Marta Lewicka, Topics in the Calculus of Variations, Fall 2012

Final Exam

— 10am - 12noon, January 9, 2013 —

Solve only two of the following three problems. Clearly indicate your choice. Only two problems will count towards the grade.

1. (30 points) State the definition of Γ-convergence in metric spaces. Prove that for the following sequence of functions $F_j : \mathbb{R} \to \mathbb{R}$:

$$F_j(x) = (-1)^j \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \text{ or } x = \frac{k}{n} \text{ and } n \in \{1 \dots j\} \\ -1 & \text{otherwise} \end{cases}$$

the Γ-limit does not exist at any point $x \in \mathbb{R}$.

2. (30 points) Let S be a smooth and compact hypersurface (a manifold of codimension 1) in $\mathbb{R}^n$. A tangent vector field $v \in W^{1,2}(S, \mathbb{R}^n)$ is called a *Killing field* if its covariant derivative is a skew-symmetric tensor on T_xS, for almost every $x \in S$:

$$\forall a.e.\ x \in S \qquad D(v)(x) := \frac{1}{2}\Big((\nabla v)_{tan} + (\nabla v)_{tan}^T\Big) = 0.$$

One can prove (there is an elementary proof) that the space:

$$\mathcal{I}(S) = \{v \in W^{1,2}(S, \mathbb{R}^n);\ v \text{ is a Killing field}\}$$

is finitely dimensional and that every $v \in \mathcal{I}(S)$ is automaticall smooth.

Prove the following Korn inequality on S: For every tangent vector field $u \in W^{1,2}(S, \mathbb{R}^n)$ there exists $v \in \mathcal{I}(S)$ such that:

$$\|u - v\|_{W^{1,2}(S)} \leq C_S \|D(u)\|_{L^2(S)}$$

and the constant C_S depends only on S.

Hint: Consider the extension:

$$\tilde{u}(x + t\vec{n}(x)) = (\mathrm{Id} + t\Pi(x))^{-1}u(x) \qquad \forall x \in S \quad \forall t \in (-\frac{h_0}{2}, \frac{h_0}{2}).$$

Prove that:

$$\|\tilde{u}_n\|_{L^2(S^{h_0})} \approx h_0^{1/2}\|u_n\|_{L^2(S)},$$
$$\|\nabla u_n\|_{L^2(S)} \leq C h_0^{-1/2}\|\tilde{u}_n\|_{W^{1,2}(S^{h_0})},$$
$$\|D(\tilde{u}_n)\|_{L^2(S^{h_0})} \leq C h_0^{1/2}\big(\|u_n\|_{L^2(S)} + \|D(u_n)\|_{L^2(S)}\big).$$

and then use Korn's inequality on the thin shell S^{h_0}.

3. (30 points) State the compactness and Γ-liminf theorem for nonlinear elasticity in the von Kármán regime (scaling $\beta = 4$) on thin shells with mid-surface given by a 2d surface $S \subset \mathbb{R}^3$. Deduce the simplified form of the Γ-limit (von Kármán energy) for plates i.e. when $S \subset \mathbb{R}^2$. The energy should be given in terms of the out-of-plane displacement v and the in-plane displacement w.