Introduction

to

Complex Spaces

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15 Lectures

Contents

Preface

This short paper has its origin in 15 lectures given at the Mathematical Institute of the Jagiellonian University in the frame of the International PhD Programme at the Jagellonian University (2010–2015). The author thanks Prof. Zwonek to invite him to this programme.

The intention of these lectures was to give some introduction to complex spaces, a topic which is rarely taught nowadays. To fulfil this goal I mainly concentrated to discuss the Cartan coherence theorem which is one of the basics in the discussion of complex spaces. Moreover, since this result is often quoted in various lecture but only rarely a proof is presented I tried to give a full proof of it. To do so we had to introduce the notion of sheaves which was unknown to the audience. The lecture ended with a discussion of cohomology and the theorems A and B of Cartan on Stein manifolds. Because of lack of time no proof was given but applications of these results were discussed.

These notes contain almost the material presented during these lectures. At many places in theses notes the reader will find the word EXERCISE which intends to ask the reader to write down a full proof by himself (or to contact the literature) to become more familiar with this topic.

After all I like to thank the audience of PhD students for their active collaboration during these lectures.

A final comment: all of the presented results are taken from books and lecture notes listed at the end of these notes. Nevertheless, I am responsible for all mistakes in this script.

Chapter 1
A bit of local analytic geometry

1.1 The Weierstrass theorems

If $D \subset \mathbb{C}^n$ is an open subset, then, as usual, we write

$$\mathcal{O}(D) := \{F : D \longrightarrow \mathbb{C} : f \text{ holomorphic on } D\}.$$

Let $D' \subset \mathbb{C}^{n-1}$ be a domain. We are interested in functions which are holomorphic on $D := D' \times \mathbb{D}(0,R)$, where $\mathbb{D}(a,r) := \{\lambda \in \mathbb{C} : |\lambda| < r\}$.

Theorem 1.1.1. *Let D be as above and let $f \in \mathcal{O}(D)$ such that f has no zeros in $D' \times \mathbb{A}(0,r,R)$, where $\mathbb{A}(0,r,R) := \{\lambda \in \mathbb{C} : r < |\lambda| < R\}$. Then:*

(a) *there exists a $k \in \mathbb{N}_0$ such that for any $z' \in D'$ the function $f(z',\cdot)$ has exactly k zeros (counted with multiplicities) in $\mathbb{D}(0,R)$;*

(b) *there exist an $h \in \mathcal{O}(D)$ without zeros and a monic "pseudo-polynomial" $P \in \mathcal{O}(D')[u]$ such that*

$$f(z) = h(z)P(z',z_n), \quad z = (z',z_n) \in D;$$

(c) *for any $g \in \mathcal{O}(D)$ there exist a $q \in \mathcal{O}(D)$ and a pseudo-polynomial $p \in \mathcal{O}(D')[u]$ of degree less than k such that*

$$g(z) = q(z)f(z) + r(z',z_n), \quad z = (z',z_n) \in D.$$

The statement in (b) is a global version of the *Weierstrass preparation theorem* while (c) is a global counterpart to the *Weierstrass division theorem*.

Proof. (a) One has only to observe that the number of zeros of $f(z',\cdot)$ is given by the number $\frac{1}{2\pi i}\int_{|\zeta|=s}\frac{f_{z_n}(z',\zeta)}{f(z',\zeta)}d\zeta$, where $r < s < R$. Since this integral is $\mathbb{Z}_+$-valued and continuous, it is identically constant, say it equals k.

(b) (The proof is due to Stickelberger.) Note that the function $\widetilde{f}$

$$D' \times \mathbb{A}(0,r,R) \ni (z',z_n) \stackrel{\widetilde{f}}{\longmapsto} \log\frac{f(z',z_n)}{z_n^k}$$

is well defined and holomorphic. Using Laurent separation one may write $\widetilde{f} = f_1 + f_2$, where $f_1 \in \mathcal{O}(D)$ and $f_2 \in \mathcal{O}(D' \times (\mathbb{C} \setminus \overline{\mathbb{D}}(0,r))$. Moreover, $f_2(z',\cdot)$ vanishes at ∞. Thus we obtain

$$f(z) = e^{f_1(z)}(z_n^k e^{f_2(z)}).$$

Note that the first factor is zero free on $\mathbb{D}(0,R)$. Using Laurent series the second part leads to a monic pseudo-polynomial $P \in \mathcal{O}(D')[z_n]$ of degree k and a holomorphic function $\widehat{f}$ on $\mathbb{C} \setminus \overline{\mathbb{D}}(0,r)$. Thus $f = e^{f_1}P + e^{f_1}\widehat{f}$. Again by the Laurent separation for the function identical zero one gets the desired equation.

(c) The proof is left as an EXERCISE. One argues via Laurent separation as in (b) but now using the function g/P on $D' \times \mathbb{A}(0,r,R)$. □

Recall that a function $f \in \mathcal{O}(U)$, U an open subset in $\mathbb{C}^n$ with $0 \in U$, is said to be *z_n-regular of order k*, if $f(0',\cdot)$ is vanishing at 0 of order k. Then the above result implies

Corollary 1.1.2. *Let f be as before. Then there exist positive numbers r, R, a non-vanishing function $h \in \mathcal{O}(\mathbb{P})$, where $\mathbb{P} := \mathbb{P}' \times \mathbb{D}(0,r) := \mathbb{D}(0,r)^{n-1} \times \mathbb{D}(0,R)$, and a monic pseudo-polynomial $P \in \mathcal{O}(\mathbb{P}')[u]$ of degree k such that*

- *$P(z',u) = u^k + a_1(z')u^{k-1} + \cdots + a_{k-1}(z')u + a_k(z')$ and $a_j(0) = 0$, $j = 1,\dots,k$;*
- *$f = hP$ on $\mathbb{P}$;*
- *any $g \in \mathcal{O}(\mathbb{P})$ can be written as $g = qf + p$ on $\mathbb{P}$, where $q \in \mathcal{O}(\mathbb{P})$ and $p \in \mathcal{O}(\mathbb{P}')[u]$ with $\deg p < k$.*

Recall that a pseudo-polynomial like the one in the above corollary is called a *Weierstrass polynomial*.

Note that the ring of convergent power series in n complex variables can be identified with the ring $\mathcal{O}_{\mathbb{C}^n,0} = \mathcal{O}_0$ of germs of holomorphic functions at 0. Let (U,f) and (V,g) be given *local holomorphic functions at* $0 \in \mathbb{C}^n$, i.e. U and V are open neighborhoods of the origin and $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$. We say that $(U,f) \sim (V,g)$ if there exist an open neighborhood $W = W(0) \subset U \cap V$ such that $f|_W = g|_W$. "$\sim$" is an equivalence relation and the corresponding class is denoted by $\mathbf{f} = \mathbf{f}_0 := [(U,f)] = [(U,f)]_0$. $\mathbf{f}$ is the *germ at* 0 *induced by the local holomorphic function* (U,f).

It is clear how the standard operations like "$+$" and "$\cdot$" are defined in $\mathcal{O}$ in order to have the structure of an integral domain. Then it is a simple EXERCISE to verify the above mentioned ring isomorphism.

We should mention that the same construction as before may be done for an arbitrary point $a \in \mathbb{C}^n$; in such a situation we write, of course, $\mathcal{O}_{\mathbb{C}^n,a}$ or simply $\mathcal{O}_a$.

Remark 1.1.3. (a) Given a finite number of germs $\mathbf{f}_1,\dots,\mathbf{f}_k \in \mathcal{O}_0 \setminus \{\mathbf{0}\}$ represented by local holomorphic functions (U,f_j). Then there exists a non-singular linear transformation $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $f_1 \circ \Phi,\dots,f_k \circ \Phi$ are holomorphic functions on $\Phi^{-1}(U)$ and they a z_n-regular (EXERCISE).

(b) Using the Baire theorem the same result is even true for a countable family of non zero germs in $\mathcal{O}_0$ (EXERCISE).

(c) Then the Weierstrass theorem can be formulated in the language of $\mathcal{O}_0$ (EXERCISE) (see [Jac-Jar01].

1.2 Algebraic consequences

Let us first recall the notion of a *Noetherian module*.

Definition 1.2.1. Let R be a commutative ring with 1. An R-module M is called *Noetherian* if any of its submodules is finitely generated over R, i.e. for any submodule $M' \subset M$ there exist vectors $v_1, \dots, v_k \in M'$ such that any vector $v \in M'$ can be written as $v = \sum_{j=1}^{k} r_j v_j$, where the r_j's belong to R.

Then one has the following characterization of Noetherian modules.

Proposition 1.2.2. *Let R be as above and let M be an R-module.. Then the following properties are equivalent:*

(i) *M is Noetherian;*
(ii) *any increasing sequence $(M_j)_{j\in\mathbb{N}}$ of submodules $M_j \subset M$ becomes stationary, i.e. $M_{k_0} = M_j$ for a certain index k_0 and all $j \geq k_0$;*
(iii) *any non empty family of submodules of M contains a maximal one (w.r.t. inclusion).*

Proof. EXERCISE □

From the above theorems (see Remark 1.1.3(c)) the following well known algebraic consequences may be derived.

Theorem 1.2.3. *(a) The ring $\mathcal{O}_{\mathbb{C}^n,0} = \mathcal{O}_0$ of germs of holomorphic functions at $0 \in \mathbb{C}^n$ is a factorial ring.*
(b) The ring $\mathcal{O}_0$ is a Noetherian ring.

The proofs of these statements can be found in [Jac-Jar01]. Both use induction, the Weierstrass theorems, and the corresponding results for polynomial-rings like the Gauss–Lemma and the fact that a polynomial ring over a Noetherian ring is again a Noetherian ring.

1.3 Germs of holomorphic functions and analytic sets

(1) Recall the following equivalence relation: two sets $A, B \subset \mathbb{C}^n$ are called equivalent (with respect to the point $0 \in \mathbb{C}^n$) if there is an open neighborhood U of 0 such that $A \cap U = B \cap U$. We denote such an equivalence class associated to a set A by $[A]_0 =: \mathbf{A}$. $\mathbf{A}$ is called an *analytic germ* if it has a representant $A \subset U$, where A is an analytic subset of the open set U. Let $\mathfrak{A}_0 = \mathfrak{A}$ denote the set of all analytic germs at the origin. (Obviously, the same construction may be done for any point $a \in \mathbb{C}^n$ resulting in $\mathfrak{A}_a$.)

The following notions are well defined: let $\mathbf{A}$ and $\mathbf{B}$ be two analytic germs at 0 represented by (U, A) and (V, B), respectively. Take an open neighborhood $W \subset U \cap V$ of 0 and define $\mathbf{A} \cap \mathbf{B}$ as the germ represented by $(W, (A \cap W) \cap (B \cap W))$. In a similar way, $\mathbf{A} \cup \mathbf{B}$ may be understood. Then we say that $\mathbf{A} \subset \mathbf{B}$ if $\mathbf{A} = \mathbf{A} \cap \mathbf{B}$.

(2) Let $\mathfrak{a} \subset \mathcal{O}_0$ be an ideal. Recall that $\mathfrak{a}$ is finitely generated by, say, germs $\mathbf{f}_1, \dots, \mathbf{f}_k \in \mathcal{O}$. Then there are an open neighborhood $U = U(0)$ and holomorphic functions $f_1, \dots, f_k \in \mathcal{O}(U)$ representing the corresponding germs. Thus $A := \{z \in U : f_1(z) = \cdots = f_k(z) = 0\}$ is an analytic set in U and it defines an analytic germ $\mathbf{A} = [A]_0$. Observe that $\mathbf{A}$ is well-defined, i.e. it does not depend on the choice of the representatives (EXERCISE).

Definition 1.3.1. Let $\mathfrak{a}$ be as above. The associated analytic germ from before is called the *locus of* $\mathfrak{a}$. We write $\mathbf{V}(\mathfrak{a}) := \mathbf{A}$.

To summarize, we have a mapping

$$\{\mathfrak{b} \subset \mathcal{O}_0 : \mathfrak{b} \text{ an ideal }\} \ni \mathfrak{a} \longrightarrow \mathbf{V}(\mathfrak{a}) \in \mathfrak{A}_0 = \mathcal{A}.$$

(3) *A geometric-algebraic dictionary*:

Given an analytic germ $\mathbf{A} \in \mathfrak{A}$ represented by (U, A) and a germ $\mathbf{f} \in \mathcal{O}$ represented by the local holomorphic function (V, f). We say that $\mathbf{f}$ *vanishes on* $\mathbf{A}$ ($\mathbf{f}|_{\mathbf{A}} = 0$) if there exists an open neighborhood $W = W(0) \subset U \cap V$ such that $f|_{W \cap A} = 0$.

Definition 1.3.2. Let $\mathbf{A} \in \mathfrak{A}$ be an analytic germ. Put

$$Id_0(\mathbf{A}) = Id(\mathbf{A}) := \{\mathbf{f} \in \mathcal{O} : \mathbf{f}|_{\mathbf{A}} = 0\}.$$

Then it is easy to see that $Id(\mathbf{A})$ is an ideal in $\mathcal{O}$. Hence we have a mapping

$$\mathfrak{A} \ni \mathbf{A} \longrightarrow Id(\mathbf{A}) \subset \{\mathfrak{b} \subset \mathcal{O} : \mathfrak{b} \text{ an ideal }\}.$$

While the first mapping translates algebraic objects into geometric ones the second one interprets geometric objects via algebraic ones. There is the following dictionary how all these mappings are related.

Proposition 1.3.3. *Let $\mathbf{A}, \mathbf{B} \in \mathfrak{A}$ and let $\mathfrak{a}, \mathfrak{b}$ be ideals from $\mathcal{O}_0$. Then the following properties hold:*

(a) *if $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$, then $Id(\mathbf{A}) \supset Id(\mathbf{B})$, $Id(\mathbf{A}) \neq Id(\mathbf{B})$;*
(b) $\mathbf{V}(Id(\mathbf{A})) = \mathbf{A}$*;*
(c) $Id(\mathbf{A} \cup \mathbf{B}) = Id(\mathbf{A}) \cap Id(\mathbf{B})$*;*
(d) $Id(\mathbf{A} \cap \mathbf{B}) \supset Id(\mathbf{A}) + Id(\mathbf{B})$*;*
(e) *if $\mathfrak{a} \subset \mathfrak{b}$, then $\mathbf{V}(\mathfrak{a}) \supset \mathbf{V}(\mathfrak{b})$;*
(f) $\mathbf{V}(\mathfrak{a}) = \mathbf{V}(\sqrt{\mathfrak{a}})$*;*

(g) $Id(\mathbf{V}(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}$;
(h) $\mathbf{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathbf{V}(\mathfrak{a}\mathfrak{b}) = \mathbf{V}(\mathfrak{a}) \cup \mathbf{V}(\mathfrak{b})$;
(i) $\mathbf{V}(\mathfrak{a} + \mathfrak{b}) = \mathbf{V}(\mathfrak{a}) \cap \mathbf{V}(\mathfrak{b})$.

The proof is left as an EXERCISE. Moreover, we should mention that in (d) there is, in general, the strict inclusion (EXERCISE). Finally, observe that, in general, $\sqrt{\mathfrak{a}}$ is strictly larger than $\mathfrak{a}$; for example, take $\mathfrak{a} := [(\mathbb{C}, z^2)]_0\mathcal{O}_{\mathbb{C},0}$.

Remark 1.3.4. We will later see that, in fact, equality is always true in (g); this is the so called *Hilbert Nullstellensatz*.

(4) *Partition of analytic germs*:
The following results show how to use the above dictionary to receive geometric results via algebraic properties.

Proposition 1.3.5. *(a) Let $(\mathbf{A}_j)_{j\in\mathbb{N}} \subset \mathfrak{A}$ be a decreasing sequence of analytic germs. Then there exists an index j_0 such that $\mathbf{A}_j = \mathbf{A}_{j_0}$ for all $j \geq j_0$, i.e. such a sequence becomes stationary.*
(b) Let $\mathcal{F} \subset \mathfrak{A}$ be a non empty family of analytic germs. Then $\mathcal{F}$ contains a minimal element w.r.t. inclusion.

Proof. Use the above dictionary and the fact that $\mathcal{O}_0$ is a Noetherian ring. Details are left to the reader. □

Definition 1.3.6. An analytic germ $\mathbf{A} \in \mathfrak{A}$ is called to be *irreducible* if whenever $\mathbf{A} = \mathbf{B} \cup \mathbf{C}$ with $\mathbf{B}, \mathbf{C} \in \mathfrak{A}$, then $\mathbf{B} = \mathbf{A}$ or $\mathbf{C} = \mathbf{A}$, i.e. $\mathbf{A}$ cannot be written as the union of two strictly smaller analytic germs.

With this notion in mind one can write any analytic germ as union of simpler, namely irreducible, analytic germs.

Proposition 1.3.7. *Let $\mathbf{A} \in \mathfrak{A}$. Then there exist a finite number of irreducible analytic germs $\mathbf{B}_1, \ldots, \mathbf{B}_k \in \mathfrak{A}$ such that $\mathbf{A} = \bigcup_{j=1}^k \mathbf{B}_j$.*

Proof. Put $\mathcal{F} := \{\mathbf{A} \in \mathfrak{A} : \mathbf{A}$ does not have such a representation $\}$. We have to show that $\mathcal{F} = \varnothing$. So assume that $\mathcal{F} \neq \varnothing$. Then, in virtue of the former result, $\mathcal{F}$ has a minimal element, say $\mathbf{B}$. Then $\mathbf{B}$ is not irreducible meaning it can be written as $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2$ with strictly smaller analytic germs. Since $\mathbf{B}$ was minimal, the $\mathbf{B}_j$ can be written as a finite union of irreducible analytic germs and therefore the same is true for $\mathbf{B}$; a contradiction. □

In fact one can make this representation in some sense a unique one. To see this we need the following lemma.

Lemma 1.3.8. *Let* $\mathbf{A} \subset \bigcup_{j=1}^{k} \mathbf{A}_j$*, where the germ* $\mathbf{A}$ *is irreducible. Then there exits an index* j_0 *such that* $\mathbf{A} \subset \mathbf{A}_{j_0}$.

Proof. Obviously we may assume that $k \geq 2$. Suppose now that $\mathbf{A}$ is not a subset of any $\mathbf{A}_j$. Put $\mathbf{A'}_j := \mathbf{A} \cap \mathbf{A}_j$. Then $\mathbf{A} = \bigcup_{j=1}^{k} \mathbf{A'}_j$. Since $\mathbf{A'}_j \neq \mathbf{A}$, we have that $\mathbf{A}$ is not irreducible; a contradiction. □

Definition 1.3.9. Let $\mathbf{A} = \bigcup_{j=1}^{k} \mathbf{A}_j$ be a partition by irreducible analytic germs. It is called to be *minimal*, if for any $m = 1, \dots, k$ the germ $\mathbf{A}_m$ is not contained in $\bigcup_{j=1, j\neq m}^{k} \mathbf{A}_j$.

Remark 1.3.10. (a) Such a minimal partition always exists.

(b) A minimal partition is uniquely determined up to its numeration (EXERCISE: use the former lemma).

(5) *Irreducible germs and prime ideals*:

There is a good characterization of irreducible analytic germs via algebraic properties of the associated ideal.

Proposition 1.3.11. *Let* $\mathbf{A} \in \mathfrak{A}$ *be an analytic germ. Then the following properties are equivalent:*

(i) $\mathbf{A}$ *is irreducible;*
(ii) $Id(\mathbf{A})$ *is a prime ideal.*

Proof. EXERCISE □

1.4 The Remmert–Stein theorem on parametrization of analytic germs

The idea in this section is to show that any irreducible analytic germ allows a coordinate system such that its representant can be written as a branched analytic covering over a certain $\mathbb{C}^k$. Recall that any analytic germ can be written as the union of such irreducible analytic germs.

Thus we start with an irreducible analytic germ $\mathbf{A} \in \mathfrak{A}$ with $0 \in A \neq \mathbb{C}^n \cap U$, where U is a neighborhood of 0 and $A \subset U$ a representant of $\mathbf{A}$. Then $0 \neq Id(\mathbf{A}) =: \mathfrak{p} \neq \mathcal{O}_0$ is a prime ideal and $\mathbf{V}(\mathfrak{p}) = \mathbf{A}$ according to our dictionary.

Note that if $\mathfrak{a} \subset \mathcal{O}_0$ is a given ideal and $\Psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ a linear non-singular coordinate transform, then $\Psi^*(\mathfrak{a}) := \{\mathbf{f} \circ \Psi : f \in \mathfrak{a}\}$, where $\mathbf{f} \circ \psi := [U, f \circ \Psi]$ when $[(V, f)] = \mathbf{f}$ and $\psi(U) \subset V$. If $\mathfrak{a}$ is a prime ideal, then $\psi^*(\mathfrak{a})$ is also a prime ideal (EXERCISE).

Step 1^o *There exist a* $k \in \{0, \dots, n-1\}$*, a linear coordinate transform* $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ *such that*

- $\Phi^*(\mathfrak{p}) \cap \mathcal{O}_{\mathbb{C}^k,0} = \{0\}$;
- *there exist Weierstrass polynomials* $p_{k+1} \in \mathcal{O}_{\mathbb{C}^k,0}[z_{k+1}]$, …, $p_n \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_n]$ *such that their germs belong to* $\Phi^*(\mathfrak{p})$.

In the future we will write also $\mathcal{O}' := \mathcal{O}_{\mathbb{C}^k,0}$.

Proof. Indeed, by assumption there exists a germ $\mathbf{f}_n = [(U_n, f_n)] \in \mathfrak{p} \setminus \{0\}$. Using Remark 1.1.3(a) there exists a linear isomorphism $\varphi_n : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $f_n \circ \varphi_n$ is a z_n-regular function on $\varphi_n^{-1}(U_n)$ of order $s_n \geq 1$. Hence, we may write

$$f_n \circ \varphi_n(z) = h_n(z)\big(z_n^{s_n} + \sum_{j=1}^{s_n} a_{n,j}(z')z_n^{s_n-j}\big) = h_n(z)p_n(z_1, \dots z_{n-1}, z_n),$$
$$z = (z', z_n) \in V_1 = V_1' \times V_1'' \subset \varphi_1^{-1}(U) \subset \mathbb{C}^{n-1} \times \mathbb{C},$$

where h_n is zero-free on V_n and the pseudo-polynomial p_n is a Weierstrass polynomial. Then $\mathbf{f}_n \circ \varphi_n = [(V_n, h_n)][(V_n, p_n)] \in \varphi_n^*(\mathfrak{p}) =: \mathfrak{p}_n$. Since the first factor is a unit in $\mathcal{O}_0$ one concludes that $[(V_n, p_n)] \in \mathfrak{p}_n$.

Now there are two alternatives, namely: either $\varphi_n^*(\mathfrak{p}) \cap \mathcal{O}_{\mathbb{C}^{n-1},0} = \varnothing$ or there exists a non trivial germ in this intersection. In the first case we are done; we take $k = n - 1$. In the second case we start the same procedure as before but now for the first $n - 1$ variables. Thus, in the second case, we take a non-trivial germ $\mathbf{f}_{n-1} \in \varphi_n^*(\mathfrak{p}) \cap \mathcal{O}_{\mathbb{C}^{n-1},0}$ which may be represented by the local function (U_{n-1}, f_{n-1}), U_{n-1} an open neighborhood of $0 \in \mathbb{C}^{n-1}$. Using the Weierstrass preparation theorem there exists a linear isomorphism $\varphi_{n-1} : \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1}$ such that $f_{n-1} \circ \varphi_{n-1}$ is a z_{n-1}-regular function of order s_{n-1} on $\varphi_{n-1}^{-1}(U_{n-1}) \subset \mathbb{C}^{n-1}$.

Hence, we may write

$$\begin{aligned} f_{n-1} \circ \varphi_{n-1}(z_1, \dots, z_{n-1}) &= h_{n-1}(z_1, \dots, z_{n-1}) \\ &\quad \big(z_{n-1}^{s_{n-1}} + \sum_{j=1}^{s_{n-1}} a_{n-1,j}(z_1, \dots, z_{n-2})z_{n-1}^{s_{n-1}-j}\big) \\ &= h_{n-1}(z_1, \dots, z_{n-1})p_{n-1}(z_1, \dots z_{n-2}, z_{n-1}), \\ z = (z', z_{n-1}) \in V_{n-1} &= V_{n-1}' \times V_{n-1}'' \subset \varphi_{n-1}^{-1}(U_{n-1}) \subset \mathbb{C}^{n-2} \times \mathbb{C}, \end{aligned}$$

where h_{n-1} is zero-free on V_{n-1} and the pseudo-polynomial p_{n-1} is a Weierstrass polynomial. Then

$$\mathbf{f}_{n-1} \circ \varphi_{n-1} = [(V_{n-1}, h_{n-1})][(V_{n-1}, p_{n-1})] \in (\varphi_{n-1} \times \operatorname{id}|_{\mathbb{C}})^*(\varphi_n^*(\mathfrak{p})).$$

Since the first factor is a unit in $\mathcal{O}_{\mathbb{C}^{n-1},0}$ one concludes that $[(V_{n-1}, p_{n-1})] \in (\varphi_{n-1} \times \operatorname{id}_{\mathbb{C}})^*(\varphi_n^*(\mathfrak{p})) =: \mathfrak{p}_{n-1}$.

Note that the former Weierstrass polynomial p_n is transform as $p_n \circ (\varphi_{n-1} \times \mathrm{id}_{\mathbb{C}})(z) = z_n^{s_n} + \sum_{j=1}^{s_n} a_{n,j}(\varphi_{n-1}(z'))z_n^{s_n-j}$; i.e. after this second transformation the former Weierstrass polynomial remains to be a Weierstrass polynomial.

As before if $\mathcal{O}_{\mathbb{C}^{n-2},0} \cap \mathfrak{p}_{n-1} = \{0\}$, then we take $k = n-2$. Otherwise we start the same procedure as above. □

Definition 1.4.1. If $\mathfrak{p} \subset \mathcal{O}$ is a prime ideal satisfying the properties in Step 1^o, we say that $\mathfrak{p}$ is $(z_k, \dots, z_n)$-regular.

Summarizing the former statement in Step 1^o, we know that a given prime ideal $\mathfrak{p}$ after the above coordinate transforms is a prime ideal which is $(z_{k+1}, \dots, z_n)$-regular. Therefore from now on we assume that $\mathfrak{p}$ is $(z_{k+1}, \dots, z_n)$-regular.

Remark 1.4.2. Looking at the proof above it could occur that the number k there depends on the procedure we had chosen. In fact, one can prove that k is an independent number, namely the *Krull dimension* of $\mathfrak{p}$.

Recall that the Krull dimension of a prime ideal $\mathfrak{p}$ in an integral domain R is given as the supremum over all $r \in \mathbb{N}_0$ such that there exists a chain

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \subsetneq R$$

of prime ideals $\mathfrak{p}_j$.

Step 2^o *Let* $0 \neq \mathfrak{p} \subsetneq \mathcal{O}_0$ *be a prime ideal which is* $(z_k, \dots, z_n)$*-regular. Then:*

(a) $\varphi : \mathcal{O}_{\mathbb{C}^k,0} \longrightarrow \mathcal{O}_0/\mathfrak{p}$ *is an injective ring homomorphism (i.e. the right ring may be thought as a ring extension of the left ring);*

(b) $\mathcal{O}_0/\mathfrak{p}$ *is a finitely generated* $\mathcal{O}_{\mathbb{C}^k,0}$*-module.*

Proof. (a) is trivial because of the regularity condition.

(b) Take an element $[\mathbf{f}] \in \mathcal{O}_0/\mathfrak{p}$, where $\mathbf{f} = [(U, f)]$ and (U, f) a local holomorphic function. Applying the Weierstrass division we get

$$f = qp_n + r_n \text{ on a small neighborhood of } 0,$$

where $r_n(z_1, \dots, z_{n-1}, z_n) = \sum_{j_n=0}^{m_n-1} f_{j_n}(z_1, \dots, z_{n-1})z_n^{j_n}$, $m_n := \deg p_n$, and the f_{j_n} are holomorphic functions near the origin. Going first to germs and then to the classes in $\mathcal{O}_0/\mathfrak{p}$ and applying the fact that $\mathbf{p}_n \in \mathfrak{p}$ (1) we have

$$[\mathbf{f}] = \sum_{j_n}^{m_n-1} \mathbf{f}_{j_n}[z_n]^{j_n}.$$

(1) $\mathbf{p}_n$ denotes the germ induced by p_n; moreover, note that one should write $[\mathbf{z_n}]$ but, for simplicity, we keep the letter z_n etc.

In the case where $k = n-1$, we see that $[1], [z_n], [z_n]^2, \dots, [z_n]^{m_n-1}$ generate $\mathcal{O}_0/\mathfrak{p}$ as an $\mathcal{O}_{\mathbb{C}^k,0}$-module.

In the case when $k < n-1$ one proceeds as above using division of the f_{j_n}'s by the Weierstrass polynomial p_{n-1}, etc.

Finally, we get

$$[f] = \sum_{j_{k+1}}^{m_{k+1}-1} \cdots \sum_{j_n}^{m_n-1} \mathbf{f}_{j_n, j_{n-1}, \dots, j_{k+1}} [z_{k+1}]^{j_{k+1}} \dots [z_n]^{j_n},$$

with germs $\mathbf{f}_{j_n, j_{n-1}, \dots, j_{k+1}} \in \mathcal{O}_{\mathbb{C}^k,0}$ and $m_j := \deg p_j$. Hence $\mathcal{O}_0/\mathfrak{p}$ is generated by the following elements $[z_{k+1}]^{j_{k+1}} \dots [z_n]^{j_n}$ which appear in the above sum. □

Remark 1.4.3. From Step 2^o one concludes that the ring $\mathcal{O}_0/\mathfrak{p}$ is integral over the ring $\mathcal{O}_{\mathbb{C}^k,0}$. Recall the notion "integral" w.r.t a ring extension $R \subset R'$: R' is called *integral over R*, if any $a \in R'$ is integral over R, i.e. a satisfies an equation $a^k + \alpha_1 a^{k-1} + \dots + \alpha_k$ with $\alpha_j \in R$, $j = 1, \dots, k$, and $k \in \mathbb{N}$. For further properties of the notion "integral over" the reader is sent to books on algebra.

Step 3^o Denote by $K := \mathbb{Q}(\mathcal{O}_{\mathbb{C}^k,0})$ and $L := \mathbb{Q}(\mathcal{O}_0/\mathfrak{p})$ the total quotient field of the integral domain $\mathcal{O}_{\mathbb{C}^k,0}$ and $\mathcal{O}_0/\mathfrak{p}$, respectively. Under the assumption that $\mathfrak{p}$ is $(z_k, \dots, z_n)$-regular the Step 2^o and the former remark imply that

$$L = K([z_{k+1}], \dots, [z_n]),$$

i.e. L is a finitely algebraic field extension of K.

Step 4^o Now we come to the last coordinate transform which in fact only changes the $(k+1)$-th coordinate. Applying the theorem of the existence of a primitive element we find an element $a \in L$ such that $K[a] = L$, where $a = a_{k+1}[z_{k+1}] + \dots + a_n[z_n]$, $a_j \in \mathbb{C}$ and $a_{k+1} = 1$. With this information we use the following coordinate transform:

$$(z_1, \dots, z_n) \longrightarrow (z_1, \dots, z_k, z_{k+1} + a_{k+2}z_{k+2} + \dots + a_n z_n, z_{k+2}, \dots, z_n).$$

After transforming our $\mathfrak{p}$ and denoting the new one again with the letter $\mathfrak{p}$ we have now (in this new coordinate system) that $L = K([z_{k+1}])$, i.e. L is the simple field extension by the algebraic element $[z_{k+1}] \in \mathbb{Q}(\mathcal{O}_0/\mathfrak{p})$.

Summary: So far we have seen that for a prime ideal $0 \neq \mathfrak{p} \subsetneq \mathcal{O}_0$ there exist a coordinate system in which $\mathfrak{p}$ is $(z_{k+1}, \dots, z_n)$-regular such that $L = K([z_{k+1}])$ is an algebraic field extension, where $K = \mathbb{Q}(\mathcal{O}_{\mathbb{C}^k,0})$ and $L = \mathbb{Q}(\mathcal{O}_0/\mathfrak{p})$.

Step 5^o Fix a $j \in \{k+1, \dots, n\}$. From the former step we know that $[z_j]$ is integral over $\mathcal{O}_{\mathbb{C}^k,0}$. Thus there exists a monic polynomial $g_j \in \mathcal{O}_{\mathbb{C}^k,0}[z_j]$ with $g([z_j]_0) = 0$. Since the ring $\mathcal{O}_{\mathbb{C}^k,0}$ is a factorial ring, we may assume that g_j is an irreducible monic polynomial which is z_j-regular as an element in $\mathcal{O}_{\mathbb{C}^{k+1},0}$. Therefore, $g_j =$ unit $\cdot P_j$, where P_j is a Weierstrass polynomial in $\mathcal{O}_{\mathbb{C}^k,0}[z_j]$. Then P_j is irreducible in $\mathcal{O}_{\mathbb{C}^k,0}[z_j]$, and so irreducible in $\mathbb{Q}(\mathcal{O}_{\mathbb{C}^k,0})[z_j] = K[z_j]$. Then this Weierstrass polynomial P_j is the uniquely given minimal polynomial for the algebraic element $[z_j]$ (EXERCISE) over K.

Step 6^o *Let* $\mathfrak{p}$ *be as above in the new coordinate system and let* $\mathbf{f} = [(U, f)] \in \mathcal{O}$. *Then there exist a polynomial* $R_f \in \mathcal{O}_{\mathbb{C}^k,0}[z_{k+1}]$ *of degree less than* $\deg P$ *such that*

$$\textit{germ of}\left(\Delta \cdot f - R_f(z', z_{k+1})\right) \in \mathfrak{p},$$

where Δ *denotes the discriminant of the polynomial* $P := P_{k+1}$ *from Step* 5^o*, i.e.* Δ *is a holomorphic function in a small neighborhood of* $0 \in \mathbb{C}^k$.

Proof. We will apply the theorem of the universal denominator:

Let $R \subset R'$ *a ring extension and* $\alpha \in R'$*, where* R *is an integral domain. Assume that* R *is integrally closed on* $\mathbb{Q}(R)$*. Moreover, let* $p \in R[u]$ *be a monic polynomial of minimal degree with* $p(\alpha) = 0$*. If now* $f \in K(\alpha)$ *be integral over* R*, then*

$$f = \frac{b_0 + b_1\alpha + \cdots + b_{m-1}\alpha^{m-1}}{\Delta},$$

where the b_j*'s belong to* R *and* $m := \deg p$.

In our situation we have $R = \mathcal{O}_{\mathbb{C}^k,0}$, $R' = \mathcal{O}/\mathfrak{p}$, $p = P$, and $\alpha = [z_{k+1}]$. Therefore,

$$[\mathbf{f}] = \frac{\mathbf{b}_0 + \mathbf{b}_1[z_{k+1}] + \cdots + \mathbf{b}_{m-1}[z_{k+1}]^{m-1}}{\Delta},$$

where the $\mathbf{b}_j$'s belong to $\mathcal{O}_{\mathbb{C}^k,0}$, $m := \deg P$, and Δ is the discriminant of P. Hence the germ of $\Delta \cdot f - R_f(z_{k+1})$ belongs to $\mathfrak{p}$. □

Step 7^o *Applying Step* 6^o *for the germs defined by the coordinate function* z_j, $j = k+1, \dots, z_n$ *there exist polynomials* $q_j \in \mathcal{O}_{\mathbb{C}^k,0}[z_{k+1}]$, $\deg q_j < \deg P$*, such that the germs induced by the local functions* $\Delta \cdot z_j - q_j$ *belong to* $\mathfrak{p}$.

After all these preparation we formulate a part of the Remmert–Stein result.

Theorem 1.4.4. *Let* $0 \neq \mathfrak{p} \subsetneq \mathcal{O}_{\mathbb{C}^n,0}$ *be a prime ideal. Assume that* $\mathfrak{p}$ *is* $(z_{k+1}, \dots, z_n)$*-regular. Then there exist a neighborhood* $U = U(0) = U' \times U'' \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ *and an analytic set* $A \subset U$ *with an irreducible germ in* 0 *such that:*

(a) $[A]_0 = \mathbf{V}(\mathfrak{p})$,

(b) *the set*

$$\begin{aligned} B := \{z = (z', z'') \in U' \times U'' : P(z', z_{k+1}) \\ = \Delta(z')z_{k+2} - q_{k+2}(z', z_{k+1}) = \Delta(z')z_n - q_n(z', z_{k+1}) = 0\} \end{aligned}$$

is an analytic set in U *with* $A \subset B$ *and* $A \setminus N = B \setminus N$*; where* $N := \{z \in U : \Delta(z) = 0\}$ *and* P*,* Δ*, and* q_j *are from Step* 7^o *above;*

(c) *if* $(z', z'') \in U' \times \mathbb{C}^{n-k}$ *with* $P(z', z_{k+1}) = \Delta(z')z_{k+2} - q_{k+2}(z', z_{k+1}) = \cdots = \Delta(z')z_n - q_n(z', z_{k+1}) = 0$*, then* $z'' \in U''$*;*
(d) *the projection* $\pi : A \longrightarrow \mathbb{C}^k$ *is proper, discrete, and* $\pi(A) = U'$*;*
(e) $\pi|_{A \setminus N} : A \setminus N \longrightarrow U' \setminus N'$ *is surjective and locally topological, where* $N' := \{z' \in U' : \Delta(z') = 0\}$.

Remark 1.4.5. Interpretation of this result: any local analytic set A with an irreducible germ $[A]_0$ in 0 can be thought locally as a branched analytic covering over $\mathbb{C}^k$ which is locally biholomorphic outside of the thin set N.

Later we will even see that $A \setminus N$ is dense in A.

Proof. (a) Note that $\mathfrak{p}$ is generated by germs $\mathbf{f}_j \in \mathcal{O}_{\mathbb{C}^n,0}$, $j = 1, \ldots, k$. Assume that they are represented by the local holomorphic functions $(U = U' \times U'', f_j)$. Put $A := \{z \in U : f_1(z) = \cdots = f_k z) = 0\}$. Then A is an analytic set in U such that $\mathbf{V}(\mathfrak{p}) = [A]_0$.

Moreover, U may be chosen so small that P, Δ, and the q_j's are holomorphic functions on U.

(b) Observe that the germs of P, $\Delta z_j - q_j$, $j = k+2, \ldots, n$ belong to $\mathfrak{p}$. Thus, after a possible shrinking of U, these functions are linear combinations of the f_j. Hence, $A \subset B$.

It remains to prove that $B \setminus N \subset A \setminus N$:

Note that

$$f_j - \sum_{\alpha_{k+1} < m_{k+1}} \cdots \sum_{\alpha_n < m_n} f_{j;\alpha_{k+1},\ldots,\alpha_n}(z') z_{k+1}^{\alpha_{k+1}} \cdots z_n^{\alpha_n} \in (P_{k+1}, \ldots, P_n)\mathcal{O}_0.$$

Put $m := \sum_{j=k+2}^{n} m_j$. Then, or a suitable s,

$$\Delta^m f_j - \sum_{\alpha_{k+1} < m_{k+1}} \cdots \sum_{\alpha_n < m_n} f_{j;\alpha_{k+1},\ldots,\alpha_n}(z') \Delta^s z_{k+1}^{\alpha_{m_{k+1}}} (\Delta z_{k+2})^{\alpha_{k+2}} \cdots (\Delta z_n)^{\alpha_n} \in \mathfrak{a},$$

where $\mathfrak{a} \subset \mathcal{O}_0$ denotes the ideal generated by $P_{k+1}, \ldots, P_n, \Delta z_{k+2} - q_{k+2}, \ldots, \Delta z_n - q_n$. Then the former equality implies that $\Delta^m f_j \equiv R_j(z_{k+1}) \mod \mathfrak{a}$, where $R_j \in \mathcal{O}_{\mathbb{C}^k,0}[u]$. Using the division by polynomials we get that $R_j = \widetilde{q}_j P_{k+1} + r_j$, where $r_j \in \mathcal{O}_{\mathbb{C}^k,0}[u]$ with $\deg r_j < m_{k+1}$. Note that $P_{k+1} \in \mathfrak{a}$. Thus we have $\Delta^m f_j \equiv r_j \mod \mathfrak{a}$.

Next we like to kill the remainder r_j. Observe that $\mathfrak{a} \subset \mathfrak{p}$ and that $\mathbf{f}_j \in \mathfrak{p}$; thus $r_j \in \mathfrak{p}$ which implies that $r_j([z_{k+1}]) = 0$. Recall that P_{k+1} was the monic polynomial of minimal degree annulating $[z_{k+1}]$. Hence, $r_j = 0$. So we have obtained that $\Delta^m f_j \equiv 0 \mod \mathfrak{a}$; i.e. any zero of the $P_{k+1}, \ldots, P_n, \Delta z_{k+j} - q_{k+j}$, $j = 2, \ldots, n-k$, in U (U has eventually to be shrinked again)is a zero of $\Delta^m f_j$. Observe that B may be a priori larger than this zero set. So it remains to kill the $P_{k+2}, \ldots, P_n$.

Fix a $j \in \{k+2, \dots, n\}$. Then

$$P_j(z', z_j) = z_j^{m_j} + b_{m_j-1} z_j^{m_j-1} + \cdots + b_0,$$

where the b_k's belong to $\mathcal{O}_{\mathbb{C}^k,0}$. Recall that $\Delta z_j \equiv q_j \mod (\Delta z_j - q_j)$. Thus

$$\begin{aligned}\Delta^{m_j} p_j(\cdot, z_j) = (\Delta z_j)^{m_j} + b_{m_j-1}\Delta(\Delta z_j)^{m_j-1} + \cdots + \Delta^{m_j} b_0 \\ (\Delta z_j - P_{k+1}(z_{k+1}) + P_{k+1}(z_{k+1}))^{m_j} + \cdots \\ = \widetilde{B}_j(z_{k+1}) \mod (\Delta z_j - q_j),\end{aligned}$$

where $\widetilde{B}_j \in \mathcal{O}_{\mathbb{C}^k,0}[z_{k+1}]$. Following the above idea after polynomial division in $\mathcal{O}_{\mathbb{C}^k,0}[z_{k+1}]$ we have $\widetilde{B}_j = q'' P_{k+1} + B_j$, where B_j is a polynomial with degree less than m_{k+1}. Ordering this last equality in another way we see that

$$\Delta^{m_j} P_j(z_j) \equiv B_j(z_{k+1}) \mod (P_{k+1}, \Delta z_j - q_j(z_{k+1})).$$

Note that the left term belongs to $\mathfrak{p}$ and also the right ideal. Hence, $B_j([z_{k+1}]) = 0$. Again using the minimality of the degree of P_{k+1} leads to the fact that $B_j = 0$.

Summarizing we have shown that

$$\Delta^{2m} f_j \in (P_{k+1}, \Delta z_j - q_j : j = k+2, \dots, n)\mathcal{O}_0.$$

Therefore, after shrinking U again such that all representant are living on U one is lead to the following inclusion

$$B \subset \{z \in U : \Delta^{2m}(z') f_1(z) = \cdots = \Delta^{2m}(z') f_k(z) = 0\}$$

or to $B \setminus N \subset A$ inside of this U.

(c) Recall that the zeros of a polynomial, say $z^s + \sum_{j=1}^{s} a_j z^{s-j}$, can be estimated by $|z| < \max\{1, sM\}$, where $M := \max\{|a_j| : j = 1, \dots, s\}$ (EXERCISE). Then apply this information to the Weierstrass polynomial P to obtain the result by shrinking U'.

(d) Properness and discreteness follow immediately because of $A \subset \{z \in U : P(z', z_{k+1}) = 0\}$.

(e) Take a point $b = (b', b'') \in A \setminus N$. Then one may find a neighborhood $V = V(b') \subset (U' \setminus N')$. Then

$$\begin{aligned}A \cap (V' \times U'') = \{z \in V' \times U'' : P(z', z_{k+1}) = 0, \\ z_{k+2} = q_{k+2}(z', z_{k+1})/\Delta(z'), \dots, z_n = q_n(z', z_{k+1})/\Delta(z')\},\end{aligned}$$

i.e. $A \cap (V' \times U'')$ is given by $(n-k)$ equations and the rank of the associated Jacobian is $(n-k)$ since $P(z', \cdot =$ has only simple zeros for z' outside of N'. It remains to apply the implicit function theorem (EXERCISE) in order to see that the projection is locally topological outside of N.

(d) It remains to complete the proof of (d). Fix a point $a' \in U' \setminus N'$. Then there exists $a_{k+1} \in \mathbb{C}$ with $P(a', a_{k+1}) = 0$. Thus it suffices to choose

$$a = \left(a', a_{k+1}, \frac{q_{k+1}(a', a_{k+1})}{\Delta(a')}, \ldots, \frac{q_n(a', a_{k+1})}{\Delta(a')}\right) \in A$$

which lies over a' and so in U. Since π proper, we get that $\pi(A)$ is closed in U'. Applying that $U' \setminus N' \subset \pi(A)$ it follows that $\overline{(U' \setminus N')}^{U'} = U'$. Therefore,

$$U' = \overline{(U' \setminus N')}^{U'} \subset \overline{\pi(A)}^{U'} = \pi(A),$$

proving the surjectivity of π. □

1.5 The Hilbert Nullstellensatz

Now we can complete our dictionary (see Proposition 1.3.3).

Theorem 1.5.1. *If $\mathfrak{a}$ is an ideal in $\mathcal{O}_0$, then $Id(\mathbf{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. Step 1^o Since $\mathcal{O}_0$ is a Noetherian ring, we may write $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$, where the $\mathfrak{q}_j$'s are primary ideals. Then $\mathbf{V}(\mathfrak{a}) = \bigcup_{j=1}^m \mathbf{V}(\mathfrak{q}_j)$, and therefore $Id(\mathbf{V}(\mathfrak{a})) = \bigcap_{j=1}^m Id(\mathbf{V}(\mathfrak{q}_j)) = \bigcap_{j=1}^m Id(\mathbf{V}(\sqrt{\mathfrak{q}_j}))$. Recall that the $\sqrt{\mathfrak{q}_j}$'s are prime ideals. Let us assume that the Nullstellensatz is true for prime ideal. Then we conclude $Id(\mathbf{V}(\mathfrak{a})) = \bigcap_{j=1}^m \sqrt{\mathfrak{q}_j} = \sqrt{\mathfrak{a}}$. So the proof of the Nullstellensatz is reduced to prime ideals.

Step 2^o Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_0$.We may assume that $\mathfrak{p}$ is $(z_{k+1}, \ldots, z_n)$-regular (EXERCISE). Then for an $\mathbf{f} \in \mathcal{O}_0$ there exist a $\mathbf{g} \in \mathcal{O}_0 \setminus \mathfrak{p}$ and a $\mathbf{h} \in \mathcal{O}_{\mathbb{C}^k,0}$ such that $\mathbf{fg} - \mathbf{h} \in \mathfrak{p}$. Indeed, if $\mathbf{f} \in \mathfrak{p}$, then take $\mathbf{g} = \mathbf{1}$ and $\mathbf{h} = \mathbf{0}$. So we may assume that $\mathbf{f} \notin \mathfrak{p}$. Recall that $\mathcal{O}_0/\mathfrak{p}$ is integral over $\mathcal{O}_{\mathbb{C}^k,0}$. Therefore, there exist an $m \in \mathbb{N}$ and germs $\mathbf{a}_j \in \mathcal{O}_{\mathbb{C}^k,0}$, $j = 0, 1, \ldots, m-1$, such that

$$\mathbf{f}^m + \mathbf{a}_{m-1}\mathbf{f}^{m-1} + \cdots + \mathbf{a}_0 \in \mathfrak{p}.$$

We take m to be minimal. Then $\mathbf{g} := \mathbf{f}^{m-1} + \mathbf{a}_{m-1}\mathbf{f}^{m-2} + \cdots + \mathbf{a}_1 \notin \mathfrak{p}$ (use the minimality of m). Moreover, put $\mathbf{h} := -\mathbf{a}_0$ which proves the above claim.

Step 3^o Let $\mathfrak{p}$ be as above. Then, by the Remmert–Stein theorem, there exist an open set $U = U' \times U'' \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$, $0 \in U$, and an analytic set $A \subset U$ such that $\mathbf{V}(\mathfrak{p}) = [A]_0$. Let $\mathbf{f} \in Id(\mathbf{V}(\mathfrak{p}))$. Now take the germs $\mathbf{g}$ and $\mathbf{h}$ constructed in Step 2^o and choose local functions (U, f), (U, g), and (U, h) (after shrinking U) representing the corresponding germs. By assumption, $f|_A = 0$ and $(gf - h)|_A = 0$. In virtue of Theorem 1.4.4 the projection $A \longrightarrow U'$ is surjective. Therefore, for any $z' \in U'$ one may find a point $z'' \in U''$ such that $(z', z'') \in A$. Thus $h(z') = g(z', z'')f(z', z'') = 0$

which implies that $h = 0$ on U'. So $\mathbf{h} = \mathbf{0}$ and $\mathbf{fg} \in \mathfrak{p}$. Since $\mathbf{g} \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, we get $\mathbf{f} \in \mathfrak{p}$. Since $\mathbf{f}$ was arbitrarily chosen, we get $Id(\mathbf{V}(\mathfrak{p})) \subset \mathfrak{p}$. □

Chapter 2

Sheaves — a short introduction

The sheaf theory was started by Leray in the form of presheaves (1949). Later, Cartan studied sheaves as “espaces étalés”.

(A) *Presheaves.*

Definition 2.0.2. A *presheaf of abelian groups on a topological space* $(X, \mathfrak{T})$ is given by a system

$$\mathcal{F} = \Big((\mathcal{F}(U))_{U \in \mathfrak{T}}, (r_V^U)_{V \subset U \subset X,\ \text{open}} \Big),$$

where $\mathcal{F}(U)$ are abelian groups, $\mathcal{F}(\varnothing) = \{0\}$, and $r_V^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ are group homomorphism with

- $r_U^U = \mathrm{id}\,|_{\mathcal{F}(U)}$;
- $r_W^U = r_W^V \circ r_V^U$ whenever $W \subset V \subset U$.

The homomorphism r_V^U is called the restriction map; we often write simply $r_V^U(f) =: f|_V$, $f \in \mathcal{F}(U)$.

Remark 2.0.3. In a similar way one may define presheaves of rings or algebras etc., where the r_V^U are then assumed to be ring homomorphisms or algebra homomorphisms. Details are left for the reader.

Example 2.0.4. (a) Let G be a given abelian group and X as above a topological space. Then the system $\mathcal{F}(U) := G$, if $U \subset X$ is a non empty open subset, $\mathcal{F}(\varnothing) := \{0\}$, $r_V^u := \mathrm{id}\,|_G$ if $V \neq \varnothing$, and $r_\varnothing^U :=$ the trivial homomorphism defines a presheaf of abelian groups which we denote by **G**; it is the constant sheaf with values in G.

(b) Let X be as above. Then the system $\mathcal{F}(U) := \mathcal{C}(U, \mathbb{C})$, $U \subset X$, with the standard restriction maps gives the presheaf $\mathcal{C}_X$ of continuous functions.

(c) Let $G \subset \mathbb{C}^n$ an open subset. Then the system $\mathcal{F}(U) := \mathcal{O}(U)$, U an open subset of G, with the standard restriction maps gives the presheaf $\mathcal{O}_G$ of holomorphic functions on G.

Observe that in (b) and (c) there are more algebraic structures on the $\mathcal{F}(U)$’s.

(B) *Morphisms between presheaves.*

Let $\mathcal{F} = \Big((\mathcal{F}(U)), (r_V^U) \Big)$ and $\mathcal{G} = \Big((\mathcal{G}(U)), (s_V^U) \Big)$ be presheaves of abelian groups over X.

Definition 2.0.5. A *presheaf morphism* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a system $(\varphi_U)_U$ of group homomorphisms $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ such that for $V \subset U \subset X$ the following equality $s_V^U \circ \varphi_U = \varphi_V \circ r_V^U$.

Remark 2.0.6. In case of presheaves with a more general algebraic structure these mappings φ_U has to respect this structure, too.

Definition 2.0.7. Let $\mathcal{F}$ and $\mathcal{G}$ be as above. One says that $\mathcal{G}$ is a *subpresheaf* of $\mathcal{F}$ ($\mathcal{G} \subset \mathcal{F}$) if, for any open $U \subset X$, $\mathcal{G}(U)$ is a subgroup of $\mathcal{G}(U)$ and $r_V^U|_{\mathcal{G}(U)} = s_V^U$ for any $V \subset U \subset X$.

Example 2.0.8. (a) If $X = G \subset \mathbb{C}^n$, then $\mathcal{O}_G \subset \mathcal{C}_G$.

(b) Let $A \subset G \subset \mathbb{C}^n$ be an analytic set in G. Then the system $\mathfrak{I}_A(U) := \{f \in \mathcal{O}(U) : f|_{A\cap U} = 0\}$ with the natural restriction maps defines a presheaf $\mathfrak{I}_A$. Obviously, $\mathfrak{I}_A \subset \mathcal{O}_G$.

A presheaf is a collection of local informations. Using the possibility of glueing one likes to get from this local information a global one which leads to the notion of a canonical presheaf.

Definition 2.0.9. Let $\mathcal{F}$ (as usual) be a presheaf of abelian groups on X. $\mathcal{F}$ is called a *canonical presheaf* if the following two properties are fulfilled:

- if $U = \bigcup_{j\in J} U_j$ and $f, g \in \mathcal{F}(U)$ with $r_{U_j}^U(f) = r_{U_j}^U(g)$ for all $j \in J$, then $f = g$;
- if $U = \bigcup_{j\in J} U_j$ and $f_j \in \mathcal{F}(U_j)$ $(j \in J)$ with $r_{U_{j,k}}^{U_j}(f_j) = r_{U_{j,k}}^{U_k}(f_k)$ for all indices $j, k \in J$, then there exists an $f \in \mathcal{F}(U)$ with $r_{U_j}^U(f) = f_j$, $j \in J$. (We will always write $U_{j,k} := U_j \cap U_k$.)

Remark 2.0.10. (a) Because of the first property in this definition the element f in the second property is uniquely defined.

(b) One should mention that in the literature a canonical presheaf is often called to be a sheaf.

Example 2.0.11. (a) $\mathcal{C}_X$ and $\mathcal{O}_G$ are canonical presheaves.

(b) Let G be a abelian group with two different elements a_1, a_2 and let X be a topological space with two disjoint non empty open subsets U_1, U_2. Then the constant sheaf **G** is not a canonical presheaf (EXERCISE).

(c) (WARNING) Let $\mathcal{G} \subset \mathcal{F}$ be two presheaves over X. Then, in general, the presheaf given by $\mathcal{F}(U)/\mathcal{G}(U)$ and the natural restriction mappings is not canonical (EXERCISE).

(C) *Sheafication of a presheaf.*

Let $\mathcal{F}$ be a presheaf of abelian groups over the topological space X. Fix an $x \in X$. Observe that the index set $J_x := \{U \subset X : x \in U \text{ open }\}$ with the semi order given by $U \leq V$ iff $V \subset U$ is a upper directed index set. Then one can define the inductive limit of the system $(\mathcal{F}(U), r_V^U)_{U \leq V, V, U \in J_x}$ (for more details contact the literature on functional analysis)

$$\mathcal{F}_x := \text{ind-}\lim_{U \in J_x} \mathcal{F}(U), \quad |\mathcal{F}| := \dot{\bigcup}_{x \in X} \mathcal{F}_x.$$

$|\mathcal{F}|$ is a set with a projection $\pi : |\mathcal{F}| \longrightarrow X$ defined by $\pi(\mathcal{F}_x) = \{x\}$. $\mathcal{F}_x$ is called the *stalk* over x.

A topology on $|\mathcal{F}|$:

Recall that an element $[f]_x \in \mathcal{F}_x$ is given via an open set $U \in J_x$ and an $f \in \mathcal{F}(U)$. A set $S \subset |\mathcal{F}|$ is called to be open if for any $[f]_x \in S$ ($f \in \mathcal{F}(U), U \in J_x$) one has $\{[f]_y : y \in U\} \subset S$.

Lemma 2.0.12. *$|\mathcal{F}|$ with the above defined system of open subsets is a topological space and $\pi : |\mathcal{F}| \longrightarrow X$ is a locally topological surjective mapping.*

Proof. EXERCISE. □

Remark 2.0.13. $(|\mathcal{F}|, \pi)$ is the *sheafication* of the presheaf $\mathcal{F}$.

(D) *Cartan sheaves.*

Definition 2.0.14. A *sheaf of abelian groups over a topological space X* is a pair (F, π) where

- F is a topological space and $\pi : F \longrightarrow X$ is locally topological and surjective,
- for any $x \in X$ the fiber $F_x := \pi^{-1}(x)$ is an abelian group,
- "+" is a continuous mapping, i.e. the mapping

$$F \times_X F := \{(a, b) \in F \times F : \pi(a) = \pi(b)\} \ni (\xi, \eta) \longrightarrow \xi + \eta \in F_{\pi(\xi)} \subset F$$

is continuous.

Remark 2.0.15. If $\mathcal{F}$ is a presheaf of abelian groups over X, then $(|\mathcal{F}|, \pi)$ is a sheaf of abelian groups over X (EXERCISE); the algebraic structure is defined as follows:

let $[f]_x, [g_x] \in \mathcal{F}_x$ be given by $f \in \mathcal{F}(U), g \in \mathcal{F}(V)$, respectively, where $U, V \in J_x$. Then $[f]_x + [g]_x := [r_W^U(f) + r_W^V(g)]_x$, where $W := U \cap V$. Note that this addition is well defined (EXERCISE)).

Now let (F, π) be a sheaf of abelian groups over X and let $U \subset X$ be open. A map $s : U \longrightarrow F$ is called a *section over U*, if s is continuous and $\pi \circ s = \mathrm{id}_U$. We denote by $\Gamma(U,F)$ all sections over U. Note that $\Gamma(U)$ is in a natural way an abelian group by $(s_1+s_2) := s_1(x)+s_2(x) \in F_x$. Then the system $(\Gamma(U,f))_U$ with the natural restriction mappings leads to a presheaf $\widehat{F}$ of abelian groups over X. One may call this presheaf the *presheafication* of F. We only mention that the sheafication of the presheafication leads to nothing new (EXERCISE).

Note that if $s_1, s_2 \in \Gamma(U,F)$ with $s_1(a) = s_2(a)$ for a point in $a \in U$, then there is a neighborhood $V \subset U$ of a such that $s_1|_V = s_2|_V$ (EXERCISE).

In the other direction, let $\mathcal{F}$ be a given presheaf as usual. Then there is a natural mapping $\mathcal{F}(U) \longrightarrow \Gamma(U,|\mathcal{F}|), U \subset X$ open:

$$\mathcal{F}(U) \ni f \overset{i_U}{\longmapsto} (U \ni x \longmapsto [f]_x \in \mathcal{F}_x) \in \Gamma(U,|\mathcal{F}|).$$

Note that i_U is a group homomorphism (EXERCISE).

Lemma 2.0.16. *Let $\mathcal{F}$ be a given presheaf. Then $\mathcal{F}$ is canonical if and only if all the i_U a group isomorphisms.*

Proof. EXERCISE. □

Remark 2.0.17. Because of the former lemma we often identify $\mathcal{F}(U)$ and $\Gamma(U,|\mathcal{F}|)$. For example, we write $\mathcal{O}(U) = \Gamma(U,\mathcal{O}) = \{f : U \longrightarrow \mathbb{C} : f \text{ holomorphic }\}$ using the fact that if $s \in \Gamma(U,\mathcal{O})$, then we may identify this section with the holomorphic function $U \ni z \longmapsto s(z)(z)$, where $\mathbf{f}_a(a) = [(U,f)]_a(a) = f(a) \in \mathbb{C}$ is the value of the germ $\mathbf{f}_a \in \mathcal{O}_{\mathbb{C}^n,a}$ at the point a. The reader is asked to check the correct interpretation during the whole text.

Definition 2.0.18. Given two sheaves (F_1,π_1), (F_2,π_2) over X. A *sheaf-morphism* $\varphi : F_1 \longrightarrow F_2$ is a continuous mapping with $\pi_2 \circ \varphi = \pi_1$ (i.e. φ maps $F_{1,x}$ into $F_{2,x}$) such that $\varphi_x := \varphi|_{F_{1,x}} : F_{1,x} \longrightarrow F_{2,x}$, $x \in X$, is a groups homomorphism.

Note that such a φ is automatically locally topological (EXERCISE).

Remark 2.0.19. As we have seen there is a way to identify a canonical presheaf and its sheafication. Therefore we will always write $\mathcal{O}_G = |\mathcal{O}_G|$ ($\mathcal{O} = \mathcal{O}_{\mathbb{C}^n}$) and $\mathfrak{I}_A = |\mathfrak{I}_A|$. In general, if it is not clear over which space this sheaf should be understood we write e.g. $\mathcal{O}_{\mathbb{C}^n}$ to emphasize that we are working over $\mathbb{C}^n$. Note that the stalk $\mathcal{O}_x$ is nothing else than the set of convergent power series at the point x. Moreover, if $A \subset G \subset \mathbb{C}^n$ is an analytic set, then $\mathfrak{I}_{A,x}; = (\mathfrak{I}_A)_x = \mathcal{O}_x$ whenever $x \notin A$. In general, $\mathfrak{I}_{A,x}$ is an ideal in the ring $\mathcal{O}_x$, $x \in G$.

Chapter 3
Coherent sheaves

In this chapter we only deal with the sheaf $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n}$ of germs of holomorphic function (a sheaf of $\mathbb{C}$-algebras) and $\mathcal{O}$-module sheaves. A sheaf (F, π) over $\mathbb{C}^n$ is a $\mathcal{O}$-module sheaf if any of its stalks F_x is a $\mathcal{O}_x$-module and every of the algebraic operation is a continuous one. Details are left to the reader.

3.1 General properties

Example 3.1.1. Define the following canonical presheaf $\mathfrak{I}$ of ideals over $\mathbb{C}$: $\mathfrak{I}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ holomorphic }, f(1/k) = 0 \text{ for those } k \in \mathbb{N}, 1/k \in U\}$, $U \subset \mathbb{C}$ open, and the restriction mappings are the natural restrictions of functions. Obviously, sheafication leads to the following sheaf $|\mathfrak{I}|$ with the following stalks

$$\mathfrak{I}_x = \begin{cases} \mathcal{O}_0, & \text{if } x \notin \{0, 1, 1/2, 1/3, \dots\} \\ 0, & \text{if } x = 0 \\ (z - x)\mathcal{O}_x, & \text{if } x = 1/k \text{ for some } k \in \mathbb{N} \end{cases} .$$

Observe that there is only one element over 0 generating the stalk $\mathfrak{I}_0$, namely the germ generated by the local function $(U, 0)$. The corresponding section is the zero section which obviously does not generate the stalks near the origin.

To overcome such difficulties we have the notion of a sheaf of finite type.

Definition 3.1.2. Let (F, π) is a $\mathcal{O}$-module sheaf over a domain $D \subset \mathbb{C}^n$.

(a) F is called to be *finitely generated over* $U \subset D$, U open, if there exist $s_1, \dots, s_k \in \Gamma(U, F)$ such that for any $a \in U$ the stalk $F_a = \pi^{-1}(a)$ is finitely generated by $s_1(a), \dots, s_k(a)$ over $\mathcal{O}_a$, i.e. any $\xi \in F_a$ can be written as $\xi = \sum_{j=1}^{k} \mathbf{f}_j s_j(a)$ with $\mathbf{f}_j \in \mathcal{O}_a$.

(b) F is said to be *finite type* if for any $a \in D$ there is an open neighborhood $U \subset D$ such that F is finitely generated over U.

Remark 3.1.3. Let $\mathfrak{I}$ be the sheaf of the former example. Then $\mathfrak{I}$ is not if finite type (EXERCISE).

Lemma 3.1.4. *Let (F, π) be a $\mathcal{O}^m$-submodule sheaf of finite type over a domain $D \subset \mathbb{C}^n$ and let the stalk F_a be generated by $s_1(a), \dots, s_k(a)$* (recall that each stalk

$F_a \subset \mathcal{O}_a^m$ is a Noetherian $\mathcal{O}_a$-module), *where* $s_j \in \Gamma(U, F)$ *and* $U = U(a) \subset D$. *Then there exists a neighborhood* $V = V(a) \subset U$ *such that* $s_1(b), \ldots, s_k(b)$ *generate the* $\mathcal{O}_b$*-module* F_b, $b \in V$.

Proof. EXERCISE. □

Corollary 3.1.5. *(a) Let* F_1, F_2 *be two* $\mathcal{O}^m$*-submodule sheaf over* $D \subset \mathbb{C}^n$. *If* $F_{1,a} \subset F_{2,a}$ *for a point* $a \in D$, *then there is a neighborhood* $V = V(a) \subset D$ *with* $F_1|_V \subset F_2|_V$.

(b) If F *is a* $\mathcal{O}^m$*-submodule sheaf of finite type over* $D \subset \mathbb{C}^n$, *then* $\mathrm{supp}(F) := \{x \in D : F_x \neq 0\}$ *is a closed subset of* D.

Proof. EXERCISE. □

Recall Example 3.1.1, where $\mathrm{supp}\, \mathfrak{I} = \mathbb{C} \setminus \{0\}$ is not closed.

In order to define the meaning of coherence we need another notation.

Definition 3.1.6. Let $F = (F, \pi)$ be an $\mathcal{O}$-module sheaf over $D \subset \mathbb{C}^n$ and $s_1, \ldots, s_k \in \Gamma(D, F)$. Then we define the following canonical presheaf: for $U \subset D$ we set $\mathcal{R}(s_1, \ldots, s_k) := \{(\varphi_1, \ldots, \varphi_k) \in \Gamma(U, \mathcal{O})^k : \sum_{j=1}^k \varphi_j s_j = 0 \text{ on } U\}$ and take the natural restriction maps. $\mathcal{R}(s_1, \ldots, s_k)$ is called the *sheaf of relations w.r.t. the* s_j*'s*.

$\mathcal{R}(s_1, \ldots, s_k)$ corresponds after sheafication to a $\mathcal{O}^k$-submodule sheaf.

With these two definitions in mind we present the main notion in this chapter.

Definition 3.1.7. A $\mathcal{O}$-module sheaf $F = (F, \pi)$ over $D \subset \mathbb{C}^n$ is called to be a *coherent sheaf* if

- F is of finite type,
- any relationsheaf for F is again of finite type.

There is an often used criterium of Serre for coherence.

Proposition 3.1.8. *Let* F_j *be* $\mathcal{O}$*-module sheaves over* $D \subset \mathbb{C}^n$ *(*$j = 1, 2, 3$*) and* $0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow{\psi} F_3 \longrightarrow O$ *be an exact sequence with sheaf-homomorphisms* φ, ψ, *i.e. for any* $x \in D$ *the following sequence* $0 \longrightarrow F_{1,x} \xrightarrow{\varphi_x} F_{2,x} \xrightarrow{\psi_x} F_{3,x} \longrightarrow 0$ *is exact. Assume that two of the sheaves* F_j *are coherent, then so the third one.*

Proof. EXERCISE. □

One of the important result in connection with complex analysis is the following one due to Oka.

Theorem 3.1.9. *Take an domain* $D \subset \mathbb{C}^n$. *Then the sheaf* $(\mathcal{O}^m)_D$ *is a coherent* $\mathcal{O}$*-module sheaf over* D.

Because of lack of time we do not present a proof of this deep result. The reader is asked to contact the corresponding literature.

In the remaining part we mention a few other examples of coherent sheaves.

Proposition 3.1.10. *Let F, G be coherent $\mathcal{O}^m$-submodule sheaves over $D \subset \mathbb{C}^n$. Then:*

(a) $F \cap G$ is a coherent $\mathcal{O}^m$-submodule sheaf.

(b) $F : G$ defined via sheafication from the presheaf

$$U \longmapsto (F : G)(U) := \{s \in \Gamma(U, \mathcal{O}) : s(a)G_a \subset F_a, a \in U\}$$

with the standard restriction maps is a coherent subsheaf of $\mathcal{O}_D$.

$F : G$ is called the *residual quotient sheaf.*

Proof. (a) It suffices to prove, since $\mathcal{O}^m$ is coherent, that $F \cap G$ is of finite type. So let us fix a point $a \in D$. By assumption there exist a neighborhood $U = U(a) \subset D$, sections $f_1, \ldots, f_k \in \Gamma(U, F)$, and sections $g_1, \ldots, g_\ell \in \Gamma(U, G)$ such that for any $z \in U$ one has

$$F_z = (f_1(z), \ldots, f_k(z))\mathcal{O}_z \text{ and } G_z = (g_1(z), \ldots, g_\ell(z))\mathcal{O}_z.$$

Using the coherence of $\mathcal{O}^m$ one knows the relation sheaf

$$\mathcal{R} := \mathcal{R}(f_1, \ldots, f_k, g_1, \ldots, g_\ell)$$

is a coherent sheaf over U. Put

$$\Phi : \mathcal{R} \longrightarrow (F \cap G)|_U \text{ as } \mathcal{R}_z \ni (\alpha_1, \ldots, \alpha_k, \beta_1 \ldots, \beta_\ell)_z \longmapsto \sum_{j=1}^{k} (\alpha_j f_j)_z \in (F \cap G)_z,$$

$z \in U$. (We use here the convention $s_z = s(z)$ for a section in an arbitrary sheaf.) Then φ is surjective (EXERCISE). Recall now that $\mathcal{R}|_U$ is of finite type. Therefore, $(F \cap G)|U$ is of finite type.

(b) Step 1^o In a first step let us assume that $G = (g)\mathcal{O}$ where $g \in \mathcal{O}(D, \mathbb{C}^m)$. Then $(F : G)(U) = \{f \in \mathcal{O}(U) : fg \in \Gamma(U, F)\}$, $U \subset D$ open. As before it suffices to verify that this quotient sheaf is of finite type. So fix a point $a \in D$. By assumption, there exist a neighborhood $U = U(a) \subset D$ and $f_1, \ldots, f_k \in \Gamma(U, F)$ such that these sections generate all fibers F_z, $z \in U$. Define the relation sheaf

$$\mathfrak{R} := \mathfrak{R}(-g|_U, f_1, \ldots, f_k)$$

over U. It is a sheaf of finite type over U, since $\mathcal{O}^m|_U$ is coherent. Put

$$\Phi : \mathfrak{R} \longrightarrow F : G, \quad \mathfrak{R}_z \ni (\mathbf{a}_0, \ldots, \mathbf{a}_k) \overset{\Phi}{\longmapsto} \mathbf{a}_0 \in (F : G)_z \quad z \in U.$$

Note that Φ is well defined. Moreover, observe that Φ is surjective (EXERCISE). So it remains to apply that $\mathfrak{R}$ is of finite type to its first component to obtain that $F : G$ is finitely generated over a neighborhood $V = V(a) \subset U$.

Step 2^o Again, fix $a \in D$. Using the finite type of G there exist a neighborhood $U = U(a) \subset D$ and sections $g_1, \ldots, g_\ell \in \Gamma(U, G)$ such that G_z is generated by $g_1(z), \ldots, g_\ell(z)$ over $\mathcal{O}_z$, $z \in U$. Then $(F : G)|_U = \cap_{j=1}^{\ell}(F : (g_j\mathcal{O}))$. Then Step 1 and (a) induce immediately that $(F : G)|_U$ is of finite type. □

Finally, we represent the gap sheaf of a sheaf w.r.t. an closed set introduced by Thimm. Let F be a $\mathcal{O}^m$-submodule sheaf over $D \subset \mathbb{C}^n$ and let $A \subset D$ be a relatively closed subset. Then the associated *gap sheaf* $F[A]$ is defined as sheafication of the following canonical presheaf

$$D \supset U \longmapsto F[A](U) := \{s \in \Gamma(U, \mathcal{O}^m) : s|_{U \setminus A} \in \Gamma(U \setminus A, f)\}$$

with the standard restriction maps. Note that the gap sheaf is collecting all sections which behave well w.r.t. F outside of A.

Theorem 3.1.11 (Thimm). *If $\mathfrak{I}$ is a coherent ideal subsheaf of $\mathcal{O}_D$ over $D \subset \mathbb{C}^n$ and $A \subset D$ is analytic* (note that A is closed in D)*, then the gap sheaf $\mathfrak{I}[A]$ is again a coherent sheaf.*

In order to be able to proof Thimm's result we need the following result.

Lemma 3.1.12. *Let $\mathfrak{I} \subset \mathcal{O}$ be a coherent ideal sheaf over $D \subset \mathbb{C}^n$, A the common zero set of the holomorphic functions $g_1, \ldots, g_k$ on D such that $\mathfrak{I}$ is generated by these functions, i.e. $\mathfrak{I}_a = ([g_1]_a, \ldots, [g_k]_a)\mathcal{O}_a$ for all $a \in D$* (as before $[(D, g_j)]_b \in \mathcal{O}_b$ denotes the holomorphic germ in b induced from the local holomorphic function (D, g_j))*. Then, for any $x \in D$, we have*

$$\mathfrak{I}[A]_x = \bigcup_{r \in \mathbb{N}} \Big(\mathfrak{I}_x : ([g_1]_x, \ldots, [g_k]_x)^r\Big).$$

Proof. Step 1^o To prove that the left hand side contains the right one it suffices to show that for an open set $U \subset D$ and a holomorphic function f on U with $f(g_1|_U, \ldots, g_k|_U)^r \in \Gamma(U, \mathfrak{I})$ one has $f|_{U \setminus A} \in \Gamma(U \setminus A, \mathfrak{I})$.

So let $a \in D \setminus A$. Then we may assume that $g_1(a) \neq 0$. Thus g is without zeros on a neighborhood $V = V(a) \subset U$. Recall that, by assumption, $g_1 f \in \Gamma(V, \mathfrak{I})$. Local division by g_1 gives that $f|_V \in \Gamma(V, \mathfrak{I})$. Then $f \in \Gamma(U \setminus A, \mathfrak{I})$.

Step 2^o Fix a point $a \in D$ and take a germ $\mathbf{f}_a := [(U, f)]_a \in \mathfrak{I}[A]_a$ where $V = V(a) \subset D$ and f holomorphic on V. Then there exist a neighborhood $W = W(a) \subset V$ such that $\mathbf{f}_b = [(W, f|_W)]_b \in \mathfrak{I}[A]_b$ for all $b \in W$.

We already know (see Proposition 3.1.10) that $(\mathfrak{I}|_W) : (f|_W \mathcal{O}_W)$ is a coherent sheaf. Therefore, shrinking W we find sections $h_1, \dots, h_l \in \Gamma(W, \mathfrak{I}|_W)$ such that $(\mathfrak{I})|_W) : (f|_W) = (h_1, \dots, h_l)\mathcal{O}_W$. Therefore one concludes that

$$\{z \in W : \Big(\mathfrak{I}|_W : (f|_W)\mathcal{O}_W\Big)_z \neq \mathcal{O}_z\} = \{z \in W : h_1(z) = \cdots = h_l(z) = 0\} =: R$$

The latter set belongs to A otherwise if there is a $b \in R \setminus A$, then $\mathfrak{I}[A]_b = \mathfrak{I}_b$ or $(\mathfrak{I} : (f))_b = \mathcal{O}_b$.

Reading this inclusion on the level of germs we see that

$$V(\mathfrak{I}_a : \mathbf{f}_a) = [R]_x \subset [A]_a.$$

Applying the Hilbert Nullstellensatz it follows that

$$([(D, g_1)]_a, \dots, [(D, g_k)]_a) \subset Id(V(\mathfrak{I}_a : ([(W, f)]_a \mathcal{O}_a))) = \sqrt{\mathfrak{I}_a : ([(W, f)]_a}.$$

Hence there exist an $r \in \mathbb{N}$ such that $\Big([(D, g_1)]_a, \dots, [(D, g_k)]_a\Big)^r \subset \mathfrak{I}_a : [(W, f)]_a$ or $[(W, f)]_a \in \mathfrak{I}_a : \Big((D, g_1)]_a, \dots, [(D, g_k)]_a\Big)^r$. □

Now we are in position to verify Theorem 3.1.11.

Proof of Theorem 3.1.11. It suffices to show that the sheaf is of finite type. So fix a point $a \in A$ (note that outside of A the gap sheaf coincides with the sheaf $\mathfrak{I}$) and choose a neighborhood $U = U(a) \subset D$ and holomorphic functions $g_1, \dots, g_k$ on U such that

$$A \cap U = \{z \in U : g_1(z) = \cdots g_k(z) = 0\}.$$

Then we have an increasing sequence of ideals $\Big(\mathfrak{I}_a : ((\mathbf{g}_{1,a}, \dots, \mathbf{g}_{k,a})\mathcal{O}_a)^r\Big)_{r \in \mathbb{N}}$, where $\mathbf{g}_{j,a} = [(U, g_j)]_a$ is the germ at a induced by the local holomorphic functions g_j, $j = 1, \dots, k$. Applying that the ring $\mathcal{O}_a$ is Noetherian one may find an exponent r_0 such that this sequence becomes stationary starting with the index r_0. Using coherence there is a small neighborhood $V = V(a) \subset U$ such that $\mathfrak{I} : ((g_1|_V, \dots, g_k|V)\mathcal{O}_V)^{r_0} = \mathfrak{I} : ((g_1|_V, \dots, g_k|V)\mathcal{O}_V)^{r_0+1}$ on V. Hence for any $b \in V$ it follows that

$$\mathfrak{I} : ((g_1|_V, \dots, g_k|V)\mathcal{O}_V)^{r_0} = \mathfrak{I} : ((g_1|_V, \dots, g_k|V)\mathcal{O}_V)^r, \quad r \geq r_0.$$

Now applying the former lemma leads to the following identity

$$\mathfrak{I}[A]_b = \bigcup_{r \in \mathbb{N}} \mathfrak{I}_b : ((\mathbf{g}_{1,b}, \dots, \mathbf{g}_{k,b})\mathcal{O}_b)^r = \mathfrak{I}_b : ((\mathbf{g}_{1,b}, \dots, \mathbf{g}_{k,b})\mathcal{O}_b)^{r_0}.$$

Therefore, it follows that over V the following equality holds

$$\mathfrak{I}[A] = \mathfrak{I} : ((g_1, \dots, g_k)\mathcal{O})^{r_0}.$$

It remains to recall that the sheaves $\mathfrak{I}$ and $((g_1|_V, \dots, g_k|_V)\mathcal{O}_V)^{r_0}$ are coherent subsheaves of $\mathcal{O}$ over V and that the residual quotient of such sheaves is again coherent. □

As a consequence of Thimm's result we get the so-called Ritt lemma.

Lemma 3.1.13. *Let $A, B \subset D \subset \mathbb{C}^n$ be analytic sets in the domain D. Then $\overline{(B \setminus A)}^D$ is again an analytic set in D.*

Proof. Because analyticity is a local property we may assume that $B = \{z \in D : f_1(z) = \cdots = f_k(z) = 0\}$ with $f_j \in \mathcal{O}(D)$. Put on D the ideal sheaf $\mathfrak{I} := (f_1, \dots, f_k)\mathcal{O}_D$. Then $\mathfrak{I}$ is of finite type and, by the Oka result, a coherent sheaf. By definition one has

$$\mathbf{V}(\mathfrak{I}) := \{z \in D : \mathfrak{I}_z \neq \mathcal{O}_z\} = B.$$

Moreover, by the Thimm result, we know that $\mathfrak{I}[A]$ is a coherent sheaf which immediately shows that $\mathbf{V}(\mathfrak{I}[A])$ is an analytic set in D.

So it remains to verify $\mathbf{V}(\mathfrak{I}[A]) = \overline{\mathbf{V}(\mathfrak{I}) \setminus A}^D$. Observe that $\mathfrak{I}|_{D \setminus A} = \mathfrak{I}[A]|_{D \setminus A}$. Thus $\mathbf{V}(\mathfrak{I}) \setminus A = \mathbf{V}(\mathfrak{I}[A]) \setminus A \subset \mathbf{V}(\mathfrak{I}[A])$. Using that $\mathbf{V}(\mathfrak{I}[A])$ is closed in D we get $\overline{\mathbf{V}(\mathfrak{I}) \setminus A}^D \subset \mathbf{V}(\mathfrak{I}[A])$. To see the inverse inclusion take a point $a \in D \setminus \overline{(\mathbf{V}(\mathfrak{I}) \setminus A)}^D$. Then there is a neighborhood $V = V(a) \subset D$ which is disjoint to $\overline{(\mathbf{V}(\mathfrak{I}) \setminus A)}^D$, thus $\mathfrak{I}[A]|_V = \mathcal{O}|_V$ implying that $a \notin \mathbf{V}(\mathfrak{I}[A])$ which gives the equality proving the result. □

Using this Ritt lemma one can prove the following additional property in Theorem 1.4.4.

Theorem 3.1.14. *Under the assumptions of Theorem 1.4.4 one has even that $A \setminus N$ is dense in A (after another possible shrinking of U).*

Proof. According to the above lemma we know that $\overline{A \setminus N}^U$ is an analytic subset of U (here we use the notation from Theorem 1.4.4). Then $A = (\overline{A \setminus N}^U) \cup N$. Recall that $[A]_0$ is an irreducible analytic germ at 0. Hence $[A]_0 = [\overline{A \setminus N}^U]_0$ or $[A]_0 = [N]_0$. Assume the latter situation is true. Then there exists a neighborhood $V = V(0) \subset U$ with $V \cap A = V \cap N$. Thus $\Delta|_{A \cap N} = 0$. Note that Δ depends only on the first k variables; so it vanishes on a non empty open set $V' \subset U'$ implying that $\Delta = 0$ in U'; a contradiction. Therefore, shrinking U we get that $A \setminus N$ is dense in A. □

3.2 Coherence of the ideal sheaf for an analytic set

Let $D \subset \mathbb{C}^n$ be a domain and let $A \subsetneq D$ an analytic subset. Recall that the canonical presheaf

$$D \supset U \longmapsto \mathfrak{I}_A(U) = \{f \in \mathcal{O}(U) : f|_A = 0\}$$

induces the sheaf of ideals $\mathfrak{I}_A \subset \mathcal{O}|_D$ over D.

Theorem 3.2.1 (Cartan 1950/51). *Let A, D be as above. Then the ideal sheaf $\mathfrak{I}_A$ is a coherent sheaf over D.*

Proof. Since the sheaf $\mathcal{O}|_D$ is coherent, it suffices to verify that $\mathfrak{I}_A$ is of finite type.

Fix a point $a \in D$.

Step 1^o Assume that $a \in D \setminus A$. Then there is a neighborhood $V = V(a) \subset D$ which is disjoint to A. Therefore, $\mathfrak{I}_A|_V = \mathcal{O}|_V$. It only needs to observe that the sheaf $\mathcal{O}|_V$ is coherent.

Step 2^o Let now $a \in A$. We may even assume that $a = 0$ (EXERCISE). Now we will use the local description of analytic sets from the former discussions.

Step $2a^o$ We know that there are a neighborhood $U = U(0)$ and analytic subsets $A_1, \dots, A_\ell \subset U$ such that

- the induced germs $A_{1,0}, \dots, A_{\ell,0}$ are irreducible,
- $A \cap U = A_1 \cup \dots \cup A_\ell$,
- $\mathfrak{I}_A|_U = \mathfrak{I}_1 \cap \dots \cap \mathfrak{I}_\ell$, where $\mathfrak{I}_j := \mathfrak{I}_{A_j}$ for $j = 1, \dots, \ell$.

Applying Proposition 3.1.10 shows that it suffices to prove that the $\mathfrak{I}_j$ are of finite type.

Thus we may also assume that the starting germ A_0 is an irreducible one which is induced from the local analytic set (U, A). Hence, the stalk $\mathfrak{p} := \mathfrak{I}_{A,0} = Id(A_0)$ is a prime ideal in $\mathcal{O}_{\mathbb{C}^n,0}$. Now we apply the local description of $\mathbf{V}(A_0)$ we recall here:

Note that

$$\begin{aligned}(A \cap U) \setminus N = \{z = (z', z'') \in (\mathbb{C}^k \times \mathbb{C}^{n-k}) \cap U :\\ P(z) = z_{k+2}\Delta(z') - q_{k+2}(z', z_{k+2}) = \dots = z_n\Delta(z') - q_n(z', z_n) = 0\} \setminus N,\end{aligned}$$

i.e. A is given outside of N as the zero set of $n-k$ holomorphic functions.

Lemma 3.2.2. *Let $\subset \mathbb{C}^n$, $f_1, \dots, f_r \in \mathcal{O}(D)$, and $A := \{z \in D : f_1(z) = \dots = f_r(z) = 0\}$. Assume that the Jacobian matrix of the mapping $F := (f_1, \dots, f_r) : D \longrightarrow \mathbb{C}^r$ has everywhere on D* rank $F' = r$. *Put*

$$\mathfrak{I} := (r_1, \dots, f_r)\mathcal{O}_D.$$

Then $\mathfrak{I}_A = \mathfrak{I}$, i.e. for every $a \in D$ any germ $\mathbf{f}_a \in \mathfrak{I}_{A,a}$ can be written as $\mathbf{f}_a = \sum_{j=1}^r \mathbf{g}_{j,a}\mathbf{f}_{j,a}$ with $\mathbf{g}_{j,a} \in \mathcal{O}_a$.

Proof. Again, as above, one needs only to discuss a case when $a \in A$. So, for simplicity, let us assume that $a = 0$. Then we may assume that for a certain neighborhood $U = U(0)$ the mapping

$$\Phi : U \longrightarrow \mathbb{P}(0, \varepsilon), \quad z \longmapsto (z_1, \dots, z_{n-r}, f_1(z), \dots, f_r(z)),$$

is a biholomorphic mapping, where $\mathbb{P}(0, \varepsilon) := \mathbb{D}(0, \varepsilon)^n$. Then

$$\Phi(A \cap U) = \{w \in \mathbb{P}(0, \varepsilon) : w_{n-r+1} = \dots = w_n = 0\}.$$

Thus for an $f \in \mathcal{O}(U)$ with $\mathbf{f}_0 \in \mathfrak{I}_0$ we may assume that $f = 0$ on $A \cap U$ (after a possible shrinking of U. Therefore, the function $f \circ \Phi^{-1} : \mathbb{P}(0, \varepsilon) \longrightarrow \mathbb{C}$ vanishes on the plane $\{w \in \mathbb{P}(0, \varepsilon) : w_{n-r+1} = \dots = w_n = 0\}$ which implies that

$$f \circ \Phi^{-1}(w) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha w^\alpha = \widetilde{a}_1(w')w_{n-r+1} + \dots + \widetilde{a}_r(w')w_n,$$

where $\widetilde{a}_j \in \mathcal{O}(\mathbb{P}_{n-r}(0, \varepsilon)) \ni w'$. In other words we have $f(z) = (\widetilde{a}_1 \circ \Phi)f_1 + \dots + (\widetilde{a}_n \circ \Phi)f_n$ on U which implies the stated property of the lemma. □

With this lemma in mind put on U

$$\mathfrak{I} := (f_{k+1}, \dots, f_n)\mathcal{O}_U,$$

where

- $f_{k+1}(z) := P_{k+1}(z', z_{k+1})$,
- $f_{k+2}(z) := z_{k+2}\Delta(z') - q_{k+2}(z', z_{k+2}), \dots, f_n(z) := z_n\Delta(z') - q_n(z', z_n)$,

where $z = (z', z_{k+1}, \dots, z_n) \in U$.

Note that on $U \setminus N$ the assumptions of the former lemma are fulfilled. Thus the lemma implies that on $U \setminus N$ the following identity is true: $\mathfrak{I} = \mathfrak{I}_A|_U$ (EXERCISE).

Recall that $\mathfrak{I}$ is a coherent ideal sheaf on the whole of U and that it coincides with $\mathfrak{I}_A$ on $U \setminus N$. So it remains to see for the coherent gap sheaf $\mathfrak{I}[N]$ that $\mathfrak{I}[N] = \mathfrak{I}_A$ on U:

Indeed, obviously one has

$$\begin{aligned} \mathfrak{I}[N](V) &= \{f \in \mathcal{O}(V) : f|_{V \setminus N} \in \mathfrak{I}(V \setminus N)\} \\ &= \{f \in \mathcal{O}(V) : f|_{V \setminus N} \in \mathfrak{I}_A(V \setminus N)\} \supset \mathfrak{I}_A(V). \end{aligned}$$

To discuss the opposite inclusion recall that $\overline{(A \cap V) \setminus N} = A \cap V$. Thus any $f \in \mathcal{O}(V)$ with $f|_{V \setminus N} \in \mathfrak{I}_A(V \setminus N)$ vanishes also on $A \cap V$ (see Theorem 3.1.14) which means that $f \in \mathfrak{I}_A(V)$. Hence the desired identity has been verified which finally proves Theorem 3.2.1. □

Chapter 4
Complex spaces

The idea here to to discuss objects that look, at least, locally as an analytic set. Recall that an analytic set $A \subset D \subset \mathbb{C}^n$ is given as $A = \operatorname{supp} \mathcal{O}/\mathfrak{I}_A$, where, in virtue of the Cartan result, the ideal sheaf $\mathfrak{I}_A$ is a coherent ideal sheaf. Moreover, $\mathcal{O}$ is a sheaf of local $\mathbb{C}$-agebras, i.e. any stalk $\mathcal{O}_a$ is Noetherian $\mathbb{C}$-algebras which have exactly one maximal ideal $\mathfrak{m}_a$ different from the whole stalk, such that $\mathbb{C} \longrightarrow \mathcal{O}_a/\mathfrak{m}_a$ is an isomorphism. So we define

Definition 4.0.3. A *local* $\mathbb{C}$*-ringed space* is a pair $(X, \mathcal{O})$ (observe we use the same letter $\mathcal{O}$ as before but with a different meaning hoping this does not lead to any confusion), where X is an arbitrary topological space and $\mathcal{O}$ is a sheaf of local $\mathbb{C}$-algebras (with the assumption that $\mathbb{C} \longrightarrow \mathcal{O}_x/\mathfrak{m}_x$, $x \in X$, is a isomorphism). The sheaf $\mathcal{O}$ is called the *structure sheaf* of $(X, \mathcal{O})$; X is the *support* of $(X, \mathcal{O})$.

In order to define the morphisms between these new objects we need the notion of the inverse image of a sheaf. Let $f : X \longrightarrow Y$ be a continuous map from topological spaces and let (F, π_Y) be a sheaf over Y. For $x \in X$ put $(f^*F)_x := F_{f(x)}$, $f^*F := \bigcup_{x\in X}(f^*F)_x$, and $\pi_X : f^*F \longrightarrow X$ as $(f^*F)_x \longmapsto x$. Define $f^* : f^*F \longrightarrow F$ by $(f^*F)_a \ni x \longmapsto x \in F_{f(a)}$. Finally, we impose on f^*F the weakest topology such that f^* and π_X are continuous. This sheaf is the *preimage sheaf of* F. If F was a sheaf of local $\mathbb{C}$-algebras over Y, then f^*F too (now over X).

Definition 4.0.4. A *morphism* between two local $\mathbb{C}$-ringed spaces $(X, \mathcal{O})_X$ and $(Y, \mathcal{O}_Y)$ is given by a pair $(f, \overline{f})$, where

- $f : X \longrightarrow Y$ is a continuous mapping,
- $\overline{f} : f^*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is a sheaf morphism (over X) which is stalkwise a local $\mathbb{C}$-algebra homomorphism, i.e. $\overline{f}_x : f^*\mathcal{O}_{Y,x} \longrightarrow \mathcal{O}_{X,x}$ maps the unique maximal ideal from the left into the unique maximal ideal of the right hand side, $x \in X$.

Let $(X, \mathcal{O}_X) \overset{(f,\overline{f})}{\longrightarrow} (Y, \mathcal{O}_Y) \overset{(g,\overline{g})}{\longrightarrow} (Z, \mathcal{O}_Z)$ be given local $\mathbb{C}$-ringed spaces and morphisms between them. Then one may define their composition $(h := g \circ f, \overline{h}) : (X, \mathcal{O}_X) \longrightarrow (Z, \mathcal{O}_Z)$ as follows:

$$(h^*\mathcal{O}_Z)_x = (\mathcal{O}_Z)_{g(f(x))} \overset{\overline{g}_{f(x)}}{\longrightarrow} (\mathcal{O}_Y)_{f(x)} \overset{\overline{f}_x}{\longrightarrow} (\mathcal{O}_X)_x, \quad x \in X.$$

Then $\overline{h}$ is a sheaf homomorphism (EXERCISE).

Definition 4.0.5. A morphism $F = (f, \overline{f}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is called to be an *isomorphism* if there exist a morphism $G = (g, \overline{g}) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ such that $F \circ G = \mathrm{id}\,|_{(Y,\mathcal{O}_Y)}$ and $G \circ F = \mathrm{id}\,|_{(X,\mathcal{O}_X)}$.

Example 4.0.6. Let $X = \{0\} \subset \mathbb{C}$ and take $\mathcal{O}_1 := \mathbb{C}$ and $\mathcal{O} := \mathbb{C}z =$ the algebra of all convergent power series at 0. Then $(X, \mathcal{O}_1)$ and $(X, \mathcal{O}_2)$ are local $\mathbb{C}$-ringed spaces and $(f, \overline{f}) : (X, \mathcal{O}_1) \longrightarrow (X, \mathcal{O}_2)$ given by $f(0) = 0$ and $(\mathcal{O}_2)_0 \ni \mathbf{h} \longmapsto h(0) \in (\mathcal{O}_1)_0$ (where $h(0)$ means the value of the function defined via the holomorphic germ $\mathbf{g}$ at the origin) is a morphism in the sense from above. It is not an isomorphism (EXERCISE).

Now we turn to define notions like sub-objects.

Definition 4.0.7. Let $(X, \mathcal{O})$ be a local $\mathbb{C}$-ringed space.

(a) An *open subspace* of $(X, \mathcal{O}_X$ is given by $(U, \mathcal{O}_X|_U)$, where $U \subset X$ is open.

(b) A *closed subspace* is a pair $(Y, \mathcal{O}_X/\mathfrak{I}|_Y)$, where $\mathfrak{I}$ is an ideal subsheaf of $\mathcal{O}_X$ and $Y := \mathrm{supp}(\mathcal{O}_X/\mathfrak{I}) =: N(\mathfrak{I})$. (WARNING: in general, Y is not a closed subset of X; recall our example.)

(c) A *local $\mathbb{C}$-ringed subspace* is a closed subspace of an open subspace of $(X, \mathcal{O}_X)$.

Let $(U, \mathcal{O}_X|_U)$ be an open subspace of the local $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$. Then it is obviously a local $\mathbb{C}$-ringed space and one has the following natural morphism $(\mathrm{id}, \overline{\mathrm{id}}) : (U, \mathcal{O}_X|_U) \longrightarrow (X, \mathcal{O}_X)$ with $\mathrm{id}(x) := x$ and $\overline{\mathrm{id}}_x : \mathcal{O}_{X,x} \longrightarrow (\mathcal{O}_X|_U)_x$ the identity, $x \in U$.

Let $(Y := N(\mathfrak{I}), \mathcal{O}_Y := \mathcal{O}_X/\mathfrak{I}|_{N(\mathfrak{I})})$ be a closed local $\mathbb{C}$-ringed subspace of $(X, \mathcal{O}_X)$. Then $(\mathrm{id}, \overline{\mathrm{id}}) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$, given by $\mathrm{id}(x) := x$, $\overline{\mathrm{id}}_x : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{I}_x$ the quotient map, $x \in Y$, is a morphism of local $\mathbb{C}$-ringed spaces.

Definition 4.0.8. Let $(X, \mathcal{O}_X)$ be a local $\mathbb{C}$-ringed space with a coherent structure sheaf. A *subspace of finite type of* $(X, \mathcal{O}_X)$ is a closed subspace $(N(\mathfrak{I}), (\mathcal{O}_X|_U/\mathfrak{I})|_{N(\mathfrak{I})})$ of an open subspace $(U, \mathcal{O}_X|_U)$, where $\mathfrak{I}$ is a coherent ideal sheaf over U.

Note that now the support of the subspace of finite type is a closed set in U.

Example 4.0.9. (a) Note that $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ (here $\mathcal{O}_{\mathbb{C}^n}$ is the sheaf of germs of holomorphic functions) is a local $\mathbb{C}$-ringed space. Let $G \subset \mathbb{C}^n$ be a domain. Then $(G, \mathcal{O}_{\mathbb{C}^n}|_G)$ is an open subspace. Denote by $\mathfrak{I}$ the coherent ideal sheaf over G generated by the zero section. Then $N(\mathfrak{I}) = G$ and $\mathcal{O}_{\mathbb{C}^n}|_G/\mathfrak{I} = \mathcal{O}_{\mathbb{C}^n}$. Thus, $(G, \mathcal{O}_{\mathbb{C}^n}|_G) = (N(\mathfrak{I}), (\mathcal{O}_{\mathbb{C}^n}|_G/\mathfrak{I})|_{N(\mathfrak{I})})$ is also a subspace of finite type of $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$.

(b) Let $\mathfrak{I} \subset \mathcal{O}_{\mathbb{C}^n}|_G$ be a coherent ideal subsheaf. Then for any $a \in G$ there exist an open neighborhood $V = V(a)$ and sections $f_1, \dots, f_k \in \Gamma(V, \mathfrak{I}|_V)$ such that their germs generate the stalks of $\mathfrak{I}$. Therefore, $N(\mathfrak{I}) \cap V = \{z \in V : f_1(z) = \cdots = f_k(z) = 0\}$ (recall that $f_j(z)$ is identified with the value of the germ at the point z). Hence, the support $N(\mathfrak{I})$ of the coherent ideal subsheaf $\mathfrak{I}$ is an analytic subset of D.

(c) Visa versa, let $A \subset D \subset \mathbb{C}^n$ be an analytic subset. Then $\mathfrak{I}_A$ is an coherent ideal subsheaf of $\mathcal{O}_{\mathbb{C}^n}|_D$ with $A = N(\mathfrak{I}_A)$, i.e. any analytic subset can be thought as the support of a closed subspace of $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ of finite type.

Definition 4.0.10. (a) A subspace of finite type of $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is called to be *a local model*.

(b) A *complex space* is a local $\mathbb{C}$-ringed space $(X, \mathcal{O})$, where

- X is a Hausdorff space,
- For any $x \in X$ there is an open neighborhood $U = U(x)$ such that $(U, \mathcal{O}|_U)$ is isomorphic to a local model. One may interpret the pair (U, the above isomorphism) as a *chart*.

(c) Morphisms between complex spaces are called *holomorphic mappings*.
(d) A *complex subspace* of a complex space $(X, \mathcal{O})$ is a subspace of finite type.
(e) The support of a complex subspace is called an *analytic subset*.

Remark 4.0.11. Let $(X, \mathcal{O})$ be a complex space and $(N(\mathfrak{I}), (\mathcal{O}|_U)/\mathfrak{I})$ be a complex subspace of $(X, \mathcal{O})$, $U \subset X$ open. Then $N(\mathfrak{I})$ is a closed subset of U and if $x \in N(\mathfrak{I})$, then there exist a neighborhood $V = V(x) \subset U$ and sections $s_1, \ldots, s_k \in \Gamma(V, \mathfrak{I})$ with $V \cap N(\mathfrak{I}) = \{z \in V : s_1(z), \ldots, s_k(z) \in \mathfrak{m}_z\}$, where $\mathfrak{m}_z$ denotes the maximal ideal in $\mathcal{O}_z$.

Let $(X, \mathcal{O})$ be as before. Then we define a scalar valued mapping red : $\mathcal{O} \longrightarrow \mathbb{C}$ via $\text{red}_x : \mathcal{O}_x \longrightarrow \mathcal{O}_x/\mathfrak{m}_x \equiv \mathbb{C}$. Then the above equations may be read as $\widehat{s}_j(x) := \text{red} \circ s_j(x) = 0$, $j = 1, \ldots, k$. Note that in the classical contact there was a bijection between the sections into the structure sheaf and the holomorphic functions. For general complex spaces it may happen that different sections induces the same function via red. For example: Let $X = \{0\} \in \mathbb{C}$ and $\mathcal{O} := \mathcal{O}_{\mathbb{C},0}/(z^2)\mathcal{O}_{\mathbb{C},0}$. Then $(X, \mathcal{O})$ is a complex space. Then the two different sections $s_1(0) := [\mathbf{z}]$ and $s_2(0) := [\mathbf{0}]$ lead to the function $\widehat{s}_1(0) = \widehat{s}_2(0)$.

Remark 4.0.12. Conversely to the former remark, any set closed set N of $U \subset X$ – $(X, \mathcal{O})$ a complex space – which can be written as in the former remark, is support of a complex subspace $(N = N(\mathfrak{I}), \mathcal{O}|_U/\mathfrak{I})$ for a coherent ideal sheaf $\mathfrak{I} \subset \mathcal{O}|_U$. This is, in fact, the coherence result by Cartan (EXERCISE).

This shows somehow that the classical notion of an analytic set can be substituted by the support of a complex subspace.

Remark 4.0.13. Let $(f, \overline{f}) : (G, \mathcal{O}_{\mathbb{C}^n}|_G) \longrightarrow (D, \mathcal{O}_{\mathbb{C}^m}|_D)$ ($G \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^m$ domains) be a morphism. Then f is holomorphic and $\overline{f}$ is completely determined by f; namely, to any germ $s_{f(z)} \in f^*(\mathcal{O}_{\mathbb{C}^m}|_D)$ is associated the germ $(s \circ f)_z \in \mathcal{O}_{\mathbb{C}^n,z}$, $z \in G$.

In this context a complex manifold may be seen as a complex space covered by charts where the models are just open subspaces of $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$.

Proposition 4.0.14. *In a complex manifold* $(X, \mathcal{O})$ *the local ring* $\mathcal{O}_x$ *is integrally closed* $x \in X$.

Proof. EXERCISE. □

Localizing the above remark we set

Definition 4.0.15. Let $(X, \mathcal{O})$ be a complex space, $x \in X$. Then X is called to be *regular in* x if $\mathcal{O}_x$ is isomorphic to the algebra of convergent power series in n variables for a certain $n \in \mathbb{N}_0$. (One also says that x is a *regular point* of X). x is called to be a *singular point* of X if it is not a regular one.

One may prove that the set of regular points in X is an open subset. Thus, all points of X are regular if and only if X is a complex manifold. Moreover, the set of irregular points is, in fact, an analytic subset of X.

As examples have shown there are complex spaces $(X, \mathcal{O})$ such that for certain $x \in X$ the stalk $\mathcal{O}_x$ contains non-trivial nilpotent elements which has lead to the following definition.

Definition 4.0.16. (a) A complex space $(X, \mathcal{O})$ is called to be *reduced in* $x \in X$ if the stalk $\mathcal{O}_x$ does not contain any non-trivial nilpotent element. (One also says that x is a *reduced point* of X.)

(b) $(X, \mathcal{O})$ is called to be *reduced* if any of its points is a reduced one.

Example 4.0.17. Let $A \subset D \subset \mathbb{C}^n$ be an analytic subset. Then $(A, (\mathcal{O}_{\mathbb{C}^n}|_D)/\mathfrak{I}|_A)$ is a reduced complex space. One may even show that any reduced complex space is locally isomorphic to such a space.

Let $(X, \mathcal{O})$ be a complex space. For $x \in X$ put $\mathfrak{n}(\mathcal{O}_x)$ the nilpotent ideal of $\mathcal{O}_x$, i.e. $\mathfrak{n}(\mathcal{O}_x) := \{f \in \mathcal{O}_x : \text{ there exists } n \in \mathbb{N} \text{ with } f^k = 0\}$. Since X is an analytic set we have that $\mathfrak{I}_X = \cup_{x \in X} \mathfrak{n}(\mathcal{O}_x)$ is a coherent sheaf (use Cartan's result). Therefore, $(X, \mathcal{O}/\mathfrak{I}_X)$ is a complex space which is called the *reduction of* X.

Starting in this general context we may repeat the discussion of local geometry as we did before in the classical situation of $\mathbb{C}^n$. For more details the reader is asked to contact the literature.

Chapter 5
Cohomology and Theorem A and B

5.1 Cohomology

(0) Recall the Cousin-I problem (an generalization of the Mittag-leffler theorem to higher dimensions) for a domain $D \subset \mathbb{C}^n$:

A *Cousin-I data* is a system $((U_j)_{j\in J}, (f_j)_{j\in J})$, where $(U_j)_j$ is an open covering of D and each f_j is a meromorphic function on U_j, such that $f_i|_{U_{i,j}} - f_j|_{U_{i,j}}$ extends to a holomorphic function $f_{i,j}$ on $U_{i,j} := U_i \cap U_j$, i.e. the pole sets of the meromorphic functions coincide. The question is whether there exists a global meromorphic function f on D such that $f|_{U_j} - f_j|U_j$ extends to a holomorphic function on U_j, $j \in J$.

Assume for a moment that there are holomorphic functions $g_j \in \mathcal{O}(U_j, j \in J$ such that $f_{i,j} = g_j - g_i$ on $U_{i,j}$, i.e. $f_i - f_j = g_j - g_i$ on $U_{i,j}$ or $f_i + g_i = f_j + g_j$ on $U_{i,j}$. Then f, given by $f_i + g_i$ on U_i, is the global meromorphic function we were looking for.

Interpretation: by the above argument we have to start with a system of holomorphic functions $f_{i,j} \in \mathcal{O}(U_{i,j})$ such that $f_{i,j} + f_{j.k} + f_{k,i} = 0$ on $U_{i,j,k}$, $i,j,k \in J$, and we are searching for a system of holomorphic functions $g_j \in \mathcal{O}(U_j)$, $j \in J$ such that $f_{i,j} = g_j - g_i$ on $U_{i,j}$.

(1) General situation (Čech cohomology): Let X be a topological space, $F = (F, \pi)$ a sheaf of abelian groups over X, and $\mathcal{U} := (U_j)_{j\in J}$ an open covering of X. For $q \in \mathbb{N}_0$ put

$$C^q(\mathcal{U}, F) := \{\xi = (\xi_{(j_0,\dots,j_q)})_{(j_0,\dots,j_q)\in J^{q+1}} : \xi_{(j_0,\dots,j_q)} \in \Gamma(U_{(j_0,\dots,j_q)}, F)\},$$

where, as usual, $\Gamma(\varnothing, F) = \{0\}$, and for negative Q put $C^q(\mathcal{U}, F) := \{0\}$.

By componentwise applying the group operation, $C^q(\mathcal{U}, F)$ becomes in an obvious way an abelian group, the so called *the group of q-cochains*.

For $q \in \mathbb{N}_0$ define

$$\delta = \delta^q : C^q(\mathcal{U}, F) \longrightarrow C^{q+1}(\mathcal{U}, F),$$

$$C^q(\mathcal{U}, F) \ni \xi \longmapsto \Big((j_0,\dots,j_{q+1}) \longmapsto \sum_{k=0}^{q+1} (-1)^k \xi_{(j_0,\dots,\widehat{j_k},\dots,j_{q+1})}|_{U_{j_0,\dots,j_{q+1}}}\Big)$$

note that, in fact, δ^q depends on the covering but we will always use the same letter which should not lead to any conflict.

Obviously, δ^q is are group homomorphisms and they satisfy

$$\delta^{q+1} \circ \delta^q = 0$$

(EXERCISE); i.e. we have a so called *complex* $\Big((C^q(\mathcal{U}, f))_q, (\delta^q)_q\Big)$ of abelian groups and group homomorphisms. Put

$$Z^q(\mathcal{U}, F) := \operatorname{Ker} \delta^q, \quad B^q(\mathcal{U}, F) := \operatorname{Im} \delta^{q-1}, \quad q \in \mathbb{Z}.$$

The first group is called the group of *q-cocycles*, while the second one is the group of *q-boundaries*.

Definition 5.1.1. The q-th Čech cohomology group for $\mathcal{U}$ with values in F is defined as

$$H^q(\mathcal{U}, F) := Z^q(\mathcal{U}, F)/B^q(\mathcal{U}, f).$$

Coming back to the Cousin-I problem we see that $\xi := (f_{i,j}) \in Z^1(\mathcal{U}, \mathcal{O})$, where $\mathcal{U} := (U_j)_{j \in J}$. To solve the problem we need to verify that $\xi \in B^1(\mathcal{U}, \mathcal{O})$, or in other words, if $H^1(\mathcal{U}, \mathcal{O}) = 0$, then the Cousin-I problem can be solved. This may show that cohomology groups are important.

Remark 5.1.2. If $\xi \in Z^0(\mathcal{U}, F) = 0$, then such a 0-cocycle defines a global section over X in F. Since $B^0(\mathcal{U}, F) = 0$, we can identify $H^0(\mathcal{U}, F) = Z^0(\mathcal{U}, f) = \Gamma(X, F)$.

(2) A few examples:

Theorem 5.1.3. *(a) If $D \subset \mathbb{C}^n$ is a domain and $\mathcal{U} = (U_j)_{j \in J}$ an open covering, then*

$$H^q(\mathcal{U}, \mathcal{E}) = 0, \quad q \geq 2,$$

where $\mathcal{E}$ is the sheaf of germs of $\mathcal{C}^\infty$-functions over D.

(b) Let $D := \mathbb{P}(0, 1)$. Then $H^1(\mathcal{U}, \mathcal{O}) = 0$.

The part (a) is verified using a partition of unity while (b) is solved using (a) and the Dolbeault lemma for a polydisc.

(3) Čech cohomology: So far the cohomology groups in (2) depend on a priori given open covering. To define such groups independent of such coverings one has to go to the inductive limit. Let $\mathcal{U} = (U_j)_{j \in J}$ and $\mathcal{V} = (V_k)_{k \in K}$ be two given open coverings of X. Say that $\mathcal{V} \geq \mathcal{U}$ if for any $k \in K$ there exist an index $\tau(k) \in J$ such that $V_k \subset U_{\tau(k)}$. Then "$\geq$" gives an upper directed relation on the family of all open coverings.

Then

$$\tau^q : C^q(\mathcal{U}, F) \longrightarrow C^q(\mathcal{V}, F),$$

$$C^q(\mathcal{U}, F) \ni \xi \overset{\tau^q}{\longmapsto} \Big(\xi_{(\tau(j_0), \ldots, \tau(j_q))}|_{V_{k_0, \ldots, k_q}}\Big) \in C^q(\mathcal{V}, F)$$

gives a group homomorphism with $\tau^{q+1} \circ \delta^q = \delta^q \circ \tau^q$, $q \in \mathbb{Z}$. Hence, τ^q maps the cocycle groups for $\mathcal{U}$ to the cocycle groups for $\mathcal{V}$ and the same is true for the coimage groups. Hence, τ induces group homomorphisms $\tau_{\mathcal{V}}^{\mathcal{U}} : H^q(\mathcal{U}, F) \longrightarrow H^q(\mathcal{V}, F)$.

Remark 5.1.4. The homomorphisms $\tau_{\mathcal{V}}^{\mathcal{U}}$ are independent of the map τ chosen above. Let $\sigma : K \longrightarrow J$ be another mapping with the property of τ. Then one has to construct a so called homotopy operator $h^q : C^q(\mathcal{U}, F) \longrightarrow C^{q-1}(\mathcal{V}, F)$ satisfying $\delta^{q-1} \circ h^q + h^{q+1} \circ \delta^q = \tau^q - \sigma^q$. In more detail, one defines

$$(h^q(\xi))_{(k_0,\ldots,k_{q-1})} := \sum_{\ell=0}^{q-1} (-1)^\ell \xi_{(\tau(k_0),\ldots,\tau(k_\ell),\sigma(k_\ell),\ldots,\sigma(k_{q-1}))}.$$

Details are left as an EXERCISE.

Definition 5.1.5. The *q-th Čech cohomology group* with values in the sheaf F is defined as ind-$\lim_{\mathcal{U}} H^q(\mathcal{U}, F)$, $q \in \mathbb{Z}$.

Remark 5.1.6. With the former observation one easily sees that $H^1(\mathcal{U}, F) \longrightarrow H^1(X, F)$ is injective.

(4) The exact cohomology sequence: Now we work with two sheaves $F = (F, \pi_F)$ and $G = (G, \pi_G)$ over the topological space X. Let $\alpha : F \longrightarrow G$ be a sheaf morphism and $\mathcal{U}$ an open covering of X as above. Then one may define

$$\widetilde{\alpha}_{\mathcal{U}}^q : C^q(\mathcal{U}, f) \longrightarrow C^q(\mathcal{U}, G),$$
$$C^q(\mathcal{U}, F) \ni \xi \overset{\widetilde{\alpha}_{\mathcal{U}}^q}{\longmapsto} \big(\alpha \circ \xi_{(j_0,\ldots,q)}\big)_{(j_0,\ldots,j_q)\in J^{q+1}} \in C^q(\mathcal{U}, G).$$

Then $\widetilde{\alpha}_{\mathcal{U}}^q$ is a group homomorphism satisfying (EXERCISE)

- $\widetilde{\alpha}_{\mathcal{U}}^q(Z^q(\mathcal{U}, F)) \subset Z^q(\mathcal{U}, G)$,
- $\widetilde{\alpha}_{\mathcal{U}}^q(B^q(\mathcal{U}, F)) \subset B^q(\mathcal{U}, G)$;

thus, α induces homomorphism $\alpha_{\mathcal{U}}^q : H^q(\mathcal{U}, F) \longrightarrow H^q(\mathcal{U}, G)$, $q \in \mathbb{Z}$.

Finally, running through all the diagrams, one may define a group homomorphism $\alpha_q^* : H^q(X, F) \longrightarrow H^q(X, G)$, $q \in ZZ$ (details ar left for the reader as EXERCISE).

Again, let X be a topological space and let F, G, H be sheaves of abelian groups over X. Moreover, let $\alpha : F \longrightarrow G$ and $\beta : G \longrightarrow H$ be sheaf morphisms such that for any $x \in X$ the sequence

$$0 \longrightarrow F_x \xrightarrow{\alpha_x} G_x \xrightarrow{\beta_x} H_x \longrightarrow 0;$$

is an exact one. For short, one says that the sequence $0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$ of sheaves is exact.

In this situation there are so called *connected morphisms* $\delta_*^q : H^q(\mathcal{U}, H) \longrightarrow H^{q+1}(\mathcal{U}, F)$ defined via the following construction:

- take a $[\xi] \in H^q(\mathcal{U}, H)$ with $\xi \in Z^q(\mathcal{U}, H)$,
- using exactness there is an $\eta \in C^q(\mathcal{U}, G)$ such that $\widetilde{\beta}^q_{\mathcal{U}}(\eta) = \xi$,
- because of $\widetilde{\beta}^{q+1}_{\mathcal{U}}(\delta^q\eta) = \delta \circ \widetilde{\beta}^q_{\mathcal{U}}(\eta) = \delta\xi = 0$, exactness leads to the existence of a $\zeta \in C^{q+1}(\mathcal{U}, F)$ with $\widetilde{\alpha}^{q+1}_{\mathcal{U}} = \delta\eta$,
- $\widetilde{\alpha}^{q+1}_{\mathcal{U}}(\delta(\zeta)) = \delta \circ \alpha^{q+1}_{\mathcal{U}}(\zeta) = \delta \circ \delta(\eta) = 0$ and the exactness implies that $\delta(\zeta) = 0$, i.e. ζ is a cocycle defining a cohomology class $[\zeta] =: \delta^q_*([\xi]) \in H^{q+1}(\mathcal{U}, F)$.

Theorem 5.1.7. *Let X be paracompact and $0 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 0$ be an exact sequence of sheaves of abelian groups over X and let $\mathcal{U}$ be a open covering of X.*

(a) Then the following sequence of groups is exact

$$0 \longrightarrow H^0(\mathcal{U}, F) \longrightarrow H^0(\mathcal{U}, G) \longrightarrow H^0(\mathcal{U}, H) \longrightarrow H^1(\mathcal{U}, F)$$
$$\longrightarrow H^1(\mathcal{U}, G) \longrightarrow H^1(\mathcal{U}, H) \longrightarrow H^2(\mathcal{U}, F) \longrightarrow \cdots$$

is exact (the homomorphisms are those from above).

(b) The analogous result is true for the Čech cohomologies.

Proof. EXERCISE. □

To conclude this section we only mention that there are also other cohomology theories like the *flabby cohomology*.

5.2 Theorems A and B

These lecture will end by a short discussion on the theorem A and B by Cartan. Let X be now a complex manifold of dimension n. Recall that X is called to be a *Stein manifold* if X is holomorphically convex, holomorphically separable, and allows global coordinates. Any domain of holomorphy in $\mathbb{C}^n$ may serve as an example of a Stein manifold.

Theorem 5.2.1 (Cartan). *Let X be a Stein manifold and $F = (F, \pi)$ be a coherent $\mathcal{O}_X$-module sheaf. Then:*

(A) the global sections in $\Gamma(X, F)$ generate the stalk F_x for all $x \in X$.

(B) $H^q(X, F) = 0$, $q \geq 2$.

Remark 5.2.2. (a) By the Nakayama lemma one can prove that (B) implies (A).

(b) For the proof of (B) see the literature.

Here we will only present a few applications of the above theorems.

Proposition 5.2.3. *Let X be a Stein manifold and let $A \subset X$ be an analytic set. Put $Id(A) := \{f \in \Gamma(X, \mathcal{O}_X) = \mathcal{O}(X) : f|_A = 0\}$. Then*

$$A = \{z \in X : f(z) = 0 \textit{ for all } f \in Id(A)\}.$$

Proof. Obviously, the inclusion "$\subset$" is true. To see the other direction take a point $a \notin A$. We have to find an $f \in Id(A)$ with $f(a) \neq 0$ to get the desired contradiction.

Note that $Id(A) = \Gamma(X, \mathfrak{I}_A)$ and $\mathfrak{I}_{A,a} = \mathcal{O}_a$. Thus, applying Theorem A for the coherent sheaf $\mathfrak{I}_A$ one may find a global function $f \in \Gamma(X, \mathfrak{I}_A)$ with $\mathbf{f}_a = \mathbf{1}_a$. Hence, $f|_A = 0$ but $f(a) \neq 0$; a contradiction. □

Example 5.2.4. Put

$$D := \{z \in \mathbb{C}^2 : |z_j| < 1, j = 1, 2\} \setminus \{z \in \mathbb{C}^2; 1/4 \leq |z_1| \leq 3/4,\ |z_2| \leq 3/4\}$$

and $A := \{z \in \mathbb{C}^2 : z_2 = 0,\ 3/4 < |z_1| < 1\}$. Then $A \subset D$ is an analytic subset of D. Take the point $a := (0, 1/2) \notin A$. Opposite to the former proposition any $f \in \mathcal{O}(D)$ with $f|_A = 0$ is vanishing also at a (EXERCISE).

Let $A \subset X$ be as above. A function $f : A \longrightarrow \mathbb{C}$ is called *holomorphic*, if any point $a \in A$ has a small open neighborhood $U = U(a)$ such that $f|_{A \cap U} = F|_{A \cap U}$ for some $F \in \mathcal{O}(U)$, i.e. f is locally extendable to a standard holomorphic function in some neighborhood.

Proposition 5.2.5. *Let X be a Stein manifold and $A \subset X$ an analytic subset. Then any holomorphic function $f : A \longrightarrow \mathbb{C}$ is the restriction of a global $F \in \mathcal{O}(X)$.*

Proof. Recall that $\mathfrak{I}_A$ and $\mathcal{O}_X$ are coherent sheaves and that $0 \longrightarrow \mathfrak{I}_A \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathfrak{I}_A \longrightarrow 0$ is an exact sequence od sheaves. Note that the sheaf $\mathcal{O}_X/\mathfrak{I}_A$ is coherent and therefore by Theorem B, $H1(X, \mathfrak{I}_A) = 0$. Hence, $H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X/\mathfrak{I}_A)$ is surjective. It only remains to mention that the last group is nothing else than the set of all holomorphic functions on A (EXERCISE). □

Proposition 5.2.6. *Let X be as before and let $f_1, \ldots, f_k \in \mathcal{O}(X)$ without a common zero. Then there exist $g_1, \ldots, g_k \in \mathcal{O}(X)$ such that $1 = \sum_{j=1}^{k} g_j f_j$.*

Proof. EXERCISE. □

Example 5.2.7. Let $D := \{z \in \mathbb{C}^n : 1 < \|z\| < 2\}$ and put $f_j(z) := z_j$, $j = 1, \ldots, n$. Prove that there are no $g_j \in \mathcal{O}(D)$ with $1 = \sum_{j=1}^{n} g_j f_j$ on D.

These few results may show how to use coherence of $\mathfrak{I}_A$ and Cartan's theorems A and B.

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