have $\lim(t_{i_j} t_{i_j}^{-1}) = (\lim t_{i_j})(\lim t_{i_j}^{-1}) = ty$. On the other hand, $\lim(t_{i_j} t_{i_j}^{-1}) = \lim e_G = e_G$. By the uniqueness of the net limits in Hausdorff spaces we have $ty = e_G$. Therefore, $y = t^{-1}$.

□

Exercise 2.5. Let S be the interval $[0,1]$ with the multiplication

$$st = \begin{cases} t, & \text{if } 0 \le t < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Show that: S is a compact right topological semigroup with $\Lambda(S) = \emptyset$. The subset $T := [0, \frac{1}{2})$ is a subsemigroup of S and $cl(T) = [0, \frac{1}{2}]$ is not a subsemigroup of S.

Proof. First of all it is straightforward to see that S is a semigroup and T its subsemigroup.

S is right topological. Because $r_t : S \to S$ is a constant function ($\mathbf{t}$ or $\mathbf{1}$) for every $r \in S$.

$\Lambda(S) = \emptyset$. Indeed, for every $s \in S$ we have that $L_s : [0,1] \to [0,1]$ has a jump discontinuity point at $\frac{1}{2}$.

$cl(T) = [0, \frac{1}{2}]$ is not a subsemigroup of S. Indeed, $cl(T) = [0, \frac{1}{2})$ and $\frac{1}{2} * \frac{1}{2} = 1 \notin T$. □

Exercise 2.6. Let $S := \mathbb{Z} \cup \{-\infty, \infty\}$ be the two-point compactification of $\mathbb{Z}$. Extend the usual addition by:

$$n + t = t + n = s + t = t \quad n \in \mathbb{Z}, \ s, t \in \{-\infty, \infty\}$$

Show: $(S, +)$ is a noncommutative compact right topological monoid having dense commutative topological centre $\Lambda(S) = \mathbb{Z}$. S is not semitopological.

Proof. First of all it is straightforward to see that $(S, +)$ is a monoid and $(\mathbb{Z}, +)$ its submonoid.

$(S, +)$ is noncommutative because $\infty + (-\infty) = -\infty$ and $(-\infty) + \infty = \infty$.

$S := \mathbb{Z} \cup \{-\infty, \infty\}$ carries the topology of the natural linear order. A natural subbase for the topology of S is the following family

$$A_n := \{x \in S : x < n\}, \ B_m := \{x \in S : \ m < x\}, \quad n, m \in \mathbb{Z}$$

Clearly, $\mathbb{Z}$ is dense in S and every $x \in \mathbb{Z}$ is an isolated point in S. The space S is homeomorphic to a closed subset

$$Y := \{-1\} \cup \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$$

of $[-1, 1]$, hence compact.

The right translations $r_t : S \to S$ are continuous. Indeed, r_∞ is the constant function $r_\infty(x) = \infty$ for every $x \in S$. $r_{-\infty}$ is the constant function $r_\infty(x) = -\infty$ for every $x \in S$. $r_k^{-1}(A_n) = A_{n-k}$, $r_k^{-1}(B_m) = B_{m-k}$ for every $k \in \mathbb{Z}$.

$\Lambda(S) = \mathbb{Z}$. Indeed, every $l_k : S \to S$ is continuous for each $k \in \mathbb{Z}$ because $l_k^{-1}(A_n) = A_{n-k}$, $l_k^{-1}(B_m) = B_{m-k}$.

$l_\infty : S \to S$ is not continuous at the **point** $s = -\infty$. Take a sequence $\{-k\}_{k \in \mathbb{N}}$. Then

$$\lim(-k) = -\infty \text{ but } \lim l_\infty(-k) = \infty \neq l_\infty(-\infty) = \infty + (-\infty) = -\infty$$

Similarly, $l_{-\infty} : S \to S$ is not continuous at the **point** $s = \infty$. □

Exercise 2.7. Show that the right topological semigroup S of the previous exercise is topologically isomorphic to the *enveloping semigroup* E of the invertible cascade $(\mathbb{Z}, [0,1])$ generated by the homeomorphism $\sigma : [0,1] \to [0,1], \sigma(x) = x^2$.

Proof. Hint: Let E be the enveloping semigroup of $(\mathbb{Z}, [0,1])$ and $j : \mathbb{Z} \to E$ be the corresponding compactification. Observe that besides the points $j(Z) = \{\widetilde{\sigma^n} \ n \in \mathbb{Z}\}$ the enveloping semigroup $E(X)$ contains two more points: a, b, where $a = \xi_{\{1\}}$ the characteristic function of $\{1\}$ and $b = 1 - \xi_0$, where $\xi_{\{0\}}$ is the characteristic function of $\{0\}$. □

Exercise 2.8. For every metric space (M,d) show that:

(1) The semigroup $\Theta(M,d)$ of all *d-contractive maps* $f: X \to X$ (that is, $d(f(x),f(y)) \leq d(x,y)$) is a topological monoid with respect to the topology of pointwise convergence;
(2) The group $\text{Iso}\,(M)$ of all onto isometries is a topological group;
(3) The evaluation map $S \times M \to M$ is a jointly continuous monoidal action for every submonoid $S \leq \Theta(M,d)$.

Proof. (1) **Algebraic part**: it is trivial to see that the composition is well defined, associative, and $\Theta := \Theta(M,d)$ is a monoid.

Continuity of the multiplication

We use the following easy reformulation of the pointwise topology:

Fact. *Let X be any nonempty set, (Y,d) be a metric space and τ_p be the pointwise (product) topology on $Y^X := \{f : X \to Y\}$. Then for every $f_0 \in Y^X$ the following family of sets is a local base at the point f_0 with respect to the topology τ_p:*

$$(f_0; x_1, \cdots, x_n; \varepsilon) := \{f \in Y^X : \ d(f_0(x_k), f(x_k) < \varepsilon \ \forall k = 1, \cdots, n\}.$$

where $x_1, \cdots, x_n$ is a finite subset in X and $\varepsilon > 0$.

Now we prove the continuity of the multiplication $m : \Theta \times \Theta \to \Theta$ at the point $(s_0, t_0) \in \Theta \times \Theta$. We have to show that st is close to s_0t_0 when s and t are close enough to s_0 and t_0, respectively. In order to get a "right idea for the proof" consider the following inequalities:

$$\begin{aligned} d(s_0t_0(x_k), st(x_k)) &\leq d(s_0t_0(x_k), st_0(x_k)) + d(st_0(x_k), st(x_k)) \\ &\leq d(s_0t_0(x_k), st_0(x_k)) + d(t_0(x_k), t(x_k)) \end{aligned}$$

Note that in the last inequality we need to use the Lipshitz-1 property for s.

Now we can easily finish the proof choosing appropriate neighborhoods for t_0 and s_0 for a given nbd $O := (s_0t_0; x_1, \cdots, x_n; \varepsilon)$ of s_0t_0. Indeed, take the following neighborhoods $U := (t_0; x_1, \cdots, x_n; \frac{\varepsilon}{2})$ and $V := (s_0; t_0(x_1), \cdots, t_0(x_n); \frac{\varepsilon}{2})$. Then for every $t \in U, s \in V$ we have $st \in O$, as desired.

Remark. *Another proof can be based on nets. Namely, to the following useful (and characterizing) property of the pointwise topology.*

a net s_i converges to s_0 in Y^X (with respect to pointwise topology) if and only if the net $s_i(x_0)$ converges to $s(x_0)$ (in Y) for every $x_0 \in X$.

(2) For the **continuity of the inversion** $\text{Iso}\,(M) \to \text{Iso}\,(M)$ at the point s_0.

In order to estimate how close can be s^{-1} to s_0^{-1} look at the following key equality (using, this time, that $s: M \to M$ is an isometry)

$$d(s^{-1}(x_k), s_0^{-1}(x_k)) = d(x_k, ss_0^{-1}(x_k)) = d(s_0(t_k), s(t_k))$$

with $x_k := s_0(t_k)$.

Now the rest is easy. For a given nbd $O(s_0^{-1}) := (s_0^{-1}; x_1, \cdots, x_n; \varepsilon)$ of s_0^{-1} choose $U(s_0) := (s_0; t_1, \cdots, t_n; \varepsilon)$ of s_0 with $t_k := s_0^{-1}(x_k)$. Now if $s \in U$ then $s^{-1} \in O$.

(3) We have to prove the continuity of the action

$$S \times X \to X$$

at every given point (s_0, x_0). We give only a key inequality (the rest will be clear):

$$d(s_0x_0, sx) \leq d(s_0x_0, sx_0) + d(sx_0, sx) \leq d(s_0x_0, sx_0) + d(x_0, x).$$

□

An action $S \times X \to X$ on a metric space (X,d) is *non-expanding* if every s-translation $\tilde{s}: X \to X$ lies in $\Theta(X,d)$. It defines a natural homomorphism $h: S \to \Theta(X,d)$.

Exercise 2.9. Let $S \times X \to X$ be a non-expanding action of S on (X,d). Show that the following conditions are equivalent:

(i) The action is continuous.

(ii) The action is separately continuous.
(iii) The natural homomorphism $h : S \to \Theta(X, d)$ of monoids is continuous.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) We have to show that $h : S \to \Theta$ is continuous. Let $s_i \to s_0$ be a converging net in S. We need to verify that $h(s_i) \to h(s_0)$. By the definition of pointwise topology it is equivalent to check that $h(s_i)(x_0) \to h(s_0)(x_0)$. By the definition of h the latter is equivalent to $s_i(x_0) \to s_0(x_0)$. This follows from separate continuity of $S \times X \to X$.

(iii) $\Rightarrow$ (i) We know by Exercise 2.8.3 that the action $\Theta \times M \to M$ is continuous. Then $S \times M \to M$ is also continuous (because $S \times X \to \Theta \times X$ is continuous). ☐

Exercise 2.10. Prove that $\Theta(V)$ and $L(V, V)$ are semitopological monoids with respect to the weak operator topology for every Banach space V.

Proof. Algebraically $\Theta(V)$ is a submonoid of $L(V, V)$. So, it is enough to show that $L(V, V)$ is a semitopological monoid with respect to the weak operator topology. Recall the definition of weak operator topology on $L(V, V)$. A net s_i τ_w-converges to s in $L(V, V)$ iff $f(s_i(v))$ converges to $f(s(v))$ in $\mathbb{R}$ for every $v \in V, f \in V^*$.

First we show that the right translations

$$\rho_t : L(V, V) \to L(V, V), \ \ \rho_t(s) := st$$

are continuous for every $t \in L(V, V)$. Indeed, let we have a convergence of nets $s_i \to s$. We have to show that $s_i t \to st$. It is equivalent to see that $f(s_i t(v))$ converges to $f(st(v))$ in $\mathbb{R}$. Or, equivalently, that $f(s_i(tv))$ converges to $f(s(tv))$ in $\mathbb{R}$. This is clear because $t(v) \in V$ (in the criterion we have the condition *for every* $v \in V$).

The case of left translations is similar by observing that $ft \in V^*$ for every $f \in V^*$ and $t \in L(V, V)$. ☐

Exercise 2.11. For every Banach space $(V, ||\cdot||)$ show that:
(1) The semigroup $\Theta(V)_s$ (with SOT) is a topological monoid.
(2) The subspace $\mathrm{Iso}\,(V)_s$ of all linear onto isometries is a topological group.

Proof. We can apply Exercise 2.8. ☐

Exercise 2.12. Let Unif(Y, Y) be the set of all uniform self-maps of a uniform space (Y, μ). Denote by $\mu_{\sup}$ the uniformity of uniform convergence on Unif(Y, Y). Show that
(1) under the corresponding topology $top(\mu_{\sup})$ on Unif(Y, Y) and the usual composition we get a topological monoid;
(2) If G is any subgroup of the monoid Unif(Y, Y) then G is a topological group;
(3) For every subsemigroup $S \subset$ Unif(Y, Y) the induced action $S \times Y \to Y$ is continuous;

Proof. (Sketch) (1) Continuity of the multiplication. The elements $(st(x), s_0t_0(x))$ are "close enough" (uniformly for every $x \in X$) because we can force the pairs

$$(st(x), s_0t(x)), \ \ (s_0t(x), s_0t_0(x))$$

be sufficiently close.

(2) Let G be any subgroup of the monoid Unif(Y, Y). For the continuity of the inversion in G note that if $(s_0(t), s(t))$ is small then $(t, s_0^{-1}s(t))$ is small for all $t \in Y$; now substituting $t = s^{-1}(x)$ we get

$$(t, s_0^{-1}s(t)) = (s^{-1}(x), s_0^{-1}(x)))$$

is small.

(3) Continuity of $S \times Y \to Y$ at point (s_0, y_0).

The elements (s_0y_0, sy) are close enough because we can force that (s_0y_0, s_0y) and (s_0y, sy) are sufficiently close. ☐

Exercise 2.13. Let Y be a compact space. Show that:
(1) The semigroup $C(Y, Y)$ endowed with the compact open topology is a topological monoid;

(2) The subset $H(Y)$ in $C(Y,Y)$ of all homeomorphisms $Y \to Y$ is a topological group;
(3) For every subsemigroup $S \subset C(Y,Y)$ the induced action $S \times Y \to Y$ is continuous;
(4) Furthermore, it satisfies the following remarkable minimality property. If τ is an arbitrary topology on S such that $(S,\tau) \times Y \to Y$ is continuous then $\tau_{co} \subset \tau$.

Proof. (1), (2) and (3) Follow directly from the previous Exercise 2.12 taking into account that the uniformity of uniform convergence for compact Y induces the compact open topology (see, for example, book of J. Kelley, General Topology).

(4) Let $(S,\tau)\times Y \to Y$ be continuous. Then by the compactness of Y it is easy to see the following

$$\forall s_0 \in S \ \forall \varepsilon \in \mu_Y \ \exists U \in N_\tau(s_0): \ (s_0 y, sy) \in \varepsilon \ \forall y \in Y.$$

This proves that the topology of compactness convergence τ_{co} is weaker than τ. □

Exercise 2.14. Let G be a countable discrete group. Show that there exists a topological group embedding $G \to \mathrm{Iso}\,(l_2)$.

Proof. It is equivalent to show that there exists a co-embedding. Indeed, for every (topological) group G the inversion map $j : G \to G, j(g) = g^{-1}$ is a co-isomorphism. So, if $h : G \to P$ is a co-embedding then $h \circ j : G \to P$ is an embedding.

Let $S_{\mathbb{N}}$ be the symmetric group. Consider the natural left action $S_{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$. It induces the natural right action

$$l_2 \times S_{\mathbb{N}} \to l_2, \ (u,\sigma) \to u\sigma$$

where $(u\sigma)(k) = u(s(k))$ (we treat (the sequence) $u \in l_2$ as a function $u : \mathbb{N} \to \mathbb{R}$). Observe that it is an action "by permutations of coordinates".

By Cayley's theorem we have an embedding of abstract (discrete) groups $\nu : G \hookrightarrow S_G \cong S_{\mathbb{N}}$. Now consider the induced action of G on l_2. More precisely, if $G := \{g_1, g_2, \cdots\}$ is an enumeration of G then we have the action of G on $\mathbb{N}$ according to its left translations $G \to G$. Consider the induced action of G on l_2

$$\pi : l_2 \times G \to l_2.$$

Then we have:

(1) π is linear.
$(u+v)\sigma = u\sigma + v\sigma, \ \ (cv)\sigma = c(v\sigma)$ for every $u, v \in l_2, c \in \mathbb{R}, \sigma \in S_{\mathbb{N}}$.
(2) π is an action by isometries.
$||u\sigma|| \leq ||u||$ for every $u \in l_2, \sigma \in S_{\mathbb{N}}$. So it follows that $||u|| = ||(u\sigma)\sigma^{-1}|| \leq ||u\sigma||$. Therefore $||u\sigma|| = ||u||$.
(3) π induces a co-homomorphism $h : S_{\mathbb{N}} \to \mathrm{Iso}\,(l_2)$
(it is similar to Exercise 2.9).
(4) h is injective.
Let $\sigma_1 \neq \sigma_2$ in $S_{\mathbb{N}}$. There exists $k \in \mathbb{N}$ such that $i = \sigma_1(k) \neq \sigma_2(k) = j$. Consider the vector $e_k \in l_2$ having the k-th coordinate $= 1$ and other coordinates $= 0$. Then $e_i = v_k\sigma_1 \neq v_k\sigma_2 = e_j$.
(5) $h(G)$ is discrete.
It is equivalent to show that the identity operator $id = h(e)$ is isolated in $h(G)$ with respect to the strong operator topology. By the definition of strong operator topology one of the possible neighborhoods of id in $h(G)$ is the following set

$$[id; e_1; \varepsilon = 1] \cap h(G) := \{h(g) \in h(G) : ||e_1 g - e_1|| < 1\}$$

where $e_1 := (1,0,0,\cdots)$. By the definition of π it is clear that $[id; e_1; 1] \cap h(G) = \{id\}$ because any *nontrivial* left translation $L_g : G \to G$ moves any point of itself. So, $h(G)$ is discrete because its neutral element is isolated.

□

Exercise 2.15. If X is (locally) fragmented by $f : X \to Y$, where (X,τ) is a Baire space and (Y,ρ) is a pseudometric space then f is continuous at the points of a dense G_δ subset of X.

Proof. For a fixed $\epsilon > 0$ consider

$$O_\epsilon := \{\text{union of all } \tau\text{-open subsets } O \text{ of } X \text{ with } \ \mathrm{diam}_\rho f(O) \leq \epsilon\}.$$

The local fragmentability implies that O_ϵ is dense in X. Clearly, $\bigcap\{O_{\frac{1}{n}} : n \in \mathbb{N}\}$ is the required dense G_δ subset of X. □

Exercise 2.16. Let (X, τ) be a separable metrizable space and (Y, ρ) a pseudometric space. Suppose that $f : X \to Y$ is a fragmented onto map. Then Y is separable. Hint: use the idea of the Cantor-Bendixon theorem.

Proof. Assume (to the contrary) that the pseudometric space (Y, ρ) is not separable. Then there exist an $\varepsilon > 0$ and an uncountable subset H of Y such that $\rho(h_1, h_2) > \varepsilon$ for all distinct $h_1, h_2 \in H$. Choose a subset A of X such that $f(A) = H$ and f is bijective on A. Since X is second countable the uncountable subspace A of X (in its relative topology) is a disjoint union of a countable set and a nonempty closed perfect set M comprising the condensation points of A (this follows from the proof of the Cantor-Bendixon theorem; see e.g. [**?**]). By fragmentability there exists an open subset O of X such that $O \cap M$ is nonempty and $f(O \cap M)$ is ε-small. By the property of H the intersection $O \cap M$ must be a singleton, contradicting the fact that no point of M is isolated. □

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