Central configurations and relative equilibria in the n–body problem

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Chapter 1

Preliminaries

1.1 Framework of the problem

The n–body problem study the motion of n–point particles in $\mathbb{R}^d$ moving under the influence of the Newton's Gravitational law, where $d = 1, 2$ or 3, assuming that the gravitational constant $G = 1$ the equations of motion can be written as

$$m_i \ddot{\mathbf{q}}_i = \sum_{i=1, i\neq j}^{n} \frac{m_i m_j}{r_{ij}} (\mathbf{q}_j - \mathbf{q}_i) = \nabla_i U = \frac{\partial U}{\partial \mathbf{q}_i} \quad i = 1, 2, \cdots n, \tag{1.1}$$

where $r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$, and $U = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$ is the Newtonian potential. We observe that the potential U only depends on the mutual distances.

Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \in \mathbb{R}^{dn}$. We define $\Delta_{ij} = \{q \in \mathbb{R}^{dn} \mid \mathbf{q}_i = \mathbf{q}_j\}$ and $\Delta{=}\bigcup \Delta_{ij}$. Then the configuration space is $X = \mathbb{R}^{dn} \setminus \Delta$. We will assume that the center of mass is fixed at the origin, that is $m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + \cdots + m_n\mathbf{q}_n = 0$.

As usual, the angular momentum is given by $\mathbf{C} = \sum \mathbf{q}_k \wedge \mathbf{p}_k$, where the momentum $\mathbf{p}_k$ is defined by $\mathbf{p}_k = m_k \dot{\mathbf{q}}_k$.

Definition 1.1. *We say that a function $U : \mathbb{R}^m \to \mathbb{R}$ is homogeneous of degree α if $U(t\mathbf{q}){=}t^\alpha U(\mathbf{q})$ $\forall t \in \mathbb{R}$, $\forall$ $\mathbf{q} \in \mathbb{R}^m$.*

Theorem 1.2 (Euler). *Let $U : \mathbb{R}^m \to \mathbb{R}$ a homogeneous function of degree α, then $\mathbf{q}\nabla U = \alpha U(\mathbf{q})$.*

We observe that the Newtonia potential U is homogeneous of degree -1. System (1.1) does not have equilibrium points, because any equilibrium point must satisfy $\nabla_i U = 0 \quad i = 1, 2, \cdots, N$, but then by Euler's theorem we should have $0 = \sum_{i=1}^{N} \nabla_i U \cdot \mathbf{q}_i = -U < 0$ which is a contradiction.

A useful definition that we are using along this course is about the size of the system, which is given by the *moment of inertia*.

Definition 1.3. *The moment of inertia is given by*

$$I = \frac{1}{2}\sum_{j=1}^{n} m_j \mathbf{q}_j^2 = \frac{1}{4\tilde{m}} \sum_{i<j} m_i m_j r_{ij}^2,$$

where $\tilde{m} = m_1 + m_2 + \cdots + m_n$.

The notation $\mathbf{q}_j^2$ means the dot product of $\mathbf{q}_j$ with itself. The second equality follows by straightforward computations using the fact that the center of mass is fixed at the origin. So, the moment of inertia is an homogeneous function of degree 2, and it can be written in terms of the mutual distances among the particles.

1.2 The two body problem

From (1.1) the equations of motion for the two body problem are

$$\begin{aligned} m_1 \ddot{\mathbf{q}}_1 &= \frac{m_1 m_2}{r_{12}^3}(\mathbf{q}_2 - \mathbf{q}_1), \\ m_2 \ddot{\mathbf{q}}_2 &= \frac{m_1 m_2}{r_{12}^3}(\mathbf{q}_1 - \mathbf{q}_2). \end{aligned} \tag{1.2}$$

Adding the above equations we obtain $m_1\ddot{\mathbf{q}}_1 + m_2\ddot{\mathbf{q}}_2 = 0$, and from here after integration we get $m_1\mathbf{q}_1 + m_2\mathbf{q}_2 = lt + m$ (Consevation of the linear momentum integral), which, fixing the center of mass at the origin we have $l = m = 0$, that is $m_1\mathbf{q}_1 + m_2\mathbf{q}_2 = 0$. Introducing these equation in (1.2) we have

$$\begin{aligned} \ddot{\mathbf{q}}_1 &= \frac{m_2}{r_{12}^3}(-\frac{m_1\mathbf{q}_1}{m_2} - \mathbf{q}_1) = -\frac{m_1+m_2}{r_{12}^3}\mathbf{q}_1 = \lambda\mathbf{q}_1, \\ \ddot{\mathbf{q}}_2 &= \frac{m_1}{r_{12}^3}(-\frac{m_2\mathbf{q}_2}{m_1} - \mathbf{q}_2) = -\frac{m_1+m_2}{r_{12}^3}\mathbf{q}_2 = \lambda\mathbf{q}_2. \end{aligned} \tag{1.3}$$

We also observe that

$$\ddot{\mathbf{q}}_2 - \ddot{\mathbf{q}}_1 = -\frac{m_1+m_2}{r_{12}^3}(\mathbf{q}_2 - \mathbf{q}_1).$$

Doing $\mathbf{r} = \mathbf{q}_2 - \mathbf{q}_1$ and $\mu = m_1 + m_2$ we obtain the classical central force problem known as the Kepler problem

$$\ddot{\mathbf{r}} = -\frac{\mu}{|\mathbf{r}|^3}\mathbf{r}. \tag{1.4}$$

Then the two body problem and the Kepler problem are equivalent, in the sense that any solution of (1.2) is a solution of (1.4) and viceversa.

Remark 1.4. *We observe that in the two body problem it is possible to uncouple the equations of motion in two central force or Kepler problems (see equation (1.3)), where the position and acceleration vectors are proportional. In general in the n–body problem such uncoupling is not always possible, except for some particular configurations called central configurations*

1.3 Central configurations

In the literature there are many definitions of a CC, of course all of them are equivalent, we start this section with the definition of a CC which we consider is the classical one.

Definition 1.5. *A central configuration (CC) in the n–body problem is a particular position of the n–particles where the position and acceleration vectors are proportional, with the same constant of proportionality. In other words if for some fixed time t_0, we have $\ddot{\mathbf{q}}_k(t_0) = \lambda \mathbf{q}_k(t_0)$ for $k = 1, 2, \cdots, n$.*

Multiplying both sides in the equation of the above definition by $\mathbf{q}_k$, adding over k and using the equations of motion (1.1), the definition of the moment of inertia and Euler's theorem for homogeneous functions, we can easily check that

$$2\lambda I = \lambda \sum_{k=1}^{n} \mathbf{q}_k^2 = \sum_{k=1}^{n} \nabla_k U \cdot \mathbf{q}_k = -U.$$

Then we obtain $\lambda = -U/2I$, that is, the constant of proportionality in a CC is always negative.

Now we would like to check if there are solutions $\mathbf{q}(t) = (\mathbf{q}_1(t), \mathbf{q}_2(t), \cdots, \mathbf{q}_n(t))$ of the n–body problem such that they form a CC for any time t, in other words we look for solutions of the form

$$\mathbf{q}_k(t) = \rho(t)\mathbf{q}_k^0, \quad \text{where} \quad \mathbf{q}_0 = (\mathbf{q}_1^0, \mathbf{q}_2^0, \cdots, \mathbf{q}_n^0) \quad \text{is a} \quad CC.$$

First we compute the value of λ at any time t. Since

$$\ddot{\mathbf{q}}_k(t) = \ddot{\rho}(t)\mathbf{q}_k^0 = \frac{\ddot{\rho}}{\rho}\rho\mathbf{q}_k^0 = \frac{\ddot{\rho}}{\rho}\mathbf{q}_k$$

we obtain that $\lambda(t) = \ddot{\rho}/\rho$.

Then, since we assume that $\mathbf{q}(t)$ is a solution of the n–body problem, $\mathbf{q}_k(t) = \rho(t)\mathbf{q}_k^0$ satisfies equations (1.1), then we must have

$$m_k\ddot{\rho}\mathbf{q}_k^0 = \frac{\partial U}{\partial \mathbf{q}_k}(\rho\mathbf{q}_1^0, \rho\mathbf{q}_2^0, \cdots, \rho\mathbf{q}_n^0) = \frac{1}{\rho^2}\frac{\partial U}{\partial \mathbf{q}_k}(\mathbf{q}_1^0, \mathbf{q}_2^0, \cdots, \mathbf{q}_n^0) = \frac{\lambda_0 m_k \mathbf{q}_k^0}{\rho^2},$$

with $\lambda_0 = \lambda(0) < 0$. Therefore $\ddot{\rho} = \lambda_0/\rho^2$.

In other words, the problem has been uncoupled in n independent problems of central force. From here, we have that a configuration $\mathbf{q}_0$ of the n–body problem is a CC if there is some solution of the n–body problem of the form

$$\mathbf{q}(t) = r(t)\mathbf{q}_0,$$

where $r(t)$ is a real valued function of t.

The inverse implication is a direct exercise that we let to the readers. With this we have that $\mathbf{q}_0(t)$ is a CC iff $\mathbf{q}(t) = r(t)\mathbf{q}_0$ is a solution of the n–body problem for some function $r(t)$. That is, we have an equivalent definition of a CC.

Proposition 1.6. *If $\mathbf{q}_0$ is a CC, then so is $c\mathbf{q}_0$ and $A\mathbf{q}_0$ for any $c \in \mathbb{R}^+$ and any $A \in SO(3)$.*

Proof. It follows directly from the symmetry of the equations under rotations and scaling. □

Remark 1.7. *By the above Proposition we also have that if $\mathbf{q}_0$ is a CC, then so is $cA\mathbf{q}^0$. Therefore, from here on we count CC modulo these Euclidean motions, that by short we remain calling them CC.*

Assuming that $d = 3$, we define the mass matrix as $M = diag\{m_1, m_1, m_1, m_2, m_2, m_2, \cdots, m_n, m_n, m_n\}$, then the equations of motion (1.1) can be written in a compact form as

$$M\ddot{\mathbf{q}} = \nabla U(\mathbf{q}),$$

then the definition of a CC is $\ddot{\mathbf{q}} = \lambda\mathbf{q}$. Applying the matrix M to both sides of this definition, and using the above equations of motion we obtain

$$\nabla U(\mathbf{q}_0) = \lambda \nabla I(\mathbf{q}_0). \tag{1.5}$$

Since the CC are invariant under scaling, we may as well normalize the CC by setting $I = 1$, in this way we obtain a normalized CC . We observe that equation (1.5) is a multiplier Lagrange problem, with λ as the Langrange multiplier. The above prove the following result, which is a geometrical interpretation of a CC.

Proposition 1.8. *The configuration $\mathbf{q}_0$ is a normalized CC if and only if it is a critical point of the restriction of $U(\mathbf{q})$ to the sphere of masses given by $I = 1$.*

Another characterization of a CC is given in the next Proposition.

Proposition 1.9. *The configuration $\mathbf{q}_0$ is a CC iff $\mathbf{q}_0$ is a critical point of IU^2.*

Proof. From the definition of a CC we have

$$\lambda \mathbf{q}_k = \ddot{\mathbf{q}}_k,$$

using the equations of motion we obtain

$$\lambda m_k \mathbf{q}_k = m_k \ddot{\mathbf{q}}_k = \frac{\partial U}{\partial \mathbf{q}_k},$$

substituting the value of $\lambda = -U/2I$ we get

$$\frac{U}{2I} m_k \mathbf{q}_k + \frac{\partial U}{\partial \mathbf{q}_k} = 0 \implies U^2 m_k \mathbf{q}_k + 2IU \frac{\partial U}{\partial \mathbf{q}_k} = \frac{\partial}{\partial \mathbf{q}_k}(IU^2) = 0.$$

□

Remark 1.10. *We observe that by one hand the function IU^2 is homogeneous of degree zero, which means that it is invariant under scaling. By the other hand, since U and I only depend on the mutual distances among the particles, it is also invariant under rotations. The function IU^2 measures how the shape of the configuration is changing. It plays a main role in the proof of Saari's conjecture that we will study ahead in these notes.*

A first interesting question is about the existence of a CC: for any choice of n masses, it is possible to guarantee the existence of at least one CC in the corresponding n–body problem?. Denoting the ellipsoid of masses as

$$S = \{\mathbf{q} \in X \mid \frac{1}{2}\mathbf{q}^t M \mathbf{q} = I = 1\}.$$

We have seen that $\mathbf{q}_0$ is a normalized CC iff $\mathbf{q}_0$ is a critical point of the restriction of the potential U to S.

Proposition 1.11. *The restriction of the potential U to S always attains its minimum at some $\mathbf{q}_0 \in S$.*

Proof. Consider the restriction

$$U|_S : S \to \mathbb{R}. \quad \text{We observe that} \quad U|_S(\mathbf{q}) \to \infty \quad \text{when} \quad \mathbf{q} \to \Delta.$$

Pick any $\mathbf{q}$ in $X \bigcap S$, and let $K = 2U(\mathbf{q})$. We choose a neighborhood V_K of Δ on which $U > K$.

Deleting V_K we get that $S \setminus V_K$ is a compact set, therefore $U|_{S \setminus V_K}$ attains its minimum.

Since $U|_{V_K} > K$ and $U_{min} \leq K/2$ we have that this minimum point is also a minimum point of $U|_S$. □

Corollary 1.12. *For any choice of masses in the n–body problem, there is at least one CC.*

Open question: For n–equal masses, who is $\mathbf{q}_{min}$. If $n \leq 6$, we know that this minimum is the regular n–gon.

One of the main open problem for planar central configurations is due to Wintner (1941) and Smale (1970): *Is the number of classes of planar central configurations finite for any choice of the (positive) masses $m_1, \ldots, m_n$?*

Hampton and Moeckel in (2006), proved this conjecture for the 4–body problem [50], in 2010, Albouy and Kaloshin proved the finiteness of planar CC in the 5–body problem [4]. The conjecture remains open for $n > 5$ and it represents one of the biggest challenges for the mathematics in the area [101]. In the classical n–body problem we always consider that all masses are positive. In the case of one negative mass, G. Roberts proved in 1999 that there exists a one–parameter not equivalent family of planar central configurations for the 5–body problem [91]. Also considering the particles endowed with masses and charges, Alfaro and Pérez-Chavela in 2002 proved the existence of a continuum of central configurations in a particular 4–body problem considering the particles endowed with a positive mass and an electrostatic charge that can have any sign [6].

Before to continue we must ask: why is important to study CC ? The answer is very simple, because there are many important reasons to do that, the great mathematician S. Smale point out that the above conjecture is one of the must interesting problems to be solved in this century (unfortunately, it is not in the list of the million dollars problems) [101]. In the following we expose just some properties of the CC that shown the main role that they play in celestial mechanics.

Importance of the CC

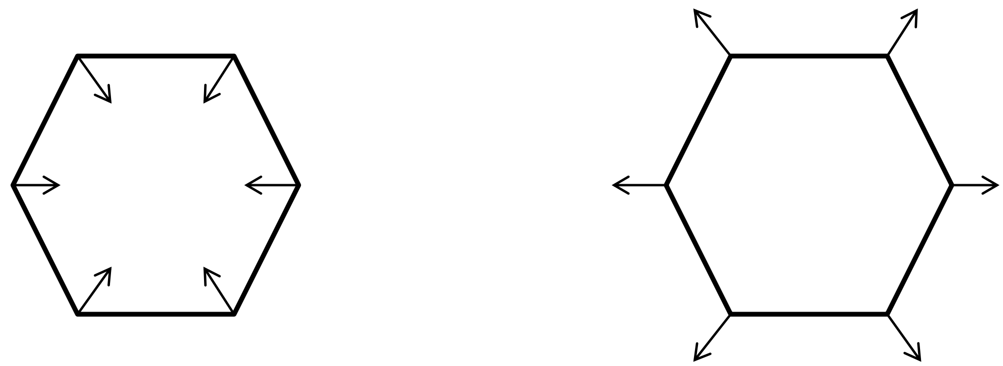

Figure 1.1: Polygonal CC

- A solution of the n–body problem is called homographic if the configuration of the particles remains similar with a CC for all time t. These are the unique explicitly solutions known in the N–body problem.
- Total collision in the N–body problem is always asymptotic to a CC. The knowledge of CC gives important insight into the dynamics near total collision.
- The set of initial conditions leading to total collision has measure zero.
- Some escapes of particles (the parabolic escapes) in the n–body problem tend to a CC.
- Bifurcations in the phase space occurs at CC. In other words, the topology of the level sets with fix energy h changes at the level sets which contain CC.

1.4 Examples of CC

In any theory it is a good custom to have some examples of the subject that we are studying. The goal of this section is to show a few examples of CC and prove the first classical results in this direction (Euler and Lagrange theorems).

1.4.1 Polygonal CC

Assuming that all masses are equal it is not difficult to verify directly from the definition that:

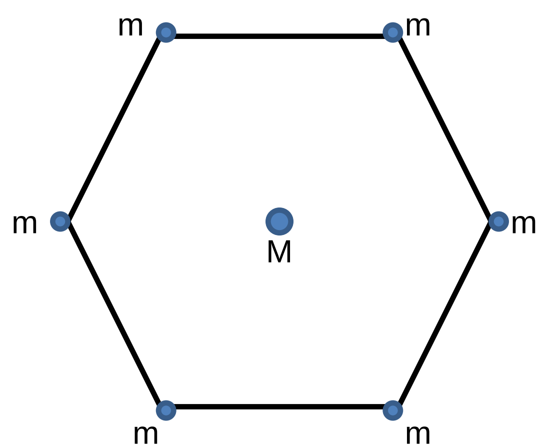

Figure 1.2: Polygonal CC with a central mass

- Any regular n–gon with a mass at each vortex or a regular polyhedron with the same mass at each edge determines a CC. In Fig. 1.1 we can see two possible motions of the particles, contractions and expansions, but of course we also can have rotations of a combination of both kind of motions.

- Any regular n–gon or regular polyhedron with equal masses at the edges and one body of any mass at the origin always form a CC. If we move the value of the central mass M, it is possible to obtain some kind of bifurcations of central configurations, in the sense that the Hessian of the potential vanish for some values of the central mass (see Fig. 1.2).

1.4.2 Euler Theorem

Here we consider the collinear three body problem. Let be three point masses m_1, m_2, m_3 located on a straight line, that we suppose is one of the coordinate axis, at the points $q_1 < q_2 < q_3$.

Theorem 1.13. *(Euler 1764) Under the above hypothesis, for any choice of the masses there are exactly 3 collinear CC. One for each ordering of the particles on the line modulus a rotation by π.*

Proof. The equations for the central configurations can be written as (remember that $m_i\ddot{q}_i = \nabla_i U(\mathbf{q}) = \lambda m_i q_i$)

$$\begin{aligned}
\lambda q_1 &= \frac{m_2}{(q_2-q_1)^2} + \frac{m_3}{(q_3-q_1)^2}, \\
\lambda q_2 &= -\frac{m_1}{(q_2-q_1)^2} + \frac{m_3}{(q_3-q_2)^2}, \\
\lambda q_3 &= -\frac{m_1}{(q_3-q_1)^2} - \frac{m_2}{(q_3-q_2)^2}.
\end{aligned}$$

Let be $x = q_2 - q_1, y = q_3 - q_2, z = q_3 - q_1 = x + y$, then the above equations can be written in terms of the variables x and y as

$$\begin{aligned}
\lambda x &= -\frac{m_1+m_2}{x^2} + \frac{m_3}{y^2} - \frac{m_3}{(x+y)^2}, \\
\lambda y &= -\frac{m_1}{(x+y)^2} + \frac{m_1}{x_2)^2} - \frac{m_2+m_3}{y^2}.
\end{aligned}$$

Since we are looking for classes of CC we can normalize the CC doing $x = 1$, in this way we obtain for the first of the above equations

$$\lambda = -(m_1+m_2) + \frac{m_3}{y^2} - \frac{m_3}{(1+y)^2},$$

substituting this expression in the second of the above equations we get

$$-(m_1+m_2)y + \frac{m_3}{y} - \frac{m_3 y}{(1+y)^2} + \frac{m_1}{(1+y)^2} - m_1 + \frac{m_2+m_3}{y^2},$$

by straightforward computations, finally we obtain the Euler polynomial

$$\begin{aligned}
p(y) &= -(m_1+m_2)y^5 - (3m_1+2m_2)y^4 - (3m_1+m_2)y^3 \\
&\quad +(2m_3+m_2)y^2 + (3m_3+2m_2)y + (m_2+m_3) = 0.
\end{aligned}$$

By Descartes rule of signs, the Euler polynomial has exactly one positive root, that is, fixing the particles 1 an 2, there is a unique possible position for the particle 3 in order to form a CC. □

Remark 1.14. *This is the translation of the original proof given by L. Euler in 1667, from here, using homothecies and rotations he got the first solution of the 3–body problem.*

1.4.3 Lagrange Theorem

Few years later from Euler's theorem, J.L. Lagrange (1772) re-discover the collinear CC and find a new class of planar CC, we state the result.

Theorem 1.15. *(Lagrange) In the planar 3–body problem, for any choice of the masses, there are exactly 2 non-collinear CC corresponding to the two possible orientations of an equilateral triangle in the plane.*

Before to start with the proof of the above theorem we write here a Lemma from calculus of several variables, which facilitates the proof. We will use it also in others chapters of these notes.

Lemma 1.16. *Let $u = f(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and let $x_1 = g_1(\mathbf{y})$, $x_2 = g_2(\mathbf{y})$,$\cdots$, $x_n = g_n(\mathbf{y})$ with $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ and $m \geqslant n$. If rank $(A) = n$, with*

$$A = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial y_m} & \cdots & \frac{\partial x_n}{\partial y_m} \end{pmatrix},$$

then $\nabla f(\mathbf{x}) = \mathbf{0}$ if and only if $\nabla u(\mathbf{y}) = \mathbf{0}$.

Proof of Lagrange Theorem. We know that the central configurations are solutions of the equation

$$\nabla U + \lambda \nabla I = 0.$$

We note that U and I depend on $\mathbf{q}$ by means of the mutual distances r_{ij}, thus we can consider $U = U(r_{12}, r_{13}, r_{23})$ where

$$(r_{12}, r_{13}, r_{23}) = (r_{12}(q_{11}, q_{12}, q_{21}, q_{22}), r_{13}(q_{11}, q_{12}, q_{31}, q_{32}), r_{23}(q_{21}, q_{22}, q_{31}, q_{32})).$$

Using Lemma 1.16, we have that if rank $(A) = 3$, with A given by

$$\begin{pmatrix} \frac{\partial r_{12}}{\partial q_{11}} & \frac{\partial r_{13}}{\partial q_{11}} & \frac{\partial r_{23}}{\partial q_{11}} \\ \frac{\partial r_{12}}{\partial q_{12}} & \frac{\partial r_{13}}{\partial q_{12}} & \frac{\partial r_{23}}{\partial q_{12}} \\ \frac{\partial r_{12}}{\partial q_{21}} & \frac{\partial r_{13}}{\partial q_{21}} & \frac{\partial r_{23}}{\partial q_{21}} \\ \frac{\partial r_{12}}{\partial q_{22}} & \frac{\partial r_{13}}{\partial q_{22}} & \frac{\partial r_{23}}{\partial q_{22}} \\ \frac{\partial r_{12}}{\partial q_{31}} & \frac{\partial r_{13}}{\partial q_{31}} & \frac{\partial r_{23}}{\partial q_{31}} \\ \frac{\partial r_{12}}{\partial q_{32}} & \frac{\partial r_{13}}{\partial q_{32}} & \frac{\partial r_{23}}{\partial q_{32}} \end{pmatrix} = \begin{pmatrix} \frac{q_{11}-q_{21}}{r_{12}} & \frac{q_{11}-q_{31}}{r_{13}} & 0 \\ \frac{q_{12}-q_{22}}{r_{12}} & \frac{q_{12}-q_{32}}{r_{13}} & 0 \\ -\frac{q_{11}-q_{21}}{r_{12}} & 0 & \frac{q_{21}-q_{31}}{r_{23}} \\ -\frac{q_{12}-q_{22}}{r_{12}} & 0 & \frac{q_{22}-q_{32}}{r_{23}} \\ 0 & -\frac{q_{11}-q_{31}}{r_{13}} & -\frac{q_{21}-q_{31}}{r_{23}} \\ 0 & -\frac{q_{12}-q_{32}}{r_{13}} & -\frac{q_{22}-q_{32}}{r_{23}} \end{pmatrix}.$$

The same argument applies to the moment of innertia I, then we have that

$$\nabla U(\mathbf{q}) = 0 \quad \text{if and only if} \quad \nabla U(r_{12}, r_{13}, r_{23}) = 0 \ ,$$

and

$$\nabla I(\mathbf{q}) = 0 \quad \text{if and only if} \quad \nabla I(r_{12}, r_{13}, r_{23}) = 0 \ .$$

After some computations we see that rank $(A) = 3$ if and only if

$$\det \begin{pmatrix} q_{11} & q_{12} & 1 \\ q_{21} & q_{22} & 1 \\ q_{31} & q_{32} & 1 \end{pmatrix} \neq 0 \ .$$

This determinant equals twice the area of the triangle formed by the 3 bodies. In short, we have proved that if $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are not collinear, then $\nabla U(r_{12}, r_{13}, r_{23}) = \mathbf{0}$ if and only if $\nabla U(\mathbf{q}) = \mathbf{0}$, and $\nabla I(r_{12}, r_{13}, r_{23}) = \mathbf{0}$ if and only if $\nabla I(\mathbf{q}) = \mathbf{0}$.

To finish the proof of the theorem we just have to solve

$$-\frac{m_i m_j}{r_{ij}^2} + \frac{\lambda}{\tilde{m}} m_i m_j r_{ij} = 0,$$

for all $1 \leq i < j \leq 3$, remember that $I = \frac{1}{2\tilde{m}} \sum_{1 \leq i < j \leq 3} r_{ij}^2$. From here we obtain $r_{ij} = (\lambda/\tilde{m})^{1/3}$. This implies that there are just one possibility for the three mutual distances among the particles, therefore the configuration is an equilateral triangle. □

1.4.4 Moulton Theorem

As you can see, try to generalize the Euler's ideas to find the total number of CC in the collinear 4–body problem turn really complicated since, we obtain a polynomial in two variables of high degree, and of course the degree of complexity increase a lot when the number of particles increase, which indicates that this is not the right way to tackle the problem. In 1910, R.F. Moulton had a clever idea to find the total number of CC in the collinear n–body problem, his proof is by induction over the number of bodies, for $n = 3$ this corresponds to Euler theorem, in this case the total number of CC is $3!/2 = 3$. Assuming that this result is true for $n = k$, he proves first that it is possible to introduce just one massless article in each interval having at least one particle at the ends, so in this way he obtain that in the collinear $k+1$–body problem (k positive masses plus one massless particle) there are exactly $(k+1)!/2$ CC. Then he proves that it is possible to extend the result when we introduce a positive mass. You can find the details of this astute proof in [77]. Here we will prove Moulton Theorem using modern techniques, the following proof is due to Moeckel [76].

Theorem 1.17. *(Moulton) In the collinear n–body problem, for any choice of n positive masses there are exactly $n!/2$ CC. One for ech ordering of the particles modulus a rotation by π.*

Proof. For clarity in the exposition we do the proof just for the case $n = 3$ (that is, we give a new proof of Euler Theorem), nevertheless the arguments are the same for the general case.

Let be $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$ the position of the 3 positive particles on a straight line, and let $S = \{\mathbf{q} \in \mathbb{R}^3 \,|\, I(\mathbf{q}) = 1\}$ the ellipsoid of masses, which topologically is a 2–dimensional sphere in the space. We define the plane given by the condition to fix the center of mass at the origin as G, that is

$$G = \{\mathbf{q} \in \mathbb{R}^3 \,|\, m_1 q_1 + m_2 q_2 + m_3 q_3 = 0\}.$$

Then $S^1 = S \bigcap G$ is a maximum circle on the the plane G. Finally we drop the singularities, that is we take $S^1 \setminus \Delta$. We observe that this set has $3! = 6$ connected components, each component corresponds to an ordering of the particles on the straight line.

Let be $\bar{U}$ the restriction of U to the set $S^1 \setminus \Delta$. As we have seen before, $\bar{U} \to \infty$ when $\mathbf{q} \to \Delta$, and then $\bar{U}$ has at least one minimum for each connected component. So there is at least 3! collinear CC.

Let be $\mathbf{c}$ a critical point of $\bar{U}$, then $\mathbf{c}$ is a CC, and each component c_i satisfies

$$F(c_i) = \sum \frac{m_i m_j}{|c_j - c_i|^3}(c_j - c_i) + \lambda m_i c_i = 0.$$

It is not difficult to verify that $D^2F(\mathbf{c})(\mathbf{v}, \mathbf{v}) > 0$ when $\mathbf{v} \neq 0$, that is the Hessian of F is positive definite at a critical point and it is a local minimum of $\bar{U}$, so this function is convex and then there exists just one critical point of $\bar{U}$ on each connected component, therefore there are 3! critical points. Taking into account the symmetry given by the reflection with respect to the origin we get the result. □

1.5 Localization of the particles in a CC

The next two results are due to C. Conley, in [76] you can find a nice geometrical demonstration of them, here our goal is to show how they give us a good idea about the localization of the particles in a CC.

Let L a fixed straight line on the plane. We denote by $\angle(\mathbf{q}_i - \mathbf{q}_j, L)$ the angle formed by L and the line segment connecting $\mathbf{q}_i$ with $\mathbf{q}_j$. Let be $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_n)$ a planar configuration of the n-body problem, and. We define

$$\Theta(\mathbf{q}, L) = max\{\angle(\mathbf{q}_i - \mathbf{q}_j, L) \,|\, i, j \in \{1, 2, \cdots, n\}, i \neq j\},$$

and let

$$\Theta(\mathbf{q}) = min_L \Theta(\mathbf{q}, L).$$

We observe that $\Theta(\mathbf{q}) = 0$ if and only if $\mathbf{q}$ is collinear. The function $\Theta(\mathbf{q})$ measures how far is a configuration $\mathbf{q}$ to be collinear.

Theorem 1.18. *(Conley 45° Theorem) There are not planar CC with* $0 < \Theta(\mathbf{q}) \leq \pi/4$.

Proof. Let $\Omega = \{\mathbf{q} \in S \,|\, 0 \leq \Theta(\mathbf{q}) \leq \pi/4\}$. The idea is to show that $\Theta(\mathbf{q})$ is a strict Lyapunov function on Ω for the flow given by $\dot{\mathbf{q}} = \nabla U(\mathbf{q})|_S$, where S is the unitary ellipsoid of masses. That is, we must prove that $\Theta(\mathbf{q})$ is strictly decreasing along the orbits of the above gradient flow. In order to prove that $\Theta(\mathbf{q}(t)) < \Theta(\mathbf{q}(t_0))$ por all $t > t_0$, we choose L such that $\Theta(\mathbf{q}_0) = \Theta(\mathbf{q}_0, L)$. Then by definition, $\Theta(\mathbf{q}(t)) \leq \Theta(\mathbf{q}(t), L)$. So it is enough to prove that $\Theta(\mathbf{q}(t), L) < \Theta(\mathbf{q}_0)$. Choose indices i, j such that

$$\Theta(\mathbf{q}(t_0), L) = \angle(q_i^0 - q_j^0, L) = \alpha.$$

If this angle decreases for each such points, then the maximum over i, j also decreases along the orbits of the above gradient flow. Let $\mathbf{v}$ a unit vector on the straight line L such that $\cos\alpha(t) = \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|} \cdot \mathbf{v}$ is non negative. Taking the derivate we obtain

$$-\sin\alpha(t)\dot{\alpha}(t) = \frac{\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j}{|\mathbf{q}_i - \mathbf{q}_j|} \cdot \left(\mathbf{v} - \frac{(\mathbf{q}_i - \mathbf{q}_j) \cdot \mathbf{v}}{|\mathbf{q}_i - \mathbf{q}_j|^2} (\mathbf{q}_i - \mathbf{q}_j) \right). \tag{1.6}$$

We observe that the vector $\mathbf{u} = \left(\mathbf{v} - \frac{(\mathbf{q}_i - \mathbf{q}_j) \cdot \mathbf{v}}{|\mathbf{q}_i - \mathbf{q}_j|^2} (\mathbf{q}_i - \mathbf{q}_j) \right)$ is the component of the projection of the vector $\mathbf{v}^\perp$ onto the plane determined by $\mathbf{v}$ and $(\mathbf{q}_i - \mathbf{q}_j)$.

Using the derivative along the gradient flow we obtain that

$$(\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) \cdot \mathbf{u} = \sum_{k \neq i,j} m_k \left(\frac{\mathbf{q}_k - \mathbf{q}_i}{|\mathbf{q}_i - \mathbf{q}_j|^3} \cdot \mathbf{u} - \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} \cdot \mathbf{u} \right).$$

At $\mathbf{q}_i$ ($\mathbf{q}_j$) we construct the cones whose central axis is on the line containing the segment $(\mathbf{q}_i - \mathbf{q}_j)$, and the amplitude from the central axis is 45°. Since $\Theta(\mathbf{q}(t_0)) \leq$

45°, the rest of the particles are in these cones, more precisely, they must be on the half part of the cones which lie on opposite sides of the segment containing the segment $(\mathbf{q}_i - \mathbf{q}_j)$. We have three possibilities for the position of the masses $\mathbf{q}_k$

- $\mathbf{q}_k$ is on the line determined by $\mathbf{q}_i, \mathbf{q}_j$

  In this case $(\mathbf{q}_k - \mathbf{q}_i) \cdot \mathbf{v} = (\mathbf{q}_k - \mathbf{q}_j) \cdot \mathbf{v} = 0$

- $\mathbf{q}_k$ is on the right cone (not on the line).

  In this case $(\mathbf{q}_k - \mathbf{q}_i) \cdot \mathbf{u} > 0$ and then

$$(\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) \cdot \mathbf{u} = \sum_{k \neq i,j} m_k (\mathbf{q}_k - \mathbf{q}_i) \cdot \mathbf{u} \left(\frac{1}{r_{ik}^3} - \frac{1}{r_{jk}^3} \right),$$

  since $\mathbf{q}_k$ is on the right cone we have that $\frac{1}{r_{ik}^3} > \frac{1}{r_{jk}^3}$.

- $\mathbf{q}_k$ is on the left cone (not on the line). In this case $(\mathbf{q}_k - \mathbf{q}_i) \cdot \mathbf{u} < 0$ and $\frac{1}{r_{ik}^3} < \frac{1}{r_{jk}^3}$.

All the above proves that the right part of the equality (1.6) is positive, which proves that the function $\Theta(\mathbf{q})$ is strictly decreasing (in [76] you can find all the geometrical arguments which complement this proof). This means that any configuration in Ω tends to be more and more collinear, and by consequence proves the statement of the Theorem. □

Remark 1.19. *The above Theorem also proves that the set Ω is positively invariant, then its complement contains all non-collinear CC, in other words, any CC is collinear or is far to be collinear.*

Corollary 1.20. *In the n–body problem, there are non-collinear CC for any choice of the masses.*

Proof. It follows directly from the fact that the potential U has a global minimum of the ellipsoid of masses S which corresponds to a CC. But we can verify easily, using the techniques of the above Theorem that the normal eigenvalues of any collinear CC are negative, that is the minimum can not be reached at a collinear CC. □

It is not difficult to verify that if $\mathbf{q}$ is an equilateral triangle, then $\Theta(\mathbf{q}) = 60°$, and that if $\mathbf{q}$ is a square, then $\Theta(\mathbf{q}) = 67.5°$. Who is $\Theta(\mathbf{q})$ if $\mathbf{q}$ is a regular pentagon, a regular hexagon, etc.?

Now we all study other result in the same direction, also due to Conley. In this way we first define the concept of a bisector.

Definition 1.21. *Considerer* $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_n)$ *a planar* CC*, given* $i \neq j \in \{1, 2, \cdots, n\}$ *by the middle point of the segment* $\mathbf{q}_i - \mathbf{q}_j$ *we construct the perpendicular line or* bisector *to the line containing the points* $\mathbf{q}_i, \mathbf{q}_j$.

Using the above definition we have divided the plane in 4 cones or quadrants, we denote the cones which are opposite to the vertex by C_I and C_{II}.

Theorem 1.22. *(Conley bisector Theorem.) Let* $\mathbf{q}$ *any planar* CC *and choose any* $i \neq j$*. Construct the open cones* C_I *and* C_{II} *using the corresponding bisector. If there is a mass* $\mathbf{q}_k$ *in one* C_I*, then there is a mass in the other* C_I *as well, the same happen for the cones* C_{II}.

Proof. Define $\mathbf{u}$ as in the proof of the previous Theorem. We observe that the terms in the sum defined by $(\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) \cdot \mathbf{u}$ coming from $\mathbf{q}_i \in C_I$ are positive, whereas the terms coming from $\mathbf{q}_i \in C_{II}$ are negative. The terms coming from $\mathbf{q}_i$ on the axis are zero. Then if only C_I were occupied we would get by one hand that $(\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) \cdot \mathbf{u} \neq 0$, but by other hand since the masses at $\mathbf{q}_i$ form a CC we have that $\dot{\mathbf{q}}_i = \dot{\mathbf{q}}_j = 0$ which is a contradiction. This proves the Theorem. $\square$

Chapter 2

CC in the 4–body problem

The question about the existence and classification of central configurations is a fascinating problem that dates back to the 18th century, with the works of L. Euler and J.L. Lagrange. The exact number of CC for any choice of the masses has been solved just for the 3–body problem and for the collinear n–body problem. Before 2004, we did not know even if the total number of CC was finite or infinite, however many numerical experiments point in the direction of the finiteness, actually C. Simó in 1978 proved numerically that the upper bound should be 50 [98]. Fortunately in 2004, Hampton and Moeckel proved that for any choice of 4 positive masses, the number of CC is finite [50]. In 2010, A. Albouy and V. Kaloshin proved a similar result in the 5–body problem [4]. Both results only shows the finiteness of the CC, but for instance the upper bound which appears in [50] is greater that 8000, which is far to be optimal according to the numerical results of C. Simó, which we state it here as a conjecture.

Conjecture *The maximum number of CC in the 4–body problem is 51, the regular tetrahedron in $\mathbb{R}^3$ and up to 50 planar CC.*

In the way to give some light in the solution of this conjecture, in 1976, J. Llibre, assuming the existence of an axis of symmetry containing two of the masses proved that there are exactly 50 planar CC [62]. In 1995, A. Albouy proves the existence of the axis of symmetry and gives a new proof of the total number of planar CC [1]. In the next subsections we state some results for planar CC in the 4–body problem with some equal masses. Previously we present here a useful characterization for the planar non–collinear central configurations of the 4–body problem with positive masses m_i, for $i = 1, \cdots, 4$, in terms of mutual distances and areas of the triangles which form three of the particles (See [48] for a complete deductions of these equations). The equations can be written as

$$
\begin{array}{l}
m_3\Delta_4(r_{13}^{-3}-r_{23}^{-3})+m_4\Delta_3(r_{14}^{-3}-r_{24}^{-3})=0,\\
m_2\Delta_4(r_{12}^{-3}-r_{23}^{-3})+m_4\Delta_2(r_{14}^{-3}-r_{34}^{-3})=0,\\
m_2\Delta_3(r_{12}^{-3}-r_{24}^{-3})+m_3\Delta_2(r_{13}^{-3}-r_{34}^{-3})=0,\\
m_1\Delta_4(r_{12}^{-3}-r_{13}^{-3})+m_4\Delta_1(r_{24}^{-3}-r_{34}^{-3})=0,\\
m_1\Delta_3(r_{12}^{-3}-r_{14}^{-3})+m_3\Delta_1(r_{23}^{-3}-r_{34}^{-3})=0,\\
m_1\Delta_2(r_{13}^{-3}-r_{14}^{-3})+m_2\Delta_1(r_{23}^{-3}-r_{24}^{-3})=0,
\end{array}
\tag{2.1}
$$

where Δ_i is the signed area of the triangle with vertices at the particles with masses (m_j, m_k, m_l), the subindices (i, j, k, l) permutes cyclically in $(1, 2, 3, 4)$.

2.1 Planar CC with three equal masses

In this section we study the planar non–collinear central configurations of the 4–body problem with three equal masses. Taking conveniently the mass unity we can assume that the three equal masses are equal to 1, and the fourth mass is $m > 0$. We say that a planar non–collinear central configuration of the 4–body problem has a *kite shape* or simply is a *kite central configuration* if it has an axis of symmetry passing through two of the masses (see Figure 2.1). The kite configuration is *convex* if none of the bodies is located in the interior of the convex hull of the other three, otherwise and if the configuration is not collinear we say that the kite configuration is *concave*.

We consider planar non–collinear central configuration of the 4–body problem with three equal masses $m_2 = m_3 = m_4 = 1$ and $m_1 = m > 0$. We assume that the configuration has an axis of symmetry containing the mass m. We also assume that the two masses outside the axis of symmetry are m_3 and m_4. By the results of Long and Sun [63], taking the hypothesis that a central configuration is kite is not restrictive if it is convex; or if it is concave and the mass m is in the interior of the convex hull of the three equal masses. Part of the results present in this section were done in collaboration with J. Lllibre and J. Bernat, in [**?**] you can find the complete proofs of the results stated here.

In order to study these kite central configurations we choose the axes of coordinates with origin at the center of mass of m_3 and m_4, the y–axis as the axis of symmetry, and the x–axis orthogonal to it. Taking conveniently the unity of length, we can suppose that the positions of the masses m_1, m_2, m_3 and m_4 are $(0, l)$, $(0, -k)$, $(-1, 0)$ and $(1, 0)$ respectively, and that always m_1 is over m_2 on the y–axis; i.e. $k + l > 0$. See Fig. 2.1.

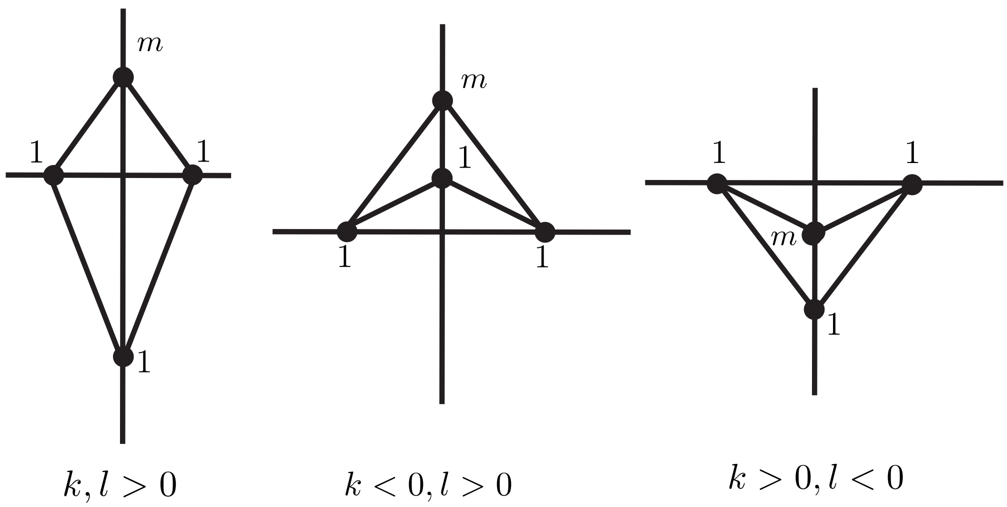

Figure 2.1: Kite central configurations with three equal masses

Using the symmetries of the kite configuration, the six equations for the planar non–collinear central configurations of the 4–body problem (2.1) reduce to the following two equations

$$\begin{array}{rcl} m(k+l)[(k+l)^{-3}-(1+l^2)^{-3/2}]+2k[(1+k^2)^{-3/2}-2^{-3}] & = & 0\ , \\ f(k,l)=(k+l)[(k+l)^{-3}-(1+k^2)^{-3/2}]+2l[(1+l^2)^{-3/2}-2^{-3}] & = & 0\ . \end{array} \tag{2.2}$$

Solutions (k,l) of equations (2.2) with $k,l\in\mathbf{R}$, and $k+l>0$ provide the planar non–collinear central configurations of the 4–body problem for the masses $m_1=m>0$ and $m_2=m_3=m_4=1$. We remark that, from the first equation of (2.2), k cannot be zero; and from the second one l also cannot be zero.

Theorem 2.1. *In the planar 4–body problem with three masses equal to 1 and the fourth one equal to m there exist $m_*=(2+3\sqrt{3})/(18-5\sqrt{3})=0.7704869545\cdots$ and $m^*=1.00265277445\cdots$ such that the total number of CC is:*

(1) 38 *if* $0<m<m_*$,

(2) 32 *if* $m=m_*$,

(3) 38 *if* $m_*<m<1$,

(4) 50 *if* $m=1$,

(5) 38 *if* $1 < m < m^*$,

(6) 32 *if* $m = m^*$,

(7) 26 *if* $m^* < m$.

Proof. We start with the following result that will permit to avoid the singular value when the mass m is written in terms of k and l.

Proposition 2.2. *The equilateral triangle, with the three masses equal to* 1 *on its vertices and the mass* $m > 0$ *at its barycenter, always is a central configuration of the planar* 4*–body problem.*

Proof. It is sufficient to check that $(k = \sqrt{3}, l = -1/\sqrt{3})$ is a solution of system (2.2) for all $m > 0$. □

The first equation of (2.2) can be written as

$$m = m(k,l) = \frac{2k[2^{-3} - (1+k^2)^{-3/2}]}{(k+l)[(k+l)^{-3} - (1+l^2)^{-3/2}]} =$$

$$\frac{k(k+l)^2(1+l^2)^{3/2}[5+k^2+2(1+k^2)^{1/2}][(1+k^2)^{1/2}-2]}{4(1+k^2)^{3/2}[1+(k+l)^2+l^2+(k+l)(1+l^2)^{1/2}][(1+l^2)^{1/2}-l-k]} .$$

When we write system (2.2) as the system

$$m = m(k,l) \ , \qquad f(k,l) = 0 \ , \tag{2.3}$$

we only lost the solutions which are described in Proposition 2.2. Now we will analyze this last system. By (2.3) we observe that a point (k,l) of the curve $f = 0$ provides a planar non–collinear central configuration of the 4–body problem for the masses $m_1 = m(k,l)$ and $m_2 = m_3 = m_4 = 1$ if and only if $m(k,l) > 0$. This is the key point for proving Theorem 2.1.

Using *Mathematica* we plot the curve $f = 0$, from here we see that this curve has only two branches in the region $k + l > 0$; one contained in the half–plane $l > 0$, and the other contained in the fourth quadrant $\{(k,l) : k > 0, l < 0\}$; see Fig. 2.2.

Using the expression of $m(k,l)$ it follows that

$$\operatorname{sign}(m) = \operatorname{sign}\left(\frac{k(\sqrt{1+k^2}-2)}{\sqrt{1+l^2}-l-k}\right) . \tag{2.4}$$

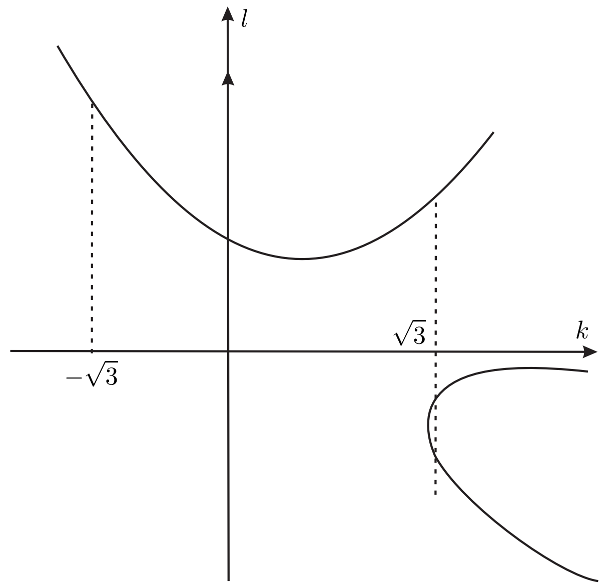

Figure 2.2: The graph of $f = 0$

We note that $k(\sqrt{1+k^2}-2)$ is positive in $(-\sqrt{3},0)\cup(\sqrt{3},\infty)$; is zero in $\{-\sqrt{3},0,\sqrt{3}\}$; and negative in the complement. The intersection of the straight line $k = \sqrt{3}$ with the curve $f = 0$ provides the three points:

$$P_2 = (\sqrt{3}, 1.29302\cdots), \quad P_5 = (\sqrt{3}, -0.09943\cdots), \quad P_6 = (\sqrt{3}, -1/\sqrt{3}).$$

While the intersection of $k = 0$ with $f = 0$ has only the point $P_4 = (0, 1.13942..)$. Finally, the intersection of $k = -\sqrt{3}$ with the curve $f = 0$ provides a unique point $P_3 = (-\sqrt{3}, 2.93734\cdots)$ in the region $k + l > 0$.

The curve $k(l) = \sqrt{1+l^2} - l$ only has points with $k > 0$, and it has a unique branch which intersects the curve $f = 0$ at the points

$$\begin{aligned} P_1 &= (k(l_1), l_1 = 0.92080\cdots), \\ P_6 &= (\sqrt{3}, -1/\sqrt{3}), \\ P_7 &= (k(l_2), l_2 = -1.14090\cdots). \end{aligned}$$

We observe in Fig. 2.3 that on the left of the curve $k(l)$ we have $\sqrt{1+l^2} - l - k$ is positive, and negative on the right.

With the above observations it follows that on $f = 0$, the function $m(k,l)$ is positive exactly on the following three arcs.

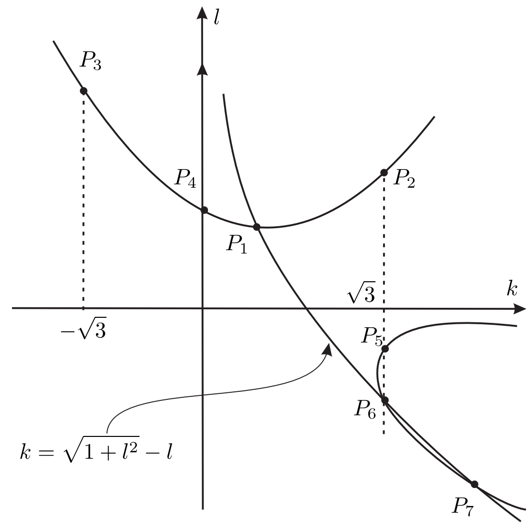

Figure 2.3: The four arcs of $f = 0$ where $m(k, l) > 0$

(i) The open arc γ_1 going from P_1 to P_2. Since on this arc k and l are positive, the corresponding central configurations associated to points (k, l) of this arc are convex. Since $m(k, l)$ on P_2 takes the value zero and on P_1 the value $+\infty$, there is at least one convex central configuration for every value of $m > 0$. It is not difficult to show that the function $m(k, l)$ is strictly increasing on the arc starting at P_2 and ending at P_1. Since the central configuration coming from one of the other two arcs will be concave, it follows that there is exactly one convex central configuration for every $m > 0$.

(ii) The open arc γ_2 going from P_3 to P_4. Since on this arc $k < 0$ and $l > 0$, the corresponding central configurations associated to points (k, l) of this arc are concave having the mass m as a vertex of the triangle formed by the convex hull of the four masses. On the points P_3 and P_4 we have that $m(k, l)$ takes the value zero. Proving that the function $m(k, l)$ has a unique critical point on this arc, which corresponds to a maximum. We compute the value of this maximum $m^* = 1.00265277445\cdots$. Therefore, for values of $m \in (0, m^*)$ there will be exactly two planar non–collinear concave configurations having the mass m as a vertex of the triangle formed by the convex hull of the four masses. This is due to the fact that the other concave central configurations will have the mass m in the interior of such a convex hull. For the value $m = m^*$ the two concave central configuration for $m \in (0, m^*)$ become the same.

(iii) The open arc γ_3 going from P_5 to P_7. Since on this arc $k > 0$ and $l < 0$, the corresponding central configurations associated to points (k, l) of this arc are concave having the mass m in the interior of the triangle formed by the convex hull of the other three masses. Since $m(k,l)$ on P_5 takes the value zero and on P_7 the value $+\infty$, there is at least one concave central configuration for every value of $m > 0$. It is not so difficult to show that the function $m(k,l)$ is strictly increasing on this arc, for every $m > 0$ this arc provides a unique concave central configuration having the mass m in the interior of the triangle formed by the convex hull of the other three masses. We note that $m(k,l)$ on the point P_6 takes the value $m_* = 0.7704869545\cdots$, and that the point P_6 corresponds to the equilateral triangle of Proposition 2.2.

For the value $m = 1$, one of the two concave central configurations obtained on the arc γ_2 coincides with the one obtained on the arc γ_3. In fact, this concave central configuration which coincides is the isosceles one. The other concave central configuration on the arc γ_2 coincide with the one described in Proposition 2.2. Of course, when $m = 1$ it is not possible to distinguish if the mass m is in a vertex of the triangle formed by the convex hull of the four masses, or if it is in its interior. All the above prove Theorem 2.1. □

At the beginning we though that all CC with three equal masses must have an axis of symmetry, so with the above we will have a complete classification of all CC in this case, but unfortunately there are CC that do not have any axis of symmetry. It is not difficult to check numerically that the 4–body problem with three equal masses has values for the fourth mass for which there are non–kite central configurations. Thus, for instance, take $m_1 = m_2 = m_3 = 1$ and $m_4 = 0.992$, then the configuration

$$\begin{array}{ll} r_{12} = 1.011826886527223\cdots\ , & r_{13} = 1.01492951542898 44\cdots\ , \\ r_{14} = 1.0238808310460596\cdots\ , & r_{23} = 1.7823680187556408\cdots\ , \\ r_{24} = 1.7546086138811445\cdots\ , & r_{34} = 1.745696989806155\cdots\ , \end{array}$$

is central and it has not any axis of symmetry.

2.2 Planar CC with two equal masses

Here we study convex CC with at least two equal masses. In 2002, Y. Long and S. Sun [63] proved that any planar convex CC with a two couple of equal masses $\delta > \alpha > 0$, and such that the diagonal corresponding to the mass α is not shorter that the one corresponding to the mass δ, must possess a line of symmetry and therefore should be a kite (see Fig. 2.4). In 2007, E. Pérez-Chavela and M. Santoprete drop the restrictions of the previous result and generalize it to the case when the CC has only two equal masses with the additional hypothesis that the smallest masses are not the

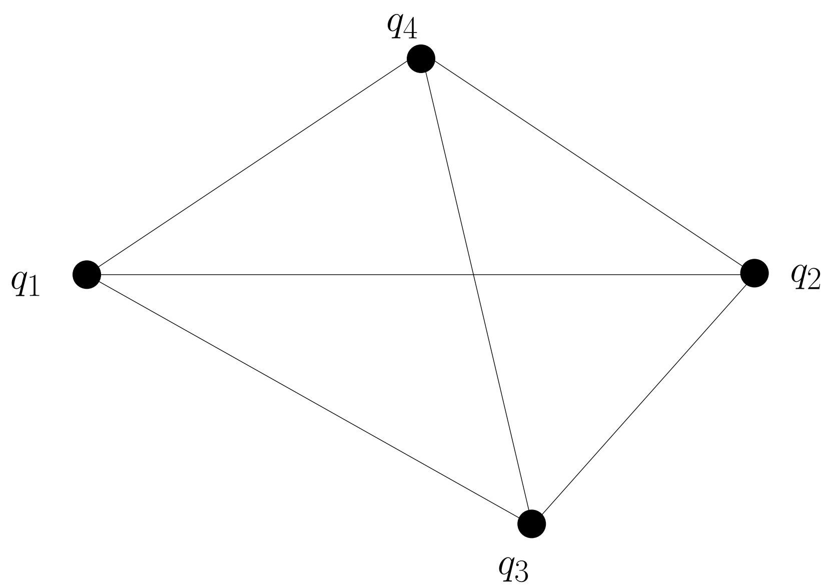

Figure 2.4: A convex CC of four masses

equal masses [85], a couple of years later A. Albouy, Y. Fu and S. Sun [3] drop this additional hypothesis. In this section we present the main results of the paper [85].

Theorem 2.3. *Let $q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4$ be a convex non-collinear central configuration with masses $(\delta, \delta, \alpha, \beta) \in (\mathbb{R}^+)^4$. Suppose that the equal masses are opposite vertices and that $\alpha \leq \delta$ or $\beta \leq \delta$. Then the configuration q must posses a symmetry, it is unique and forms a kite.*

Proof. We consider the planar 4-body problem with masses

$$m_1 = m_2 = 1, \quad m_3 = \alpha, \quad m_4 = \beta.$$

Let

$$a = r_{12}^2,\ b = r_{13}^2,\ c = r_{14}^2, d = r_{23}^2,\ e = r_{24}^2,\ f = r_{34}^2.$$

For $1 \leq i \leq 4$ let $|\Delta_i|$ be the area of the sub-triangle formed by the remaining three vertices of the configuration q when deleting the point q_i. We define the oriented areas of these sub-triangles of the convex non-collinear configuration q by

$$\Delta_1 = -|\Delta_1|,\ \ \Delta_2 = -|\Delta_2|,\ \ \Delta_3 = |\Delta_3|,\ \ \Delta_4 = |\Delta_4|$$

when the masses are opposite vertices of a quadrilateral. The Δ_i above satisfy the equation

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 0 \tag{2.5}$$

The Cayley determinant

$$S = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & b & c \\ 1 & a & 0 & d & e \\ 1 & b & d & 0 & f \\ 1 & c & d & f & 0 \end{vmatrix} \tag{2.6}$$

satisfies $S = 0$.

In 1900 Dziobek proved that

$$\frac{\partial S}{\partial r_{ij}^2} = 32\Delta_i\Delta_j \qquad \forall i \neq j. \tag{2.7}$$

Let $\psi(s) = s^{-1/2}$ for $s > 0$. Then the potential function and the moment of inertia are given by

$$U = \sum_{1 \leq i < j \leq 4} m_i m_j \psi(r_{ij}^2) \tag{2.8}$$

and

$$I = \frac{1}{m'} \sum_{1 \leq i < j \leq 4} m_i m_j r_{ij}^2 \tag{2.9}$$

respectively, where $m' = \sum_{i=1}^{4} m_i$.

Using Lagrange multipliers, Dziobek characterized the central configurations of four bodies as the extreme of

$$U - \lambda S - \mu(I - I_0)$$

as a function of $\lambda, \mu, r_{12}, \cdots, r_{34}$, where λ and μ are Lagrange multipliers and I_0 is a fix moment of inertia. Thus, for any i, j with $1 \leq i < j \leq 4$, the central configurations satisfy

$$\frac{\partial U}{\partial r_{ij}^2} = \lambda \frac{\partial S}{\partial r_{ij}^2} + \mu \frac{\partial I}{\partial r_{ij}^2} \tag{2.10}$$

or

$$\frac{\partial U}{\partial r_{ij}^2} = m_i m_j \psi'(r_{ij}^2),$$

where $\psi'(s)$ denotes the derivative of the function $\psi(s)$ with respect to s.

We also have

$$\frac{\partial I}{\partial r_{ij}^2} = \frac{m_i m_j}{m'}.$$

Consequently, equation (2.10) becomes

$$m_i m_j \psi'(r_{ij}^2) = 32\lambda \Delta_i \Delta_j + \frac{m_i m_j \mu}{m'}. \tag{2.11}$$

Therefore, using our mass convention, the equations for the central configurations are:

$$\psi'(r_{12}^2) = \quad \nu \Delta_1 \Delta_2 + \xi \tag{2.12a}$$
$$\psi'(r_{13}^2) = \quad \tfrac{\nu}{\alpha} \Delta_1 \Delta_3 + \xi \tag{2.12b}$$
$$\psi'(r_{14}^2) = \quad \tfrac{\nu}{\beta} \Delta_1 \Delta_4 + \xi \tag{2.12c}$$
$$\psi'(r_{23}^2) = \quad \tfrac{\nu}{\alpha} \Delta_2 \Delta_3 + \xi \tag{2.12d}$$
$$\psi'(r_{24}^2) = \quad \tfrac{\nu}{\beta} \Delta_2 \Delta_4 + \xi \tag{2.12e}$$
$$\psi'(r_{34}^2) = \quad \tfrac{\nu}{\alpha\beta} \Delta_3 \Delta_4 + \xi \tag{2.12f}$$

where $\nu = 32\lambda$ and $\xi = \frac{\mu}{m'}$. Moreover, there are implicit relations between the r_{ij}^2 and the Δ_i:

$$t_k = \sum_{i=1}^{4} \Delta_i r_{ik}^2, \quad t_1 = t_2 = t_3 = t_4. \tag{2.13}$$

We need the next two Lemmas to prove Theorem 2.3.

Lemma 2.4. *For a central configuration, the corresponding ν in the equations (2.12a)-(2.12f) is positive.*

Lemma 2.5. *The following inequality holds:*

$$\left(\frac{\Delta_i}{m_i} - \frac{\Delta_j}{m_j}\right)(\Delta_i - \Delta_j) \geq 0. \tag{2.14}$$

Consequently $\Delta_i > \Delta_j$ if and only if $\frac{\Delta_i}{m_i} > \frac{\Delta_j}{m_j}$.

Let

$$A = \psi'(a), B = \psi'(b), \ldots, F = \psi'(f). \tag{2.15}$$

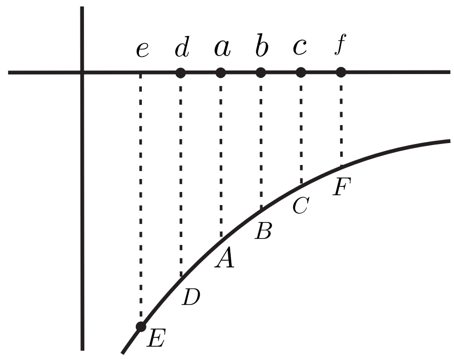

Figure 2.5: The geometric interpretation of the points A, B, C, D, E, F

In Fig. 2.5 we plot the geometric interpretation of these quantities.

After some tedious computations we obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ f-e-d & \alpha(e-d-f) & \beta(d-f-e) \\ F & E & D \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a+f & b+e & c+d \\ A & B & C \end{vmatrix} \tag{2.16}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ f-c-b & \beta(b-f-c) & \alpha(c-b-f) \\ F & B & C \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a+f & b+e & c+d \\ A & E & D \end{vmatrix} \tag{2.17}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ \beta(a-e-e) & e-c-a & c-a-e \\ A & E & C \end{vmatrix} = \alpha \begin{vmatrix} 1 & 1 & 1 \\ a+f & b+e & c+d \\ F & B & D \end{vmatrix} \tag{2.18}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha(a-d-b) & b-a-d & d-b-a \\ A & B & D \end{vmatrix} = \beta \begin{vmatrix} 1 & 1 & 1 \\ a+f & b+e & c+d \\ F & E & C \end{vmatrix} \tag{2.19}$$

These are the equations of the *balanced configurations* due to A. Albouy and A. Chenciner [2].

Observe that, in order to have symmetry, under the hypothesis of Theorem 2.3 the following inequality must hold

$$\Delta_1 = \Delta_2. \tag{2.20}$$

Note that $\Delta_3/\alpha = \Delta_4/\beta$ or $\Delta_3 = \Delta_4$ only if one is in a symmetric configurations with $\alpha = \beta$

We assume $\beta \neq \alpha$. To show that a configuration is symmetric, i.e. that (2.20) holds, one can assume that

$$\Delta_1 \neq \Delta_2 \quad \text{and} \quad \Delta_3 \neq \Delta_4.$$

and then derive a contradiction. This is the strategy of the proof.

If we assume $\Delta_1 \neq \Delta_2$ and $\Delta_3 \neq \Delta_4$ we have four cases

$$\begin{aligned} &\textbf{(a)}\ \Delta_1 < \Delta_2 < 0 < \Delta_3 < \Delta_4 \\ &\textbf{(b)}\ \Delta_2 < \Delta_1 < 0 < \Delta_3 < \Delta_4 \\ &\textbf{(c)}\ \Delta_1 < \Delta_2 < 0 < \Delta_4 < \Delta_3 \\ &\textbf{(d)}\ \Delta_2 < \Delta_1 < 0 < \Delta_4 < \Delta_3. \end{aligned} \tag{2.21}$$

Using Lemma 2.4 and Lemma 2.5 we rule out the above 4 possibilities. This proves that the configuration q must posses a symmetry.

We now show that the configuration must be a kite

Lemma 2.6. *If $\Delta_1 = \Delta_2$ the quadrilateral q is a kite.*

Proof. By equations (2.12a-2.12f) of the central configurations we have

$$\psi'(r_{13}^2) = \frac{\nu}{\alpha}\Delta_1\Delta_3 + \xi = \frac{\nu}{\alpha}\Delta_2\Delta_3 + \xi = \psi'(r_{23}^2). \tag{2.22}$$

Since $\psi'(s)$ is a monotone increasing function of s we obtain

$$r_{13} = r_{23}. \tag{2.23}$$

Similarly

$$\psi'(r_{14}^2) = \frac{\nu}{\alpha}\Delta_1\Delta_4 + \xi = \frac{\nu}{\alpha}\Delta_2\Delta_4 + \xi = \psi'(r_{24}^2). \tag{2.24}$$

and thus

$$r_{14} = r_{24}. \tag{2.25}$$

Therefore the quadrilateral is a kite. The proof is similar if we assume $\Delta_3 = \Delta_4$. $\square$

The uniqueness of this kite central configuration was proved by E. Leandro in [61]. This concludes the proof of Theorem 2.3.

Theorem 2.7. *Let $q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4$ be a convex non-collinear central configuration with masses $(\delta, \delta, \alpha, \alpha) \in (\mathbb{R}^+)^4$. Suppose that the equal masses are opposite vertices then the configuration q must posses a symmetry forms a rhombus and it is unique.*

We observe that Theorem 2.7 follows directly from Theorem 2.3 by taking $\alpha = \beta$. However, the main reason to include this result as a Theorem is by one hand , the fact that it completely answer the question formulated by Y. Long and S. Sun in [63], and by the other hand, that is not obvious from the equations of the balanced configurations, or from Dziobek equations that the kite central configuration obtained by Theorem 2.3 is a rhombus, but it follows from the uniqueness of the rhombus central configuration and a result by E. Leandro [61] that can be summarized as follows.

Lemma 2.8. *For any $\alpha > 0$ and $\beta > 0$, there exists a unique central configuration $q = (q_1, q_2, q_3, q_4)$ with masses $(1, 1, \alpha, \beta)$ where q_1 and q_2 as well as q_3 and q_4 are located at the opposite vertices of a kite shaped quadrilateral.*

This concludes the proof of Theorem 2.7. $\square$

Chapter 3

Homographic solutions and relative equilibria

Definition 3.1. *A solution $q(t) = (q_1(t), q_2(t), \cdots, q_n(t))$ of the n–body problem is called homographic if the configuration of the particles remains similar with itself for all time t. In other words, there exists a scalar function $R = R(t) > 0$ and an orthogonal matrix $\Omega(t) \in SO(3)$, such that*

$$q_k(t) = R(t)\Omega(t)q_k(0), \quad k = 1, 2, \cdots, n,$$

where $(q_1(0), q_2(0), \cdots, q_N(0))$ is a central configuration.

The two limit cases are so important kind of homographic solutions that we mention them separately

1. $\Omega(t) \equiv Id.$, in this case the solution have the form

   $q_k(t) = R(t)q_k(0)$ and is called a homothetic solution.

2. $R(t) \equiv 1$, in this case the solution have the form

   $q_k(t) = \Omega(t)q_k(0)$ and is called a relative equilibrium.

A relative equilibrium is a periodic orbit of (1.1), therefore in rotating coordinates it is a fixed point and we can study its stability. A well known result from geometry is the *Principal Axis Theorem*, which states that any rotation in $\mathbb{R}^3$ is around a fix axis. So without lose of generality we can assume that the rotation axis is the z–axis, then the matrix $\Omega(t)$ has the form

$$\Omega(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3.2. *The relative equilibria are planar solutions of the n–body problem.*

Proof. Consider the relative equilibrium

$$q_k(t) = \Omega(t) q_k(0), \tag{3.1}$$

taking the first derivative of (3.1) we get $\dot{q}_k(t) = \dot{\Omega}(t) q_k(0)$, by the form of the matrix Ω we conclude that the third component of the vectors q_k are constant. Taking the second derivative of equation (3.1) we obtain

$$\ddot{q}_k(t) = \ddot{\Omega}(t) q_k(0) = -\omega^2 \Omega(t) \hat{I} q_k(0),$$

where $\hat{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Last equation implies that the third component of each vector q_k is zero. This concludes the prove of the Proposition. □

By the above Proposition, in order to check the kind of motion for the relative equilibria, we can consider planar motions, so without loss of generality we consider the motion in $\mathbb{R}^2$ with

$$\Omega(t) = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}.$$

Then considering a planar motion, from equation (3.1) we have

$$\ddot{q}_k = -\omega^2 \Omega(t) q_k(0) = -\omega^2 q_k.$$

Which corresponds to a central configuration for all t with $\lambda = -\omega^2$. From here we obtain the value of the angular velocity in a relative equilibrium, $\omega = \pm\sqrt{U_0/2I_0}$. In the relative equilibrium there is a perfect balance between the centrifugal forces and the attractive ones. From the first properties of CC we know that $-\omega^2 = \lambda = \ddot{\rho}/\rho = \lambda_0/\rho^3$, that is $\omega^2\rho^3 = -\lambda_0 = U_0/2I_0$. This relationship corresponds to Kepler's third law.

Then we have that examples of CC are examples of relative equilibria, they are periodic solutions on the n-body problem. A relative equilibrium is a rest point in the space X/S^1.

The homographic solutions are more general solutions of the n–body problem, and in some sense the most natural, it combine rotations and homothecies of a central configurations, that is, in a homographic solution the configuration of the particles remains similar with itself for all time. We observe that $R(t)\Omega(t)$ is not a group at least that $R(t)$ is constant, and that any homographic solution which is not planar must

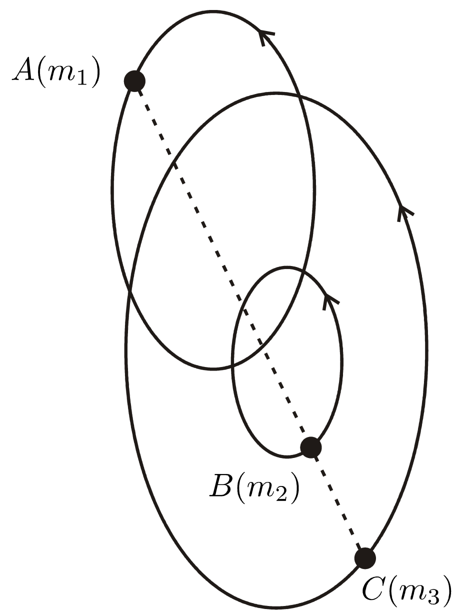

Figure 3.1: The Euler homographic solution

be a homothetic orbit. The history of the homographic solutions is very old, in 1767 L. Euler obtained a class of periodic solutions: given three arbitrary masses located initially in a central configuration, if suitable initial velocities are chosen then the masses will move periodically on ellipses maintaining the collinear configuration and the ratio among the distances (see Fig. 3.1), these were the first homographic solutions found in the history. Five years later, J.L. Lagrange rediscovered the Euler's solutions and found a new class of homographic solutions. If three arbitrary masses are located at the vertices of an equilateral triangle and if suitable initial velocities are taken, then the masses will move periodically on ellipses forming at any time an equilateral triangle (see Figure 3.2), these homographic solutions are called Langrange's solutions.

3.1 The Saari's conjecture

In late's 60's Donald G. Saari, was interested in the study of bounded motions in the n–body problem, so after a deep analysis of these kind of solutions, in 1969 postulates his famous conjecture which appears firstly in [95]: *In the Newtonian n–body problem, if along a solution the size of the configuration represented by the moment of inertia I is constant, then the solution is a relative equilibrium.* In other words: Newtonian

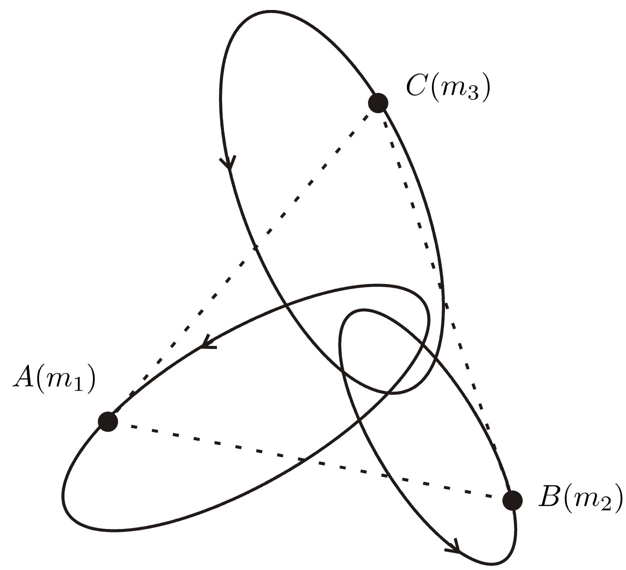

Figure 3.2: The Lagrange homographic solution

particle systems of constant moment of inertia behave like rigid bodies.

In [34], joint with F. Doacu, T. Fujiwara and M. Santoprete we do an extension of the above conjecture, that we explain briefly in these notes . Denoting by U to the potential energy of the system, the function $UI^{1/2}$ is a homogeneous function with degree of homogeneity equals zero, which means that it is invariant under contractions and expansions of the configuration, and since U and I depend only on mutual distances, $UI^{1/2}$ is also invariant under rotations, therefore this function measures how the shape of the configuration is changing, $UI^{1/2}$ is called *the configuration measure*. The configuration measure $UI^{1/2}$ is very useful for the analysis of the homographic motions. The extension of Saari's conjecture known as the homographic Saari's conjecture states that: *In the Newtonian n–body problem every solution with constant configuration measure is a homographic solution.*

The homographic Saari's conjecture is more natural since it provides a better understanding and new representations of the homographic solutions, which are the unique solutions of the n–body problem which can be described explicitly. The Saari's homographic conjecture is an extension of the previous one because if the moment of inertia is constant, then using the Lagrange–Jacobi identity of celestial mechanics is easy verify that the total energy H must be negative and that the potential U is also constant, in fact $U = -H$, therefore $UI^{1/2}$ is constant, in this case the homographic solutions correspond to relative equilibria. But of course, it is possible to have $UI^{1/2}$

constant with U and I no constants, so in this way Saari's homographic conjecture covers a huge class of orbits, the collision and escape orbits, and has sense for any value of the total energy H.

Now we shall define an important concept, which will play a main role in the rest of this paper.

Definition 3.3. *The configurational measure of the system (1.1) is given by $UI^{1/2}$.*

We observe that the configurational measure is a homogeneous function of degree zero, which means that it is independent of the size of the configuration, also since the potential U and the moment of inertia I can be written in terms of mutual distances, the configuration measure depends only on mutual distances and therefore it is invariant under rotations, so it measures how the shape of the configuration is changing in time. With all the above we can establish the first conjecture.

Conjecture[Saari's conjecture] *Any solution of the n–body problem with constant moment of inertia I, is a relative equilibrium.*

It is easy to check that if I is constant, then the potential energy is also constant and negative and also the kinetic energy T is constant, so the people thought that should be easy to prove this conjecture, in the early 70's there were many attempts to solve this conjecture, some of them led to the publication of incorrect proofs. In the late 70's the people were convinced that the proof of Saari's conjecture should be really difficult, and as usual in these cases the interest to prove this conjecture decay a lot until in the year 2000, A. Chenciner and R. Montgomery [26] gave a remarkable periodic solution of the three body problem, the figure eight solution which holds in the case of equal masses (see Figure 3.3). The moment of inertia for the figure eight solution is not exactly constant, but it is almost constant [99], so with this result the interest in Saari's conjecture grows exponentially. In 2002, C. McCord proved that the conjecture is true for the three body problem with equal masses [67] and in 2003 R. Moeckel obtained a computer assisted proof for the Newtonian three body problem for any values of the masses [75].

In these notes we will study the so called extended or homographic Saari's conjecture. **Conjecture** [Saari's homographic conjecture] *If in the n–body problem the configurational measure is constant, then the corresponding solution is homographic.*

As we have mentioned before, if the moment of inertia I is constant, then U is constant and therefore $UI^{a/2}$ is constant. In this case every homographic solution with constant moment of inertia is a relative equilibrium. In other words, Saari's conjecture is a particular case of Saari's homographic conjecture, so the results obtained in the second case also are true for the original Saari's conjecture, except of course when

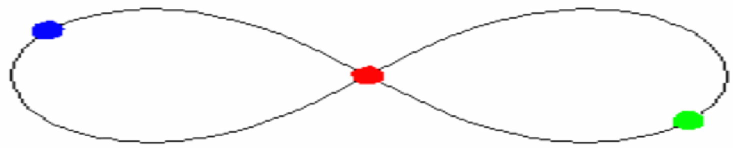

Figure 3.3: The figure eight solution

the value of the total energy $h \geq 0$ where it does not have sense because we have unbounded motions.

From here on, we assume that the configurational measure is constant, let μ the value of this constant

$$\mu = UI^{1/2}.$$

The Lagrange-Jacobi identity gives nice relationships between the second derivative of the moment of inertia and the constants of motion and it has the form:

$$\begin{aligned} \frac{d^2 I}{dt^2} &= 2\sum \frac{|\mathbf{p}_k|^2}{m_k} - 2U \\ &= 4H + 2U \\ &= 4H + 2\mu I^{-1/2} \end{aligned} \tag{3.2}$$

Integrating the Lagrange-Jacobi identity (3.2) we obtain

$$\frac{1}{2}\left(\frac{dI}{dt}\right)^2 + \left(-4HI - 4\mu I^{-1/2}\right) = -2B, \tag{3.3}$$

where B is a constant of integration. Let us write

$$\Phi(I) = -4HI - 4\mu I^{-1/2}.$$

Then the equation for the first integral (3.3), takes the form:

$$\frac{1}{2}\left(\frac{dI}{dt}\right)^2 + \Phi(I) = -2B. \tag{3.4}$$

We can check easily that the physical region for $\Phi(I)$ is $\Phi(I) \leq 0$ and that $B \geq 0$, where $B = 0$ if and only if the motion is homothetic. Using equation (3.4) we can study the evolution of the moment of inertia and the mutual distances among the particles for orbits with μ constant, in particular we can prove that any collision is a total collision and from here we obtain our first result:

Theorem 3.4. *Given n arbitrary masses, Saari's homographic conjecture is true in the n–body problem for collision orbits. That is, every collision solution in the n–body problem with constant configurational measure μ, is an homothetic orbit.*

For the collinear n–body problem we were also able to prove the Saari's homographic conjecture in the general case.

Theorem 3.5. *Given n arbitrary masses, Saari's homographic conjecture is true in the collinear n–body problem.*

A brief sketch of the proof is the following: we analyze two cases

- If the angular momentum C is zero, then the orbit necessary has a collision and therefore is a total collision orbit, then applying Theorem (3.4) we get the result.
- If $C \neq 0$, then all particles are rotating with constant angular velocity and the orbit correspond to a relative equilibrium.

In [33] you can find a complete proof of the above Theorem. On that paper we proved a more general result, that Theorem (3.5) is true when the particles are moving under the influence of potentials which only depend on the mutual distances among the particles.

Now we will introduce a new kind of coordinates, they will be the key tool for proving the Saari's conjecture for a huge set of initial conditions in the 3–body problem. We start restricting our analysis to planar motions, where we employ complex variables to express the dynamical variables on $\mathbb{R}^2$. Given (a_x, a_y), $(b_x, b_y) \in \mathbb{R}^2$, we write $a = a_x + ia_y$, $b = b_x + ib_y \in \mathbb{C}$. $a^\dagger = a_x - ia_y$ is the complex conjugate of a. $a \cdot b = a_x b_x + a_y b_y$ is the inner product and $a \wedge b = a_x b_y - +a_y b_x$ is the outer product. Then we have

$$a^{\dagger}b = a \cdot b + i a \wedge b. \tag{3.5}$$

Now we define the new coordinates:

Definition 3.6. *The Fujiwara coordinates $\{Q_k\}$ are given by*

$$Q_k = \exp\left(-iC\int_0^t \frac{dt}{I}\right)\frac{q_k}{\sqrt{I}}.$$

The Fujiwara coordinates are independent of size and of the angular momentum and therefore, in some sense, are the natural coordinates to study homographic motions, these coordinates will play a main role in our next results.

Using (3.5), we can check easily that the moment of inertia and the angular momentum in Fujiwara coordinates are constant and take the values 1 and 0 respectively, that is

$$\sum m_k |Q_k|^2 = 1, \quad C(Q) = \sum m_k Q_k \wedge \frac{dQ_k}{dt} = 0.$$

The next two Lemmas characterize the homographic solutions in Fujiwara coordinates, the first one in the general n–body problem and the second one for solutions with constant configuration measure.

Lemma 3.7. *A planar solution of the n–body problem in coordinates $q_k(t)$ is homographic if and only if $dQ_k(t)/dt = 0$ for all k and t.*

Is not difficult verify that

$$\sum m_k \left|\frac{dQ_k}{dt}\right|^2 = \frac{B - C^2}{I^2(q)},$$

and from here we obtain

Lemma 3.8. *Every planar solution of the n–body problem with constant configurational measure is homographic if and only if $B = C^2$.*

The equations of motion (1.1) in Fujiwara coordinates take the form

$$\begin{aligned} m_k \frac{d^2 Q_k}{dt^2} &= \frac{g_k(Q)}{I^{3/2}} - \left(\frac{1}{I}\frac{dI}{dt} + i\frac{2C}{I}\right) m_k \frac{dQ_k}{dt} \\ &- \left(\frac{1}{2I}\frac{d^2 I}{dt^2} - \frac{1}{4I^2}\left(\frac{dI}{dt}\right)^2 - \frac{C^2}{I^2}\right) m_k Q_k, \end{aligned} \tag{3.6}$$

where $g_k(Q) = \sum_{j\neq k} m_j m_k (Q_j - Q_k)/r_{jk}^3(Q)$ and $I = I(q)$.

In order to simplify our equations we re-scale the time variable and define the new moment (the Fujiwara moment) by

$$d\tau = \frac{dt}{I(q)}, \quad \text{and} \quad P_k = m_k \frac{dQ_k}{d\tau} = I(q) m_k \frac{dQ_k}{dt}.$$

Using that $\mu =$ constant and the new moment, after some straightforward computations the equations of motion (3.6) take the form

$$\frac{dP_k}{d\tau} = -2iCP_k + I(q)^{1/2} G_k(Q) - (B - C^2) m_k Q_k, \tag{3.7}$$

where $G_k(Q) = g_k(Q) + \mu m_k Q_k$. The kinetic energy becomes

$$\sum \frac{|P_k|^2}{m_k} = B - C^2 = \text{constant},$$

We observe that all the analysis and results given above are true for the general n–body problem, from here on we will restrict our analysis to the case of the 3–body problem. This is due to the next Lemma which is essential for the next results, but unfortunately it only holds in the case $n = 3$.

Lemma 3.9. *If six given complex quantities q_k and $p_k, k = 1, 2, 3,$ satisfy the properties:*

1. $\sum p_k = 0,$
2. $\sum q_k^\dagger p_k = 0,$
3. $\sum m_j m_k |q_j - q_k|^2 > 0,$

then, there is a complex number ζ such that $p_\ell = \zeta(q_j^\dagger - q_k^\dagger)$ where (j, k, ℓ) permutes cyclically in $(1, 2, 3)$.

The above Lemma means that in the 3–body problem, if we fix the center of mass at the origin (property 1), if the angular momentum is zero (property 2) and if the moment of inertia is constant and no triple collision occurs (property 3), then the triangle with vertices at q_1, q_2, q_3 is inversely similar to the triangle whose perimeters are p_1, p_2, p_3

Since $\sum Q_k^\dagger P_k = 0$, $\sum P_k = 0$, and $I(Q) = 1$, we can apply the above Lemma to the variables Q_k and P_k, getting that there exist a variable $\kappa \geq 0$ and a real variable ϕ such that

$$P_\ell = \kappa e^{i\phi}(Q_j^\dagger - Q_k^\dagger),$$

From the fact that

$$\sum \frac{|P_\ell^2}{m_\ell} = \frac{\kappa^2}{m_1 m_2 m_3} \sum m_j m_k |Q_j - Q_k|^2 = \frac{M\kappa^2}{m_1 m_2 m_3},$$

we can obtain the value for κ, which turns out to be constant

$$\kappa = \sqrt{\frac{m_1 m_2 m_3}{M} \sum \frac{|P_\ell^2}{m_\ell}} = \sqrt{\frac{m_1 m_2 m_3 (B - C^2)}{M}}.$$

It is easy to check that Q_k and G_k also satisfy

$$\sum Q_k^\dagger G_k(Q) = 0.$$

Then, applying again the Lemma (3.9), there exist $\rho \geq 0$ and a real variable ψ such that

$$G_\ell(Q) = \rho e^{i\psi}(Q_j^\dagger - Q_k^\dagger),$$

$$\rho = \sqrt{\frac{m_1 m_2 m_3}{M} \sum \frac{|G_\ell^2}{m_\ell}}.$$

Combining the above two results, and assuming $\rho \neq 0$, we obtain the following relationship between P_k and G_k

$$P_k = \frac{\kappa}{\rho} e^{i(\phi - \psi)} G_k.$$

Let us observe that the scale factor κ/ρ and the phase factor $\exp i(\phi - \psi)$ are the same for all $k = 1, 2, 3$. Now, coming back to the condition

$$\mu = U(q) I(q)^{a/2} = U(Q) = \text{constant},$$

differentiation with respect to τ yields

$$\begin{aligned}
0 = -\frac{dU(Q)}{d\tau} &= \sum \frac{1}{m_k} P_k \cdot g_k(Q) = \sum \frac{1}{m_k} P_k \cdot G_k(Q) \\
&= \frac{\kappa}{\rho} \sum \frac{|G_k(Q)|^2}{m_k} \cos(\phi - \psi) \\
&= \sqrt{(B - C^2) \sum \frac{|G_k(Q)|^2}{m_k}} \cos(\phi - \psi).
\end{aligned}$$

If the orbit **is not homographic**, then $B \neq C^2$ and $G_k \neq 0$, therefore $\cos(\phi - \psi) = 0$. From here we conclude that

$$P_k = i\epsilon \frac{\kappa}{\rho} G_k, \text{ with } \quad \epsilon = \pm 1,\, \kappa \neq 0,\, G_k \neq 0. \tag{3.8}$$

The above equation shows that there are only two possible values for the Fujiwara momentum P_k and $-P_k$, in order to have a possible non-homographic solution with constant configuration measure. The next step in the proof of Saari's conjecture is to introduce the above values of P_k in the equations of motion and get a contradiction, in this way we differentiate (3.8) with respect to τ getting

$$\begin{aligned} \frac{dP_\ell}{d\tau} &= \kappa e^{i\phi} \left\{ i\frac{d\phi}{d\tau}(Q_j^\dagger - Q_k^\dagger) + \left(\frac{P_j^\dagger}{m_j} - \frac{P_k^\dagger}{m_k} \right) \right\} \\ &= i\frac{d\phi}{d\tau} P_\ell - (B - C^2) m_\ell Q_\ell. \end{aligned} \tag{3.9}$$

Using the equations of motion in Fujiwara coordinates (3.7) we obtain

$$G_k \left(\epsilon \frac{d\phi}{d\tau} + 2\epsilon C + \frac{\rho}{\kappa} I^{1/2} \right) \frac{\kappa}{\rho} = 0. \tag{3.10}$$

Since we are analyzing the equations of motion with $G_k \neq 0$ and $\kappa \neq 0$, the necessary and sufficient condition for the candidate Fujiwara momenta P_k to satisfy the equations of motion is

$$\epsilon \frac{d\phi}{d\tau} + 2\epsilon C + \frac{\rho}{\kappa} I^{1/2} = 0. \tag{3.11}$$

After many tedious computations we obtain

$$e^{i\phi} = i\epsilon \frac{m_1 m_2 m_3}{\rho} \sum \frac{1}{r_{jk}^3}(Q_j - Q_k) Q_\ell$$

$$\begin{aligned} \epsilon \frac{d\phi}{d\tau} &= m_1 m_2 m_3 e^{-i\phi} \frac{d}{d\tau} \left(\frac{1}{\rho} \sum \frac{1}{r_{jk}^3}(Q_j - Q_k) Q_\ell \right) \\ &= 3 \frac{m_1^2 m_2^2 m_3^2 \Delta^2}{M} \frac{\kappa}{\rho^3} \sum \frac{m_\ell}{r_{jk}^5} \left(\frac{1}{r_{k\ell}^3} - \frac{1}{r_{\ell j}^3} \right)^2 \\ &+ \frac{\kappa}{\rho} \left(2\mu - \sum \frac{m_j + m_k}{r_{jk}^3} \right), \end{aligned}$$

where Δ is twice the oriented area of the triangle $Q_1Q_2Q_3$ that is, $\Delta = Q_1 \wedge Q_2 + Q_2 \wedge Q_3 + Q_3 \wedge Q_1$.

The condition (3.11) becomes

$$\begin{aligned} -I^{1/2}(q) &= \frac{m_1^2 m_2^2 m_3^2}{M^2}(B - C^2)\left(\frac{f_1}{\rho^4} + \frac{f_2}{\rho^2}\right) \\ &+ 2\epsilon C\sqrt{\frac{m_1 m_2 m_3}{M}(B - C^2)}\,\frac{1}{\rho}, \end{aligned} \tag{3.12}$$

with

$$f_1 = 3m_1m_2m_3\Delta^2 \sum \frac{m_\ell}{r_{jk}^5}\left(\frac{1}{r_{k\ell}^3} - \frac{1}{r_{\ell j}^3}\right)^2$$

$$f_2 = \frac{M}{m_1m_2m_3}\left(2\mu - \sum \frac{m_j + m_k}{r_{jk}^3}\right).$$

After a deep analysis of equation (3.12) and using strongly that the left hand side of (3.12) depends only on q whereas the right side depends only on Q, we obtain a contradiction for two huge classes of orbits.

Theorem 3.10. *The homographic Saari's conjecture is true for the planar 3–body problem if the initial configuration is not a central configuration, and for which $B - C^2 > 0$ is small enough.*

Theorem 3.11. *The homographic Saari's conjecture is true for the planar 3–body problem if the initial configuration is close enough to an equilateral triangle.*

In the non-negative energy case we have to study carefully the behavior of the escapes; because if the particles escape by tending to a collinear configuration, then the area of the respective configuration tends to zero and the analysis of equation (3.12) in order to obtain a contradiction for the Fujiwara momenta P_k is much more difficult. Since there are just three collinear central configurations let us denote them by (m_3, m_1, m_2), (m_1, m_2, m_3), and (m_2, m_3, m_1). Each case corresponds to a certain value of the configurational measure; we call them the critical values of μ and denote them by $\mu_c^{(1)}$, $\mu_c^{(2)}$, and $\mu_c^{(3)}$.

For each $k = 1, 2, 3$, $\mu(Q) = \mu_c^{(k)}$ defines solutions that pass through the rectilinear central configuration k, we call such solutions critical paths, then after many computations we obtain.

Theorem 3.12. *Saari's homographic conjecture is true if the initial configuration is not on any of the critical paths* $\mu = \mu_c^{(k)}$ *and* $0 < a < 2$.

For the special case when all masses are equal, we were able to solve the technical difficulties carried in the asymptotic analysis of equation (3.12) getting

Theorem 3.13. *In the Newtonian equal mass case of the three body problem, Saari's homographic conjecture is true for all non-negative values of the energy.*

Chapter 4

The charged n–body problem

The charged n–body problem is the n body problem when in addition with the mass, any point particle is endowed of an electrostatic charge $e_i \in \mathbb{R}$ that is, the charge can have any sign and the particles are moving now under the influence of Newtonian and Coulombian forces, in this case the equations of motion are given by

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \tag{4.1}$$

where the potential U takes the form

$$U = \sum_{i<j} \frac{m_i m_j - e_i e_j}{r_{ij}}, \tag{4.2}$$

defined on the same configuration space X.

Let us observe that in this case the potential given by (4.2) can have any sign depending on the parameters, this is the main difference with the potential for the classical n–body problem. The definition for the central configurations in charged problems is the same (see Definition 1.5), so we are interested in the critical points of the potential U given in equation (4.2) restricted to the ellipsoid S given by the mass matrix $M = diag\{m_1, m_1, m_1, m_2, m_2, m_2, \cdots, m_n, m_n, m_n\}$.

4.1 The charged three body problem

For the charged three body problem, the potential U takes the form

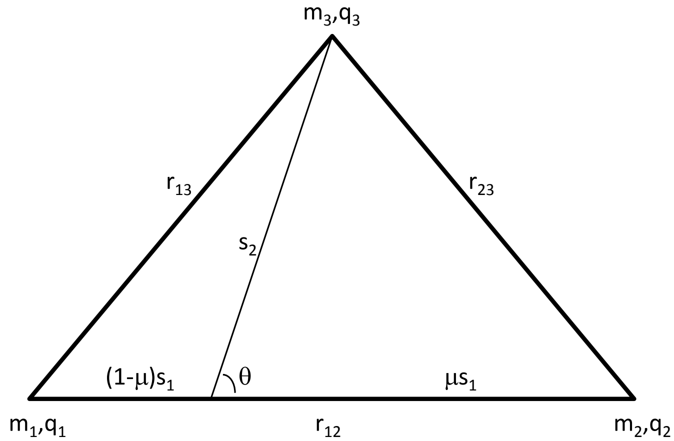

Figure 4.1: The Jacobi coordinate system J_{12}

$$U = \frac{m_1m_2 - e_1e_2}{r_{12}} + \frac{m_2m_3 - e_2e_3}{r_{23}} + \frac{m_1m_3 - e_1e_3}{r_{13}} = \frac{\lambda_3}{r_{12}} + \frac{\lambda_1}{r_{23}} + \frac{\lambda_2}{r_{13}},$$

where $\lambda_i = m_jm_k - e_je_k$ with (i, j, k) permuting cyclically in (1,2,3).

As in the Newtonian three body problem, $\vec{r} = (q_1, q_2, q_3) \in (R^3)^3$ is a CC if and only if

$$\bigtriangledown(IU^2) = 0 \text{ at } \vec{r}.$$

Since the center of mass is fixed at the origin, we can introduce Jacobi's coordinates to reduce the number of independent variables.

Let

$$\begin{aligned} \vec{s}_1 &= q_2 - q_1, \\ \vec{s}_2 &= q_3 - \frac{m_1q_1 + m_2q_2}{m_1 + m_2}. \end{aligned}$$

Using the above relationship between $\vec{s}_1, \vec{s}_2$ and q_1, q_2, q_3 and the fact that the center of mass is fixed at the origin, that is $m_1q_1 + m_2q_2 + m_3q_3 = 0$, we have (see Fig. 4.1)

$$\begin{aligned} q_1 &= -\frac{m_2}{m_1+m_2}\vec{s}_1 - \frac{m_3}{m_1+m_2+m_3}\vec{s}_2, \\ q_2 &= \frac{m_1}{m_1+m_2}\vec{s}_1 - \frac{m_3}{m_1+m_2+m_3}\vec{s}_2, \\ q_3 &= \frac{m_1+m_2}{m_1+m_2+m_3}\vec{s}_2. \end{aligned}$$

Expressing the potential U and the moment of inertia I in terms of $|\vec{s}_1| = s_1$ and $|\vec{s}_2| = s_2$ we have:

$$U = \frac{\lambda_3}{s_1} + \frac{\lambda_1}{\sqrt{s_2^2 + (\mu s_1)^2 - 2\mu s_1 s_2 \cos\theta}} + \frac{\lambda_2}{\sqrt{s_2^2 + ((1-\mu)s_1)^2 + 2(1-\mu)s_1 s_2 \cos\theta}},$$

$$I = g_1 s_1^2 + g_2 s_2^2.$$

Where θ is the angle between the vectors $\vec{s}_1$ and $\vec{s}_2$ and

$$\mu = \frac{m_1}{m_1+m_2}, \qquad d_1 = \frac{m_1 m_2}{m_1+m_2}, \qquad d_2 = \frac{m_3(m_1+m_2)}{m_1+m_2+m_3}.$$

So we have

$$\frac{1}{d_2} I U^2 = (\beta + \rho^2)\left(\lambda_3 + \frac{\lambda_1}{\sqrt{\rho^2 + \mu^2 - 2\mu\rho\cos\theta}} + \frac{\lambda_2}{\sqrt{\rho^2 + (1-\mu)^2 + 2(1-\mu)\rho\cos\theta}}\right)^2,$$

where $\delta = d_1/d_2$ and $\rho = s_2/s_1$.

Now that IU^2 is expressed in terms of the parameters $\lambda_1, \lambda_2, \lambda_3, \beta, \mu$ and the two independent variables ρ, and θ, the formula for the central configurations of the charged three body problem is

$$\begin{aligned} \frac{\partial}{\partial \rho}\left(\frac{1}{d_2} I U^2\right) &= 0, \\ \frac{\partial}{\partial \theta}\left(\frac{1}{d_2} I U^2\right) &= 0\ . \end{aligned} \tag{4.3}$$

4.1.1 Symmetries

The Jacobi System of the previous section, where the base vector is $\vec{s_1} = q_2 - q_1$, is denoted by J_{12} (see Fig. 4.1). Denote the selected Jacobi System, where $\vec{s_1} = q_3 - q_1$ is the base vector, and $\vec{s_2}$ is the vector which goes from mass center of masses m_1 and m_3 to m_2, as J_{13}. The remain Jacobi System is denoted by J_{21}. Because there exists an one to one correspondence between these Jacobi Systems, all our analysis is in the Jacobi System J_{12}. An analysis of the collinear central configurations (CCC) in the Jacobi System J_{12} reduces to whether particle m_3, is between the other two, or exterior to them.

Let p_i be the position on a fixed line of particle m_i for $i = 1, 2, 3$. We will see that it is enough to study the case where p_1, p_2 and p_3 are aligned in this order, which will be called Case I. In the system J_{12} this corresponds to $\theta = 0, \rho > \mu$. The other possible CCC correspond to the alignments p_3, p_1, p_2 (Case II), and p_1, p_3, p_2 (Case III).

For Case II, in system J_{12} we have that $\theta = \pi$ and $\rho > 1 - \mu$. Which corresponds to study the Case I in the system J_{21}. To express the potential U in these variables it is necessary to introduce the changes $\bar{m}_1 = m_2,\ \bar{m}_2 = m_1,\ \bar{m}_3 = m_3,\ \bar{e}_1 = e_2,\ \bar{e}_2 = e_1,\ \bar{e}_3 = e_3$, and with these changes we have $\bar{\lambda}_1 = \lambda_2,\ \bar{\lambda}_2 = \lambda_1,\ \bar{\lambda}_3 = \lambda_3$.

For the Case III, in the System J_{12}, we have two possibilities

- $\theta = 0, \qquad \rho < \mu \ ,$
- $\theta = \pi, \qquad \rho < 1 - \mu \ .$

Both possibilities correspond to study the Case I in the system J_{13}, where now the respective changes are $\hat{m}_1 = m_1,\ \hat{m}_2 = m_3,\ \hat{m}_3 = m_2, \hat{e}_1 = e_1,\ \hat{e}_2 = e_3,\ \hat{e}_3 = e_2$, and with these changes we have $\hat{\lambda}_1 = \lambda_1,\ \hat{\lambda}_2 = \lambda_3,\ \hat{\lambda}_3 = \lambda_2$.

4.1.2 Noncollinear Central Configurations

We start our analysis studying the planar or noncollinear CC. Let R_{ij} be the relative distance between p_i and p_j with respect to $s_1 = r_{12}$, *i.e.*

$$R_{13} = \frac{r_{13}}{s_1}, \qquad R_{23} = \frac{r_{23}}{s_1}, \qquad R_{12} = \frac{r_{12}}{s_1} = 1.$$

So

$$\frac{1}{d_2} I U^2 = (\beta + \rho^2) \left[\lambda_3 + \frac{\lambda_1}{R_{23}} + \frac{\lambda_2}{R_{13}} \right]^2 . \tag{4.4}$$

From system 4.3, the central configurations must satisfy

$$\rho\left(\lambda_3+\frac{\lambda_1}{R_{23}}+\frac{\lambda_2}{R_{13}}\right)+(\rho^2+\beta)\left[-\frac{\lambda_1(\rho-\mu\cos\theta)}{R_{23}^3}-\frac{\lambda_2(\rho+(1-\mu)\cos\theta)}{R_{13}^3}\right]=0,$$
$$-\frac{\lambda_1\rho\mu\sin\theta}{R_{23}^3}+\frac{\lambda_2(\rho(1-\mu)\sin\theta)}{R_{13}^3}=0. \tag{4.5}$$

Solving the second equation in (4.5), we have

$$\rho\sin\theta=0 \qquad \text{or} \qquad \frac{m_2\lambda_2}{R_{13}^3}=\frac{m_1\lambda_1}{R_{23}^3}.$$

In the case of $\rho\sin\theta=0$, we have $\rho=0$ or $\sin\theta=0$ which corresponds to collinear CC. We will study this case later.

Now consider the second case and let

$$\frac{m_2\lambda_2}{R_{13}^3}=\frac{m_1\lambda_1}{R_{23}^3}=z^{-3}.$$

Then we can solve for R_{13} and R_{23} and substitute them into the first equation of (4.5), obtaining

$$z^3=\frac{1}{m_3\lambda_3}.$$

This means that the unique solution is

$$\frac{m_1\lambda_1}{R_{23}^3}=\frac{m_2\lambda_2}{R_{13}^3}=\frac{m_3\lambda_3}{R_{12}^3}=m_3\lambda_3. \tag{4.6}$$

Only when $\lambda_1,\lambda_2,\lambda_3$, have the same sign does this solution define an unique central configuration. Physically this means that all charges have the same polarity. Using the above relation we can check that this central configuration is a triangle with one body at each vertex if the parameters satisfy the constraint

$$(m_i\lambda_i)^{1/3}+(m_j\lambda_j)^{1/3}>(m_k\lambda_k)^{1/3}, \tag{4.7}$$

where (i,j,k) permutes cyclically in $(1,2,3)$. Note that this triangle could be of any shape depending on the choice of the parameters.

Remark 4.1. *We observe that if all λ_i have the same sign and the constraint (4.7) holds, then we can always construct the relation (4.6), which guaranties the existence of non-collinear CC.*

Observe that in the Jacobi System J_{12}, if the angle θ is replaced by $-\theta$, we get the same result about the existence of non-collinear CC. This just reflets the symmetry of the potential function U,

$$U(s_1, s_2, \theta) = U(s_1, s_2, -\theta).$$

Because three points in R^3 define a plane, in order to count the total number of classes of planar CC, we only consider rotations in the plane. Therefore the existence of one non-collinear CC implies the existence of the symmetric one, which must be considered different because is not obtained by a planar rotation of the original one. One special case is when all charges are zero. Then $m_1\lambda_1 = m_2\lambda_2 = m_3\lambda_3 = m_1m_2m_3$. The solution will be

$$\frac{1}{R_{23}^3} = \frac{1}{R_{13}^3} = \frac{1}{R_{12}^3} = 1,$$

or

$$R_{12} = R_{13} = R_{23} = 1,$$

we observe that in this case the constraint 4.7 holds in a trivial way, so here we get the Lagrange's configurations for the classical Newtonian 3-body problem. This is a big difference with the classical 3–body problem, where the planar CC are the just the equilateral triangles. In the charged problem, choosing adequately the parameters any triangle shape could be a CC.

If any two of $\lambda_1, \lambda_2, \lambda_3$ have different signs, then the solution does not define any configuration, which means that non-collinear CC does not exist. In this way, we have shown that the total number of non-collinear CC is either 0 or 2, according to the values of the parameters (or more precisely, the signs of some parameters must satisfy the conditions (4.6) and (4.7).

4.1.3 Collinear Central Configurations

We have seen that the two possible conditions in order to have collinear central configurations (CCC) are $\rho = 0$ and $\sin\theta = 0$. We now study them in detail.

- The condition $\rho = 0$.

  Under this hypothesis, the solution for all θ of equation 4.5 becomes

$$\frac{\lambda_1}{\mu^2} = \frac{\lambda_2}{(1-\mu)^2} \tag{4.8}$$

Thus, only when the parameters satisfy (4.8) it is possible to have a CC with m_3 sitting at the center of masses of the particles with masses m_1 and m_2. This is intuitive because equation (4.8) requires the attracting force on m_3 by m_2 (repelling force if λ_1 is negative) to be equal to the attracting force on m_3 by m_1 , so m_3 can remains at the mass center forever. This corresponds to case III in our symmetries.

- The condition $\sin\theta = 0$.

  Here we have again two possibilities $\theta = 0$ and $\theta = \pi$.

  For $\theta = 0$, we only need to consider $\rho > \mu$, the other possibility $0 < \rho < \mu$ corresponds to case III. Similarly $\theta = \pi$, causes $0 < \rho < 1 - \mu$ or $\rho > 1 - \mu$ which correspond to case III and case II respectively.

We start analyzing the case $\theta = 0, \quad \rho > \mu$ (case I), when $\lambda_i \neq 0, i = 1, 2, 3$. Using $r_{23} = s_2 - \mu s_1$, the first equation of (4.5) becomes

$$F(\rho) = \lambda_3\rho - \frac{\lambda_1\mu(\rho + \frac{\beta}{\mu})}{(\rho-\mu)^2} + \frac{\lambda_2(1-\mu)(\rho + \frac{\beta}{1-\mu})}{(\rho + 1 - \mu)^2} = 0. \tag{4.9}$$

To simplify this function, we introduce new parameters. Let

$$\begin{aligned} \alpha_1 &= m_1\lambda_1, \\ \alpha_2 &= m_2\lambda_2, \\ \alpha_3 &= (m_1 + m_2)\lambda_3, \end{aligned}$$

and

$$\begin{aligned} \beta_1 &= \tfrac{\alpha_1}{\alpha_2}, \\ \beta_3 &= \tfrac{\alpha_3}{\alpha_2}. \end{aligned}$$

Also let $x = \rho - \mu$ so in this case, $x > 0$ With these new parameters, we have

$$F(\rho) = \frac{f(x)\alpha_2}{m_1 + m_2},$$

where

$$f(x) = \beta_3 x + \beta_3\mu - \frac{\beta_1(x + 1 + \frac{m_2}{m_3})}{x^2} + \frac{x - \frac{m_1}{m_3}}{(x+1)^2}. \tag{4.10}$$

By choosing m_3 as the unit of mass (so $m_3 = 1$), the above equation becomes

$$f(x) = \beta_3 f_3(x) - \beta_1 f_1(x) + f_2(x) = 0, \tag{4.11}$$

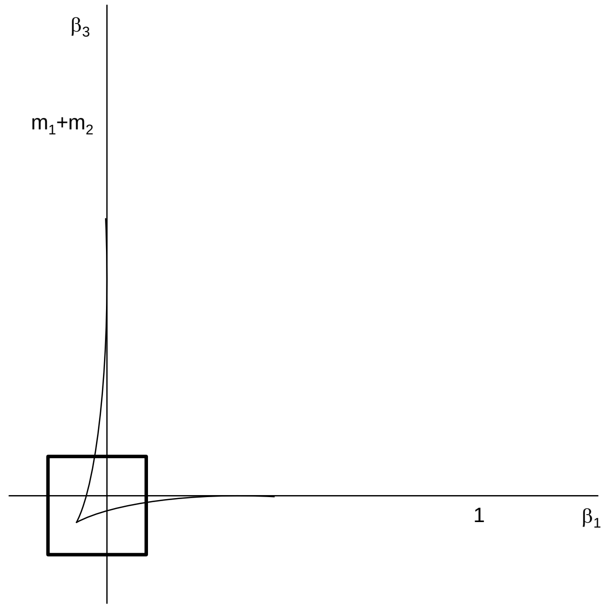

Figure 4.2: The bifurcation curve for $m_1 = m_2 = m_3 = 1$

where

$$f_3(x) = x + \frac{m_1}{m_1 + m_2} > 0,$$

$$f_1(x) = \frac{x + 1 + m_2}{x^2} > 0,$$

$$f_2(x) = \frac{x - m_1}{(x+1)^2}.$$

From (4.11) we have

$$\beta_3 = \beta_1 g_1(x) + g_2(x), \tag{4.12}$$

where

$$g_1(x) = \frac{f_1(x)}{f_3(x)},$$

$$g_2(x) = -\frac{f_2(x)}{f_3(x)}.$$

In the $\beta_1\beta_3$ plane equation (4.12) is the equation of a straight line for fixed x and fixed masses. Here $g_1(x)$ is the slope and $g_2(x)$ is the β_3–intercept. For any fixed point (β_1, β_3), the number of lines passing through this point when x is moving from zero to infinity, determines the number of CCC for that particular set of parameters.

The next step is to examine how the line changes as x varies from 0 to $+\infty$. First of all,

$$g_1'(x) = -\frac{2x^2 + [3(m_2+1) + \frac{m_1}{m_1+m_2}]x + \frac{2m_1(m_2+1)}{m_1+m_2}}{x^3(x + \frac{m_1}{m_1+m_2})^2} < 0,$$

and $g_1(0) = +\infty, g_1(+\infty) = 0$. So the slope of the line monotonically decreases from $+\infty$ to 0 as x goes from 0 to $+\infty$.

Secondly,

$$g_2'(x) = -\frac{-2x^2 + (3m_1 - \frac{m_1}{m_1+m_2})x + \frac{m_1(3m_1+m_2)}{m_1+m_2} + \frac{m_1}{m_1+m_2}}{(x+1)^3(x + \frac{m_1}{m_1+m_2})^2},$$

and $g_2(0) = m_1 + m_2, \quad g_2(+\infty) = 0$. So we have

$$\begin{cases} g_2'(x) & < 0 \text{ when } 0 < x < x_+, \\ g_2'(x) & > 0 \text{ when } x_+ < x, \end{cases}$$

where

$$x_+ = \frac{(3m_1 - \frac{m_1}{m_1+m_2}) + \sqrt{(3m_1 - \frac{m_1}{m_1+m_2})^2 + 8(\frac{m_1(3m_1+m_2)}{m_1+m_2} + \frac{m_1}{m_1+m_2})}}{4}.$$

So the β_3–intercept of the line decreases from $m_1 + m_2$ when $0 < x < x_+$ and then increases to 0 when $x > x_+$.

Based on the analysis of $g_1(x)$ and $g_2(x)$, we can sketch in the $\beta_1\beta_3$ plane the changes of the line as x goes from 0 to $+\infty$. Thus we get some regions in the $\beta_1\beta_3$ plane and in each region the number of times the line passes through any point in that region is the same, *i.e.* the number of CCC remains fixed (see Fig.4.3).

We can also determine the bifurcation curve between regions computing the envelope of the family of lines given by the equation 4.12. For this, we derivate 4.12 with respect to x, to obtain the parametric form of the bifurcation curve

$$\begin{aligned} \beta_1(x) &= -\frac{g_2'(x)}{g_1'(x)}, \\ \beta_3(x) &= -\frac{g_2'(x)}{g_1'(x)} g_1(x) + g_2(x), \end{aligned} \tag{4.13}$$

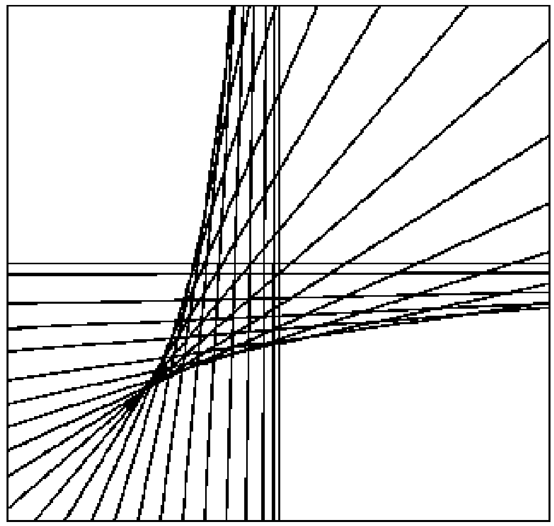

Figure 4.3: Detail of the family of lines associated to the bifurcation curve for $m_1 = m_2 = m_3 = 1$

which is a piecewise smooth curve. It is easy to check that $\lim_{x\to\infty} \beta_1(x) = 1$ and $\lim_{x\to\infty} \beta_3(x) = 0$. The bifurcation curve given by (4.13), and the coordinate axis in the $\beta_1\beta_3$ plane determine the regions where we obtain the same number of CCC, this number could be 0,1,2 or 3 (see Fig. 4.4).

In order to determine the total number of CCC we need to work with different Jacobi systems. The key point is to use the bifurcation curve (Eq. 4.13) in different Jacobi system, in this way we need that all the parameters $\lambda_i \neq 0, \quad i = 1, 2, 3$. In order to point the relation between two different Jacobi systems, we introduce the rate charge-mass $\delta_i = \frac{q_i}{m_i}, i = 1, 2, 3$. So

$$\lambda_i = m_j m_k (1 - \delta_j \delta_k) \text{ where } (i, j, k) \text{ permutes cyclically in } (1,2,3).$$

With this notation

$$\beta_1 = \frac{1 - \delta_2\delta_3}{1 - \delta_1\delta_3}\ , \qquad \beta_3 = \left(\frac{m_1 + m_2}{m_3}\right) \frac{1 - \delta_1\delta_2}{1 - \delta_1\delta_3}. \tag{4.14}$$

We denote by $\bar{\beta}_1$ and $\bar{\beta}_3$ the analogous variables in the Jacobi system J_{21} (case II), and by $\hat{\beta}_1$ and $\hat{\beta}_3$ the respective ones in the Jacobi system J_{13} (case III). By the

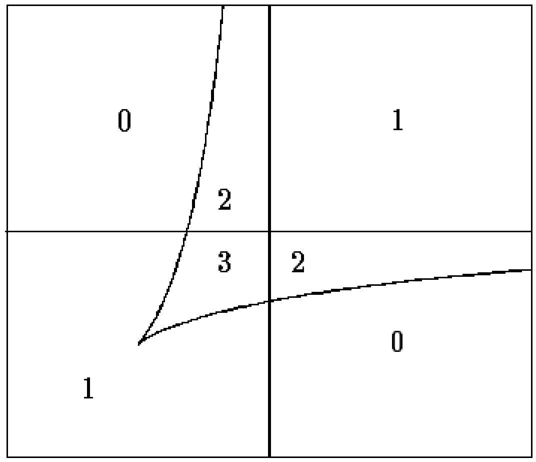

Figure 4.4: Different regions according to the number of CCC

symmetries studied previously, we get

$$\bar{\beta}_1 = \frac{1}{\beta_1}, \qquad \hat{\beta}_1 = \left(\frac{m_1+m_2}{m_3}\right)\frac{\beta_1}{\beta_3}, \tag{4.15}$$

$$\bar{\beta}_3 = \frac{\beta_3}{\beta_1}, \qquad \hat{\beta}_3 = \frac{(m_1+m_2)(m_1+m_3)}{m_2 m_3}\frac{1}{\beta_3}. \tag{4.16}$$

We observe that in order to get the $\bar{\beta}$s and $\hat{\beta}$s we need $\lambda_i \neq 0$ for $i = 1, 2, 3$. Then in order to determine the total number of CCC, we must to analyze two casses depending if $\lambda_1\lambda_2\lambda_3 \neq 0$ or not. The second case is simpler, since the potential function U reduces a lot if some of the parameters are zero. For instance, if all $\lambda_i = 0$, the potential $U \equiv 0$ and we do not have any CCC. If two of the $\lambda' s$ are zero, the charged three body problem reduces to a Kepler problem, attractive or repulsive, depending of the sign of the $\lambda_i \neq 0$, so in this case we neither have any CCC. If just one $\lambda_i = 0$, the corresponding bifurcation diagrams are much simpler that when all $\lambda' s$ are different from zero. Using the symmetries it is not difficult to show that in this case we can have $0, 1, 2$ or 3 CCC.

If all $\lambda' s$ do not vanish, then according to the bifurcation diagram plotted in Fig. 4.4, in principle we can have up to 3 different CCC for each ordering of the particles, so in principle we can obtain up to 9 different CCC, but doing a deeper analysis, it is possible to prove that if you are in a region with 3 different CCC in a certain Jacobian coordinate system, then when change the ordering of the particles, it is impossible to obtain again 3 different CCC (see [83] for a complete deduction of the total number of CCC in the charged three body problem).

We summarize this Section with the count of all possible CC in the charged three body problem.

Theorem 4.2. *Given $m_1, m_2, m_3 \in R^+$ and $e_1, e_2, e_3 \in R$ we can construct the respective $\lambda_i = m_j m_k - e_j e_k$, with $i = 1, 2, 3$ permuting cyclically in $(1, 2, 3)$. So*

1. *If at least two λ_i vanish, we have 0 CC.*
2. *If just one λ_i vanishes, then we can have 0,1,2 or 3 CC.*
3. *If all the λ_i have the same sign, and $(m_i\lambda_i)^{1/3} + (m_j\lambda_j)^{1/3} > (m_k\lambda_k)^{1/3}$ then we have exactly 5 CC (2 non-collinear and 3 collinear), in the other cases we only have the three collinear CC.*
4. *If the λ_i's do not vanish and do not have the same sign, then we can have 1, 2, 3, 4 or 5 CC (all of them are collinear).*

4.2 Relative equilibria in charged problems

In order to study relative equilibria, we use rotating coordinates. Given

$$\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n) \in \Omega$$

$$\mathbf{p} = M\dot{\mathbf{r}} \qquad M = diag\{m_1, m_1, m_1, \cdots, m_n, m_n, m_n\}$$

Let

$$\begin{aligned} x &= R\,\mathbf{r}\,, \\ y &= R\,\mathbf{p}\,, \end{aligned}$$

where R is the $3n \times 3n$ block diagonal matrix, with blocks given by the rotation matrix

$$A(wt) = \begin{pmatrix} \cos wt & -\sin wt & 0 \\ \sin wt & \cos wt & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where w is the constant angular velocity.

The equations of motion in rotating coordinates take the form

$$\begin{aligned} \dot{x} &= K\,x + M^{-1}y, \\ \dot{y} &= \nabla U(x) + K\,y, \end{aligned} \tag{4.17}$$

where K is a $3n \times 3n$ block diagonal matrix, with blocks of the form

$$\begin{pmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The equilibrium points of (4.17) are given by

$$\{(x,y) \mid M^{-1}\nabla U(x) + w^2 E\, x = 0, \quad y = -M\, K\, x\}, \tag{4.18}$$

where E is the block diagonal matrix, with blocks

$$I_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With the above we obtain an equivalent definition of a realtive equilibria.

Definition 4.3. *A configuration $x \in \Omega$ is called a relative equilibrium if it satisfies the equation*

$$M^{-1}\nabla\, U(x) + w^2 E\, x = 0\,. \tag{4.19}$$

Multiplying the equation (4.19) by the factor $x^t M$, and using Euler's theorem for homogeneous functions we get

$$w^2 = \frac{U(x)}{x^t M E x}. \tag{4.20}$$

Remark 4.4. *A C.C. generates a relative equilibrium if $U(x) > 0$.*

4.3 Some curiosities

The following two results give us examples of continua of CC, the basic idea to prove the next Theorems is to play with the parameters, for especial choice of them we get the results (see [6] for a complete proof of these facts).

Theorem 4.5. *In the rhomboidal charged problem, generically there are $0, 1$ or 2 CC, for the special case $m_1 = m_2 = m_3 = m_4 = m$ and charges $e_1 = e_2 = m, e_3 = e_4 = -m$, there is a continuum of central configurations.*

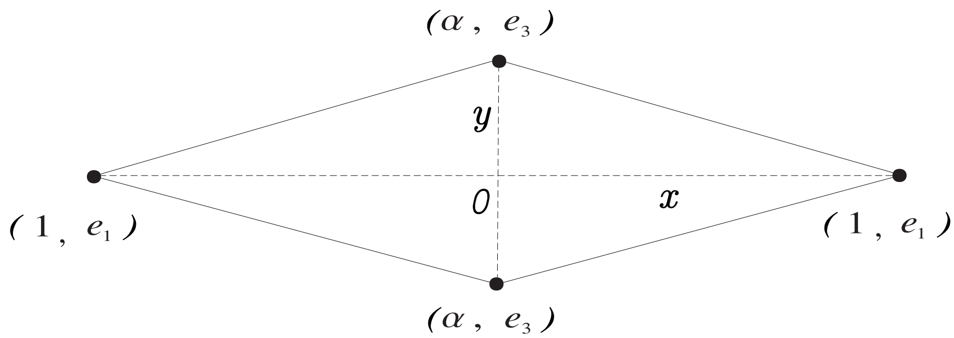

Figure 4.5: The rhomboidal charged problem

Theorem 4.6. *In the nested charged problem of the $n_1 + n_2 + 1$, for some values of the parameters there exists a continuum of central configurations.*

In this result again, choosing adequately the parameters it is possible to obtain a continua of CC, where any position of the first polygon with respect to the second one is a CC (in [6] you can find the details of the proof).

In the following example we show that in the charged problems it is possible to have spatial relative equilibria, a surprising result since as we have seen in the previous Chapter, in the classical Newtonian n–body problem all relative equilibria are planar.

Consider a bi-piramidal configuration (see Fig. 4.7), where the masses and charges have the following values

$$\begin{array}{lll} m_1 = m_2 = 1, & m_3 = m_4 = m, & m_5 = m_6 = \mu, \\ e_1 \ = \ e_2 = e, & e_3 \ = \ e_4 = m + 1 - e, & e_5 \ = \ e_6 = \mu\,, \end{array}$$

whith $m \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$, and $q \in \mathbb{R}$.

It is not difficult to check that there are values of μ, m and e for which the rotating bi-piramidal configuration is a relative equilibria.

4.4 Stability of relative equilibria

The linearized vector field of (4.17) around an equilibrium point determines a linear Hamiltonian system $\dot{z} = \mathcal{L}z; \quad z \in \mathbb{R}^{6n}$, where

$$\mathcal{L} = \begin{pmatrix} K & M^{-1} \\ D\nabla U(x) & K \end{pmatrix}. \tag{4.21}$$

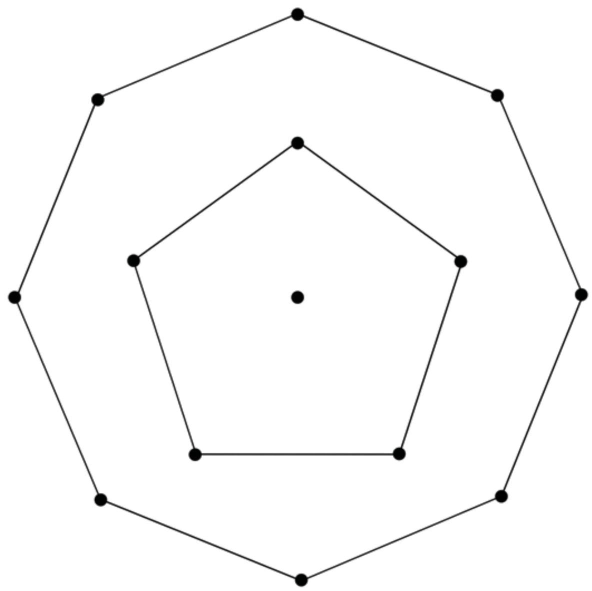

Figure 4.6: The nested charged polygons

The characteristic polynomial

$$p(\lambda) = p(-\lambda) = p(\overline{\lambda}) = p(-\overline{\lambda})$$

Definition 4.7. *A relative equilibrium is spectrally stable if all roots of the characteristic polynomial $p(\lambda)$ satisfy $\lambda^2 \leq 0$.*

From here on we will restrict our analysis to planar relative equilibria.

Proposition 4.8. *For any planar relative equilibria, the characteristic polynomial can be factorized as*

$$p(\lambda) \;=\; \lambda^2(\lambda^2 + w^2)^3 Q_3\,(\lambda)$$

where Q_3 is a polynomial of degree $4(n-2)$.

If all roots of Q_3 satisfy $\lambda^2 < 0$, we say that planar spectral stability is non degenerated.

Now we will apply the previous concepts to analyze the stability of relative equilibria in the charged 3–body problem, in this way, let

$$\delta_{jk} \;=\; \frac{\lambda_{jk}}{m_j m_k} \;=\; 1 - \delta_j\delta_k, \text{ for } j \neq k,$$

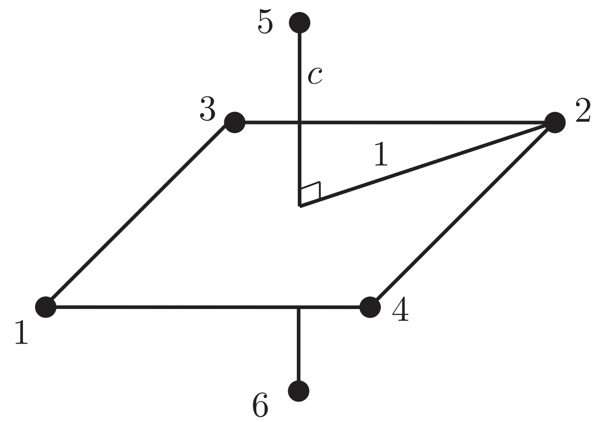

Figure 4.7: An example of a spatial CC

where $\delta_i = q_i/m_i,\ i = 1, 2, 3,$

A necessary condition to have planar relative equilibria is

$$\delta_{12} > 0, \qquad \delta_{13} > 0, \qquad \delta_{23} > 0. \tag{4.22}$$

Planar relative equilibria exist just for the attractive charged problem, and the mutual distances satisfy

$$r_{jk} = \sqrt[3]{\frac{\delta_{jk}}{\delta_{12}}}, \qquad j \neq k. \tag{4.23}$$

Proposition 4.9. *In the charged three body problem, given a triangle of any shape* Γ, *there exist masses and charges such that,* Γ *represents a relative equilibrium.*

In the classical Newtonian 3-body problem, it is well known that the collinear relative equilibria are unstable, whereas the equilateral triangle or Lagrangian relative equilibria are linearly stable if the masses satisfy the Routh condition

$$27(m_1m_2 + m_1m_3 + m_2m_3) < (m_1 + m_2 + m_3)^2.$$

To obtain Q_3 we start getting the six eigenvalues of the matrix L, they are given by

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 2w^2, \quad \lambda_4 = -w^2, \quad \lambda_5 = a, \quad \lambda_6 = b, \tag{4.24}$$

where a and b are real numbers to be determined.

We know that

$$\operatorname{tr} L = \sum_{i=1}^{6} \lambda_i = w^2 + a + b, \tag{4.25}$$

therefore

$$\operatorname{tr}^2 L \;=\; \sum_{k=1}^{6} \lambda_k^2 \;+\; 2\sum_{j<k} \lambda_j \lambda_k \;=\; \operatorname{tr} L^2 \;+\; 2\sum_{j<k} \lambda_j \lambda_k, \tag{4.26}$$

Using (4.24)

$$\operatorname{tr}^2 L = \operatorname{tr} L^2 - 4w^4 + 2w^2(a+b) + 2ab. \tag{4.27}$$

By (4.25), $a + b = \operatorname{tr} L - w^2$, so

$$2ab = (\operatorname{tr} L - w^2)^2 + 5w^4 - \operatorname{tr} L^2. \tag{4.28}$$

Finally, after some computations we get

$$Q_3 \;=\; \lambda^4 \;+\; (3w^2 - \operatorname{tr} L)\,\lambda^2 \;+\; (3w^4 + \frac{1}{2}\operatorname{tr}^2 L \;-\; \frac{1}{2}\operatorname{tr} L^2). \tag{4.29}$$

Definition 4.10. *Let*

$$\begin{array}{rcl} c_{jk} &=& m_j^{-1}\lambda_{jk}\, r_{jk}^{-3}, \qquad \textit{if} \qquad j \neq k, \\ c_{jj} &=& -\sum_{k\neq j} c_{jk}. \end{array} \tag{4.30}$$

We define the matrix C as $C = (c_{jk})$.

The first result is about stability of collinear relative equilibria in the charged 3–body problem (remember that in the classical Newtonian 3..body problem all collinear relative equilibria are unstable.

Theorem 4.11. *In the charged 3–body problem a collinear relative equilibrium is linearly stable and non degenerated if*

$$w^{-2} tr C \in (-2, -17/9] \cup [-1, -1/2). \tag{4.31}$$

For $w^{-2} tr C = -2$ or $w^{-2} tr C = -1/2$, linear stability is degenerated.

Let $(s_1, s_2, s_3) \in \mathbb{R}^3$, a collinear configuration such that $s_1 < s_2 < s_3$. Without loss of generality we assume $s_3 - s_1 = 1$. We define $x = (s_2 - s_1)^{-1}$ and $y = (s_3 - s_2)^{-1}$,

$$x^{-1} + y^{-1} = 1, \qquad x > 1, \;\; y > 1. \tag{4.32}$$

$$w^{-2} tr C \;=\; -\,2 \;-\; \frac{c_1\, p(x) \;+\; c_2\, p(y) \;+\; c_3\, q(x) \;+\; c_4\, q(y)}{m_2\, \delta_{12}\, x^2 \;+\; m_2\, \delta_{23}\, y^2 \;+\; m_1\, \delta_{13} \;+\; m_3\, \delta_{13}}, \tag{4.33}$$

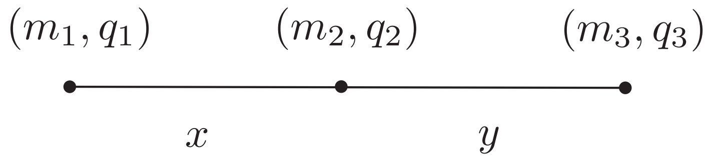

Figure 4.8: Collinear CC in the charged 3–body problem

where

$$p(x) \;=\; x^2 + x + 1, \qquad q(x) \;=\; x - 1,$$

and

$$c_1 = m_1\delta_{12}, \quad c_2 = m_3\delta_{23}, \quad c_3 = m_3(\delta_{23} - \delta_{13}), \quad c_4 = m_1(\delta_{12} - \delta_{13}).$$

As a first example we consider masses and charges such that $\delta_{12} = \delta_{13} = \delta_{23} > 0$. (we observe that the Newtonian case corresponds to $\delta_{jk} = 1$)

$$w^{-2} tr C \;=\; -\,2 \;-\; \frac{m_1\, p(x) \;+\; m_3\, p(y)}{m_2\, x^2 \;+\; m_2\, y^2 \;+\; m_1 \;+\; m_3}.$$

Since p and q are positive functions, we have

$$w^{-2} tr C < -2, \tag{4.34}$$

Therefore the corresponding collinear relative equilibria are unstable.

As a second example we suppose a symmetric distribution of masses and charges

$$m_1 = m_3 = 1, \quad q_1 = q_3 \in \mathbb{R}, \quad m_2 = m \in \mathbb{R}^+, \quad q_2 \in \mathbb{R}, \tag{4.35}$$

then $\delta_{12} = \delta_{23}$.

The relative equilibria must satisfy

$$\begin{aligned} &(y - x)\,[\,\delta_{12}\,(m+1)\,(x^2 + xy + y^2) \;+\; \delta_{12}\, xy \;+\; \delta_{13}\,] \;=\; 0, \\ &U \;=\; m\,\delta_{12}\,(x + y) \;+\; \delta_{13} \;>\; 0, \end{aligned} \tag{4.36}$$

x and y satisfy $x^{-1} + y^{-1} \;=\; 1,$ $x \;>\; 1,$ $y \;>\; 1$. Symmetric solution $x = y = 2$,

Equation (4.33) becomes

$$w^{-2} tr C \;=\; -\,\frac{8\,(m+1)\,\delta_{12} \;+\; \delta_{13}}{4\,m\,\delta_{12} \;+\; \delta_{13}}.$$

Applying Theorem 4.11 to the corresponding relative equilibria, they are linearly stable and non degenerated if $w^{-2} tr C \in [-1, -1/2)$, which holds provided

$$\delta_{12} < 0, \qquad \delta_{13} > 0, \qquad -\frac{\delta_{13}}{\delta_{12}} > 12\,m + 16;$$

another possibility is $w^{-2} tr C \in (-2, -17/9]$, which holds if masses and charges satisfy

$$\delta_{12} > 0, \qquad \delta_{13} > 0, \qquad 8 < \frac{\delta_{13}}{\delta_{12}} \leq 9 + \frac{m}{2}.$$

The non symmetric solutions of (4.36) correspond to $x \neq y$ and parameters δ_{12} and δ_{13} of opposite sign. In this case, (4.36) implies

$$x + y = x\,y = P > 4\,,$$

where

$$P = \frac{m + \sqrt{m^2 + 4\,(m+1)\,Q}}{2(m+1)}, \qquad Q = -\frac{\delta_{13}}{\delta_{12}} > 0. \tag{4.37}$$

$$w^{-2} tr C = -\frac{1}{2} + \frac{(P-4)[\,m\,(P-\frac{1}{2}) + P\,]}{m\,(\,P+2\,)} > -\frac{1}{2}.$$

Therefore the non symmetric collinear relative equilibria are unstable. □

The second Theorem corresponds to the stability of the non-collinear relative equilibria in the charged 3–body problem.

Theorem 4.12. *In the charged* 3*–body problem a non collinear relative equilibrium is linearly stable and non degenerated, if and only if, the masses and charges satisfy the condition*

$$36\,(\,m_1\,m_2\,\sin^2\theta_3 + m_1\,m_3\,\sin^2\theta_2 + m_2\,m_3\,\sin^2\theta_1\,)$$
$$\leq (\,m_1 + m_2 + m_3\,)^2.$$

Where θ_i *is the interior angle of the triangle formed by three particles* i, j, k *corresponding to the vertex containing the* i*–th particle (see Fig. 4.9).*

As an application of the above Theorem we consider masses and charges such that $\delta_{12} = \delta_{13} = \delta_{23} > 0$. (The Newtonian case corresponds to $\delta_{jk} = 1$.)

The triangular shape of the respective relative equilibrium is equilateral. Applying Theorem 4.12, we get that non collinear relative equilibria are stable if holds the inequality

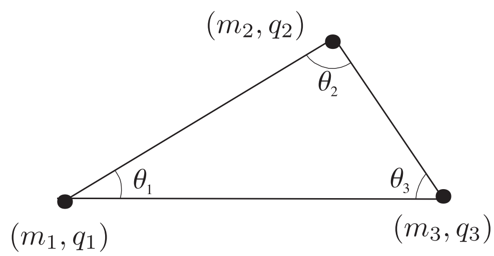

Figure 4.9: A planar CC in the charged 3–body problem

$$27\,(\,m_1m_2\,+\,m_1m_3\,+\,m_2m_3\,)\;<\;(m_1\,+\,m_2\,+\,m_3\,)^2. \tag{4.38}$$

The next result is an imediate consequence of Theorem 4.12

Proposition 4.13. *In the charged three body problem, given any triangular configuration* Γ, *there exist masses and charges such that,* Γ *represents a relative equilibrium linearly stable and non degenerated.*

Proof. We have seen in the first part of this chapter that the shape of a triangle in a CC depends on the ratios of masses and charges. Even more, we know that given any triangular shape, we can find values for the masses and charges such that this configuration leads to a relative equilibria, then taking a dominant mass in the above configuration, by Theorem 4.12 we get the result. □

A famous conjecture about linear stability for relative equilibria in the Newtonian n–body problem was given by R. Moeckel in 1955, it states that in order to have linear stability one of the masses must be significantly bigger than the others [73]. To finish this chapter we will show that this is the conjecture is not true for charged problems.

Suppose $m_1 = m_2 = m_3$, (no dominant mass) the respective relative equilibria are spectral stable if

$$\sin^2\theta_1\,+\,\sin^2\theta_2\,+\,\sin^2\theta_3\;\leq\;1/4.$$

Then, choosing the charges such that the interior angles of the triangle satisfy the above inequality, the respective relative equilibrium is linearly stable (see . It is not difficult to verify that in this case the triangle must be almost collinear see Fig. 4.10,

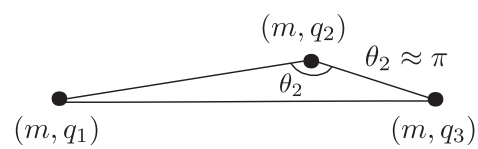

Figure 4.10: A planrar relative equilibrium with equal masses

Chapter 5

The curved n–body problem

The curved n–body problem generalizes the Newtonian n–body problem of celestial mechanics to spaces of curvature $\kappa =$ constant. In these notes we study just the bi-dimensional case, but it can be studied in any dimension. The history of this problem is really old, it starts with the works of Nikolai Lobachevsky and János Bolyai co-discoverers of the first non-Euclidean geometry around the year 1835. Lobachevsky propose to study a Kepler problem in the 3-dimensional hyperbolic space, $\mathbf{H}^3$, by defining an attractive force proportional to the inverse area of the 2-dimensional sphere of the same radius as the distance between bodies. In 1860, P.J. Serret extended the gravitational force to the sphere $\mathbf{S}^2$ and solved the corresponding Kepler problem. Ten years later, E. Schering revisited Lobachevsky's law for which he obtained an analytic expression given by the cotangent potential that we will study study in these notes. In 1873, R. Lipschitz considered the same problem in $\mathbf{S}^3$, but defined a potential proportional to $1/\sin(r/R)$, where r denotes the distance between bodies and R is the curvature radius. In 1885, W. Killing adapted Lobachevsky's idea to $\mathbf{S}^3$ and defined an extension of the Newtonian force given by the inverse area of a 2-dimensional sphere (in the spirit of Schering), for which he proved a generalization of Kepler's three laws. Some years later, H. Liebmann proved an analogues of Bertrand's theorem in $\mathbf{S}^2$ and $\mathbf{H}^2$ which states that there exist only two analytic central potentials in the Euclidean space (the harmonic oscillator and the Kepler's problem), for which all bounded orbits are periodic.

Unfortunately, this direction of research was neglected in the decades following the birth of special and general relativity. But even in these years, E. Schrödinger developed a quantum-mechanical analogue of the Kepler problem in $\mathbf{S}^2$, [**?**]. Schrödinger used the same cotangent potential of Schering and Liebmann, which he deemed to be the natural extension of Newton's law to the sphere. In [], the reader can find an wide history of these facts, as well as an extensive analysis of the relative equilibria in curved spaces. However before to finish this small introductory history of this subject we must mention the work of J. Cariñena, M. Rañada, and M. Santander, who provided a unified approach in the framework of differential geometry, emphasizing the dynamics of the cotangent potential in $\mathbf{S}^2$ and $\mathbf{H}^2$, [22]. And the work of A. Shchepetilov, [97], who in 1996 proved the non-integrability of the two body problem in spaces of constant curvature.

The history of the problem shows that there is no unique way of extending the classical idea of gravitation to spaces of constant curvature, but that the cotangent potential is the most natural candidate. Therefore we take this potential as a starting point of our approach

We assume that the laws of classical mechanics hold for point masses moving on manifolds, so we can apply the results of constrained Lagrangian dynamics to derive the equations of motion. Thus two kinds of forces act on bodies: (i) those given by the mutual interaction between particles, represented by the gradient of the potential, and (ii) those that occur due to the constraints, which involve both position and velocity terms.

In 1992, Kozlov and Harin showed that the only central potential that satisfies the fundamental properties (i) and (ii) in $\mathbf{S}^2$ and has meaning in celestial mechanics is the cotangent of the distance, [59].

The material present in these notes has been done in collaboration with several authors, with F. Diacu and M. Santoprete we obtained the equations of motion for the curved n–body problem in spaces of constant curvature $\mathbf{K}$, for any sign of the curvature, with J.G. Reyes we studied the positive curvature case in intrinsic coordinates; and with F. Diacu and J.G. Reyes we did the analysis of the negative curvature case using intrinsic coordinates.

5.1 Notations and equations of motion

In order to obtain the equations of motion we introduce the notations that we will be using. We consider n bodies of masses $m_1, \cdots, m_n$ moving on a surface of constant curvature $\mathbf{K}$.

- If $\mathbf{IK} > 0$, the surfaces are spheres of radii $R = \mathbf{K}^{-1/2}$ given by the equation $x^2 + y^2 + z^2 = \mathbf{K}^{-1}$
- If $\mathbf{K} = 0$, we recover the Euclidean plane

- If $\mathbf{K} < 0$, we consider the Weierstrass model of hyperbolic geometry , which is devised on the sheets with $z > 0$ of the hyperboloid of two sheets $x^2 + y^2 - z^2 = \mathbf{K}^{-1}$.

The coordinates of the body of mass m_i are given by $\mathbf{q}_i = (x_i, y_i, z_i)$ and a constraint, depending on $\mathbf{K}$, that restricts the motion of this body to one of the above described surfaces.

$\widetilde{\nabla}_{\mathbf{q}_i}$ denotes either of the gradient operators

- $\nabla_{\mathbf{q}_i} = (\partial_{x_i}, \partial_{y_i}, \partial_{z_i})$, for $\mathbf{K} \geq 0$,
- $\overline{\nabla}_{\mathbf{q}_i} = (\partial_{x_i}, \partial_{y_i}, -\partial_{z_i})$, for $\mathbf{K} < 0$,

with respect to the vector $\mathbf{q}_i$.

$$\widetilde{\widetilde{\nabla}} \quad \text{stands for the operator} \quad (\widetilde{\nabla}_{\mathbf{q}_1}, \ldots, \widetilde{\nabla}_{\mathbf{q}_n})$$

For $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ in $\mathbf{R}^3$, we define $\mathbf{a} \odot \mathbf{b}$ as either of the inner products

- $\mathbf{a} \cdot \mathbf{b} := (a_x b_x + a_y b_y + a_z b_z)$ for $\mathbf{K} \geq 0$,
- $\mathbf{a} \boxdot \mathbf{b} := (a_x b_x + a_y b_y - a_z b_z)$ for $\mathbf{K} < 0$, (the Lorentz inner product).

We also define $\mathbf{a} \otimes \mathbf{b}$ as either of the cross products

- $\mathbf{a} \times \mathbf{b} := (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x)$ for $\mathbf{K} \geq 0$,
- $\mathbf{a} \boxtimes \mathbf{b} := (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_y b_x - a_x b_y)$ for $\mathbf{K} < 0$.

We define the distance between two points on the sphere as the usual arc length between them for $\mathbf{K} > 0$, and as the hyperbolic arc length if $\mathbf{K} > 0$, that is $d_{\mathbf{K}}(\mathbf{a}, \mathbf{b}) := R\Theta$ where $R = \mathbf{K}^{-1/2}$. We use $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \Theta$

$$d_{\mathbf{K}}(\mathbf{a}, \mathbf{b}) := \begin{cases} \mathbf{K}^{-1/2} \cos^{-1}(\mathbf{K}\mathbf{a} \cdot \mathbf{b}), & \mathbf{K} > 0 \\ |\mathbf{a} - \mathbf{b}|, & \mathbf{K} = 0 \\ (-\mathbf{K})^{-1/2} \cosh^{-1}(\mathbf{K}\mathbf{a} \boxdot \mathbf{b}), & \mathbf{K} < 0. \end{cases}$$

Notice that d_0 is the limiting case of $d_{\mathbf{K}}$ when $\mathbf{K} \to 0$.

The potential function is defined as

$$U_{\mathbf{K}}(\mathbf{q}) := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} m_i m_j \mathrm{ctn}_{\mathbf{K}}(d_{\mathbf{K}}(\mathbf{q}_i, \mathbf{q}_j))$$

where $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ is the configuration of the system. Notice that $\mathrm{ctn}_0(d_0(\mathbf{q}_i, \mathbf{q}_j)) = |\mathbf{q}_i - \mathbf{q}_j|^{-1}$, which means that we recover the Newtonian potential in the Euclidean case. Therefore the potential $U_{\mathbf{K}}$ varies continuously with the curvature $\mathbf{K}$.

The potential can be written in terms of the dot product among the position vector as

$$U_{\mathbf{K}}(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{m_i m_j (\sigma \mathbf{K})^{1/2} \frac{\mathbf{K}\mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\mathbf{K}\mathbf{q}_i \odot \mathbf{q}_i}\sqrt{\mathbf{K}\mathbf{q}_j \odot \mathbf{q}_j}}}{\sqrt{\sigma - \sigma\left(\frac{\mathbf{K}\mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\mathbf{K}\mathbf{q}_i \odot \mathbf{q}_i}\sqrt{\mathbf{K}\mathbf{q}_j \odot \mathbf{q}_j}}\right)^2}}, \quad \mathbf{K} \neq 0, \tag{5.1}$$

where

$$\sigma = \begin{cases} +1, & \text{for } \mathbf{K} > 0 \\ -1, & \text{for } \mathbf{K} < 0. \end{cases}$$

The Lagrangian of the n-body system has the form

$$L_{\mathbf{K}}(\mathbf{q}, \dot{\mathbf{q}}) = T_{\mathbf{K}}(\mathbf{q}, \dot{\mathbf{q}}) + U_{\mathbf{K}}(\mathbf{q}),$$

where $T_{\mathbf{K}}(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\mathbf{K}\mathbf{q}_i \odot \mathbf{q}_i)$ is the kinetic energy of the system.

Then, according to the theory of constrained Lagrangian dynamics the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L_{\mathbf{K}}}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L_{\mathbf{K}}}{\partial \mathbf{q}_i} - \lambda_{\mathbf{K}}^{i}(t)\frac{\partial f_i}{\partial \mathbf{q}_i} = \mathbf{0}, \quad i = 1, \ldots, n, \tag{5.2}$$

where $f_{\mathbf{K}}^{i} = \mathbf{q}_i \odot \mathbf{q}_i - \mathbf{K}^{-1}$

Finally, using that

$$\dot{\mathbf{q}}_i \odot \mathbf{q}_i = 0 \quad \text{and} \quad \mathbf{q}_i \odot \ddot{\mathbf{q}}_i = -\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i,$$

the equations of motion become

$$m_i \ddot{\mathbf{q}}_i = \widetilde{\nabla}_{\mathbf{q}_i} U_{\mathbf{K}}(\mathbf{q}) - m_i \mathbf{K}(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i, \quad \mathbf{q}_i \odot \mathbf{q}_i = \mathbf{K}^{-1}, \qquad \mathbf{K} \neq 0, \ i = 1, \ldots, n. \tag{5.3}$$

5.2 The positive curvature case

We have seen in the previous sections that the simplest orbits in the n–body problem are the relative equilibria, that is the solutions where the particles move uniformly as a rigid body, so the first step to understand the relative equilibria in the curved n–body problem is to describe all rigid motions. A basic result of geometry is *The Principal Axis Theorem* which states that any rotation in $\mathbf{R}^3$ is around any fixed axis, that we can fix without loos of generality as the z–axis. We remember that this rotation is represented by the one.parameter subgroup given by the matrix

$$A(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.4}$$

A first approach is of course try to study the solutions $q(t)$ which are invariant by (5.4), that is solutions of the form $Q(t) = A(t)q(t)$. But unfortunately the resultant equation in terms of $A(t), \dot{A}(t), \ddot{A}(t)$ is a mess and we were not able to obtain any additional information. One of the main obstacles are the constrains, so the approach in this paper consist in studying the equations of motion written without any constraint, in this way we use the stereographic projection through the north pole of the sphere, and translate the equations of motion to the complex plane $\mathbb{C}$.

Let Π the stereographic projection through the north pole

$$\begin{aligned} \Pi \quad : \quad & S^2_R \to \mathbb{R}^2 \\ & Q \to q \end{aligned} \tag{5.5}$$

If $Q = (x, y, z)$, then $q = (u, v)$ where

$$u = \frac{Rx}{R - z} \qquad \textit{and} \qquad v = \frac{Ry}{R - z}.$$

The inverse function is given by

$$x = \frac{2R^2 u}{u^2 + v^2 + R^2}, \quad y = \frac{2R^2 v}{u^2 + v^2 + R^2},$$
$$z = \frac{R(u^2 + v^2 - R^2)}{u^2 + v^2 + R^2}.$$

The metric of the sphere S^2_R is transformed into the metric

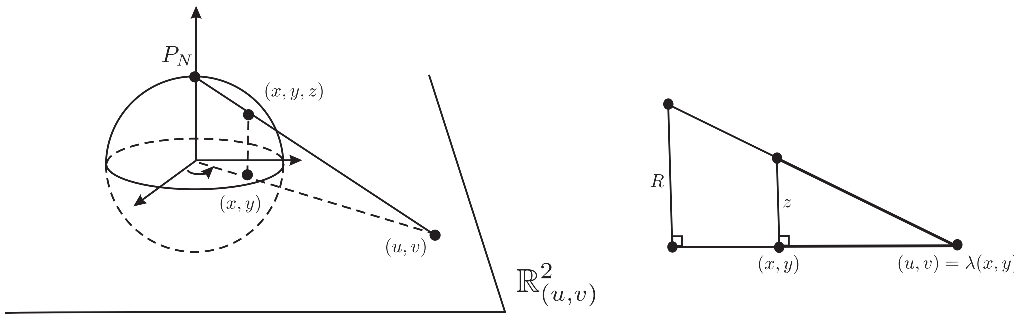

Figure 5.1: The stereographic projection

$$ds^2 = \frac{4R^4}{(u^2 + v^2 + R^2)^2}(du^2 + dv^2). \tag{5.6}$$

Under the above transformations the geodesics of S^2_R are transferred in: the equator of S^2_R into the geodesic circle or radius R; the meridians into straight lines through the origin; the other maximum circles in S^2_R into ellipses.

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can write the equations of motion in complex variable

$$m_k \ddot{z}_k = \frac{2m_k \bar{z}_k \dot{z}_k^2}{R^2 + |z_k|^2} + \frac{(R^2 + |z_k|^2)^2}{4R^4}\frac{1}{2}\frac{\partial U_R}{\partial \bar{z}_k}, \tag{5.7}$$

where

$$\frac{\partial U_R}{\partial \bar{z}_k} = \Sigma^n_{k \neq j} \frac{4m_k m_j R^2\, P_{1,(k,j)}(z, \bar{z})}{[Q_{1,(k,j)}(z, \bar{z})]^{3/2}} \tag{5.8}$$

and

$$\begin{aligned} P_{1,(k,j)}(z, \bar{z}) \;\; &= \;\; (R^2 + |z_k|^2)(|z_j|^2 + R^2)^2 \\ & \quad \big[(|z_j|^2 - (z_k \bar{z}_j + z_j \bar{z}_k) - R^2) z_k + (|z_k|^2 + R^2) z_j\big]. \end{aligned} \tag{5.9}$$

Theorem 5.1. *The equations of motion for the n–body problem on $\mathbb{M}^2_R$ and the corresponding equations on the sphere $S^2_{\mathbf{K}}$ are equivalent.*

Let $\mathrm{Iso}(\mathbb{M}^2_R)$ be the group of isometries of $\mathbb{M}^2_R$, and let us denote by $\{G(t)\}$ an one-parametric subgroup of $\mathrm{Iso}(\mathbb{M}^2_R)$, which acts coordinatewise in $\mathbb{M}^{2n}_R$.

Definition 5.2. *A relative equilibrium of the curved n–body problem is a solution $z(t)$ of (5.3) which is invariant relative to the subgroup $\{G(t)\}$. In other words, the function obtained by the action denoted by $w(t) = G(t)z(t)$ is also a solution of (5.3).*

From differential geometry we know that the *group of proper isometries* of $\mathbb{S}^2_R$ is the quotient $\mathrm{SU}(2)\,/\,\{\pm I\}$ of the special unitary subgroup

$$\mathrm{SU}(2) = \{A \in \mathrm{GL}(2,\mathbb{C}) \mid \bar{A}^T A = I\}$$

where each matrix $A \in \mathrm{SU}(2)$ has the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$.

Every $A \in \mathrm{SU}(2)$ is associated to a unique Moebius transformation (fractional linear transformation) $f_A : \mathbb{S}^2_R \to \mathbb{S}^2_R$

$$f_A(z) = \frac{az + b}{-\bar{b}z + \bar{a}}.$$

We observe that $f_A = f_{-A}$.

The *Lie algebra* of SU(2) is the 3-dimensional real linear space

$$\mathrm{su}(2) = \{X \in \mathrm{M}(2,\mathbb{C}) \mid \bar{X}^T = -X,\ \mathrm{traceX} = 0\}$$

spanned by the basis of complex Pauli's matrix,

$$\left\{ h_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, h_2 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, h_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$$

Let us consider the standard exponential map of matrices from the Lie algebra into its Lie group

$$\exp : \mathrm{su}(2) \to \mathrm{SU}(2)$$

applied to the one-parametric additive subgroups (straight lines in su(2)) $\{t\,h_1\}$, $\{t\,h_2\}$, and $\{t\,h_3\}$, for obtaining the respective one-parametric subgroups of SU(2), given by

- The subgroup

  $$H_1(t) = \exp(t\,h_1) = \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix},$$

  which defines the one-parametric family of acting Moebius transformations

  $$f_{H_1}(z) = \frac{z\cos(t/2) + \sin(t/2)}{-z\sin(t/2) + \cos(t/2)}.$$

- The subgroup

$$H_2(t) = \exp(t\,h_2) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix},$$

  which defines the one-parametric family of acting Moebius transformations

$$f_{H_2}(z) = e^{it}\,z.$$

- The subgroup

$$H_3(t) = \exp(t\,h_3) = \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ i\sin(t/2) & \cos(t/2) \end{pmatrix},$$

  which defines the one-parametric family of acting Moebius transformations

$$f_{H_3}(z) = \frac{z\cos(t/2) + i\sin(t/2)}{zi\sin(t/2) + \cos(t/2)}.$$

Proposition 5.3. *If $G : SU(2)/\pm I \to SO(3)$ is the isomorphism between the above groups of isometries of the sphere, then $G(H_2(t)) = A(t)$.*

We observe that the rotation around z–axis corresponds, under the isomorphism G to the subgroup H_2. Actually the subgroups H_1 and H_3 represent rotations under G around the x and y-axis respectively, which when are applied in $\mathbf{M}_R^2$ with the metric derived from the stereographic projection become quite complicated. Thanks to the Principal Axis Theorem, we can ignore them, so from here on we will use only the subgroup H_2. So, for the study of relative equilibria in M_R^2 it is enough just to analyze the second class of Moebius transformations $\{H_2(t)\}$ obtained by the exponential process:

$$w_k(t) = e^{it} z_k(t),$$

where $z(t) = (z_1(t), \cdots, z_n(t))$ is a solution of equation (5.7).

Some results about relative equilibria:

Theorem 5.4. *A solution $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ form a relative equilibria if the coordinates satisfy a system of algebraic equations and $|z_j(t)| = r_j$ for $j = 1, 2, \cdots n$.*

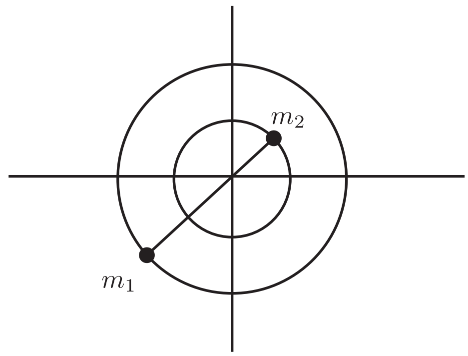

Figure 5.2: CC in the curved 2–body problem

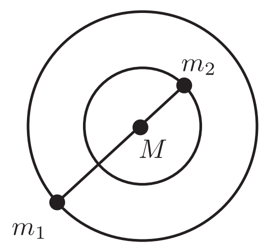

Figure 5.3: CC in the curved 3–body problem

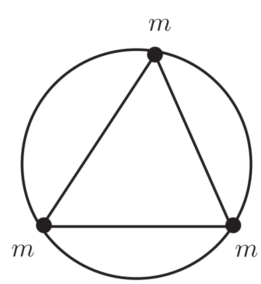

Figure 5.4: Lagrangian CC in the curved 3–body problem

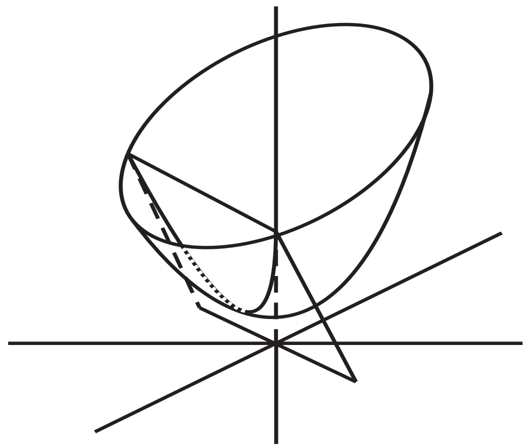

Figure 5.5: buscar

5.3 The negative curvature case

Consider a connected and simply connected two dimensional surface of constant negative curvature $\mathbf{K} = -\dfrac{1}{R^2}$, it is well known that this surface is locally represented by the upper leaf the two dimensional hyperbolic sphere of radius R denoted by $\mathbb{L}^2_R$, and contained in the Minkowski space $\mathbb{R}^3_1$ (see Fig. 5.5). In such space the inner product is defined by

$$Q_1 \odot Q_2 = x_1x_2 + y_1y_2 - z_1z_2,$$

for any pair of points $Q_1, Q_2 \in \mathbb{R}^3_1$.

The Principal Axis Theorem for $\mathbb{L}^2_R$ states that any $A \in \mathrm{Lor}(\mathcal{M}^3)$ can be written in some basis as:

$$A = P \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1},$$

or

$$A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{bmatrix} P^{-1},$$

or

$$A = P \begin{bmatrix} 1 & -t & t \\ t & 1 - t^2/2 & t^2/2 \\ t & -t^2/2 & 1 + t^2/2 \end{bmatrix} P^{-1},$$

As in the case of positive curvature, here we use the stereographic projection in order to have the equations written in complex coordinates. In this way let Π be the stereographic projection of $\mathbb{L}^2_R$ through the north pole $(0,0,-R)$ contained in the lower leaf of the hyperbolic sphere into the planar disk of radio R with center at the origin of coordinates denoted by $\mathbb{D}^2_R$ (see Fig **??**)

$$\begin{aligned} \Pi \ : \ & \mathbb{L}^2_R \to \mathbb{D}^2_R \\ & Q \to q. \end{aligned} \tag{5.10}$$

It is easy to verify that the image $\mathbb{L}^2_R$ under Π is the whole open disk $\mathbb{D}^2_R$, known as the Poincaré disk, and that if $Q = (x, y, z)$, then $q = (u, v)$ where

$$u = \frac{Rx}{R+z} \qquad \textit{and} \qquad v = \frac{Ry}{R+z}. \tag{5.11}$$

The inverse function is given by

$$x = \frac{2R^2 u}{R^2 - u^2 - v^2} \quad y = \frac{2R^2 v}{R^2 - u^2 - v^2} \quad z = \frac{R(R^2 + u^2 + v^2)}{R^2 - u^2 - v^2}.$$

The metric (distance) of the sphere $\mathbb{L}^2_R$ is transformed into the metric on the Poincaré disk

$$ds^2 = \frac{4R^4}{(R^2 - u^2 - v^2)^2}(du^2 + dv^2). \tag{5.12}$$

From the stereographic projection, we have that the geodesics are the diameters of the Poincaré disk $\mathbb{D}^2_R$ and circles intersecting orthogonally the circle $|z| = R$.

Definition 5.5. *A relative equilibrium of the negatively curved n–body problem is a solution $z(t)$ of the equations of motion which is invariant respect to the subgroups $\{G(t)\}$ of* $\mathrm{Iso}(\mathbb{D}^2_R)$*, that is, the function obtained by the action $w(t) = G(t)\, z(t)$ is also a solution of the equations of motion.*

Let

$$\mathrm{SU}(1,1) = \{A \in \mathrm{GL}(2,\mathbb{C}) \mid \bar{A}^T A = \tilde{I}\}$$

be the *special ortochronic unitary group* where

$$\tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

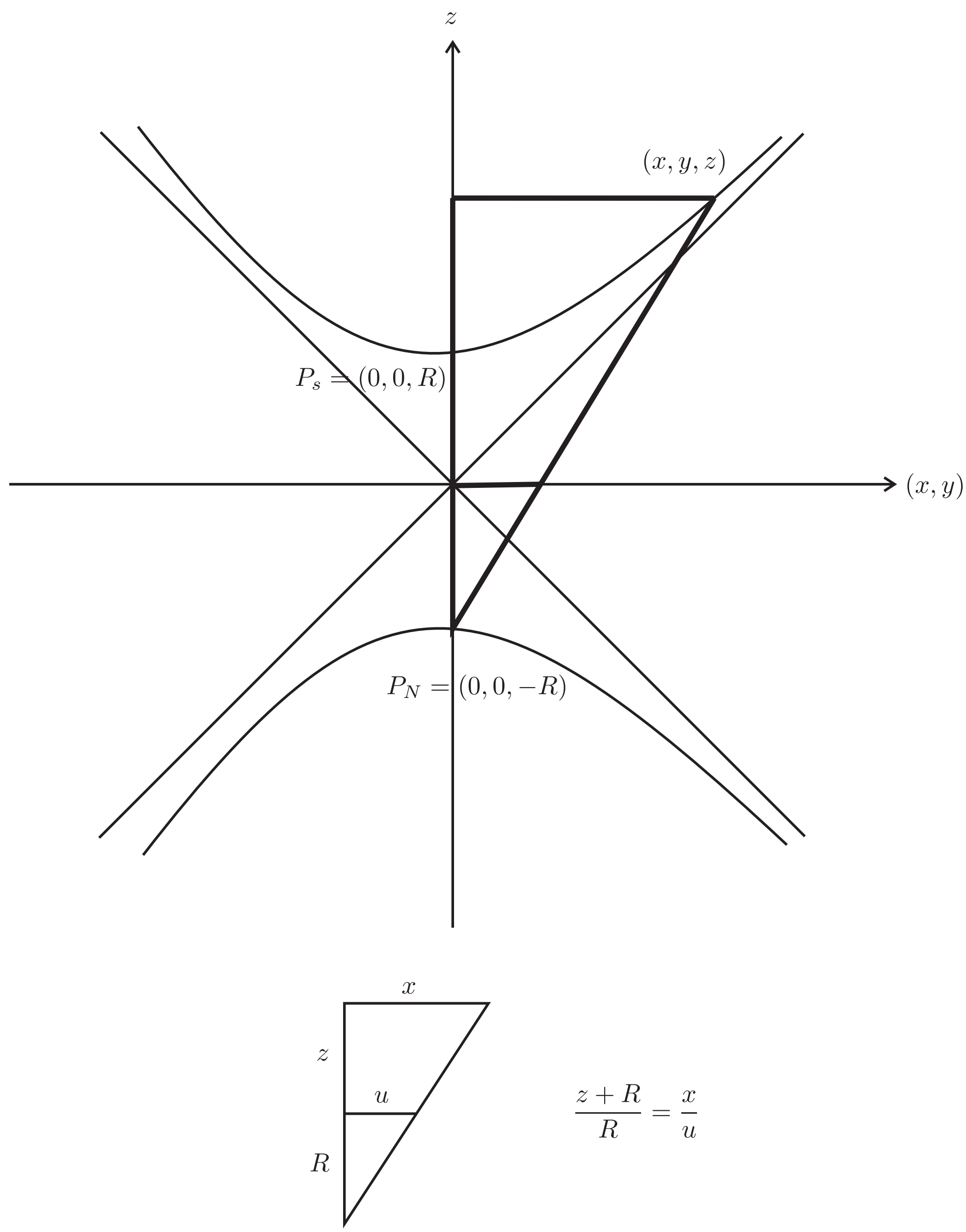

Figure 5.6: Stereographic projection of the hyperboloid

and any matrix $A \in \mathrm{SU}(1,1)$ has the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$

with $a, b \in \mathbb{C}$ satisfying $|a|^2 - |b|^2 = 1$.

The *group of proper isometries* of $\mathbb{D}^2_R$ is the quotient group $\mathrm{SU}(1,1)/\{\pm I\}$.

On the other hand, every class $A \in \mathrm{SU}(1,1)/\{\pm I\}$ has associated one Moebius transformation $f_A : \mathbb{D}^2_R \to \mathbb{D}^2_R$, where

$$f_A(z) = \frac{az + b}{\bar{b}z + \bar{a}},$$

from here we obtain the necessity to take the identification by $\pm I$ since $f_{-A}(z) = f_A(z)$.

We know from differential geometry that the *Lie algebra* of $\mathrm{SU}(1,1)$ is the 3-dimensional real linear space

$$\mathrm{su}(1,1) = \{X \in \mathrm{M}(2,\mathbb{C}) \mid \tilde{I}\,\bar{X}^T = -X\,\tilde{I},\ \mathrm{traceX} = 0\}$$

spanned by the Killing vector fields in $\mathbb{D}^2_R$ associated to the Pauli matrices, that is well known form a basis of $\mathrm{su}(1,1)$

$$\left\{ g_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, g_2 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, g_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}$$

As in the case of positive curvature, we consider the exponential map of matrices,

$$\exp : \mathrm{su}(1,1) \to \mathrm{SU}(1,1)$$

applied to the one-parametric additive subgroups (straight lines) $\{t\,g_1\}$, $\{t\,g_2\}$, and $\{t\,g_3\}$, for obtaining the one-parametric subgroups of $\mathrm{SU}(1,1)$, in this way we obtain

- The subgroup

$$G_1(t) = \exp(t\,g_1) = \begin{pmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{pmatrix},$$

which defines in $\mathbb{D}^2_R$ the one-parametric family of acting Moebius transformations

$$f_{G_1}(z) = \frac{\cosh(t/2)\,z + \sinh(t/2)}{\sinh(t/2)\,z + \cosh(t/2)}. \tag{5.13}$$

- The subgroup
$$G_2(t) = \exp(t\, g_2) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix},$$
which defines the one-parametric family of Moebius transformations
$$f_{G_2}(z) = e^{it}\, z. \tag{5.14}$$

- The subgroup
$$G_3(t) = \exp(t\, g_3) = \begin{pmatrix} \cosh(t/2) & i \sinh(t/2) \\ -i \sinh(t/2) & \cosh(t/2) \end{pmatrix},$$
which defines the one-parametric family of acting Moebius transformations
$$f_{G_3}(z) = \frac{\cosh(t/2)\, z + i \sinh(t/2)}{-i \sinh(t/2)\, z + \cosh(t/2)}. \tag{5.15}$$

Proposition 5.6. *Let $H : \mathrm{SU}(1,1)/\{\pm I\} \to SO(1,2)$ the isomorphism between the groups of proper isometries of the Poincare disk and of the hyperbolic sphere, then $H(G_2(t)) = A(t)$.*

Results for relative equilibria in $\mathbb{D}^2_R$ The results for elliptic relative equilibria in $\mathbb{D}^2_R$ are similar to those on $\mathbb{M}^2_R$.

In order to study the other kinds of relative equilibria in the negative curved n–body problem, it was necessary to use another model of the hyperbolic geometry, the so called the Poincaré upper half plane $\mathbb{H}^2_R$.

We translate the main problem to the space $\mathbb{H}^2_R$ by doing a global isometric fractional linear transformation $z : \mathbb{H}^2_R \to \mathbb{D}^2_R$, defined by
$$z = z(w) = \frac{-Rw + iR^2}{w + iR}, \tag{5.16}$$
where w is the complex variable in the upper half plane
$$\mathbb{H}^2_R = \{w \in \mathbb{C} \,|\, \mathrm{Im}\,(w) > 0\}.$$

Since,
$$dz = \frac{-2R^2 i}{(w + iR)^2}\, dw, \quad \text{and} \quad d\bar{z} = \frac{2R^2 i}{(\bar{w} - iR)^2}\, d\bar{w},$$

we have that the metric (**??**) of the disk $\mathbb{D}^2_R$ becomes into the metric, conventionally written as

$$ds^2 = -\frac{4R^2 dw\, d\bar{w}}{(w-\bar{w})^2} \tag{5.17}$$

for the space $\mathbb{H}^2_R$. Such space endowed with the metric (5.17) will be called the Poincaré model of the hyperbolic geometry.

The geodesic curves in $\mathbb{H}^2_R$ become into half circles which are orthogonal to the real axis (y=0), and into vertical straight lines also orthogonal to such axis. All of these curves have infinite arc length.

By applying the transformation (5.16) to the equation (**??**), we obtain the new potential in the coordinates $(w, \bar{w})$ given by

$$V_R(w,\bar{w}) = \frac{1}{R}\sum_{1\le k<j\le n}^{n} m_k m_j \frac{(\bar{w}_k + w_k)(\bar{w}_j + w_j) - 2(|w_k|^2 + |w_j|^2)}{(\Theta_{3,(k,j)}(w,\bar{w}))^{1/2}}, \tag{5.18}$$

where

$$\Theta_{3,(k,j)}(w,\bar{w}) = [(\bar{w}_k + w_k)(\bar{w}_j + w_j) - 2(|w_k|^2 + |w_j|^2)]^2 - (\bar{w}_k - w_k)^2(\bar{w}_j - w_j)^2. \tag{5.19}$$

Let

$$\mathrm{SL}(2,\mathbb{R}) = \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det A = 1\}$$

be the *special linear real two dimensional group*, which is a 3-dimensional smooth real manifold.

It is known from differential geometry that the *group of proper isometries* of $\mathbb{H}^2_R$ is the projective quotient group $\mathrm{SL}(2,\mathbb{R})/\{\pm I\}$.

The *Lie algebra* of $\mathrm{SL}(2,\mathbb{R})$ is the 3-dimensional real linear space

$$\mathrm{sl}(2,\mathbb{R}) = \{X \in \mathrm{M}(2,\mathbb{R}) \mid \mathrm{traceX} = 0\}$$

spanned by the following suitable set of Killing vector field basis,

$$\left\{ X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Again we consider the exponential map of matrices,

$$\exp : \mathrm{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$$

applied to the one-parametric additive subgroups (straight lines) $\{t\, X_1\}$, $\{t\, X_2\}$, and $\{t\, X_3\}$, for obtaining the one-parametric subgroups of $\mathrm{SL}(2,\mathbb{R})$ given by

- The isometric dilatation subgroup

$$\phi_1(t) = \exp(t\,X_1) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

  which defines the one-parametric family of acting Moebius transformations

$$f_1(w,t) = e^t w(t). \tag{5.20}$$

- The isometric shift subgroup

$$\phi_2(t) = \exp(t\,X_2) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

  which defines in $\mathbb{H}^2_R$ the one-parametric family of acting Moebius transformations

$$f_2(w,t) = w(t) + t. \tag{5.21}$$

- The isometric rotation subgroup

$$\phi_3(t) = \exp(t\,X_3) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

  which defines the one-parametric family of acting Moebius transformations

$$f_3(w,t) = \frac{(\cos t)\,w(t) + \sin t}{(-\sin t)\,w(t) + \cos t}. \tag{5.22}$$

From the *Theorem of invariance of domain*, the isometry (5.16) carries the interior of $\mathbb{D}^2_R$ into the interior of $\mathbb{H}^2_R$ and therefore simple closed curves contained in $\mathbb{D}^2_R$ and given by the Moebius group (5.14) into simple closed curves contained in $\mathbb{H}^2_R$. Moreover, for the circle $z(t) = z_0 e^{it}$ in the Poincare disk the corresponding curve

$$w(t) = \frac{iR(R - z_0 e^{it})}{R + z_0 e^{it}}$$

in the upper half plane satisfies $w(0) = w(2\pi)$ and therefore this last curve is also closed and must belong to the class of the Moebius rotation group (5.22).

We study here the relative equilibria associated to the subgroup (5.20) which defines the one-parametric family of acting Moebius transformations

$$f_1(w,t) = e^t w(t)$$

in the Poincaré plane $\mathbb{H}^2_R$. First we give the necessary and sufficient conditions in order to have hyperbolic relative equilibria.

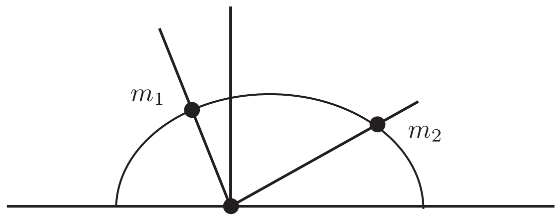

Figure 5.7: buscar

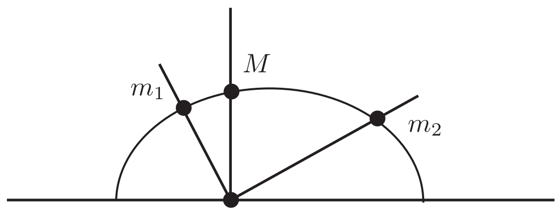

Figure 5.8: buscar

Theorem 5.7. *Consider n–point particles with masses $\{m_1, m_2, \cdots, m_n\}$ moving on $\mathbb{H}^2_R$. A necessary and sufficient condition for the solution $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))$ to be a relative equilibrium associated to the Killing vector field X_1 is that the coordinates hold the system of algebraic rational equations*

$$\frac{R(w_k + \bar{w}_k)\, w_k}{8(w_k - \bar{w}_k)^4} = \sum_{1\le k<j\le n} \frac{m_j(\bar{w}_j - w_j)^2(w_k - w_j)(\bar{w}_j - w_k)}{[\Theta_{3,(k,j)}(w, \bar{w})]^{3/2}} \tag{5.23}$$

where

$$\begin{aligned} \Theta_{3,(k,j)}(w, \bar{w}) &= [(\bar{w}_k + w_k)(\bar{w}_j + w_j) - 2(|w_k|^2 + |w_j|^2)]^2 \\ &- (\bar{w}_k - w_k)^2(\bar{w}_j - w_j)^2. \end{aligned}$$

Results for hyperbolic relative equilibria in $\mathbb{H}^2_R$

Finally we study the parabolic relative equilibria associated to the subgroup

$$\phi_2(t) = \exp(t\, X_2) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

generated by the Killing vector field X_2 and which defines the one-parametric family of acting Moebius transformations

$$f_2(w, t) = w(t) + t$$

in the upper half plane $\mathbb{H}^2_R$.

By using the condition to have parabolic relative equilibria in the equations of motion, we get necessary and sufficient conditions which characterize these kind of motions.

Theorem 5.8. *Consider n–point particles with masses $\{m_1, m_2, \cdots, m_n\}$ moving on $\mathbb{H}^2_R$. A necessary and sufficient condition for the solution $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))$ to be a relative equilibrium associated to the Killing vector field X_2 is that the coordinates hold the system of algebraic rational equations*

$$\frac{-R}{4(w_{k,0} - \bar{w}_{k,0})^4} = \sum_{1 \le k < j \le n} \frac{m_j(\bar{w}_{j,0} - w_{j,0})^2(w_{k,0} - w_{j,0})(\bar{w}_{j,0} - w_{k,0})}{(\tilde{\Theta}_{3,(k,j)}(w_0, \bar{w}_0))^{3/2}} \tag{5.24}$$

where

$$\begin{aligned} \tilde{\Theta}_{3,(k,j)}(w_0, \bar{w}_0) &= [(\bar{w}_{k,0} + w_{k,0})(\bar{w}_{j,0} + w_{j,0}) - 2(|w_{k,0}|^2 + |w_{j,0}|^2)]^2 \\ &- (\bar{w}_{k,0} - w_{k,0})^2(\bar{w}_{j,0} - w_{j,0})^2. \end{aligned} \tag{5.25}$$

Theorem 5.9. *For the n–body problem in one space of negative constant Gaussian curvature there are not parabolic relative equilibria.*

Acknowledgements

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