Marta Lewicka, Topics in the Calculus of Variations, Fall 2012

Final Exam: Solutions

— 10am - 12noon, January 9, 2013 —

Solve only two of the following three problems. Clearly indicate your choice. Only two problems will count towards the grade.

1. (30 points) State the definition of Γ-convergence in metric spaces. Prove that for the following sequence of functions $F_j : \mathbb{R} \to \mathbb{R}$:

$$F_j(x) = (-1)^j \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \text{ or } x = \frac{k}{n} \text{ and } n \in \{1 \dots j\} \\ -1 & \text{otherwise} \end{cases}$$

the Γ-limit does not exist at any point $x \in \mathbb{R}$.

Given a sequence of functions $F_j : (X, d) \to \overline{\mathbb{R}}$, we say that it Γ-converges to a function $F : (X, d) \to \overline{\mathbb{R}}$, if the following two conditions hold:

(i) *For every sequence x_j converging to some $x \in X$, we have:*

$$\liminf_{n\to\infty} F_j(x_j) \geq F(x).$$

(ii) *For every $x \in X$ there exists a sequence x_j converging to x, such that*

$$\lim_{j\to\infty} F_j(x_j) = F(x).$$

To prove the non-existence of Γ-limit in the problem, fix $x \in \mathbb{R}$ and assume, by contradiction, that $F(x)$ exists. From property (ii) it follows that $F(x) \in \{0, 1, -1\}$.

If $F(x) = 1$, then by (i) above we must have $F_j(x_j) \geq 1$ for j large, if $\lim x_j = x$. This is impossible, since $F_j(x)$ is either 0 or -1 for even values of j and all x.

Another option is that $F(x) = 0$. But, for any given x, we can easily construct an approximating sequence of rational numbers x_k such that $x_{2k} = p_{2k}/n_{2k}$ and $n_{2k} > 2k$. For this sequence $F_{2k}(x_{2k}) = -1$, which brings a contradiction. The existence of such a sequence is trivially true by the Archemidean property of real numbers: let $n_{2k} = 2k + 1$ and choose integer p_{2k} such that $x_{2k} = p_{2k}/n_{2k}$ is the closest to x. Since the steps of size $1/n_{2k}$ are arbitrarily small as k increases, x_{2k} converges to x.

So the only option is that $F(x) = -1$. But this is impossible, since there is no recovery sequence x_j in property (ii) for which $F_j(x_j) = -1$ for all (large) j.

2. (30 points) Let S be a smooth and compact hypersurface (a manifold of codimension 1) in $\mathbb{R}^n$. A tangent vector field $v \in W^{1,2}(S, \mathbb{R}^n)$ is called a *Killing field* if its covariant derivative is a skew-symmetric tensor on T_xS, for almost every $x \in S$:

$$\forall a.e.\ x \in S \qquad D(v)(x) := \frac{1}{2}\Big((\nabla v)_{tan} + (\nabla v)_{tan}^T\Big) = 0.$$

One can prove (there is an elementary proof) that the space:

$$\mathcal{I}(S) = \{v \in W^{1,2}(S, \mathbb{R}^n);\ v \text{ is a Killing field}\}$$

is finitely dimensional and that every $v \in \mathcal{I}(S)$ is automaticall smooth.

Prove the following Korn inequality on S: For every tangent vector field $u \in W^{1,2}(S, \mathbb{R}^n)$ there exists $v \in \mathcal{I}(S)$ such that:

$$\|u - v\|_{W^{1,2}(S)} \leq C_S \|D(u)\|_{L^2(S)}$$

and the constant C_S depends only on S.

Hint: Consider the extension:

$$\tilde{u}(x + t\vec{n}(x)) = (\mathrm{Id} + t\Pi(x))^{-1}u(x) \qquad \forall x \in S \quad \forall t \in (-\frac{h_0}{2}, \frac{h_0}{2}).$$

Prove that:

$$\begin{aligned} \|\tilde{u}_n\|_{L^2(S^{h_0})} &\approx h_0^{1/2}\|u_n\|_{L^2(S)}, \\ \|\nabla u_n\|_{L^2(S)} &\leq Ch_0^{-1/2}\|\tilde{u}_n\|_{W^{1,2}(S^{h_0})}, \\ \|D(\tilde{u}_n)\|_{L^2(S^{h_0})} &\leq Ch_0^{1/2}\big(\|u_n\|_{L^2(S)} + \|D(u_n)\|_{L^2(S)}\big). \end{aligned} \tag{1}$$

and then use Korn's inequality on the thin shell S^{h_0}.

By $\mathcal{I}(S)^\perp$ let us denote the orthogonal complement of $\mathcal{I}(S)$ in the Hilbert space E of all $W^{1,2}(S)$ tangent vector fields on S. We now prove that:

$$\|u\|_{W^{1,2}(S)} \leq C_S\|D(u)\|_{L^2(S)} \qquad \forall u \in \mathcal{I}(S)^\perp \tag{2}$$

which clearly implies the result in the problem. If (2) was not true, there would be a sequence $u_n \in \mathcal{I}(S)^\perp$ such that:

$$\|u_n\|_{W^{1,2}(S)} = 1, \qquad \|D(u_n)\|_{L^2(S)} \longrightarrow 0 \quad \textit{as } n \to \infty.$$

Without loss of generality, u_n converges weakly in $W^{1,2}(S)$ to some u belonging to the closed space $\mathcal{I}(S)^\perp$. Moreover $D(u) = 0$ so $u \in \mathcal{I}(S)$, and hence there must be $u = 0$. Therefore, the sequence u_n converges to 0 (strongly) in $L^2(S)$. This contradicts $\|u_n\|_{W^{1,2}(S)} = 1$, because:

$$\|u_n\|_{W^{1,2}(S)} \leq C_S\left(\|u_n\|_{L^2(S)} + \|D(u_n)\|_{L^2(S)}\right).$$

The above inequality results from Korn's inequality on the open set S^{h_0} applied to the extensions $\tilde{u}_n \in W^{1,2}(S^{h_0})$ defined in the Hint, after noting (1).

Now (1) follows by direct calculations; for each $z = x + t\vec{n}(x) \in \tilde{S}$ and $\tau_1 \in T_xS$:

$$\begin{aligned} \partial_{\tau_1}\tilde{u}(z) = &\Big\{\nabla\left[(Id + t\Pi(x))^{-1}\right](Id + t\Pi(x))^{-1}\tau_1\Big\}u(x) \\ &+ (Id + t\Pi(x))^{-1}\nabla u(x)(Id + t\Pi(x))^{-1}\tau_1. \end{aligned}$$

The first component above is bounded by $C|tu(x)|$ (we may assume that $|\tau_1| = 1$). Taking the scalar product of the second component with any $\tau_2 \in T_xS$ gives:

$$\big((Id + t\Pi(x))^{-1}\tau_2\big) \cdot \nabla u(x)(Id + t\Pi(x))^{-1}\tau_1.$$

Since $(Id + t\Pi(x))(T_xS) = T_xS$ we obtain:

$$\begin{aligned} \tau_2^T D(\tilde{u})(z)\tau_1 &= \big((Id + t\Pi(x))^{-1}\tau_2\big) \cdot D(u)(x)(Id + t\Pi(x))^{-1}\tau_1 \\ &\quad + Z(t,x) \cdot u(x), \\ |Z(t,x)| &\leq C. \end{aligned}$$

On the other hand, $\vec{n}(x) \cdot \tilde{u}(z) = 0$, so for any $\tau \in T_xS$:

$$\begin{aligned} \vec{n} \cdot \partial_\tau\tilde{u}(z) &= -\Big(\Pi(x)(Id + t\Pi(x))^{-1}\tau\Big) \cdot \tilde{u}(z) \\ &= -\Big((Id + t\Pi(x))^{-1}\Pi(x)(Id + t\Pi(x))^{-1}u(x)\Big) \cdot \tau = \tau \cdot \partial_{\vec{n}}\tilde{u}(z). \end{aligned}$$

Hence:

$$\vec{n}^T D(\tilde{u})(z)\tau = -\Big((Id + t\Pi(x))^{-1}\Pi(x)(Id + t\Pi(x))^{-1}u(x)\Big)\cdot\tau,$$
$$\vec{n}^T D(\tilde{u})(z)\vec{n} = 0.$$

3. (30 points) State the compactness and Γ-liminf theorem for nonlinear elasticity in the von Kármán regime (scaling $\beta = 4$) on thin shells with mid-surface given by a 2d surface $S \subset \mathbb{R}^3$. Deduce the simplified form of the Γ-limit (von Kármán energy) for plates i.e. when $S \subset \mathbb{R}^2$. The energy should be given in terms of the out-of-plane displacement v and the in-plane displacement w.

Define thin shells S^h of thickness h by:

$$S^h = \{z = x + t\vec{n}(x);\ x \in S,\ -h/2 < t < h/2\}, \qquad 0 < h < h_0.$$

For a $W^{1,2}$ deformation $u^h : S^h \to \mathbb{R}^3$, we assume that its elastic energy (scaled per unit thickness) is given by the nonlinear functional:

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h).$$

The stored-energy density function $W : \mathbb{R}^{3\times 3} \to [0,\infty]$ is assumed to satisfy the conditions of normalization, frame indifference and nondegeneracy:

$$\forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad W(R) = 0, \quad W(RF) = W(F),$$
$$W(F) \geq c\ \mathrm{dist}^2(F, SO(3)),$$

with a uniform constant $c > 0$.

Let $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ be a sequence of deformations such that the scaled energies $\{\frac{1}{h^4}I^h(y^h)\}$ are uniformly bounded. Then there exist rigid motions of $\mathbb{R}^3$, given through proper rotations $Q^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the normalized deformations:

$$y^h(x + t\vec{n}) = (Q^h)^T u^h(x + t\frac{h}{h_0}\vec{n}) - c^h \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$$

the following hold.

(i) *y^h converge in $W^{1,2}(S^{h_0})$ to the projection $\pi(x + t\vec{n}) = x$.*

(ii) *The averaged displacements $V^h[y^h] = \frac{1}{h}\fint_{-h_0/2}^{h_0/2} y^h(x + t\vec{n}) - x\ dt$ converge (up to a subsequence) in $W^{1,2}(S)$ to some first order infinitesimal isometry $V \in \mathcal{V}$, i.e. $V \in W^{2,2}(S, \mathbb{R}^3)$ and $\mathrm{sym}(\nabla V)_{tan} = 0$.*

(iii) *$\frac{1}{h}\mathrm{sym}\nabla V^h[y^h]$ converge (up to a subsequence) weakly in $L^2(S)$ to some symmetric matrix field $B \in \mathcal{B}$, where the space of finite strains $\mathcal{B}$ contains all limits of symmetrized gradients of vector fields $w \in W^{1,2}(S, \mathbb{R}^3)$.*

(iv) *There holds:*

$$\liminf_{h\to 0} \frac{1}{h^4} I^h(y^h) \geq I(V, B),$$

where:

$$I(V,B) = \frac{1}{2}\int_S \mathcal{Q}_2\left(B - \frac{1}{2}(A^2)_{tan}\right) + \frac{1}{24}\int_S \mathcal{Q}_2\left((\nabla(A\vec{n}) - A\Pi)_{tan}\right). \tag{3}$$

Above, the matrix (tensor) field $A \in W^{1,2}(S, so(3))$ is uniquely defined by: $\partial_\tau V(x) = A(x)\tau$ for all $\tau \in T_x S$ and almost all $x \in S$. The quadratic forms $\mathcal{Q}_2$ are defined by means of the energy density W.

Let now $S \subset \mathbb{R}^2$. Given $V \in \mathcal{V}$ we see that: $0 = \mathrm{sym}\nabla V = \mathrm{sym}\nabla V_{tan}$, so by Korn's inequality there must be: $V(x) = Cx + c + v(x)e_3$ for some $C \in so(2)$, $c \in \mathbb{R}^2$ and the out-of-plane displacement $v \in W^{2,2}(S, \mathbb{R})$. Assuming without loss of generality that C and c are 0, we get: $A(x) = \begin{bmatrix} 0 & 0 & -\partial_1 v \\ 0 & 0 & -\partial_2 v \\ \partial_1 v & \partial_2 v & 0 \end{bmatrix}$. Consequently: $(A^2)_{tan} = -\nabla v \otimes \nabla v$ and $\nabla(A\vec{n})_{tan} = -\nabla^2 v$.

We now turn to identifying the space $\mathcal{B}$. Let $w_n \in W^{1,2}(S, \mathbb{R}^2)$ be such that $B = \lim_{n\to\infty} \mathrm{sym}\nabla w_n$ in $L^2(S)$. Since $\|\mathrm{sym}\nabla w_n\|_{L^2(S)}$ is bounded, by Korn's inequality it follows that there exist $C_n \in so(2)$ and $c_n \in \mathbb{R}^2$ such that the sequence $v_n = w_n - (C_n x + c_n)$ is bounded in $W^{1,2}(S, \mathbb{R}^2)$. Therefore, up to a subsequence, v_n converges weakly in $W^{1,2}$ to some in-plane-displacement $w \in W^{1,2}(S, \mathbb{R}^2)$, and:

$$B \leftharpoondown \mathrm{sym}\nabla w_n = \mathrm{sym}\nabla v_n \rightharpoonup \mathrm{sym}\nabla w \qquad \textit{weakly in } L^2(S).$$

Hence $B = \mathrm{sym}\nabla w$.

We thus see that the functional in (3) can be written as:

$$I(V,B) = I(v,w) = \frac{1}{2}\int_S \mathcal{Q}_2\left(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v\right) + \frac{1}{24}\int_S \mathcal{Q}_2\left(\nabla^2 v\right),$$

for $v \in W^{2,2}(S, \mathbb{R})$ and $w \in W^{1,2}(S, \mathbb{R}^2)$.