Introduction to non-Newtonian fluid mechanics

Josef Málek

October 2012

In this lecture we will try to answer the following questions.

Questions

Q1. What is mechanics?

Q2. What is a fluid?

Q3. What is a Newtonian fluid?

Q4. What are non-Newtonian fluids?

Q5. Why mechanics of non-Newtonian fluids is important?

Q6. What are the effects that can not be described by classical models?

Q7. What will be the content of this course?

Q8. Is J.M. appropriate lecturer?

Q9. What are the unique features of the course?

Q10. Are there other approaches to describe such ”strange” features of materials?

Classical mechanics

Classical mechanics describes the evolution of mass particles which follow three Newton laws:

Law of inertia If the external force $\mathbf{F} = \mathbf{0}$ then object in a state of uniform motion tends to remain in that state of motion.

Law of force If there is an external force, it is equal to the time change of the momentum $m\mathbf{v}$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F}.$$

Law of action and reaction For every action there is an equal and opposite reaction.

We show four examples of one-dimensional linear spring.

1. Linear spring The displacement of the spring y satisfies ODE

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F_E + F_S,$$

where F_S is the spring force and F_E is the external force. If the spring force $F_S = -ky$ and F_E is gravitational force, then

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = mg - ky.$$

The initial-value problem has to be solved

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{k}{m}y &= 0\\ y(0) &= y_0\\ \frac{\mathrm{d}y}{\mathrm{d}t}(0) &= y_1.\end{aligned}$$

2. Spring with damping force We can add the damping force for the spring $F_d = -b\frac{\mathrm{d}y}{\mathrm{d}t}$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{k}{m}y = 0.$$

3. Generalized non-linear spring Spring force is a nonlinear function of $y : F_S = F_S(y)$, for example Duffing spring $F_S(y) = \alpha y + \beta y^3$. The well-known Duffing oscillator with the dumping satisfies

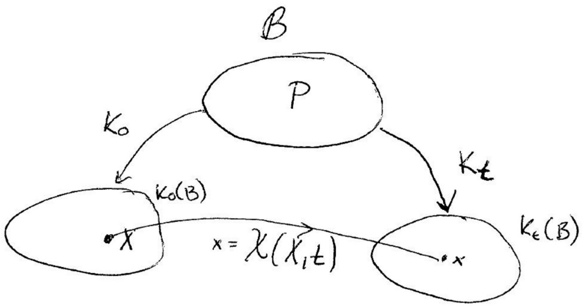
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{m}(\alpha y + \beta y^3) = 0.$$

4. Further generalizations – implicit relation Finally, we can have a fully implicit relation for the between the spring force F_S and the displacement y, and between the damping force F_D and the velocity $\frac{\mathrm{d}y}{\mathrm{d}t}$

$$\tilde{f}\left(F_S, y\right) = 0$$
$$\tilde{g}\left(F_D, \frac{\mathrm{d}y}{\mathrm{d}t}\right) = 0.$$

Continuum mechanics

Instead of describing only a mass point we would like to describe whole body. To do this we describe it as a continuum. Let B is a body, $K_0(B)$ a configuration in the initial state and $K_t(B)$ the configuration at time t (see Figure).



The we define a motion χ which maps from $K_0(B)$ to $K_t(B) : x = \chi(X, t)$. Further we define the following kinematical quantities

Velocity

$$\mathbf{v}(X, t) = \frac{\partial \chi(X, t)}{\partial t} \Longrightarrow \mathbf{v}(x, t) := \mathbf{v}(\chi^{-1}(x, t), t)$$

Deformation gradient

$$\mathbf{F} = \frac{\partial \chi(X, t)}{\partial X} = \frac{\partial x}{\partial X}$$

Left Cauchy-Green tensor

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T$$

Right Cauchy-Green tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}.$$

Balance equations

Balance of mass This balance is derived in the following way: (ϱ density, $\mathbf{v}$ fluid velocity)

$$m(P_t) = \int_{P_t} \varrho \, \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} m(P_t) = 0 \quad \forall P_t \subset K_t(B)$$

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t} m(P_t) &= \int_{P_t} \varrho(x,t) \, \mathrm{d}x = \int_{P_0} \frac{\mathrm{d}}{\mathrm{d}t} \varrho(\chi(X,t),t) \det \nabla_X \chi(X,t) \, \mathrm{d}X \\ &= \int_{P_0} \left(\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial \chi_k} \frac{\partial \chi_k}{\partial t} + \varrho \mathrm{div}\, \mathbf{v} \right) \det \nabla_X \chi(X,t) \, \mathrm{d}X \\ &= \int_{P_t} \left(\frac{\partial \varrho}{\partial t} + \nabla_x \varrho \mathbf{v} + \varrho \mathrm{div}\, \mathbf{v} \right) \, \mathrm{d}x = \int_{P_t} \frac{\partial \varrho}{\partial t} + \mathrm{div}\, (\mathbf{v}\varrho) \, \mathrm{d}x \end{aligned}$$

$$\varrho_{,t} + \mathrm{div}\, (\varrho \mathbf{v}) = 0$$

Balance of linear momentum

$$(\varrho \mathbf{v})_{,t} + \mathrm{div}\, (\varrho \mathbf{v} \otimes \mathbf{v}) - \mathrm{div}\, \mathbf{T} = \varrho \mathbf{b},$$

$\mathbf{b}$ volume force, $\mathbf{T}$ Cauchy stress tensor.

Balance of angular momentum

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}}$$

Balance of energy

$$\frac{\partial (\varrho E)}{\partial t} + \mathrm{div}\, (\varrho E \mathbf{v}) + \mathrm{div}\, \mathbf{q} - \mathrm{div}\, (\mathbf{T}\mathbf{v}) = \varrho \mathbf{b} \cdot \mathbf{v} + \varrho r,$$

E energy, $\mathbf{q}$ thermal flux, r thermal source.

Together with these balance equations we need to add some information to be able to solve the problem:

1. Constitutive equations – characterize the response of class of materials at certain class of processes. So far we have 8 equations and 14 unknowns still necessary to add some constitutive equations.
2. Boundary conditions
3. Initial conditions

Comments:

- Balance equations are formulated on averages (similar to calculus of variation) it suggests to work with weak formulation.
- The last formulation is not equivalent in a weak sense (only classical sense)

What is a Newtonian fluid?

We have to distinguish between compressible and incompressible fluid.
Constitutive equation for the Cauchy stress tensor $\mathbf{T}$ for compressible Newtonian fluid

$$\begin{aligned}\mathbf{T} &= -p(\varrho)\mathbf{I} + 2\mu(\varrho)\mathbf{D}(\mathbf{v}) + \lambda(\varrho)(\operatorname{div}\mathbf{v})\,\mathbf{I} \\ &= -p(\varrho)\mathbf{I} + 2\mu(\varrho)\mathbf{D}^d(\mathbf{v}) + \frac{2\mu(\varrho)+3\lambda(\varrho)}{3}(\operatorname{div}\mathbf{v})\,\mathbf{I}\end{aligned} \tag{1}$$

where $\mathbf{A}^d := \mathbf{A} - \frac{1}{3}(\operatorname{tr}\mathbf{A})\,\mathbf{I}$.

Incompressibility

Definition 1. *Volume of any chosen subset (at initial time $t = 0$) remains constant during the motion.*

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}|P_t| = \frac{\mathrm{d}}{\mathrm{d}t}\int_{P_t} \mathrm{d}x = \int_{P_t} \operatorname{div}\mathbf{v}\ \mathrm{d}x \Longrightarrow \operatorname{div}\mathbf{v} = 0$$

in solid mechanics we can write $\det\mathbf{F} = 1$

$$\int_{P_0} \det\mathbf{F}\ \mathrm{d}X = \int_{P_t} \mathrm{d}x$$

Consequences of incompressibility, balance of mass + incompressibility

$$\begin{cases} \frac{\partial\varrho}{\partial t} + \mathbf{v}\cdot\nabla\varrho + \varrho\operatorname{div}\mathbf{v} = 0 \\ \operatorname{div}\mathbf{v} = 0 \end{cases} \Longrightarrow \begin{cases} \frac{\partial\varrho}{\partial t} + \mathbf{v}\cdot\nabla\varrho = 0 \\ \operatorname{div}\mathbf{v} = 0 \end{cases}$$

Non-homogeneous incompressible Newtonian fluid

$$\mathbf{T} = -m\mathbf{I} + 2\mu(\varrho)\mathbf{D}^d(\mathbf{v}), \quad \mu(\varrho) > 0. \tag{2}$$

Homogeneous incompressible Newtonian fluid

$$\mathbf{T} = -m\mathbf{I} + 2\mu^*\mathbf{D}^d(\mathbf{v}), \quad \mu^* > 0. \tag{3}$$

The mean normal stress $m = -(\operatorname{tr}\mathbf{T})/3$ is the unknown. The pressure p in the compressible model is in a similar situation as m in the incompressible model, but in the compressible model the pressure has to be constitutively defined, and it is not the unknown.

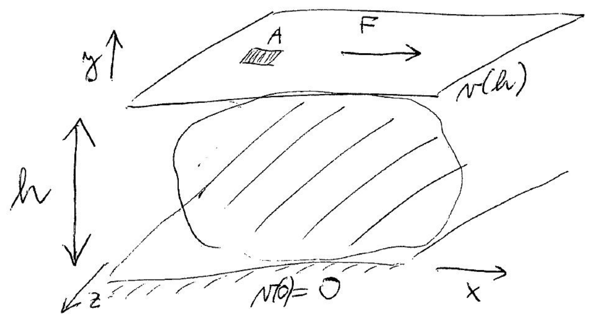
What is non-Newtonian fluid?

A fluid is non-Newtonian if its response to mechanical loading can not be describe either (1), (2) or (3).

Concept of viscosity

Newton (1687):

> "The resistance arising from the want of lubricity in parts of the fluid, **other things being equal**, is proportional to the velocity with which the parts are separated from one another."



Definition 2 (Viscosity)**.** *Viscosity is the coefficient proportionality between the shear stress and the shear-rate*

Generalized viscosity

$$T_{xy} = \frac{F_z}{A} \sim \frac{v(h+y) - v(y)}{h} \sim v'(y) = 2D_{xy}$$

$$\mu_g(\kappa) := \frac{T_{xy}(\kappa)}{\kappa}, \quad \text{where } \kappa = v'$$

Simple shear flow: $\mathbf{v}(x,y,z) = \begin{pmatrix} v(y) \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{D} = \frac{1}{2}\begin{pmatrix} 0 & v' & 0 \\ v' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

$$bT_{xy} = \mu v'(y).$$

Experimental data show that the viscosity depends on the shear-rate, pressure, etc.

One can understand the word 'proportional' in the sense of arbitrary denpendence. Then we can understand Newton's statement in the sense of implicit relation

$$G(T_{xy}, D_{xy}, \dots) = 0.$$

Physical dimension of the dynamic viscosity

$$[\mu] = \frac{\text{N}}{\text{m}^2} \frac{\text{m}}{\text{m s}^{-1}} = \frac{\text{kg m s}^{-2}}{\text{m}^2 s} = \text{kg m}^{-1}\text{s}^{-1} =: 1\,000 \text{ cP},$$

and kinematic viscosity

$$\left[\frac{\mu^*}{\varrho^*}\right] = \text{m}^2\text{s}^{-1}.$$

Kinematic viscosity of some fluids

Fluid	Viscosity [cP]
Air (at 18° C)	0.02638
Benzene	0.5
Water (at 18)	1
Olive oil (at 20)	84
Motor oil SAE 50	540
Honey	2000–3000
Ketchup	50000–70000
Peanut butter	150000–250000
Tar	3×10^{10}
Earth lower mantle	3×10^{25}

Non-Newtonian phenomena and model for non-Newtonian fluids

Definition 3. *The fluid is non-Newtonian if and only if its behavior can not be described by the model of Navier-Stokes fluid*

compressible	$\mathbf{T} = -p(\varrho,\theta)\mathbf{I} + 2\mu(\varrho,\theta)\mathbf{D} + \lambda(\varrho,\theta)(\operatorname{div}\mathbf{v})\mathbf{I}$
incompressible	$\mathbf{T} = -p\mathbf{I} + 2\mu(\varrho,\theta)\mathbf{D}.$

This definition is not useful. We describe the non-Newtonian phenomena.

Non-Newtonian phenomena

(1) *Shear thinning, shear thickening property*

(2) *Pressure thickening property*

(3) *Presence of activation/deactivation criteria* (connected with the stress or with the shear rate)

(4) *Presence of non-zero normal stress differences in a simple shear flow*

(5) *Stress relaxation*

(6) *Non-linear creep*

Consider a fluid velocity $\mathbf{v} = (u(y), 0, 0)$ with the stress tensor in the form

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}.$$

We are interested in the shear rate $|D_{12}|$ and the shear stress $|S_{12}|$. The shear rate $|D_{12}|$ is in the literature denoted for example as κ neboor $\dot{\gamma}$, shear stress $|S_{12}|$ is denoted as σ or τ.

Ad (1) **Shear thinning, shear thickening property**

We are interested in the dependence of $|S_{12}|$ on $|D_{12}|$. Let us define a generealized viscosity μ_g as

$$\mu_g(|D_{12}|) := \frac{|S_{12}(|D_{12}|)|}{|D_{12}|}.$$

Newtonian fluid

For the Newtonian fluid it holds $S_{12} = \mu D_{12}$ and $D_{12} = u'(y)$, so $|S_{12}|$ is a linear function of $|D_{12}|$ with the proportionality coefficient μ (see Figure 1).

In the case of Newtonian fluid $\mu_g = \mu = konst.$, the graph μ_g vs. $\dot{\gamma}$ is in the Figure 2.

Non-Newtonian fluid

Shear thickening property (dilatant fluids) means that $|S_{12}|$ is a superlinear function of $|D_{12}|$ (Figure 3). The generalized viscosity μ_g is a increasing function, in the Figure 4 is the example where the generalized viscosity is degenerate at the beginning, usually the generealized visosity is positive in zero.

Shear thinning property (pseudoplastic fluids) means that $|S_{12}|$ is a sublinear function of $|D_{12}|$ (Figure 5). The generalized viscosity μ_g is a decreasing function, in the Figure 6 is the example where the generalized viscosity is singular at the beginning, usually the generealized visosity is finite in zero.

Generally we can say that there is no reason that $|S_{12}|$ is monotone function of $|D_{12}|$ [1].

[1] It does not have to be even a function.

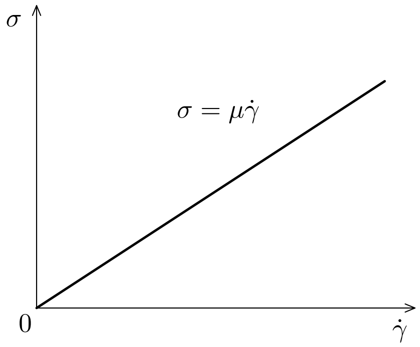


Figure 1: Newtonian fluid,
(shear stress / shear rate)

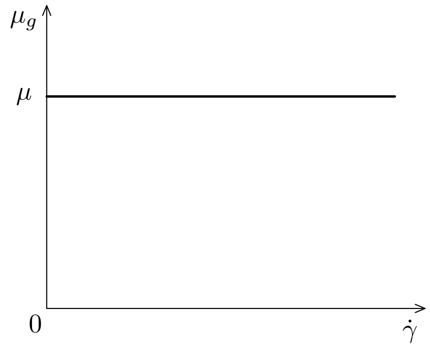


Figure 2: Newtonian fluid,
(generalized viscosity / shear rate)

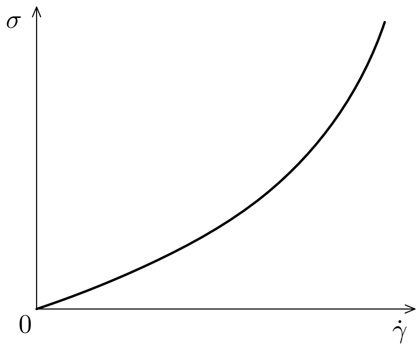


Figure 3: Shear thickening,
(shear stress / shear rate)

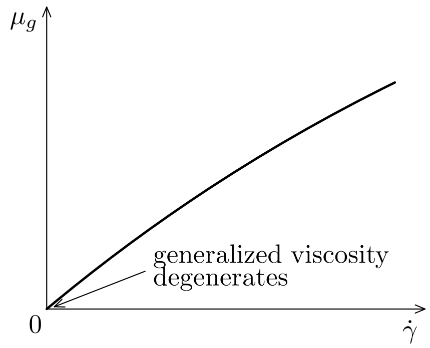


Figure 4: Shear thickening,
(generealized viscosity / shear rate)

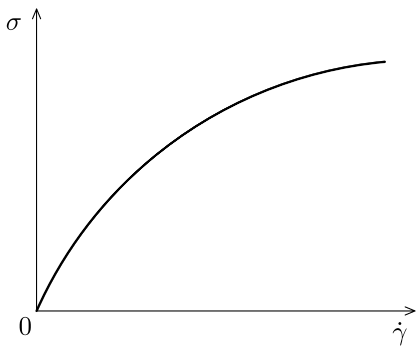


Figure 5: Shear thinning,
(shear stress / shear rate)

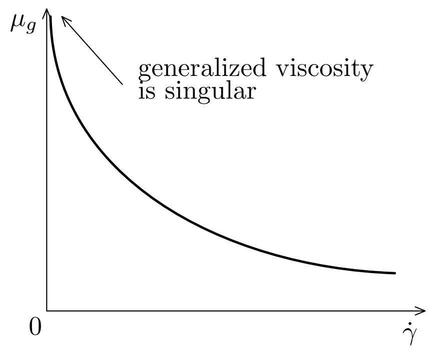


Figure 6: Shear thinning,
(generalized viscosity / shear rate)

The best known models are power-law fluids where the stress tensor look like

$$\mathbf{T} = -p\mathbf{I} + \underbrace{2\mu^*|\mathbf{D}(\mathbf{v})|^{r-2}}_{\mu_g(|\mathbf{D}|)=\tilde{\mu}_g(|\mathbf{D}|^2)} \mathbf{D}(\mathbf{v}).$$

For $r = 2$ (3) we speak about the Newtonian fluid, for $r > 2$ (1) shear thickening fluid and for $1 < r < 2$ (2) shear thinning fluid, see Figures 7 and 8.

Shear thinning fluid is for example the blood. Shear thickenning fluid are for example some painting colors, sauces (dressings), ketchups. Viscosity is declared as a quality parameter for some industrial products.

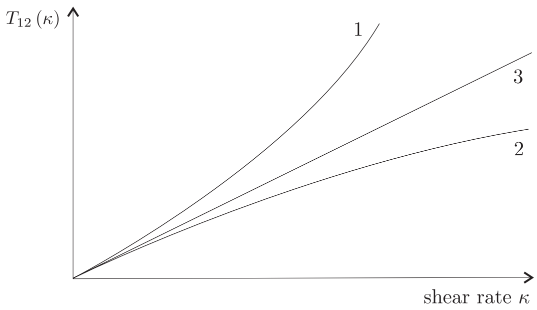


Figure 7: Shear stress / shear rate.

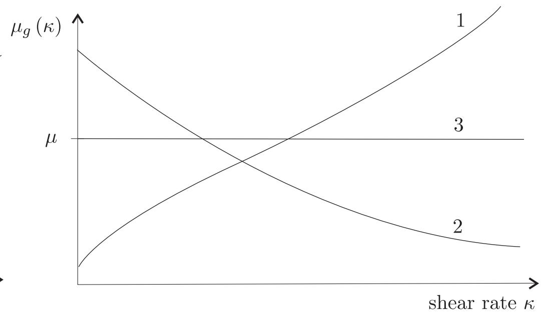


Figure 8: Generalized viscosity / shear rate.

Ad (2) **Pressure thickening property** The generalized viscosity is not constant as in the case of Newtonian fluids[2] (see Figure 9) but it is a function of pressure p (see Figure 10). Experimental data say that the viscosity is an incresing function of the pressure, not decreasing.

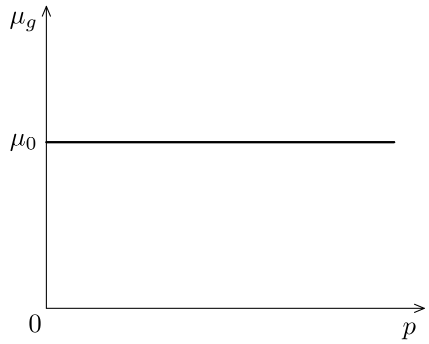


Figure 9: Newtonian fluid

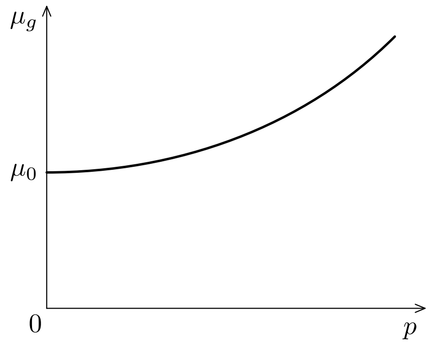


Figure 10: Viscosity dependent on pressure

Around the year 1890 Barus proposed a model

$$\mu(p) = \mu_0 \exp(\alpha p), \ \alpha > 0.$$

The Nobel prize was given for the experimental validation of the pressure dependence. These models are used for example in journal bearings where the pressure differences are very high.

Compressible vs. incompressible fluid For the compressible fluid the pressure $p = \tilde{p}(\varrho)$ is a function of the density ϱ. Let us suppose that we can invert this relation $\varrho = \tilde{\varrho}(p)$, then the formulation

$$\mathbf{T} = -\tilde{p}(\varrho)\mathbf{I} + 2\mu(\varrho)\mathbf{D} + \lambda(\varrho)(\operatorname{div}\mathbf{v})\mathbf{I}$$

is equivalent with

$$\mathbf{T} = -p\mathbf{I} + 2\tilde{\mu}(p)\mathbf{D} + \tilde{\lambda}(p)(\operatorname{div}\mathbf{v})\mathbf{I},$$

which is the model with the viscosity dependent on the pressure.

Een if the material is compressible (but not extremely), the incompressible pressure dependence model is used. Such models are difficult to treat both analytically and numerically.

[2] Already Stokes in his work wrote about the dependence of the viscosity on the pressure. For the simplicity he further supposed the viscosity to be constant.

Ad (3) **Presence of activation/deactivation criteria**

The fluid starts to flow when it reaches the critical value of the stress τ – called the yield stress. in the case that the following dependence of the stress on the shear rate is linear, respectively nonlinear, we call the fluid Bingham fluid, respectively Herschel-Bulkley fluid[3].

The standard formulation is the following:

$$\mathbf{D} = 0 \Leftrightarrow |\mathbf{S}| \le \tau,$$
$$\mathbf{T} = -p\mathbf{I} + 2\tau\frac{\mathbf{D}}{|\mathbf{D}|} + \tilde{\mu}_g(|\mathbf{D}|^2)\mathbf{D} \Leftrightarrow |\mathbf{S}| > \tau.$$

If $\tilde{\mu}_g$ is constant, it is the Bingham fluid, if not, it is Herschel-Bulkley fluid (see Figure 11).

Further example is the fluid where the response is connected with the chemical processes. At some value $|D_{12}|$ the fluid locks and does not flow, see Figure 12.

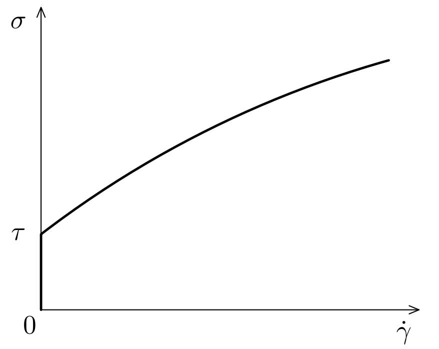


Figure 11: Herschel-Bulkley

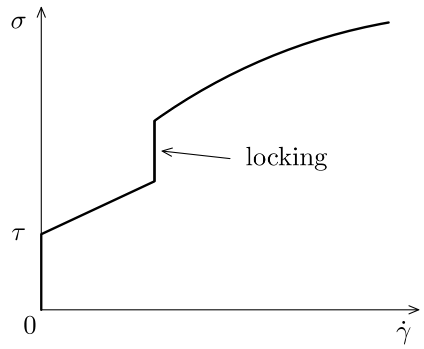


Figure 12: Locking

Except these explicit relations between the shear stress $|S_{12}|$ and shear rate $|D_{12}|$ we can consider a general implicit relation in the form

$$g(|S_{12}|, |D_{12}|) = 0.$$

We do not have to just compute the tensor $\mathbf{S}$ from the knowledge of $\mathbf{D}$ and insert it into the balance of linear momentum, it is also possible to define a tensor function $\mathbf{G}$

$$\mathbf{G}(\mathbf{D}, \mathbf{S}) = 2\mu(|\mathbf{D}|^2)\left(\tau^* + (|\mathbf{S}| - \tau^*)^+\right)\mathbf{D} - (|\mathbf{S}| - \tau^*)^+\mathbf{S}$$

that maps

$$\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$$

and this equation add to the system of balance equations. The advantage is that $\mathbf{G}$ is continuous and no special tools has to be used. The disadvantage is that the system we are solving is bigger.

Ad (4) **Presence of non-zero normal stress differences in a simple shear flow**

In the three dimensional space we define three viscometric functions:

μ	viscosity
$N_1 := T_{11} - T_{22}$	first normal stress difference
$N_2 := T_{22} - T_{33}$	second normal stress difference

In the simple shear flow we suppose a fluid velocity in the form $\mathbf{v} = (u(y), 0, 0)$.

[3]We call it fluid although the definition of the fluid says that the fluid can not sustain the shear stres.

Newtonian fluid Let us compute the Cauchy stress $\mathbf{T}$

$$\mathbf{D} = \frac{1}{2}\begin{pmatrix} 0 & u'(y) & 0 \\ u'(y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} = \begin{pmatrix} -p & \mu u'(y) & 0 \\ \mu u'(y) & -p & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

We can see that $N_1 = N_2 = 0$, and so there are no normal stress differences in the Newtonian.

Non-Newtonian fluid Non-Newtonian fluids like for example the amyl dissolved in the water, exhibit the presence of non-zero stress differences in the simple shear flow. If we press the fluid in one direction, it reacts in other direction, usually perpendicular. These effects are connected with this property:

- Rod climbing
- Die swell
- Delayed die swell
  Let us denote D the diameter of the pipe and D_E the biggest diameter of the fluid after flowing out from the pipe. For the Newtonian fluids it was experimentally found that
  $$\frac{D_E}{D} = 1.13.$$
  For non-Newtonian fluids is this ratio equal even to four and it holds
  $$\frac{D_E}{D} = \left[0.13 + \left(1 + \frac{1}{2}\left(\frac{N_1}{2S}\right)_{\mathrm{w}}\right)^2\right]^{1/6},$$
  where S is the shear stress and a subscript w means that it is a value on the wall.
- Flow through the sloping channel
  In the case of Newtonian fluid the surface of the fluid is smooth (free boundary), in the case of non-Newtonian fluid the surface is not smooth, there are bumps.
- Inverted secondary flow
  Consider a container with a fluid, the fluid is covered with no gap between the fluid and the cover. Now we start to rotate with the cover. The secondary flow creates in the fluid, this secondary flow can be seen in the cross-section. In the case of non-Newtonian fluid is the direction of the secondary flor oposite than in the case of Newtonian fluid and the direction depends on the size of N_2.

Ad (5) **Stress relaxation**

Stress relaxation test is the following (see Figure 13): At time $t = 0$ we deform the material with constant relative prolongation ε and then we study the shear stress σ.

Consider two basic materials: linear elastic spring and linear dashpot (two coaxial cylinders with almost same radii, between these cylinders is a Newtonian fluid with the viscosity μ).

Linear spring is described with Hooke law $\sigma = E\varepsilon$, where E is the elastic modulus. Linear dashpot is described by the relation $\sigma = \mu\dot{\varepsilon}$, where μ is the viscosity.

The result of the stress relaxation test is in the case of linear spring in the Figure 14, in the case of linear dashpot in the Figure 15. In the case of dashpot is the stress singular in zero.

Most of materials are the combination of these two basic materials, their response is depicted in the Figure 16

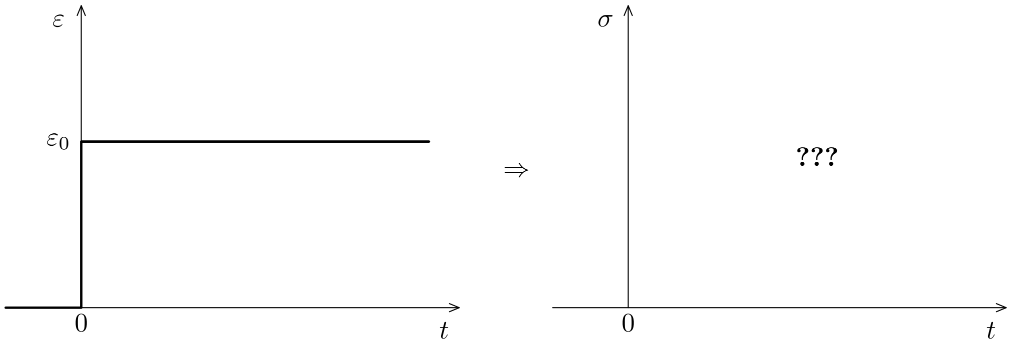


Figure 13: Stress relaxation test

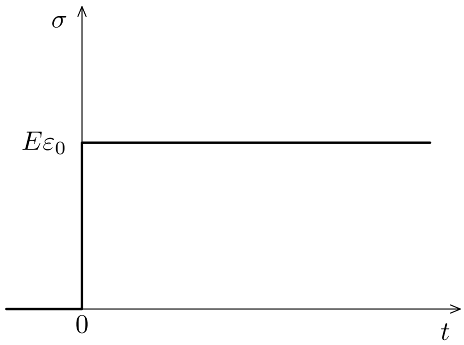


Figure 14: Linear spring

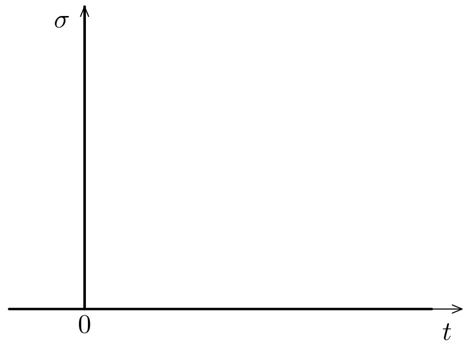


Figure 15: Linear dashpot

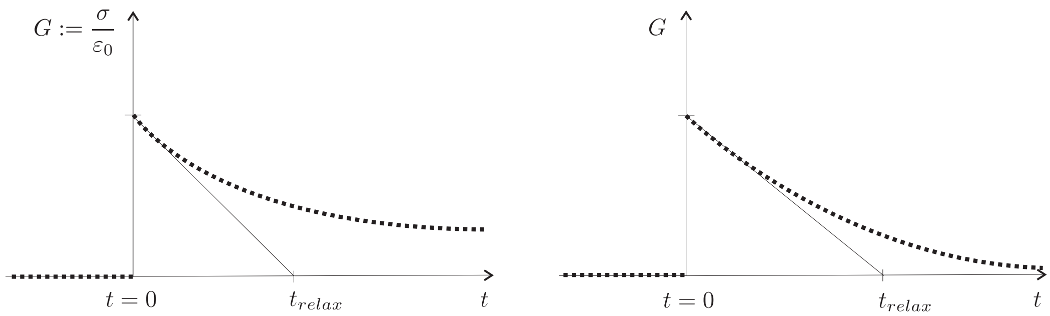


Figure 16: Viscoelastic solid Viscoelastic fluid

Ad (6) **Non-linear creep**

Non-linear creep test is the following (see Figure 17): At time $t = 0$ we deform the material with constant stress σ, at time $t = t^*$ we turn off the stress and study the relative prolongation ε.

The result of the creep test is in the case of linear spring in Figure 18, in the case of linear dashpot in Figure 19.

Again, most of materials are the combination of two basic materials, in Figure 20 is the resultfor the material similar to viscoelastic solid a viscoelastic fluid.

Except the algebraic models there exist also rate-type models, integral models and stochastic models. A detailed overview of non-Newtonian models is in the following section.

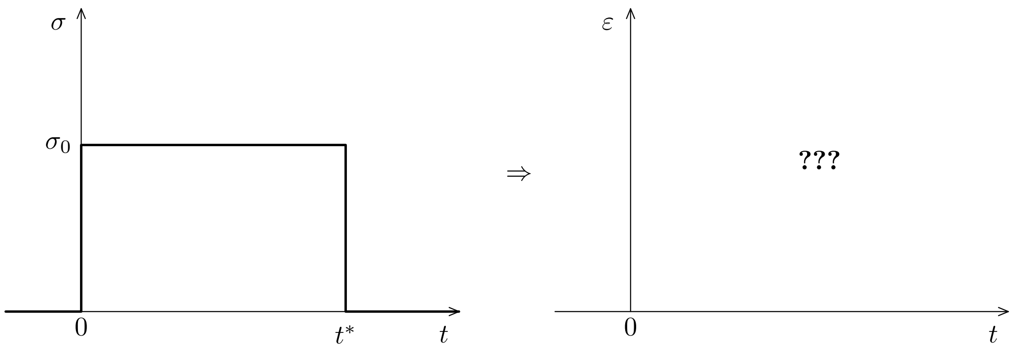


Figure 17: Non-linear creep test

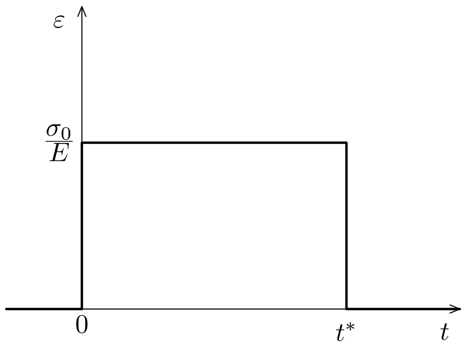


Figure 18: Linear spring

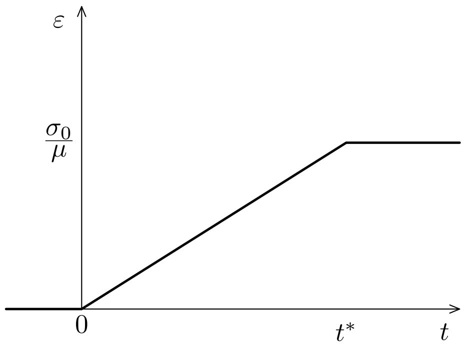


Figure 19: Linear dashpot

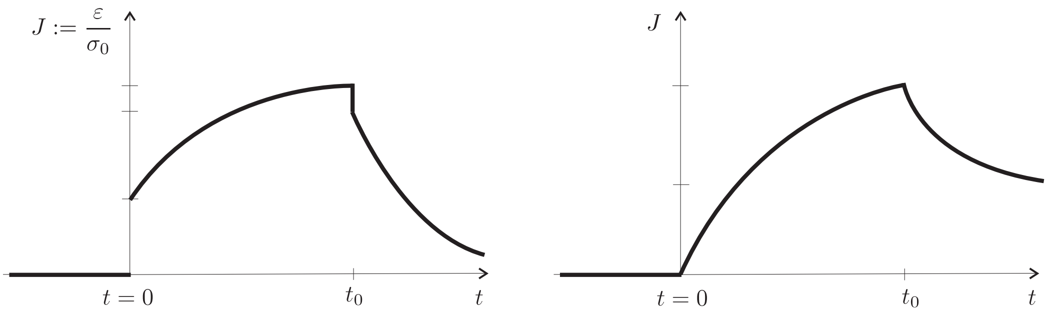


Figure 20: Viscoelastic solid Viscoelastic fluid

Non-Newtonian models

Models with variable viscosity

General form:

$$\mathbf{T} = -p\mathbf{I} + \underbrace{2\mu(\mathbf{D}, \mathbf{T})\mathbf{D}}_{\mathbf{S}} \tag{4}$$

Particular models mainly developed by chemical engineers.

Shear dependent viscosity

Ostwald–de Waele power law [24], [12]

$$\mu(\mathbf{D}) = \mu_0 |\mathbf{D}|^{n-1} \tag{5}$$

Fits experimental data for: ball point pen ink, molten chocolate, aqueous dispersion of polymer latex spheres

Carreau Carreau–Yasuda [8], [36]

$$\mu(\mathbf{D}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1 + \alpha|\mathbf{D}|^2)^{\frac{n}{2}}} \tag{6}$$

$$\mu(\mathbf{D}) = \mu_\infty + (\mu_0 - \mu_\infty)\left(1 + \alpha|\mathbf{D}|^a\right)^{\frac{n-1}{a}} \tag{7}$$

Fits experimental data for: molten polystyrene

Eyring [14] , [29]

$$\mu(\mathbf{D}) = \mu_\infty + (\mu_0 - \mu_\infty)\frac{\operatorname{arcsinh}(\alpha|\mathbf{D}|)}{\alpha|\mathbf{D}|} \tag{8}$$

$$\mu(\mathbf{D}) = \mu_0 + \mu_1\frac{\operatorname{arcsinh}(\alpha_1|\mathbf{D}|)}{\alpha_1|\mathbf{D}|} + \mu_2\frac{\operatorname{arcsinh}(\alpha_2|\mathbf{D}|)}{\alpha_2|\mathbf{D}|} \tag{9}$$

Fits experimental data for: napalm (coprecipitated aluminum salts of naphthenic and palmitic acids; jellied gasoline), 1% nitrocelulose in 99% butyl acetate

Cross [11]

$$\mu(\mathbf{D}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + \alpha|\mathbf{D}|^n} \tag{10}$$

Fits experimental data for: aqueous polyvinyl acetate dispersion, aqueous limestone suspension

Sisko [35]

$$\mu(\mathbf{D}) = \mu_\infty + \alpha|\mathbf{D}|^{n-1} \tag{11}$$

Fits experimental data for: lubricating greases

Models with pressure dependent viscosity

Barus [1]

$$\mu(\mathbf{T}) = \mu_{\text{ref}} \exp\beta\,(p - p_{\text{ref}}) \tag{12}$$

Fits experimental data for: mineral oils[4], organic liquids[5]

Models with stress dependent viscosity

Ellis [21]

$$\mu(\mathbf{T}) = \frac{\mu_0}{1 + \alpha|\mathbf{T}^\delta|^{n-1}} \tag{13}$$

Fits experimental data for: 0.6% w/w carboxymethyl cellulose (CMC) solution in water, poly(vynil chloride)[6]

[4] [20]
[5] [6]
[6] [33]

Glen [16]

$$\mu(\mathbf{T}) = \alpha |\mathbf{T}^\delta|^{n-1} \tag{14}$$

Fits experimental data for: ice

Seely [34]

$$\mu(\mathbf{T}) = \mu_\infty + (\mu_0 - \mu_\infty) \exp\left(-\frac{|\mathbf{T}^\delta|}{\tau_0}\right) \tag{15}$$

Fits experimental data for: polybutadiene solutions

Blatter [25] , [5]

$$\mu(\mathbf{T}) = \frac{A}{(|\mathbf{T}^\delta|^2 + \tau_0^2)^{\frac{n-1}{2}}} \tag{16}$$

Fits experimental data for: ice

Models with discontinuous rheology

Bingham Herschel–Bulkley [4], [17]

$$\begin{aligned} |\mathbf{T}^\delta| > \tau^* \quad &\text{if and only if} \quad \mathbf{T}^\delta = \tau^* \frac{\mathbf{D}}{|\mathbf{D}|} + 2\mu(|\mathbf{D}|)\mathbf{D} \\ |\mathbf{T}^\delta| \leq \tau^* \quad &\text{if and only if} \quad \mathbf{D} = 0 \end{aligned} \tag{17}$$

Fits experimental data for: paints, toothpaste, mango jam[7]

Differential type models

Rivlin–Ericksen fluids

Rivlin–Ericksen [31], [32]

General form:

$$\mathbf{T} = -p\mathbf{I} + \mathfrak{f}(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \dots) \tag{18}$$

where

$$\mathbf{A}_1 = 2\mathbf{D} \tag{19a}$$

$$\mathbf{A}_n = \frac{\mathrm{d}\mathbf{A}_{n-1}}{\mathrm{d}t} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{n-1} \tag{19b}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the usual Lagrangean time derivative and $\mathbf{L}$ is the velocity gradient.

Criminale–Ericksen–Filbey [10]

$$\mathbf{T} = -p\mathbf{I} + \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_1^2 \tag{20}$$

Fits experimental data for: polymer melts (explains mormal stress differences)

Reiner–Rivlin [30]

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + \mu_1\mathbf{D}^2 \tag{21}$$

Fits experimental data for: N/A

[7] [2]

Rate type models

Maxwell, Oldroyd, Burgers

Maxwell [22]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \tag{22a}$$

$$\mathbf{S} + \lambda_1 \overset{\triangledown}{\mathbf{S}} = 2\mu\mathbf{D} \tag{22b}$$

$$\overset{\triangledown}{\mathbf{M}} := \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} - \mathbf{L}\mathbf{M} - \mathbf{M}\mathbf{L}^T \tag{23}$$

Fits experimental data for: N/A

Oldroyd-B [23]

$$\mathbf{T} = -\pi\mathbf{I} + \mathbf{S} \tag{24a}$$

$$\mathbf{S} + \lambda \overset{\triangledown}{\mathbf{S}} = \eta_1 \mathbf{A}_1 + \eta_2 \overset{\triangledown}{\mathbf{A}}_1 \tag{24b}$$

Fits experimental data for: N/A

Oldroyd 8-constants

[23]

$$\mathbf{T} = -\pi\mathbf{I} + \mathbf{S} \tag{25a}$$

$$\mathbf{S} + \lambda_1 \overset{\triangledown}{\mathbf{S}} + \frac{\lambda_3}{2}\left(\mathbf{D}\mathbf{S} + \mathbf{S}\mathbf{D}\right) + \frac{\lambda_5}{2}\left(\operatorname{tr}\mathbf{S}\right)\mathbf{D} + \frac{\lambda_6}{2}\left(\mathbf{S} : \mathbf{D}\right)\mathbf{I} \tag{25b}$$

$$= -\mu\left(\mathbf{D} + \lambda_2 \overset{\triangledown}{\mathbf{D}} + \lambda_4 \mathbf{D}^2 + \frac{\lambda_7}{2}\left(\mathbf{D} : \mathbf{D}\right)\mathbf{I}\right) \tag{25c}$$

Fits experimental data for: N/A

Burgers [7]

$$\mathbf{T} = -\pi\mathbf{I} + \mathbf{S} \tag{26a}$$

$$\mathbf{S} + \lambda_1 \overset{\triangledown}{\mathbf{S}} + \lambda_2 \overset{\triangledown\triangledown}{\mathbf{S}} = \eta_1 \mathbf{A}_1 + \eta_2 \overset{\triangledown}{\mathbf{A}}_1 \tag{26b}$$

Fits experimental data for: N/A

Giesekus [15]

$$\mathbf{T} = -\pi\mathbf{I} + \mathbf{S} \tag{27a}$$

$$\mathbf{S} + \lambda \overset{\triangledown}{\mathbf{S}} - \frac{\alpha\lambda_2}{\mu}\mathbf{S}^2 = -\mu\mathbf{D} \tag{27b}$$

Fits experimental data for: N/A

Phan-Thien–Tanner [26], [27]

$$\mathbf{T} = -\pi\mathbf{I} + \mathbf{S} \tag{28a}$$

$$Y\mathbf{S} + \lambda\overset{\triangledown}{\mathbf{S}} + \frac{\lambda\xi}{2}\left(\mathbf{DS} + \mathbf{SD}\right) = -\mu\mathbf{D} \tag{28b}$$

$$Y = \exp\left(-\varepsilon\frac{\lambda}{\mu}\operatorname{tr}\mathbf{S}\right) \tag{28c}$$

Fits experimental data for: N/A

Johnson–Segalman [19]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \tag{29a}$$

$$\mathbf{S} = 2\mu\mathbf{D} + \mathbf{S}' \tag{29b}$$

$$\mathbf{S}' + \lambda\left(\frac{\mathrm{d}\mathbf{S}'}{\mathrm{d}t} + \mathbf{S}'\left(\mathbf{D} - a\mathbf{D}\right) + \left(\mathbf{D} - a\mathbf{D}\right)^T\mathbf{S}'\right) = 2\eta\mathbf{D} \tag{29c}$$

Fits experimental data for: spurt

Johnson–Tevaarwerk [18]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \tag{30a}$$

$$\overset{\triangledown}{\mathbf{S}} + \alpha\sinh\frac{\mathbf{S}}{s_0} = 2\mu\mathbf{D} \tag{30b}$$

Fits experimental data for: lubricants

Integral type models

Kaye–Bernstein–Kearsley–Zapas [3], [9]

$$\mathbf{T} = \int_{\xi=-\infty}^{t}\frac{\partial\mathbf{W}}{\partial I}\mathbf{C} + \frac{\partial\mathbf{W}}{\partial II}\mathbf{C}^{-1}d\xi \tag{31}$$

Fits experimental data for: polyisobutylene, vulcanised rubber

Entropy and Newtonian fluids

Definition 4. *There exists a specific density of entropy η (further just entropy) which is a function of state variables,*

$$\eta = \tilde{\eta}(y_0, y_1, \dots),$$

where $y_0 = e$ is a density of internal energy (further just energy), and it holds:

(i) $\dfrac{\partial \tilde{\eta}}{\partial e} > 0 \;\Rightarrow\; e = \tilde{e}(\eta, y_1, y_2, \dots)$, *denote a temperature* $\theta := \dfrac{\partial \tilde{e}}{\partial \eta}$

(ii) $\eta \to 0_+$, *if* $\theta \to 0_+$

(iii) $S(t) = \int_\Omega (\varrho\eta)(t,x)\,\mathrm{d}x \to S_{\max}$ *for* $t \to +\infty$, *if* Ω *is mechanically and thermally (energetically) isolated*

For Newtonian fluids it holds

$$\eta = \tilde{\eta}(e, \varrho) \Leftrightarrow e = \tilde{e}(\eta, \varrho).$$

Balance equations:

$$\begin{aligned} \dot{\varrho} &= -\varrho \operatorname{div} \mathbf{v}, \\ \varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T} + \varrho \mathbf{b}, \\ \varrho \dot{E} &= \operatorname{div}(\mathbf{T}\mathbf{v} + \mathbf{q}) + \varrho \mathbf{b} \cdot \mathbf{v}. \end{aligned}$$

Differentiate e, multiply by ϱ,

$$\varrho \dot{e} = \varrho \frac{\partial \tilde{e}}{\partial \eta} \dot{\eta} + \varrho \frac{\partial \tilde{e}}{\partial \varrho} \dot{\varrho}$$

and insert it into the balance of linear momentum,

$$\varrho \mathbf{b} \cdot \mathbf{v} + \operatorname{div}(\mathbf{T}\mathbf{v} + \mathbf{q}) - (\operatorname{div} \mathbf{T}) \cdot \mathbf{v} - \varrho \mathbf{b} \cdot \mathbf{v} = \varrho \frac{\partial \tilde{e}}{\partial \eta} \dot{\eta} - \varrho^2 \frac{\partial \tilde{e}}{\partial \varrho} \operatorname{div} \mathbf{v}.$$

Now denote

$$\theta = \frac{\partial \tilde{e}}{\partial \eta}, \quad p = \varrho^2 \frac{\partial \tilde{e}}{\partial \varrho}$$

and obtain

$$\begin{aligned} \varrho \theta \dot{\eta} &= \mathbf{T} \cdot \nabla \mathbf{v} + \operatorname{div} \mathbf{q} + p \operatorname{div} \mathbf{v} \\ &\overset{\text{symetrie } \mathbf{T}}{=} \mathbf{T} \cdot \mathbf{D} + \operatorname{div} \mathbf{q} + p \operatorname{div} \mathbf{v}. \end{aligned}$$

Using the deviatoric parts we get

$$\begin{aligned} \varrho \theta \dot{\eta} &= \mathbf{T}^d \cdot \mathbf{D}^d + \frac{1}{3}(\operatorname{tr} \mathbf{T}) \frac{1}{3} (\operatorname{div} \mathbf{v}) \underbrace{\mathbf{I} \cdot \mathbf{I}}_{3} + \operatorname{div} \mathbf{q} + p \operatorname{div} \mathbf{v} \\ &= \mathbf{T}^d \cdot \mathbf{D}^d + \left(\frac{1}{3} (\operatorname{tr} \mathbf{T}) + p \right) \operatorname{div} \mathbf{v} + \operatorname{div} \mathbf{q}. \end{aligned}$$

Dividing by θ we get

$$\varrho \dot{\eta} - \operatorname{div} \left(\frac{\mathbf{q}}{\theta} \right) = \underbrace{\frac{1}{\theta} \left[\mathbf{T}^d \cdot \mathbf{D}^d + \left(\frac{1}{3} (\operatorname{tr} \mathbf{T}) + p \right) \operatorname{div} \mathbf{v} + \frac{\mathbf{q} \cdot \nabla \theta}{\theta} \right]}_{\xi}.$$

ξ is called rate of entropy production. The second law of thermodynamics says that $\xi \geq 0$, and so

$$\zeta := \frac{\xi}{\theta} = \varrho\dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) \geq 0.$$

If we denote $m = (\operatorname{tr}\mathbf{T})/3$ the mean normal stress, then it holds

$$\xi = \mathbf{T}^d \cdot \mathbf{D}^d + \left(\frac{1}{3}(\operatorname{tr}\mathbf{T}) + p\right)\operatorname{div}\mathbf{v} + \frac{\mathbf{q}\cdot\nabla\theta}{\theta} = \underbrace{\{\mathbf{T}^d, m+p, \mathbf{q}\}}_{\text{termodynamical fluxes}} \cdot \underbrace{\left\{\mathbf{D}^d, \operatorname{div}\mathbf{v}, \frac{\nabla\theta}{\theta}\right\}}_{\text{termodynamical afinities}} \tag{32}$$

What is ξ for Navier-Stokes equations?

Compressible Navier-Stokes-Fourier equations

$$\begin{aligned}\mathbf{T} &= -p(\varrho,\eta)\mathbf{I} + 2\mu(\varrho,\eta)\mathbf{D} + \lambda(\varrho,\eta)(\operatorname{div}\mathbf{v})\mathbf{I},\\ \mathbf{q} &= K(\varrho,\eta)\nabla\theta.\end{aligned}$$

It holds

$$m + p = \frac{2\mu + 3\lambda}{3}\operatorname{div}\mathbf{v}$$

and

$$\mathbf{T}^d = 2\mu\mathbf{D}^d.$$

Insert it into (32).

Option 1

$$\xi = 2\mu|\mathbf{D}^d|^2 + \frac{2\mu+3\lambda}{3}(\operatorname{div}\mathbf{v})^2 + K\frac{|\nabla\theta|^2}{\theta}.$$

If $\mu \geq 0$, $2\mu + 3\lambda \geq 0$ a $K \geq 0$, then $\xi \geq 0$.

Option 2

$$\xi = \frac{1}{2\mu}|\mathbf{T}^d|^2 + \frac{3}{2\mu+3\lambda}(m+p)^2 + \frac{1}{K}\frac{|\mathbf{q}|^2}{\theta}.$$

If $\mu > 0$, $2\mu + 3\lambda > 0$ a $K > 0$, then $\xi > 0$.

We ask: When is $\xi = 0$?

Option 1 We do not want any restriction for the velocity or temperature, that is why $\mu = 2\mu + 3\lambda = K = 0$.

Option 2 It is possible only if $\mathbf{T}^d = \mathbf{0}, m = -p, \mathbf{q} = \mathbf{0}$.

Incompressible Navier-Stokes-Fourier equations

Option 1

$$\xi = 2\mu|\mathbf{D}^d|^2 + K\frac{|\nabla\theta|^2}{\theta}.$$

If $\mu \geq 0$ a $K \geq 0$, then $\xi \geq 0$.

Option 2

$$\xi = \frac{1}{2\mu}|\mathbf{T}^d|^2 + \frac{1}{K}\frac{|\mathbf{q}|^2}{\theta}.$$

If $\mu > 0$ a $K > 0$, then $\xi > 0$.

Remark

$$\theta = \frac{\partial e}{\partial \eta}, \quad p = \varrho^2 \frac{\partial e}{\partial \varrho}.$$

Gibbs equation

$$\theta \, \mathrm{d}\eta = \mathrm{d}e + p \, \mathrm{d}\left(\frac{1}{\varrho}\right),$$

from this equation the relation for the pressure can be derived. This equation holds for compressible gas, there is no reason why it should hold in general for more complicated materials.

Question: Is there a way how from the knowledge for constitutive relations for η and ξ get the constitutive relations for $\mathbf{T}$ and $\mathbf{q}$?

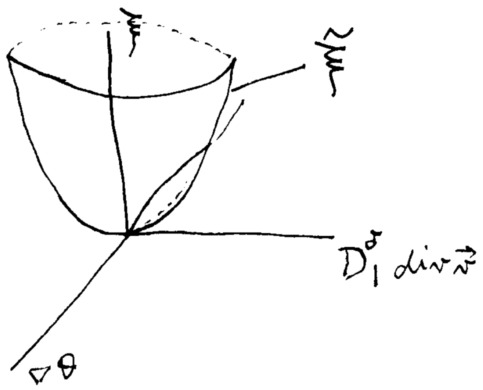
Principle of maximimzation of the rate of entropy production

Our aim is to derive Navier-Stokes equations. It holds

$$\xi = \mathbf{T}^d \cdot \mathbf{D}^d + (m+p) \operatorname{div} \mathbf{v} + \frac{\mathbf{q} \cdot \nabla\theta}{\theta}, \tag{33}$$

we choose a constitutive relation for $\tilde{\xi}$

$$\xi = \tilde{\xi}(\mathbf{D}^d, \operatorname{div} \mathbf{v}, \nabla\theta) = 2\mu|\mathbf{D}^d|^2 + \frac{2\mu + 3\lambda}{3}(\operatorname{div} \mathbf{v})^2 + K\frac{|\nabla\theta|^2}{\theta}.$$



There is many possibilities when $\xi = \tilde{\xi}$, we choose the one when $\tilde{\xi}$ is maximal. That is why we maximize it over $\mathbf{D}^d, \operatorname{div} \mathbf{v}, \nabla\theta$ such that (33) holds

$$\max_{(33)+\mathbf{D}^d, \operatorname{div} \mathbf{v}, \nabla\theta} \tilde{\xi}(\mathbf{D}^d, \operatorname{div} \mathbf{v}, \nabla\theta).$$

This heuristic principle is called *principle of maximimzation of the rate of entropy production.*

The maximization is done using Lagrangeo multipiers. We define the Lagrange function

$$L = \tilde{\xi}(\mathbf{D}^d, \operatorname{div} \mathbf{v}, \nabla\theta) + \lambda\left(\tilde{\xi}(\mathbf{D}^d, \operatorname{div} \mathbf{v}, \nabla\theta) - \left(\mathbf{T}^d \cdot \mathbf{D}^d + (m+p) \operatorname{div} \mathbf{v} + \frac{\mathbf{q} \cdot \nabla\theta}{\theta}\right)\right),$$

and we are looking for its extreme , i.e.

$$\frac{\partial L}{\partial \mathbf{D}^d} = \mathbf{0}, \ \frac{\partial L}{\partial \operatorname{div} \mathbf{v}} = 0, \ \frac{\partial L}{\partial \nabla\theta} = \mathbf{0}.$$

We obtain

$$
\begin{aligned}
(1+\lambda)\frac{\partial\tilde{\xi}}{\partial\mathbf{D}^d} &= \lambda\mathbf{T}^d, \\
(1+\lambda)\frac{\partial\tilde{\xi}}{\partial\operatorname{div}\mathbf{v}} &= \lambda(m+p), \\
(1+\lambda)\frac{\partial\tilde{\xi}}{\partial\nabla\theta} &= \lambda\frac{\mathbf{q}}{\theta}.
\end{aligned}
\tag{34}
$$

Now we eliminate λ in this way. We multiply the first equation by $\mathbf{D}^d$, the second by $\operatorname{div}\mathbf{v}$ and the third by $\nabla\theta$,

$$
\frac{1+\lambda}{\lambda} = \frac{\tilde{\xi}}{\frac{\partial\tilde{\xi}}{\partial\mathbf{D}^d}\cdot\mathbf{D}^d + \frac{\partial\tilde{\xi}}{\partial\operatorname{div}\mathbf{v}}\operatorname{div}\mathbf{v} + \frac{\partial\tilde{\xi}}{\partial\nabla\theta}\cdot\nabla\theta} = \frac{1}{2}.
$$

If we insert this result into (34) and do the differentiation, we obtain Navier-Stokes equations

$$
\begin{aligned}
\mathbf{T}^d &= 2\mu\mathbf{D}^d, \\
m+p &= \frac{2\mu+3\lambda}{3}\operatorname{div}\mathbf{v}, \\
\frac{\mathbf{q}}{\theta} &= K\frac{\nabla\theta}{\theta}.
\end{aligned}
$$

Maximization $\tilde{\xi}$ for incompressible Navier-Stokes equations

For isothermal incompressible materials it holds

$$
\operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v} = 0,\ \theta = \theta^* = const.
$$

Constraint for ξ is

$$
\xi = \mathbf{T}\cdot\mathbf{D}. \tag{35}
$$

First, let $\tilde{\xi}$ is given by

$$
\tilde{\xi} = 2\mu|\mathbf{D}|^2.
$$

We use the principle of maximization of the rate of entropy production

$$
\max_{(35)+\mathbf{D}+\operatorname{tr}\mathbf{D}=0} \tilde{\xi}(\mathbf{D}).
$$

We define the Lagrange function

$$
L = \tilde{\xi}(\mathbf{D}) + \lambda_1\Big(\tilde{\xi}(\mathbf{D}) - (\mathbf{T}\cdot\mathbf{D})\Big) + \lambda_2\operatorname{tr}\mathbf{D}
$$

and maximize it with respect to $\mathbf{D}$,

$$
(1+\lambda_1)\frac{\partial\tilde{\xi}}{\partial\mathbf{D}} = \lambda_1\mathbf{T} - \lambda_2\mathbf{I}.
$$

Multiply this equation by $\mathbf{D}$ and get

$$
\frac{1+\lambda_1}{\lambda_1} = \frac{\mathbf{T}\cdot\mathbf{D}}{\frac{\partial\tilde{\xi}}{\partial\mathbf{D}}\cdot\mathbf{D}} = \frac{1}{2},
$$

further take the trace of this equations and get

$$
3\lambda_2 = \lambda_1\operatorname{tr}\mathbf{T} \ \Rightarrow\ \frac{\lambda_2}{\lambda_1} = \frac{1}{3}\operatorname{tr}\mathbf{T}.
$$

We have

$$\mathbf{T} = 2\mu\mathbf{D} + \frac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I}.$$

Incompressible Navier-Stokes equations can be also derived in other way. Let

$$\tilde{\xi} = \frac{1}{2\mu}|\mathbf{T}^d|^2 \quad (\mathbf{q} = \mathbf{0}) \tag{36}$$

and we are maximizing $\tilde{\xi}$

$$\max_{(36)+\mathbf{T}^d} \tilde{\xi}(\mathbf{T}^d).$$

We obtain

$$\mathbf{D}^d = \mathbf{D} = \frac{1}{2\mu}\mathbf{T}^d = \frac{1}{2\mu}\left(\mathbf{T} - \frac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I}\right),$$

which is less usual, but equivalent formulation of incompressible Navier-Stokes equations.

Integral form of entropy production

Integrate the balance of energy ($\mathbf{b} = 0$, $r = 0$, $\mathbf{v}\cdot\mathbf{n} = 0$) over Ω

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho\left(e + \frac{|\mathbf{v}|^2}{2}\right)\,\mathrm{d}x = \int_\Omega \operatorname{div}\,(\mathbf{T}\mathbf{v} + \mathbf{q})\,\mathrm{d}x = \int_{\partial\Omega}(\mathbf{T}\mathbf{v} + \mathbf{q})\cdot\mathbf{n}\,\mathrm{d}S = 0$$

In order to have conservation of energy it suffices to assume pointwise

$$\mathbf{q}\cdot\mathbf{n} = -\mathbf{T}\mathbf{v}\cdot\mathbf{n} = -\mathbf{T}\mathbf{n}\cdot\mathbf{v} = -\left((\mathbf{T}\mathbf{n})_\tau + (\mathbf{T}\mathbf{n})_n\right)\cdot(\mathbf{v}_\tau + (\mathbf{v}\cdot\mathbf{n})\mathbf{n}) = -(\mathbf{T}\mathbf{n})_\tau\cdot\mathbf{v}_\tau$$

For the total entropy $S = \int_\Omega \varrho\eta\,\mathrm{d}x$ we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = \int_\omega \frac{1}{\theta}\left[\mathbf{T}^\delta : \mathbf{D}^\delta + \dots\right] - \int_{\partial\Omega}\frac{\mathbf{q}\cdot\mathbf{n}}{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = \left(\mathbf{T}^\delta, \mathbf{D}^\delta\right)_{L^2(\Omega)} + (m + p, \operatorname{div}\mathbf{v})_{L^2(\Omega)} + \left(\mathbf{q}, \frac{\nabla\theta}{\theta}\right)_{L^2(\Omega)} + (-(\mathbf{T}\mathbf{n})_\tau, \mathbf{v}_\tau)_{L^2(\Omega)}$$

The simplest proportional relation

$$\begin{aligned}\mathbf{T}^d &= 2\mu\mathbf{D}^d,\\ m + p &= \frac{2\mu + 3\lambda}{3}\operatorname{div}\mathbf{v},\\ \frac{\mathbf{Q}}{\theta} &= K\frac{\nabla\theta}{\theta}\\ (\mathbf{T}\mathbf{n})_\tau &= -\gamma\mathbf{v}_\tau\end{aligned}$$

with

$$\mu \geq 0, \quad 2\mu + 3\lambda \geq 0, \quad K \geq 0, \quad \gamma \geq 0$$

guarantee the second law of thermodynamics.

In general we have

$$\begin{aligned}&\mathcal{G}_1\left(\mathbf{T}^d, \mathbf{D}^d\right) = 0,\\ &\mathcal{G}_2\left(m + p, \operatorname{div}\mathbf{v}\right) = 0,\\ &\mathcal{G}_3\left(\frac{\mathbf{Q}}{\theta}, \frac{\nabla\theta}{\theta}\right) = 0,\\ &\mathcal{G}_4\left((\mathbf{T}\mathbf{n})_\tau, \mathbf{v}_\tau\right) = 0.\end{aligned}$$

If $\mathbf{D}^\delta = 0$ or $\operatorname{div}\mathbf{v} = 0$ or $\nabla\theta = 0$ or $\mathbf{v}_\tau = 0$ we get constrained material (rigid), resp. incompressible, resp. isothermal process, resp. no-slip boundary condition.

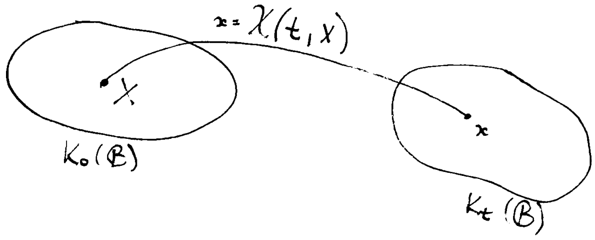
No dissipation if $\mathbf{T}^\delta = 0$ and $\mathbf{q} = 0$ and $m = -p$ and $(\mathbf{T}\mathbf{n})_\tau = 0$.

Remark: If you have dissipation you have energy estimates

Cauchy model for finite elasticity

Kinematical quantities

$$\mathbf{F}_{K_0} = \frac{\partial \chi}{\partial X},$$
$$\dot{\mathbf{F}}_{K_0} = \mathbf{L}\mathbf{F}_{K_0} \;\Rightarrow\; \mathbf{B}_{K_0} = \mathbf{F}_{K_0}\mathbf{F}_{K_0}^{\mathrm{T}},$$



where $\mathbf{B}_{K_0}$ is the left Cauchy-Green tensor, we can compute its time derivative

$$\dot{\mathbf{B}}_{K_0} = \mathbf{L}\mathbf{B}_{K_0} + \mathbf{B}_{K_0}\mathbf{L}^{\mathrm{T}} \text{a } \operatorname{tr}\dot{\mathbf{B}}_{K_0} = 2\mathbf{D}\cdot\mathbf{B}_{K_0}.$$

Let us have the entropy

$$\eta = \tilde{\eta}(e, \mathbf{B}_{K_0}) \Leftrightarrow e = \tilde{e}(\eta, \mathbf{B}_{K_0}),$$

For example the neo-Hookean material fullfils

$$\tilde{e} = e_0(\eta) + \frac{\mu}{2}(\operatorname{tr}\mathbf{B}_{K_0} - 3).$$

Insert e into the balance of energy

$$\begin{aligned}
\varrho\theta\dot{\eta} &= \operatorname{div}(\mathbf{T}\mathbf{v} + \mathbf{q}) - \operatorname{div}\mathbf{T}\cdot\mathbf{v} - \mu\mathbf{B}_{K_0}\cdot\mathbf{D} \\
&= \mathbf{T}\cdot\mathbf{D} - \mu\mathbf{B}_{K_0}\cdot\mathbf{D} + \operatorname{div}\mathbf{q} \\
&= (\mathbf{T} - \mu\mathbf{B}_{K_0})\cdot\mathbf{D} + \operatorname{div}\mathbf{q}.
\end{aligned}$$

We get

$$\varrho\dot{\eta} + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \frac{1}{\theta}\underbrace{\left[(\mathbf{T} - \mu\mathbf{B}_{K_0})\cdot\mathbf{D} + \frac{\mathbf{q}\cdot\nabla\theta}{\theta}\right]}_{\text{produkce entropie}}.$$

Elastic material does not disipate, that is why the rate of entropy production has to be zero, and so

$$\mathbf{T} = \mu\mathbf{B}_{K_0},$$

and for the incompressible material we have

$$\mathbf{T} = -\Phi\mathbf{I} + \mu\mathbf{B}_{K_0}.$$

Korteweg fluids

We will use the principle of maximization of the rate of entropy production to derive the model derived in 1901 by Dutch mathematician Korteweg. This model is used for describing capilar effects, multiphase and granulated materials. The model is formulated by the following equations

$$\begin{aligned}
\dot{\varrho} &= -\varrho\operatorname{div}\mathbf{v}, \\
\varrho\dot{\mathbf{v}} &= \operatorname{div}\mathbf{T}, \\
\mathbf{T} &= -p(\varrho)\mathbf{I} + 2\mu(\varrho)\mathbf{D}(\mathbf{v}) + \lambda(\varrho)(\operatorname{div}\mathbf{v})\mathbf{I} + \mathbf{K}, \\
\mathbf{K} &= \frac{\kappa}{2}(\Delta(\varrho^2) - |\nabla\varrho|^2)\mathbf{I} - \kappa(\nabla\varrho\otimes\nabla\varrho).
\end{aligned}$$

In the paper [13] Dunn and Serrin showed that this model is thermodynamically compatible. We know that if we suppose this constitutive relation

$$\eta = \tilde{\eta}(e, \varrho),$$

then for compressible Navier-Stokes equations we obtain

$$\begin{aligned}\varrho\dot{\eta} + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) &= \frac{1}{\theta}\left[\mathbf{T}^d \cdot \mathbf{D}^d + (m+p)\operatorname{div}\mathbf{v} + \frac{\mathbf{q}\cdot\nabla\theta}{\theta}\right] \\ &= \frac{1}{\theta}\left[\mathbf{T}^d_{\mathrm{dis}} \cdot \mathbf{D}^d + t_{\mathrm{dis}}\operatorname{div}\mathbf{v} + \frac{\mathbf{q}_{\mathrm{dis}}\cdot\nabla\theta}{\theta}\right].\end{aligned}$$

We introduce disipative terms (with subscript dis) and write the constitutive relation for the rate of entropy production

$$\xi = \tilde{\xi}(\mathbf{T}^d_{\mathrm{dis}}, t_{\mathrm{dis}}, \mathbf{q}_{\mathrm{dis}}) = \frac{1}{2\mu}|\mathbf{T}^d_{\mathrm{dis}}|^2 + \frac{2}{2\mu+3\lambda}t^2_{\mathrm{dis}} + \frac{1}{K}|\mathbf{q}_{\mathrm{dis}}|^2$$

and using the principle of maximization of the rate of entropy production we get Navier-Stokes-Fourier system.

For the Korteweg model we consider the entropy dependent on the gradient of density ϱ

$$\eta = \tilde{\eta}(e, \varrho, \nabla\varrho) \leftrightarrow e = \tilde{e}(\eta, \varrho, \nabla\varrho).$$

Differentiate

$$\varrho\dot{e} = \varrho\frac{\partial\tilde{e}}{\partial\eta}\dot{\eta} + \varrho\frac{\partial\tilde{e}}{\partial\varrho}\dot{\varrho} + \varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\dot{\overline{\nabla\varrho}},$$

we know that $\dot{\varrho} = -\varrho\operatorname{div}\mathbf{v}$, and need to compute $\dot{\overline{\nabla\varrho}}$,

$$\dot{\overline{\nabla\varrho}} = -\nabla(\varrho\operatorname{div}\mathbf{v}) - (\nabla\mathbf{v})\nabla\varrho.$$

Let us denote

$$\theta = \frac{\partial\tilde{e}}{\partial\eta}, p = \varrho^2\frac{\partial\tilde{e}}{\partial\varrho},$$

therefore we have

$$\begin{aligned}\varrho\theta\dot{\eta} &= \varrho\dot{E} - \varrho\dot{\mathbf{v}}\cdot\mathbf{v} - p\frac{\dot{\varrho}}{\varrho} - \varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\dot{\overline{\nabla\varrho}} = \\ &= \operatorname{div}(\mathbf{T}\mathbf{v}+\mathbf{q}) - \operatorname{div}\mathbf{T}\cdot\mathbf{v} + p\operatorname{div}\mathbf{v} + \varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\cdot\nabla(\varrho\operatorname{div}\mathbf{v}) + \varrho\left(\frac{\partial\tilde{e}}{\partial\nabla\varrho}\otimes\nabla\varrho\right)\cdot\nabla\mathbf{v} = \\ &= \mathbf{T}\cdot\nabla\mathbf{v} + \left(\varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\otimes\nabla\varrho\right)\cdot\nabla\mathbf{v} + \operatorname{div}\mathbf{q} + p\operatorname{div}\mathbf{v} + \operatorname{div}\left(\varrho^2\operatorname{div}\mathbf{v}\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right) - \varrho(\operatorname{div}\mathbf{v})\operatorname{div}\left(\varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right).\end{aligned}$$

The principle of objectivity says that e is a function of the size of $\nabla\varrho$,

$$e = \tilde{e}(\eta, \varrho, |\nabla\varrho|),$$

this implies that

$$\frac{\partial\tilde{e}}{\partial\nabla\varrho}\otimes\nabla\varrho$$

is a symmetric tensor. We obtain

$$\begin{aligned}\varrho\theta\dot{\eta} &= \left(\mathbf{T} + \varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\otimes\nabla\varrho\right)\cdot\mathbf{D} + \left(p - \varrho\operatorname{div}\left(\varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right)\right)\operatorname{div}\mathbf{v} + \operatorname{div}\left(\mathbf{q} + \varrho^2(\operatorname{div}\mathbf{v})\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right) = \\ &= \underbrace{\left(\mathbf{T}^d + \varrho\left(\frac{\partial\tilde{e}}{\partial\nabla\varrho}\otimes\nabla\varrho\right)^d\right)}_{\mathbf{T}^d_{\mathrm{dis}}}\cdot\mathbf{D}^d + \underbrace{\left(p - \varrho\operatorname{div}\left(\varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right) + m + \frac{\varrho}{3}\frac{\partial\tilde{e}}{\partial\nabla\varrho}\cdot\nabla\varrho\right)}_{t_{\mathrm{dis}}}\operatorname{div}\mathbf{v} + \operatorname{div}\underbrace{\left(\mathbf{q} + \varrho^2(\operatorname{div}\mathbf{v})\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right)}_{\mathbf{q}_{\mathrm{dis}}}.\end{aligned}$$

We get the equation

$$\varrho\dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}_{\text{dis}}}{\theta}\right) = \frac{1}{\theta}\left[\mathbf{T}^d_{\text{dis}} \cdot \mathbf{D}^d + t_{\text{dis}} \operatorname{div} \mathbf{v} + \frac{\mathbf{q}_{\text{dis}} \cdot \nabla\theta}{\theta}\right].$$

$\xi = 0$, pokud $\mathbf{T}^d_{\text{dis}} = \mathbf{0}, t_{\text{dis}} = 0, \mathbf{q}_{\text{dis}} = \mathbf{0}$, pak

$$\begin{aligned}\mathbf{T} = \mathbf{T}^d + m\mathbf{I} &= -\varrho\left(\frac{\partial\tilde{e}}{\partial\nabla\varrho} \otimes \nabla\varrho\right)^d - p(\dots)\mathbf{I} - \frac{\varrho}{3}\left(\frac{\partial\tilde{e}}{\partial\nabla\varrho} \cdot \nabla\varrho\right)\mathbf{I} + \varrho \operatorname{div}\left(\varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right)\mathbf{I} \\ &= -p(e,\varrho,\nabla\varrho)\mathbf{I} + \varrho \operatorname{div}\left(\varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho}\right)\mathbf{I} - \varrho\left(\frac{\partial\tilde{e}}{\partial\nabla\varrho} \otimes \nabla\varrho\right).\end{aligned}$$

The special choice of internal energy

$$e = \tilde{e}(\eta,\varrho,\nabla\varrho) = e_0(\eta,\varrho) + \frac{\beta}{2\varrho}|\nabla\varrho|^2 \;\Rightarrow\; \varrho\frac{\partial\tilde{e}}{\partial\nabla\varrho} = \beta\nabla\varrho$$

gives the Korteweg model

$$\begin{aligned}\mathbf{T} &= -p(\dots)\mathbf{I} + \varrho\beta \operatorname{div} \Delta\varrho\mathbf{I} - \beta(\nabla\varrho \otimes \nabla\varrho) \\ &= -p\mathbf{I} + \frac{\beta}{2}\left(\Delta(\varrho^2) - |\nabla\varrho|^2\right)\mathbf{I} - \beta(\nabla\varrho \otimes \nabla\varrho).\end{aligned}$$

From the choice of constitutive equations for the rate of entropy production

$$\xi = \tilde{\xi}(\mathbf{T}^d_{\text{dis}}, t_{\text{dis}}, \mathbf{q}_{\text{dis}}) = \frac{1}{2\mu}|\mathbf{T}^d_{\text{dis}}|^2 + \frac{3}{2\mu + 3\lambda}t^2_{\text{dis}} + \frac{1}{K}|\mathbf{q}_{\text{dis}}|^2$$

and the choice[8]

$$\begin{aligned}\mathbf{T}^d_{\text{dis}} &= 2\mu\mathbf{D}^d, \\ t_{\text{dis}} &= \frac{2\mu + 3\lambda}{3} \operatorname{div} \mathbf{v}, \\ \mathbf{q}_{\text{dis}} &= K\nabla\theta\end{aligned}$$

we see that $\xi \geq 0$ and the second law of thermodynamics is satisfied. Then we get the Korteweg model

$$\begin{aligned}\mathbf{q} &= K\nabla\theta - \varrho^2 \operatorname{div} \mathbf{v}\frac{\partial\tilde{e}}{\partial\nabla\varrho} = K\nabla\theta - \beta\varrho(\operatorname{div} \mathbf{v})\nabla\varrho, \\ \mathbf{T} &= -p\mathbf{I} + 2\mu\mathbf{D} + \lambda(\operatorname{div} \mathbf{v})\mathbf{I} + \frac{\beta}{2}\left(\Delta(\varrho^2) - |\nabla\varrho|^2\right)\mathbf{I} - \beta(\nabla\varrho \otimes \nabla\varrho).\end{aligned}$$

Viscoelastic materials

Our aim is to model materials capable of describing non-Newtonian phenomena like nonlinear stress relaxation, nonlinear creep or the normal stress differences in simple shear flow. Such materials are for example geomaterials, biological materials, polymers and other chemical products like food for example.

Repeat what is the stress relaxation test and creep test.

Stress relaxation test Stress relaxation test is the following (see Figure 21): At time $t = 0$ we deform the material with constant relative prolongation ε and then we study the shear stress σ. For linear spring it holds

$$\sigma = E\varepsilon,$$

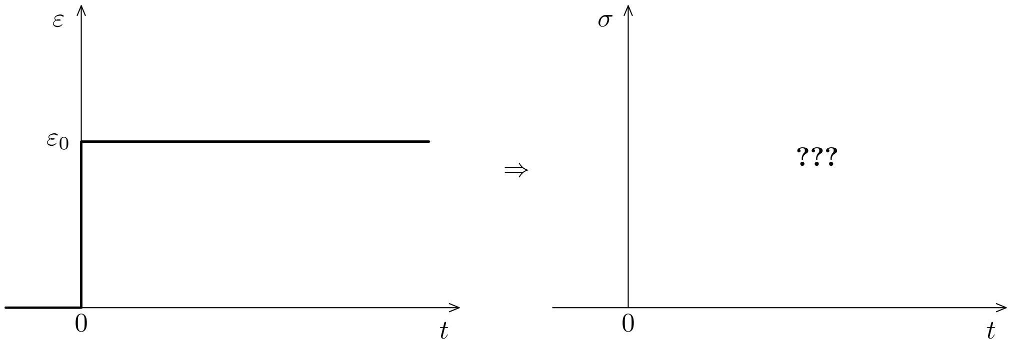


Figure 21: Stress relaxation test

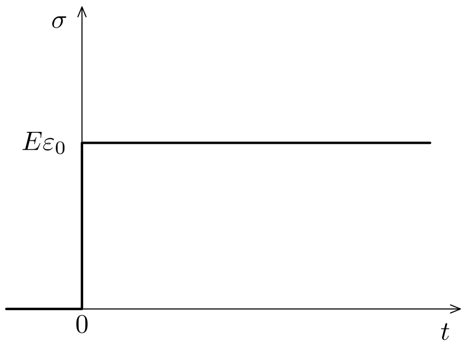


Figure 22: Linear spring

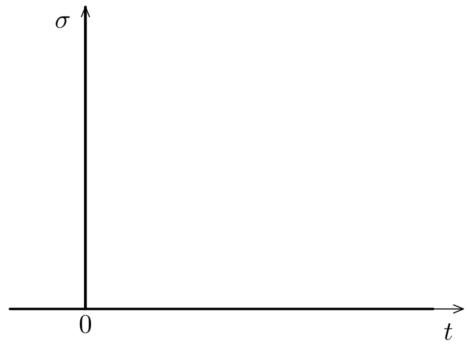


Figure 23: Linear dashpot

for the linear dashpot it holds

$$\sigma = \mu\dot{\varepsilon}.$$

Let us define the stress relaxation function

$$G(t) = \frac{\sigma(t)}{\varepsilon_0}.$$

Real materials exhibit the following bahavior

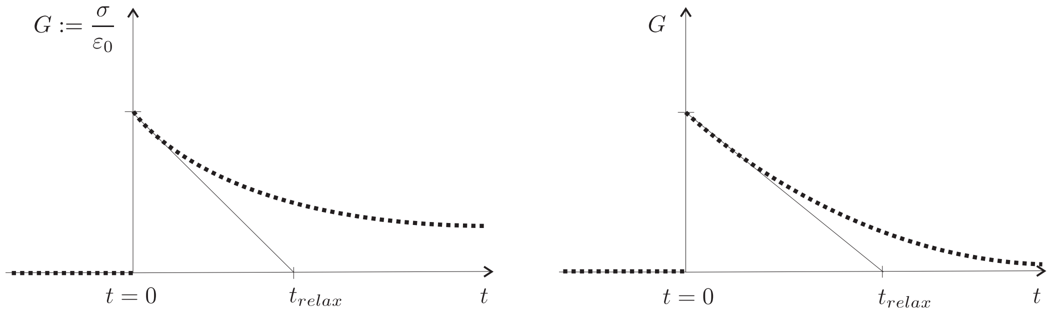


Figure 24: Viscoelastic solid Viscoelastic fluid

Creep test Creep test is the following (see Figure 25): At time $t = 0$ we deform the material with constant stress σ, at time $t = t^*$ we turn off the stress and study the relative prolongation ε.

[8] It is possible to use the principle of maximization of the rate of entropy production, but for the simplicity we can guess it, because the rate of entropy production is quadratic.

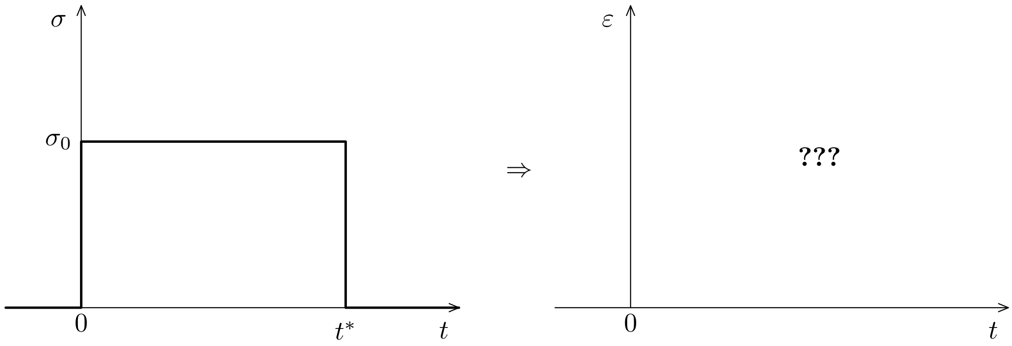


Figure 25: Creep test

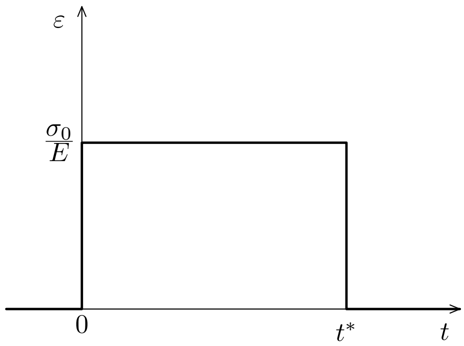


Figure 26: Linear spring

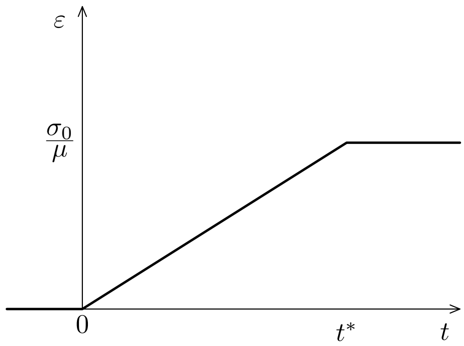


Figure 27: Linear dashpot

Let us define the creep test function

$$J(t) = \frac{\varepsilon(t)}{\sigma_0}.$$

Real materials exhibit the following bahavior

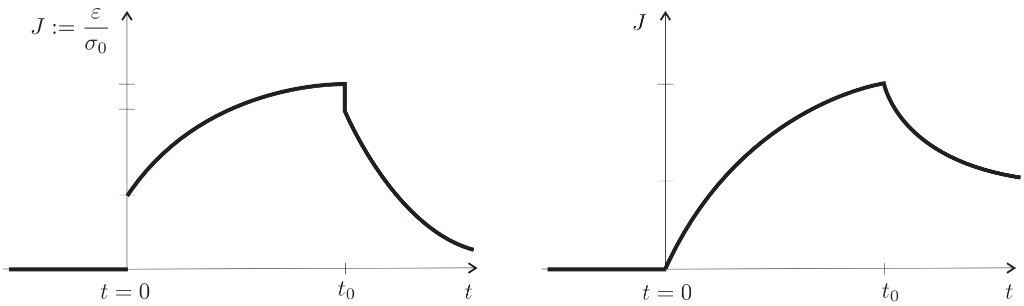


Figure 28: Viscoelastic solid Viscoelastic fluid

The result of stress relaxation test is in the case of linear spring in Figure 26, in the case of linear dashpot in Figure 27.

Creation of simple models by combining linear springs and linear dashpots

Maxwell element We get the Maxwell element if we connect the linear spring and linear dashpot in series. Denote F resp. F_S resp. F_D the force in the whole element resp. in the

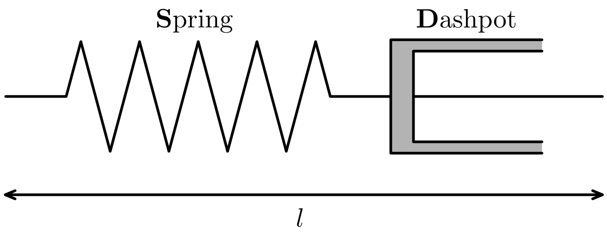


Figure 29: Maxwell element

spring resp. in the dashpot, and Δ resp. Δ_S resp. Δ_D o the prolongation of the whole element resp. of the spring resp. of the dashpot. Then for the linear spring holds

$$F_\mathrm{S} = E\Delta_\mathrm{S}, \sigma_\mathrm{S} = E\varepsilon_\mathrm{S},$$

for the linear dashpot

$$F_\mathrm{D} = \mu\dot{\Delta}_\mathrm{D}, \sigma_\mathrm{D} = \mu\dot{\varepsilon}_\mathrm{D}.$$

For the Maxwell element we do the computation in details. First we derive the constitutive relation. In the series connection is the stress in the spring and dashpot the same, i.e. $\sigma_\mathrm{D} = \sigma_\mathrm{S}$, the whole prolongation is the sum of the prolongation of the spring and the dashpot, i.e. $\Delta = \Delta_\mathrm{S} + \Delta_\mathrm{D}$. We substitute from the constitutive relation for the linear spring and the linear dashpot

$$\dot{\Delta} = \dot{\Delta}_\mathrm{S} + \dot{\Delta}_\mathrm{D} = \frac{\dot{F}_\mathrm{S}}{E} + \frac{F_\mathrm{D}}{\mu},$$

and get the constitutive equation fot the Maxwell element

$$E\dot{\varepsilon} = \dot{\sigma} + \frac{E}{\mu}\sigma \;\Leftrightarrow\; p_1\dot{\sigma} + p_0\sigma = q_1\dot{\varepsilon}.$$

From the initial condition we obtain

$$p_1\sigma(0+) = q_1\varepsilon(0+).$$

Now copmute how the Maxwell element behaves in the creep test. Let $\sigma(t) = \sigma_0$, then

$$\varepsilon(t) = \frac{\sigma_0}{q_1}(p_1 + p_0 t)$$

and the creep function is equal to

$$J(t) = \frac{\varepsilon(t)}{\sigma_0} = \frac{1}{q_1}(p_1 + p_0 t).$$

The response is linear it is not a satisfacory result.

Further compute the stress relaxation test for Maxwell element. Let $\varepsilon(t) = \varepsilon_0 H(t)$, where $H(t)$ is the Heavisideova function

$$\sigma(t) = \frac{q_1}{p_1}\varepsilon_0 e^{-\frac{p_0}{p_1}t}$$

and the stress relaxation function is equal to

$$G(t) = \frac{\sigma(t)}{\varepsilon_0} = \frac{q_1}{p_1} e^{-\frac{p_0}{p_1}t}.$$

This response looks reasonably.

Now compute how the stress σ depends on th deformation ε

$$\overline{\left(\sigma(t)e^{\frac{p_0}{p_1}t}\right)}^{\boldsymbol{\cdot}} = \frac{q_1}{p_1}\dot{\varepsilon}e^{\frac{p_0}{p_1}t}$$

$$\begin{aligned}\sigma(t) &= \sigma(0+)e^{-\frac{p_0}{p_1}t} + \frac{q_1}{p_1}\int_0^t \dot{\varepsilon}(t)e^{-\frac{p_0}{p_1}(t-\tau)}\,\mathrm{d}\tau \\ &\overset{\text{initial cond.}}{=} \frac{q_1}{p_1}\varepsilon(0+)e^{-\frac{p_0}{p_1}t} + \frac{q_1}{p_1}\int_0^t \dot{\varepsilon}(t)e^{-\frac{p_0}{p_1}(t-\tau)}\,\mathrm{d}\tau \\ &= \varepsilon(0+)G(t) + \frac{q_1}{p_1}\int_0^t \dot{\varepsilon}(t)G(t-\tau)\,\mathrm{d}\tau.\end{aligned}$$

In the higher dimension this result corresponds to the fact that we obtained the integral relation between $\mathbf{T}$ and $\mathbf{D}$.

Now compute how ε depends on σ.

$$\begin{aligned}q_1\varepsilon(t) &= q_1\varepsilon(0+) + p_1\int_0^t \dot{\sigma}(\tau)\,\mathrm{d}\tau + p_0\int_0^t \sigma(\tau)\,\mathrm{d}\tau \\ &= q_1\varepsilon(0+) + \int_0^t \dot{\sigma}(\tau)(p_1 + p_0(t-\tau))\,\mathrm{d}\tau + p_0\sigma(0+)t.\end{aligned}$$

Using initial condition we obtain

$$\begin{aligned}\varepsilon(t) &= \left(\frac{p_1}{q_1} + \frac{p_0}{q_1}t\right)\sigma(0+) + \int_0^t \dot{\sigma}(\tau)\left(\frac{p_1}{q_1} + \frac{p_0}{q_1}(t-\tau)\right)\mathrm{d}\tau \\ &= J(t)\sigma(0+) + \int_0^t \dot{\sigma}(\tau)J(t-\tau)\,\mathrm{d}\tau.\end{aligned}$$

We obtained the rate-type model and two integral type models, all equivalent. All models hold in one dimensional space, but it is not clear how to generalize them into three dimensional space.

1. Linear dashpot is not an approximative theory, but the linear spring is.
2. What to do if the dashpot or the spring depend on the deformation in nonlinear way?
3. Partial time derivative is neither material time derivative nor objective time derivative, the generalizations are not unique.
4. Many nonlinear 3D models can reduce in 1D to the same equation. So many models can be used to capture 1D experimental data.

Kelvin-Voigt element We get the Kelvin-Voigt element if we paralelly connect the spring and the dashpot.

The prolongations are the same for the spring and the dashpot. The total force is equal to the sum of the forces in spring and dashpot, so

$$F = F_{\mathrm{S}} + F_{\mathrm{D}}, \quad \Delta = \Delta_{\mathrm{S}} = \Delta_{\mathrm{D}}.$$

Using constitutive relations for the linear spring and the linear dashpot we get

$$\begin{aligned}p_0\sigma &= q_0\varepsilon + q_1\dot{\varepsilon}, \\ \varepsilon(0+) &= 0.\end{aligned}$$

The stress relaxation function is equal to

$$G(t) = \frac{q_0}{p_0} + \frac{q_1}{p_0}\delta(t),$$

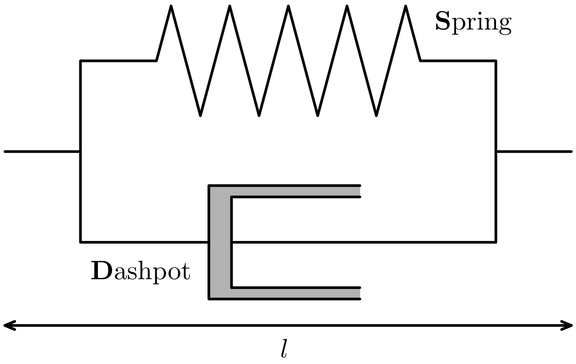


Figure 30: Kelvin-Voigtů element

this is disadvantage for Kelvin-Voigt. The creep function is

$$J(t) = \frac{p_0}{q_0}\left(1 - e^{-\frac{q_0}{q_1}t}\right),$$

which is the advantage of Kelvin-Voigt element

Both derived models were onedimensional, we show teh modification into the higher dimension. Suppose that the fluid is incompressible, $\operatorname{tr}\mathbf{D} = 0$. Consider the stress tensor in the form

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}$$

and say that $\mathbf{S}$ satisfies the same equation as derived earlier, let us try

$$\mathbf{S} + \lambda\dot{\mathbf{S}} = 2\mu\mathbf{D}.$$

Suppose we have two coordinate systems – starred and unstarred, the starred one is related to the unstarred by

$$x^* = \mathbf{Q}(x - x_0) + c,$$

where $\mathbf{Q}$ is orthogonal. We want the tensor $\mathbf{S}$ to transform as

$$\mathbf{S}^* = \mathbf{Q}\mathbf{S}\mathbf{Q}^{\mathrm{T}}.$$

In the starred system the material satisfies the equation

$$\mathbf{S}^* + \lambda\dot{\overline{\mathbf{S}}}^* = 2\mu\mathbf{D}^*,$$

which is equivalent to

$$\mathbf{Q}\mathbf{S}\mathbf{Q}^{\mathrm{T}} + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{Q}\mathbf{S}\mathbf{Q}^{\mathrm{T}}\right) = 2\mu\mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}}.$$

If it holded

$$\mathbf{Q}\mathbf{S}\mathbf{Q}^{\mathrm{T}} + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{Q}\mathbf{S}\mathbf{Q}^{\mathrm{T}}\right) = \mathbf{Q}\left(\mathbf{S} + \lambda\dot{\mathbf{S}}\right)\mathbf{Q}^{\mathrm{T}},$$

everything would be OK, but this does not hold because

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{Q}\mathbf{S}\mathbf{Q}^{\mathrm{T}}\right) = \dot{\mathbf{Q}}\mathbf{S}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{S}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{S}\dot{\mathbf{Q}}^{\mathrm{T}}$$

and the sum of the first and last term can never be equal zero. So, our try is bad – $\dot{\mathbf{S}}$ is not objective. Let us denote the objective derivative of $\mathbf{S}$ by $\overset{\circ}{\mathbf{S}}$ satisfying

$$\overset{\circ}{\overline{\mathbf{S}^*}} = \mathbf{Q}\overset{\circ}{\mathbf{S}}\mathbf{Q}^{\mathrm{T}}.$$

Define now

$$\overset{\circ}{\mathbf{S}} = \frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T + (\mathrm{tr}\,\mathbf{L})\mathbf{S}$$

and verify that this derivative is objective. We want

$$\overline{\overset{\circ}{\mathbf{S}^*}} = \mathbf{Q}\,\overset{\circ}{\mathbf{S}}\,\mathbf{Q}^\mathrm{T} = \mathbf{Q}\left(\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T + (\mathrm{tr}\,\mathbf{L})\mathbf{S}\right)\mathbf{Q}^\mathrm{T}.$$

Compute.

$$\begin{aligned}\frac{\mathrm{d}\mathbf{S}^*}{\mathrm{d}t} - \mathbf{L}^*\mathbf{S}^* - \mathbf{S}^*(\mathbf{L}^\mathrm{T})^* + (\mathrm{tr}\,\mathbf{L}^*)\mathbf{S}^* &= \overset{\circ}{\overline{\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T}}} = \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T}\right) - \mathbf{L}^*\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T} - \mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T}(\mathbf{L}^\mathrm{T})^* + (\mathrm{tr}\,\mathbf{L}*)\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T}.\end{aligned}$$

We need to know what is $\mathbf{L}^*$. After some algebraic manipulation it can be computed that

$$\mathbf{L}^* = \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\mathbf{L}\mathbf{Q}^\mathrm{T}.$$

Inserting this into the previous calculation we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T}\right) - \mathbf{L}^*\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T} - \mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T}(\mathbf{L}^\mathrm{T})^* + (\mathrm{tr}\,\mathbf{L}*)\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T} &= \\ = \dot{\mathbf{Q}}\mathbf{S}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\dot{\mathbf{S}}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\mathbf{S}\dot{\mathbf{Q}}^\mathrm{T} - \left(\dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\mathbf{L}\mathbf{Q}^\mathrm{T}\right)\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T} - \\ \mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T}\left(\dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\mathbf{L}\mathbf{Q}^\mathrm{T}\right) + \mathrm{tr}\left(\dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\mathbf{L}\mathbf{Q}^\mathrm{T}\right)\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T} &= \\ = \mathbf{Q}\dot{\mathbf{S}}\mathbf{Q}^\mathrm{T} - \mathbf{Q}\mathbf{L}\mathbf{S}\mathbf{Q}^\mathrm{T} - \mathbf{Q}\mathbf{S}\mathbf{L}^\mathrm{T}\mathbf{Q}^\mathrm{T} + \mathrm{tr}\left(\dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\mathbf{L}\mathbf{Q}^\mathrm{T}\right)\mathbf{Q}\mathbf{S}\mathbf{Q}^\mathrm{T} &= \\ = \mathbf{Q}\left(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^\mathrm{T}\right)\mathbf{Q}^\mathrm{T} + \mathbf{Q}(\mathrm{tr}\,\mathbf{L})\mathbf{S}\mathbf{Q}^\mathrm{T},\end{aligned}$$

because the trace is cyclic and

$$\mathbf{0} = \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} + \mathbf{Q}\dot{\mathbf{Q}}^\mathrm{T}.$$

The derivative is objective even if there is not the last term with the trace

$$\overset{\triangledown}{\mathbf{S}} = \frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T.$$

This derivative is called Oldroyd upper convected derivative. If we use it in the Maxwell element we obtain the Maxwell model

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda\,\overset{\triangledown}{\mathbf{S}} = 2\mu\mathbf{D}.$$

Incompressible rate-type fluid models – derivation of thermodynamically compatible models

Motivaation and the aim Combining two linear dashpots and one linear spring we obtain the relation between the shear stress and shear rate

$$p_1\dot{T}_{xy} + p_0T_{xy} = q_1\dot{D}_{xy} + q_0D_{xy}.$$

In the previous section we explained why it is not trivial to generalize this model into three dimensional space. A similar model is a Yeleswerapu model for blood:

$$\begin{aligned}
\operatorname{div}\mathbf{v} &= 0,\\
\varrho\dot{\mathbf{v}} &= \operatorname{div}\mathbf{T} + \varrho\mathbf{b},\\
\mathbf{T} &= -p\mathbf{I} + \mathbf{S},\\
\mathbf{S} + \lambda_1\left(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^{\mathrm{T}}\right) &= \mu(|\mathbf{D}|)\mathbf{D} + \lambda_2(\dot{\mathbf{D}} - \mathbf{L}\mathbf{D} - \mathbf{D}\mathbf{L}^{\mathrm{T}}),\\
\mu(|\mathbf{D}|) &= \mu_\infty + (\mu_0 - \mu_\infty)\left[\frac{1 + \ln(1 + \Lambda|\mathbf{D}|)}{1 + \Lambda|\mathbf{D}|}\right].
\end{aligned}$$

But is this model thermodynamically compatible satisfying the second law of thermodynamics?

First we derive three dimensional Kelvin-Voigt model.

Kelvin-Voigt model

Suppose the entropy in the form $\eta = \tilde{\eta}(e, \varrho, \mathbf{B}_{k_R}) \Leftrightarrow e = \tilde{e}(\eta, \varrho, \mathbf{B}_{k_R}) = \hat{e}(\eta, \varrho, \operatorname{tr}\mathbf{B}_{k_R})$ and define kinematical quantities. First the deformation gradient

$$\mathbf{F}_{k_R} = \frac{\partial \chi_{k_R}}{\partial X}$$

left and right Cauchy-Green tensor

$$\mathbf{B}_{k_R} = \mathbf{F}_{k_R}\mathbf{F}_{k_R}^{\mathrm{T}}, \quad \mathbf{C}_{k_R} = \mathbf{F}_{k_R}^{\mathrm{T}}\mathbf{F}_{k_R}.$$

Compute

$$\varrho\dot{E} - \varrho\dot{\mathbf{v}}\cdot\mathbf{v} = \varrho\dot{e} = \varrho\frac{\partial\hat{e}}{\partial\eta}\dot{\eta} + \varrho\frac{\partial\hat{e}}{\partial\varrho}\dot{\varrho} + \varrho\frac{\partial\hat{e}}{\partial(\operatorname{tr}\mathbf{B}_{k_R})}\mathbf{I}\cdot\dot{\mathbf{B}}_{k_R}.$$

We know that $\dot{\mathbf{F}}_{k_R} = \mathbf{L}\mathbf{F}_{k_R}$ which implies

$$\dot{\mathbf{B}}_{k_R} = \mathbf{L}\mathbf{B}_{k_R} + \mathbf{B}_{k_R}\mathbf{L}^{\mathrm{T}} \;\Rightarrow\; \mathbf{I}\cdot\dot{\mathbf{B}}_{k_R} = 2\mathbf{B}_{k_R}\cdot\mathbf{D}.$$

Continue computing

$$\begin{aligned}
\varrho\theta\dot{\eta} - \operatorname{div}\mathbf{q} = \mathbf{T}\cdot\mathbf{D} + p\operatorname{div}\mathbf{v} - 2\varrho\frac{\partial\hat{e}}{\partial(\operatorname{tr}\mathbf{B}_{k_R})}\mathbf{B}_{k_R}\cdot\mathbf{D} = \\
= \left(\mathbf{T} - 2\varrho\frac{\partial\hat{e}}{\partial(\operatorname{tr}\mathbf{B}_{k_R})}\mathbf{B}_{k_R}\right)^d\cdot\mathbf{D}^d + \left(m - \frac{2}{3}\varrho\frac{\partial\hat{e}}{\partial(\operatorname{tr}\mathbf{B}_{k_R})}\operatorname{tr}\mathbf{B}_{k_R} + p\right)\operatorname{div}\mathbf{v}.
\end{aligned}$$

Divide by temperature θ and get

$$\varrho\dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \frac{1}{\theta}\underbrace{\left[\mathbf{T}^d_{\mathrm{dis}}\cdot\mathbf{D}^d + t_{\mathrm{dis}}\operatorname{div}\mathbf{v} + \frac{\mathbf{q}\cdot\nabla\theta}{\theta}\right]}_{\xi}.$$

The incompressibility and constant temperature gives

$$\xi = \mathbf{T}^d_{\mathrm{dis}}\cdot\mathbf{D}^d = \left(\mathbf{T} - 2\varrho\frac{\partial\hat{e}}{\partial(\operatorname{tr}\mathbf{B}_{k_R})}\mathbf{B}_{k_R}\right)^d\cdot\mathbf{D}^d.$$

Further for a general $\xi = \tilde{\xi}(\mathbf{T}_{\mathrm{dis}}, \mathbf{D})$ choose the constitutive relation

$$\tilde{\xi} = \frac{1}{2\nu}|\mathbf{T}_{\mathrm{dis}}|^2,$$

or

$$\tilde{\xi} = 2\nu|\mathbf{D}|^2.$$

In the first case we maximize ξ with respect to $\mathbf{T}_{\text{dis}}$ with no other constrains, in the second case we maximize with respect to $\mathbf{D}$ with the incompressibility condition $\operatorname{tr}\mathbf{D} = 0$. We obtain

$$\mathbf{T} = -m\mathbf{I} + 2\nu\mathbf{D} + 2\varrho\frac{\partial \hat{e}}{\partial\left(\operatorname{tr}\mathbf{B}_{k_R}\right)}\mathbf{B}_{k_R}^d.$$

If we use neo-Hookean free energy

$$\varrho\psi = \frac{\mu}{2}\left(\operatorname{tr}\mathbf{B}_{k_R} - 3\right),$$

the it holds

$$\frac{\partial \hat{e}}{\partial\left(\operatorname{tr}\mathbf{B}_{k_R}\right)} = \frac{\mu}{2}\mathbf{I}$$

and we obtain Kelvin-Voigt model

$$\mathbf{T} = -m\mathbf{I} + 2\nu\mathbf{D} + \mu\mathbf{B}_{k_R}^d.$$

If the rate of entropy production is zero, then

$$\begin{aligned}\mathbf{T}_{\text{dis}}^d &= \mathbf{0},\\ p + m + \mu\frac{1}{3}\operatorname{tr}\mathbf{B}_{k_R} &= 0,\\ \mathbf{q} &= \mathbf{0}\end{aligned}$$

and we obtain

$$\mathbf{T} = \mathbf{T}^d + m\mathbf{I} = \mathbf{T}_{\text{dis}}^d + \mu\mathbf{B}_{k_R}^d + m\mathbf{I} = -p\mathbf{I} + \mu\mathbf{B}_{k_R}.$$

If we do not require the incompressibility and use the relations as in Navier-Stokes-Fourier

$$\begin{aligned}\mathbf{T}_{\text{dis}}^d &= 2\nu\mathbf{D}^d,\\ p + m + \mu\frac{1}{3}\operatorname{tr}\mathbf{B}_{k_R} &= \frac{2\nu + 3\lambda}{3}\operatorname{div}\mathbf{v},\\ \mathbf{q} &= k\nabla\theta,\end{aligned}$$

we get

$$\begin{aligned}&\mathbf{T} = -p\mathbf{I} + 2\nu\mathbf{D} + \lambda(\operatorname{div}\mathbf{v})\mathbf{I} + \mu\mathbf{B}_{k_R},\\ &\mathbf{q} = k\nabla\theta.\end{aligned}$$

Viscoelastic model (K.R. Rajagopal, A.R. Srinivasa [28])

The body made from the viscoelastic material is at time t in configuration $\kappa(t)$, this configuration is called the current configuration. We will compare this configuration with the configuration $\kappa(r)$ which is a configuration of the system in rest at the beginning, so called the reference configuration.

We define now the natural configuration $\kappa(p(t))$. It is a configuration of the system corresponding to the current configuration $\kappa(t)$ the body would go to on the sudden removal of external stimuli. Then the only difference between the natural and current configuration is purely elastic deformation. By defining the natural configuration we split the whole deformation into purely elastic part and the dissipative part.

We motivate the notion of the natural configuration using a simple example. Suppose we have a simple linear viscoelastic material where in every material point is Maxwell element. The spring represent the elastic part and the dashpot the viscous part In Figure 31a) jthe material is at the beginning in rest, relaxed. The spring and the dashpot are not stretched.

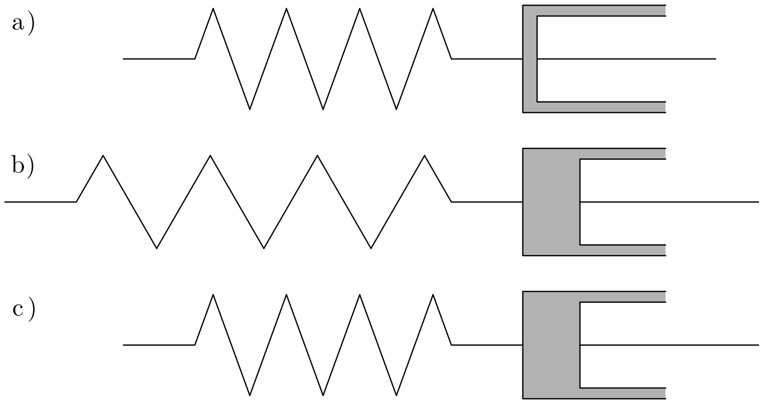


Figure 31: Motivational example for the natural configuration

The system is in the reference configuration $\kappa(r)$. Then we stretch it (Figure 31b)) and the system goes to the current configuration $\kappa(t)$. Now we let the system relax, the spring shrinks to its reference length but the dashpot remains stretched. (Figure 31c)). The system is in the natural configuration $\kappa(p(t))$. Because during the relaxation the deformation of the dashpot does not change, we can think that the whole free energy of the system is hidden only in the deformation of the natural configuration. konfigurace.

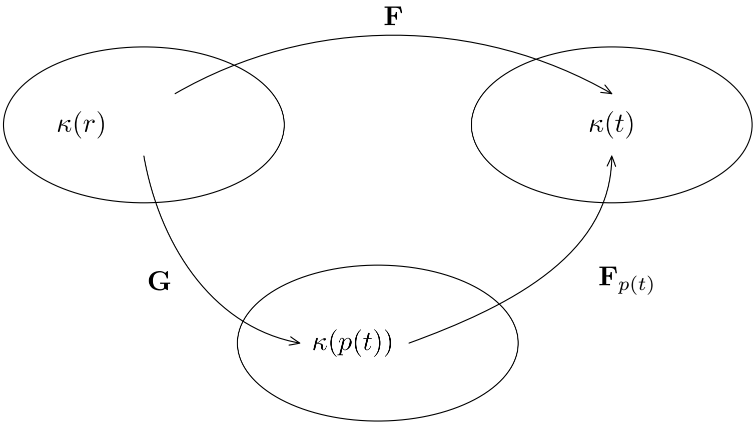


Figure 32: Přirozená konfigurace

We define some kinematical quantities. First we define the deformation gradient (in Figure 32 denoted by $\mathbf{F}$)

$$\mathbf{F}_{k_R} = \nabla\chi$$

that maps infinitesimal element from $\kappa(r)$ to $\kappa(t)$. We define standard left and right Cauchy-Green tensors.

$$\mathbf{B}_{k_R} = \mathbf{F}_{k_R}\mathbf{F}_{k_R}^{\mathrm{T}}, \quad \mathbf{C}_{k_R} = \mathbf{F}_{k_R}^{\mathrm{T}}\mathbf{F}_{k_R},$$

Further we define $\mathbf{F}_{k_{p(t)}}$ as a mapping of the infinitesimal element from $\kappa(p(t))$ to $\kappa(t)$ (in Figure. 32 denoted by $\mathbf{F}_{p(t)}$) and $\mathbf{G}$ that maps from $\kappa(r)$ to $\kappa(p(t))$. These two new variables are related through the deformation gradient

$$\mathbf{G} = \mathbf{F}_{k_{p(t)}}^{-1}\mathbf{F}_{k_R}.$$

Define leftand right Cauchy-Green tensor corresponding to $\kappa(p(t))$

$$\mathbf{B}_{k_{p(t)}} = \mathbf{F}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}}, \quad \mathbf{C}_{k_{p(t)}} = \mathbf{F}^{\mathrm{T}}_{k_{p(t)}}\mathbf{F}_{k_{p(t)}}$$

and the rate of the deformation of $\kappa(p(t))$ (in the analogy with the usual relation between the velocity gradient and deformation gradient), resp. its symmetric part

$$\mathbf{L}_{k_{p(t)}} = \dot{\mathbf{G}}\mathbf{G}^{-1}, \quad \mathbf{D}_{k_{p(t)}} = \frac{\mathbf{L}_{k_{p(t)}} + \mathbf{L}^{\mathrm{T}}_{k_{p(t)}}}{2}.$$

The material time derivative of $\mathbf{B}_{k_{p(t)}}$ is equal to

$$\begin{aligned}\dot{\mathbf{B}}_{k_{p(t)}} = \dot{\mathbf{F}}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}} + \mathbf{F}_{k_{p(t)}}\dot{\mathbf{F}}^{\mathrm{T}}_{k_{p(t)}} = \\ \dot{\mathbf{F}}_{k_R}\mathbf{G}^{-1}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}} + \mathbf{F}_{k_R}\dot{\overline{\mathbf{G}^{-1}}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}} + \mathbf{F}_{k_{p(t)}}\dot{\overline{\mathbf{G}^{-\mathrm{T}}}}\mathbf{F}^{\mathrm{T}}_{k_R} + \mathbf{F}_{k_{p(t)}}\mathbf{G}^{-\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}_{k_R} = \\ \mathbf{L}\mathbf{B}_{k_{p(t)}} + \mathbf{B}_{k_{p(t)}}\mathbf{L}^{\mathrm{T}} - 2\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}},\end{aligned}$$

where we used

$$\dot{\overline{\mathbf{A}^{-1}}} = -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}.$$

And we naturally obtained the Oldroyd upper convected time derivative

$$\overset{\triangledown}{\mathbf{B}}_{k_{(p(t))}} = \dot{\mathbf{B}}_{k_{p(t)}} - \mathbf{L}\mathbf{B}_{k_{p(t)}} - \mathbf{B}_{k_{p(t)}}\mathbf{L}^{\mathrm{T}} = -2\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}}.$$

Note that $\overset{\triangledown}{\mathbf{I}} = -2\mathbf{D}$. It holds that

$$\mathrm{tr}\,\dot{\mathbf{B}}_{k_{p(t)}} = 2\mathbf{B}_{k_{p(t)}} \cdot \mathbf{D} - 2\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}} \cdot \mathbf{F}_{k_{p(t)}}.$$

Choose the entropy in the form $\eta = \tilde{\eta}(e, \varrho, \mathbf{B}_{k_{p(t)}})$, we can invert it and use the internal energy $e = \tilde{e}(\eta, \varrho, \mathbf{B}_{k_{p(t)}})$, concretely choose

$$e = e_0(\eta, \varrho) + \frac{\mu}{2\varrho}\left(\mathrm{tr}\,\mathbf{B}_{k_{p(t)}} - 3\right)$$

and as in the case for Kelvin-Voigt we obtain

$$\xi = (\mathbf{T} - \mu\mathbf{B}_{k_{p(t)}}) \cdot \mathbf{D} + (p + \tilde{m})\,\mathrm{div}\,\mathbf{v} + \frac{\mathbf{q} \cdot \nabla\theta}{\theta} + \mu\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}} \cdot \mathbf{F}_{k_{p(t)}}.$$

For simplicity $\mathbf{q} = \mathbf{0}$ and $\mathrm{div}\,\mathbf{v} = 0$ and like in [28], we suppose that the elastic response is also incompressible, i.e. $\det\mathbf{B}_{k_{p(t)}} = 1$. From this and incompressibility $\mathbf{F}_{k_R}$ results that $\det\mathbf{G} = 1$. Differentiating $\det\mathbf{G}$ gives $\mathrm{tr}\,\mathbf{D}_{k_{p(t)}} = 0$.

For the rate of entropy production holds (we the zero traces of $\mathbf{D}$ and $\mathbf{D}_{k_{p(t)}}$)

$$\begin{aligned}\xi &= (\mathbf{T} - \mu\mathbf{B}_{k_{p(t)}})^d \cdot \mathbf{D} + \mu\mathbf{C}^d_{k_{p(t)}} \cdot \mathbf{D}_{k_{p(t)}} \\ &= \mathbf{T}^d_{\mathrm{dis}} \cdot \mathbf{D} + \mu\mathbf{C}^d_{k_{p(t)}} \cdot \mathbf{D}_{k_{p(t)}}.\end{aligned}$$

We choose the following constitutive relation for the rate of entropy production

$$\xi = \tilde{\xi}(\mathbf{D}, \mathbf{D}_{k(p(t)}, \mathbf{C}_{k_{p(t)}}) = 2\nu_0|\mathbf{D}|^2 + 2\nu_1\mathbf{D}_{k_{p(t)}} \cdot \mathbf{C}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}.$$

We observe that $\mathbf{D}_{k_{p(t)}} \cdot \mathbf{B}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}} = |\mathbf{D}_{k_{p(t)}}\mathbf{F}_{k_{p(t)}}|^2 \geq 0$, and the second law of thermodynamics is satisfied. We define the Lagrange function

$$L(\mathbf{D}, \mathbf{D}_{k_{p(t)}}) = \tilde{\xi}(\dots) + \lambda_1(\tilde{\xi}(\dots) - \mathbf{T}^d_{\mathrm{dis}} \cdot \mathbf{D} - \mu\mathbf{C}^d_{k_{p(t)}} \cdot \mathbf{D}_{k_{p(t)}}) + \lambda_2\,\mathrm{tr}\,\mathbf{D} + \lambda_3\,\mathrm{tr}\,\mathbf{D}_{k_{p(t)}}.$$

and maximize. The necessary conditions are

$$\frac{\partial L}{\partial \mathbf{D}} = 0, \quad \frac{\partial L}{\partial \mathbf{D}_{k_{p(t)}}} = 0.$$

We substite for L

$$\frac{1+\lambda_1}{\lambda_1}\frac{\partial\tilde{\xi}}{\partial\mathbf{D}} = \mathbf{T}^d_{\text{dis}} + \frac{\lambda_2}{\lambda_1}\mathbf{I}, \tag{37}$$

$$\frac{1+\lambda_1}{\lambda_1}\frac{\partial\tilde{\xi}}{\partial\mathbf{D}_{k_{p(t)}}} = \mu\mathbf{C}^d_{k_{p(t)}} + \frac{\lambda_3}{\lambda_1}\mathbf{I}. \tag{38}$$

First we do the differentiation

$$\frac{\partial\tilde{\xi}}{\partial\mathbf{D}} = 4\nu_0\mathbf{D}, \quad \frac{\partial\tilde{\xi}}{\partial\mathbf{D}_{k_{p(t)}}} = 4\nu_1\mathbf{C}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}.$$

and eliminate the Lagrange multipliers. To do this we add $(37)\cdot\mathbf{D} + (38)\cdot\mathbf{D}_{k_{p(t)}}$, and use the constrains of incompressibility

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\mathbf{T}^d_{\text{dis}}\cdot\mathbf{D} + \mu\mathbf{C}^d_{k_{p(t)}}\cdot\mathbf{D}_{k_{p(t)}}}{\frac{\partial\tilde{\xi}}{\partial\mathbf{D}}\cdot\mathbf{D} + \frac{\partial\tilde{\xi}}{\partial\mathbf{D}_{k_{p(t)}}}\cdot\mathbf{D}_{k_{p(t)}}} = \frac{\tilde{\xi}}{2\tilde{\xi}} = \frac{1}{2}.$$

We take the trace of (37) and obtain

$$0 = \operatorname{tr}\mathbf{T}^d_{\text{dis}} - 3\frac{\lambda_2}{\lambda_1} \;\Rightarrow\; \frac{\lambda_2}{\lambda_1} = 0.$$

If we take the trace of (38), we obtain

$$\frac{\lambda_3}{\lambda_1} = \frac{2}{3}\nu_1\mathbf{D}_{k_{p(t)}}\cdot\mathbf{C}_{k_{p(t)}}.$$

We get

$$\mathbf{T}^d_{\text{dis}} = 2\nu_0\mathbf{D}, \tag{39}$$

$$2\nu_1\left(\mathbf{C}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}} - \frac{1}{3}(\mathbf{D}_{k_{p(t)}}\cdot\mathbf{C}_{k_{p(t)}})\mathbf{I}\right) = \mu\mathbf{C}^d_{k_{p(t)}}. \tag{40}$$

From the equations (39) it implies that

$$\mathbf{T} = -p\mathbf{I} + 2\nu_0\mathbf{D} + \mu\mathbf{B}^d_{k_{p(t)}}.$$

Before the further calculation we remind that

$$\dot{\mathbf{B}}_{k_{p(t)}} = \mathbf{L}\mathbf{B}_{k_{p(t)}} + \mathbf{B}_{k_{p(t)}}\mathbf{L}^{\mathrm{T}} - 2\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}},$$

and compute the Oldroyd upper convected time derivative of the deviatoric part of $\mathbf{B}_{k_{p(t)}}$.

$$\begin{aligned}\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p(t)}} &= \overset{\triangledown}{\overline{\left(\mathbf{B}_{k_{p(t)}} - \frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{I}\right)}} = \overset{\triangledown}{\mathbf{B}}_{k_{p(t)}} - \frac{1}{3}(\operatorname{tr}\dot{\mathbf{B}}_{k_{p(t)}})\mathbf{I} - \frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\overset{\triangledown}{\mathbf{I}} = \\ &= -2\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}} - \frac{1}{3}\operatorname{tr}(\mathbf{L}\mathbf{B}_{k_{p(t)}} + \mathbf{L}^{\mathrm{T}}\mathbf{B}_{k_{p(t)}})\mathbf{I} + \\ &\quad + \frac{2}{3}\operatorname{tr}(\mathbf{F}^{\mathrm{T}}_{k_{p(t)}}\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}})\mathbf{I} + \frac{2}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{D} = \\ &= -2\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}} - \frac{2}{3}(\mathbf{D}\cdot\mathbf{B}_{k_{p(t)}})\mathbf{I} + \frac{2}{3}(\mathbf{C}_{k_{p(t)}}\cdot\mathbf{D}_{k_{p(t)}})\mathbf{I} + \frac{2}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{D}.\end{aligned}$$

Multiply (40) from the left by $\mathbf{F}^{-\mathrm{T}}_{k_{p(t)}}$ and from the right by $\mathbf{F}^{\mathrm{T}}_{k_{p(t)}}$, we obtain

$$2\nu_1\left(\mathbf{F}_{k_{p(t)}}\mathbf{D}_{k_{p(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p(t)}} - \frac{1}{3}(\mathbf{C}_{k_{p(t)}}\cdot\mathbf{D}_{k_{p(t)}})\mathbf{I}\right) = \mu\mathbf{C}^d_{k_{p(t)}}.$$

From the relations for the Oldroyd derivative of the deviatoric part of $\mathbf{B}_{k_{p(t)}}$ we get

$$2\nu_1\left(-\frac{1}{2}\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p(t)}}-\frac{1}{3}(\mathbf{D}\cdot\mathbf{B}^d_{k_{p(t)}})\mathbf{I}+\frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{D}\right)=\mu\mathbf{B}^d_{k_{p(t)}}.$$

Note that the trace both sides of the equation is zero, in the three dimensional space it is five scalar equations for six unknowns. We complete the system of equations with the equation for incompressibility of the elastic response

$$\det\left(\mathbf{B}_{k_{p(t)}}+\frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{I}\right)=1.$$

We obtain the set of equations for nonlinear viscoelastic rate-type fluid model

$$\begin{aligned}
\operatorname{div}\mathbf{v}&=0,\\
\varrho\dot{\mathbf{v}}&=\operatorname{div}\mathbf{T},\\
\mathbf{T}&=-p\mathbf{I}+2\nu_0\mathbf{D}+\mu\mathbf{B}^d_{k_{p(t)}},\\
2\nu_1\left(-\frac{1}{2}\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p(t)}}-\frac{1}{3}(\mathbf{D}\cdot\mathbf{B}^d_{k_{p(t)}})\mathbf{I}+\frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{D}\right)&=\mu\mathbf{B}^d_{k_{p(t)}}, \qquad (41)\\
\det\left(\mathbf{B}_{k_{p(t)}}+\frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{I}\right)&=1. \qquad (42)
\end{aligned}$$

Linearization of the viscoelastic model

By linearization of the elastic response we show that this model reduces to Oldroyd-B model. Suppose that the left Cauchy-Green tensor is in the form $\mathbf{B}_{k_{p(t)}}=\mathbf{I}+\mathbf{A}$, where $\|\mathbf{A}\|=\varepsilon,\quad 0<\varepsilon\ll 1$. By linearization we understand that after substitution into the nonlinear viscoelastic model we will neglect all term of the order $O(\varepsilon^2)$ and higher. We use the last equation for incompressibility (42) and do the Taylor expansion of determinant. We obtain

$$1=\det\left(\mathbf{B}_{k_{p(t)}}\right)=\det(\mathbf{I}+\mathbf{A})=1+\operatorname{tr}(\mathbf{A})+\frac{1}{2}\left((\operatorname{tr}\mathbf{A})^2-\operatorname{tr}\left(\mathbf{A}^2\right)\right)+O(\varepsilon^3)\ \Rightarrow$$
$$\operatorname{tr}(\mathbf{A})=\underbrace{\frac{1}{2}\left(\operatorname{tr}\left(\mathbf{A}^2\right)-(\operatorname{tr}\mathbf{A})^2\right)+O(\varepsilon^3)}_{O(\varepsilon^2)}.$$

From this it implies that

$$\operatorname{tr}\mathbf{B}_{k_{p(t)}}=3+\frac{1}{2}\left(\operatorname{tr}\left(\mathbf{A}^2\right)-(\operatorname{tr}\mathbf{A})^2\right)+O(\varepsilon^3),$$

we obtain the relation for $\mathbf{B}^d_{k_{p(t)}}$,

$$\mathbf{B}^d_{k_{p(t)}}=\mathbf{B}_{k_{p(t)}}-\frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p(t)}})\mathbf{I}=\mathbf{A}-\frac{1}{6}\left(\operatorname{tr}\left(\mathbf{A}^2\right)-(\operatorname{tr}\mathbf{A})^2\right)\mathbf{I}+O(\varepsilon^3)\mathbf{I}.$$

Further we need to compute $\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p(t)}}$.

$$\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p(t)}}=\dot{\mathbf{B}}^d_{k_{p(t)}}-\mathbf{L}\mathbf{B}^d_{k_{p(t)}}-\mathbf{B}^d_{k_{p(t)}}\mathbf{L}^{\mathrm{T}}=\overset{\triangledown}{\mathbf{A}}-\frac{1}{3}(\dot{\mathbf{A}}\cdot\mathbf{A}-\underbrace{(\operatorname{tr}\mathbf{A})}_{O(\varepsilon^2)}(\operatorname{tr}\dot{\mathbf{A}}))\mathbf{I}+O(\varepsilon^2).$$

Inserting all into (41) we get

$$\mu\mathbf{A}+\nu_1\overset{\triangledown}{\mathbf{A}}=2\nu_1\mathbf{D}-\frac{1}{3}\nu_1\left[\left(2\mathbf{D}-\dot{\mathbf{A}}\right)\cdot\mathbf{A}\right]\mathbf{I}+O(\varepsilon^2). \qquad (43)$$

This model is ver similar to Oldroyd-B model, the difference is in the last term. We show that this term is of the order $O(\varepsilon^2)$, and so we can neglect it. Take the scala product of (43) with tensor $\mathbf{A}$, we get

$$\mu \underbrace{|\mathbf{A}|^2}_{O(\varepsilon^2)} + \nu_1 \dot{\mathbf{A}} \cdot \mathbf{A} - \nu_1 \underbrace{\left(\mathbf{LA} \cdot \mathbf{A} + \mathbf{AL}^{\mathrm{T}} \cdot \mathbf{A}\right)}_{O(\varepsilon^2)} = 2\nu_1 (\mathbf{D} \cdot \mathbf{A}) - \frac{1}{3}\nu_1 \left[\left(2\mathbf{D} - \dot{\mathbf{A}}\right) \cdot \mathbf{A}\right] \underbrace{\operatorname{tr} \mathbf{A}}_{O(\varepsilon^2)} .$$

From this it implies that[9]

$$\left(2\mathbf{D} - \dot{\mathbf{A}}\right) \cdot \mathbf{A} = O(\varepsilon^2).$$

We showed that the nonlinear viscoelatic model linearizes to Oldroyd-B model.

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T}, \\ \mathbf{T} &= -p\mathbf{I} + 2\nu_0 \mathbf{D} + \mu \mathbf{A}, \\ \mu \mathbf{A} + \nu_1 \overset{\triangledown}{\mathbf{A}} &= 2\nu_1 \mathbf{D}, \end{aligned}$$

and so derived thermodynamically compatible nonlinear viscoelastic model is in some sense consistent with the well-known viscoelastic model.

Nonlinear two-component viscoelastic model

Let consider the material consisting of two components, to each component corresponds one natural configuration $k_{p_1(t)}$ a $k_{p_2(t)}$ (see Figure (33)). We choose the entropy $\eta =$

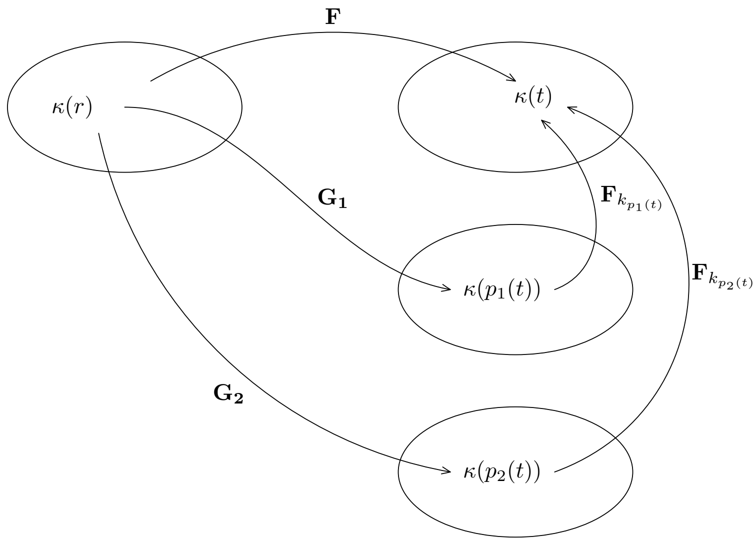


Figure 33: Two natural configurations

$\tilde{\eta}(e, \varrho, \mathbf{B}_{k_{p_1(t)}}, \mathbf{B}_{k_{p_2(t)}})$, we can invert it and use the internal energy $e = \tilde{e}(\eta, \varrho, \mathbf{B}_{k_{p_1(t)}}, \mathbf{B}_{k_{p_2(t)}})$ and concretely we choose

$$e = e_0(\eta, \varrho) + \frac{\mu_1}{2\varrho}\left(\operatorname{tr} \mathbf{B}_{k_{p_1(t)}} - 3\right) + \frac{\mu_2}{2\varrho}\left(\operatorname{tr} \mathbf{B}_{k_{p_2(t)}} - 3\right),$$

[9] Note that from this we can read

$$\dot{\mathbf{A}} = 2\mathbf{D} + O(\varepsilon),$$

which is expected.

again we consider only isothermal processes and the incompressible material $\operatorname{tr}\mathbf{D} = \operatorname{tr}\mathbf{D}_{k_{p_1(t)}} = \operatorname{tr}\mathbf{D}_{k_{p_2(t)}} = 0$. After some calculation we obtain a relation for the rate of entropy production

$$\begin{aligned}\xi &= (\mathbf{T} - \mu_1\mathbf{B}_{k_{p_1(t)}} - \mu_2\mathbf{B}_{k_{p_2(t)}}) \cdot \mathbf{D} + \mu_1\mathbf{C}^d_{k_{p_1(t)}} \cdot \mathbf{D}_{k_{p_1(t)}} + \mu_2\mathbf{C}^d_{k_{p_2(t)}} \cdot \mathbf{D}_{k_{p_2(t)}} \\ &= \mathbf{T}^d_{\text{dis}} \cdot \mathbf{D} + \mu_1\mathbf{C}^d_{k_{p_1(t)}} \cdot \mathbf{D}_{k_{p_1(t)}} + \mu_2\mathbf{C}^d_{k_{p_2(t)}} \cdot \mathbf{D}_{k_{p_2(t)}}.\end{aligned}$$

We choose the following constitutive relation for the rate of entropy production

$$\begin{aligned}\xi = \tilde{\xi}(\mathbf{D}, \mathbf{D}_{k_{p_1(t)}}, \mathbf{D}_{k_{p_2(t)}}, \mathbf{B}_{k_{p_1(t)}}, \mathbf{B}_{k_{p_2(t)}}) = 2\nu_0|\mathbf{D}|^2 + 2\nu_1\mathbf{D}_{k_{p_1(t)}} \cdot \mathbf{B}_{k_{p_1(t)}}\mathbf{D}_{k_{p_1(t)}} + \\ 2\nu_2\mathbf{D}_{k_{p_2(t)}} \cdot \mathbf{B}_{k_{p_2(t)}}\mathbf{D}_{k_{p_2(t)}}.\end{aligned}$$

Again we observe that the rate of entropy production is non-negative and so the second law of thermodynamics is satified. We define the Lagrange function

$$\begin{aligned}L(\mathbf{D}, \mathbf{D}_{k_{p_1(t)}}, \mathbf{D}_{k_{p_2(t)}}) = \tilde{\xi}(\dots) + \lambda_1(\tilde{\xi}(\dots) - \mathbf{T}^d_{\text{dis}} \cdot \mathbf{D} - \mu_1\mathbf{C}^d_{k_{p_1(t)}} \cdot \mathbf{D}_{k_{p_1(t)}} - \mu_2\mathbf{C}^d_{k_{p_2(t)}} \cdot \mathbf{D}_{k_{p_2(t)}}) + \\ \lambda_2 \operatorname{tr}\mathbf{D} + \lambda_3 \operatorname{tr}\mathbf{D}_{k_{p_1(t)}} + \lambda_4 \operatorname{tr}\mathbf{D}_{k_{p_2(t)}}.\end{aligned}$$

Necessary conditions for the maximum are

$$\frac{\partial L}{\partial \mathbf{D}} = 0, \quad \frac{\partial L}{\partial \mathbf{D}_{k_{p_1(t)}}} = 0, \quad \frac{\partial L}{\partial \mathbf{D}_{k_{p_2(t)}}} = 0.$$

We substitute for L

$$\frac{1+\lambda_1}{\lambda_1}\frac{\partial \tilde{\xi}}{\partial \mathbf{D}} = \mathbf{T}^d_{\text{dis}} + \frac{\lambda_2}{\lambda_1}\mathbf{I}, \tag{44}$$

$$\frac{1+\lambda_1}{\lambda_1}\frac{\partial \tilde{\xi}}{\partial \mathbf{D}_{k_{p_1(t)}}} = \mu_1\mathbf{C}^d_{k_{p_1(t)}} + \frac{\lambda_3}{\lambda_1}\mathbf{I}, \tag{45}$$

$$\frac{1+\lambda_1}{\lambda_1}\frac{\partial \tilde{\xi}}{\partial \mathbf{D}_{k_{p_2(t)}}} = \mu_1\mathbf{C}^d_{k_{p_2(t)}} + \frac{\lambda_4}{\lambda_1}\mathbf{I}. \tag{46}$$

Do the differentiation

$$\frac{\partial \tilde{\xi}}{\partial \mathbf{D}} = 4\nu_0\mathbf{D}, \quad \frac{\partial \tilde{\xi}}{\partial \mathbf{D}_{k_{p_1(t)}}} = 4\nu_1\mathbf{C}_{k_{p_1(t)}}\mathbf{D}_{k_{p_1(t)}}, \quad \frac{\partial \tilde{\xi}}{\partial \mathbf{D}_{k_{p_2(t)}}} = 4\nu_2\mathbf{C}_{k_{p_2(t)}}\mathbf{D}_{k_{p_2(t)}}.$$

and eliminate the Lagrange multipliers. To do this we add $(44)\cdot\mathbf{D} + (45)\cdot\mathbf{D}_{k_{p_1(t)}} + (46)\cdot\mathbf{D}_{k_{p_2(t)}}$, use the constrains of incompressibility and get

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\mathbf{T}^d_{\text{dis}} \cdot \mathbf{D} + \mu_1\mathbf{C}^d_{k_{p_1(t)}} \cdot \mathbf{D}_{k_{p_1(t)}} + \mu_2\mathbf{C}^d_{k_{p_2(t)}} \cdot \mathbf{D}_{k_{p_2(t)}}}{\frac{\partial \tilde{\xi}}{\partial \mathbf{D}} \cdot \mathbf{D} + \frac{\partial \tilde{\xi}}{\partial \mathbf{D}_{k_{p_1(t)}}} \cdot \mathbf{D}_{k_{p_1(t)}} + \frac{\partial \tilde{\xi}}{\partial \mathbf{D}_{k_{p_2(t)}}} \cdot \mathbf{D}_{k_{p_2(t)}}} = \frac{\tilde{\xi}}{2\tilde{\xi}} = \frac{1}{2}.$$

Now take the trace of (44) and obtain

$$0 = \operatorname{tr}\mathbf{T}^d_{\text{dis}} - 3\frac{\lambda_2}{\lambda_1} \;\Rightarrow\; \frac{\lambda_2}{\lambda_1} = 0.$$

By taking the traces of (45) and (46), we obtain

$$\begin{aligned}\frac{\lambda_3}{\lambda_1} &= \frac{2}{3}\nu_1\mathbf{D}_{k_{p_1(t)}} \cdot \mathbf{C}_{k_{p_1(t)}}, \\ \frac{\lambda_4}{\lambda_1} &= \frac{2}{3}\nu_2\mathbf{D}_{k_{p_2(t)}} \cdot \mathbf{C}_{k_{p_2(t)}}.\end{aligned}$$

We get

$$\mathbf{T}^d_{\text{dis}} = 2\nu_0\mathbf{D}, \tag{47}$$

$$2\nu_1\left(\mathbf{C}_{k_{p_1(t)}}\mathbf{D}_{k_{p_1(t)}} - \frac{1}{3}(\mathbf{D}_{k_{p_1(t)}}\cdot\mathbf{C}_{k_{p_1(t)}})\mathbf{I}\right) = \mu_1\mathbf{C}^d_{k_{p_1(t)}}, \tag{48}$$

$$2\nu_2\left(\mathbf{C}_{k_{p_2(t)}}\mathbf{D}_{k_{p_2(t)}} - \frac{1}{3}(\mathbf{D}_{k_{p_2(t)}}\cdot\mathbf{C}_{k_{p_2(t)}})\mathbf{I}\right) = \mu_2\mathbf{C}^d_{k_{p_2(t)}}. \tag{49}$$

From (47) it implies that

$$\mathbf{T} = -p\mathbf{I} + 2\nu_0\mathbf{D} + \mu_1\mathbf{B}^d_{k_{p_1(t)}} + \mu_2\mathbf{B}^d_{k_{p_2(t)}}.$$

As in the case of model with one natural configuration it holds

$$\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p_1(t)}} = -2\mathbf{F}_{k_{p_1(t)}}\mathbf{D}_{k_{p_1(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p_1(t)}} - \frac{2}{3}(\mathbf{D}\cdot\mathbf{B}_{k_{p_1(t)}})\mathbf{I} + \frac{2}{3}(\mathbf{C}_{k_{p_1(t)}}\cdot\mathbf{D}_{k_{p_1(t)}})\mathbf{I} + \frac{2}{3}(\operatorname{tr}\mathbf{B}_{k_{p_1(t)}})\mathbf{D},$$

$$\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p_2(t)}} = -2\mathbf{F}_{k_{p_2(t)}}\mathbf{D}_{k_{p_2(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p_2(t)}} - \frac{2}{3}(\mathbf{D}\cdot\mathbf{B}_{k_{p_2(t)}})\mathbf{I} + \frac{2}{3}(\mathbf{C}_{k_{p_2(t)}}\cdot\mathbf{D}_{k_{p_2(t)}})\mathbf{I} + \frac{2}{3}(\operatorname{tr}\mathbf{B}_{k_{p_2(t)}})\mathbf{D}$$

Multiplying (48) from the left by $\mathbf{F}^{-\mathrm{T}}_{k_{p_1(t)}}$ and from the right by $\mathbf{F}^{\mathrm{T}}_{k_{p_1(t)}}$, and (49) from the left by $\mathbf{F}^{-\mathrm{T}}_{k_{p_2(t)}}$ and from the right by $\mathbf{F}^{\mathrm{T}}_{k_{p_2(t)}}$, we get

$$2\nu_1\left(\mathbf{F}_{k_{p_1(t)}}\mathbf{D}_{k_{p_1(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p_1(t)}} - \frac{1}{3}(\mathbf{B}_{k_{p_1(t)}}\cdot\mathbf{D}_{k_{p_1(t)}})\mathbf{I}\right) = \mu_1\mathbf{B}^d_{k_{p_1(t)}},$$

$$2\nu_2\left(\mathbf{F}_{k_{p_2(t)}}\mathbf{D}_{k_{p_2(t)}}\mathbf{F}^{\mathrm{T}}_{k_{p_2(t)}} - \frac{1}{3}(\mathbf{B}_{k_{p_2(t)}}\cdot\mathbf{D}_{k_{p_2(t)}})\mathbf{I}\right) = \mu_2\mathbf{B}^d_{k_{p_2(t)}}.$$

From the relations for Oldroyd derivatives of $\mathbf{B}^d_{k_{p_1(t)}}$ a $\mathbf{B}^d_{k_{p_2(t)}}$ we obtain

$$2\nu_1\left(-\frac{1}{2}\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p_1(t)}} - \frac{1}{3}(\mathbf{D}\cdot\mathbf{B}^d_{k_{p_1(t)}})\mathbf{I} + \frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p_1(t)}})\mathbf{D}\right) = \mu_1\mathbf{B}^d_{k_{p_1(t)}},$$

$$2\nu_2\left(-\frac{1}{2}\overset{\triangledown}{\mathbf{B}}{}^d_{k_{p_2(t)}} - \frac{1}{3}(\mathbf{D}\cdot\mathbf{B}^d_{k_{p_2(t)}})\mathbf{I} + \frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p_2(t)}})\mathbf{D}\right) = \mu_2\mathbf{B}^d_{k_{p_2(t)}}.$$

The trace of both equations is zero, we have 2×5 scalar equations for 2×6 unknowns. We complete the system of equations with two equations for incompressibility of two elastic responses

$$\det\left(\mathbf{B}_{k_{p_1(t)}} + \frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p_1(t)}})\mathbf{I}\right) = 1,$$

$$\det\left(\mathbf{B}_{k_{p_2(t)}} + \frac{1}{3}(\operatorname{tr}\mathbf{B}_{k_{p_2(t)}})\mathbf{I}\right) = 1.$$

We obtain the governing equations for the nonlinear two-component viscoelastic rate-type fluid model

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0, \\
\varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T}, \\
\mathbf{T} &= -p\mathbf{I} + 2\nu_0 \mathbf{D} + \mu_1 \mathbf{B}^d_{k_{p_1(t)}} + \mu_2 \mathbf{B}^d_{k_{p_2(t)}}, \\
2\nu_1 \left(-\frac{1}{2} \overset{\triangledown}{\mathbf{B}}{}^d_{k_{p_1(t)}} - \frac{1}{3}(\mathbf{D} \cdot \mathbf{B}^d_{k_{p_1(t)}})\mathbf{I} + \frac{1}{3}(\operatorname{tr} \mathbf{B}_{k_{p_1(t)}})\mathbf{D} \right) &= \mu_1 \mathbf{B}^d_{k_{p_1(t)}}, \\
2\nu_2 \left(-\frac{1}{2} \overset{\triangledown}{\mathbf{B}}{}^d_{k_{p_2(t)}} - \frac{1}{3}(\mathbf{D} \cdot \mathbf{B}^d_{k_{p_2(t)}})\mathbf{I} + \frac{1}{3}(\operatorname{tr} \mathbf{B}_{k_{p_2(t)}})\mathbf{D} \right) &= \mu_2 \mathbf{B}^d_{k_{p_2(t)}}, \\
\det \left(\mathbf{B}_{k_{p_1(t)}} + \frac{1}{3}(\operatorname{tr} \mathbf{B}_{k_{p_1(t)}})\mathbf{I} \right) &= 1, \\
\det \left(\mathbf{B}_{k_{p_2(t)}} + \frac{1}{3}(\operatorname{tr} \mathbf{B}_{k_{p_2(t)}})\mathbf{I} \right) &= 1.
\end{aligned}
$$

Homework

This is not full list of homework. It is just to complete the lecture notes.

Homework 1. *Under the assumption*

$$\xi = \mathbf{T}^d \cdot \mathbf{D}^d + (m+p)\operatorname{div}\mathbf{v} + \frac{\mathbf{q}\cdot\nabla\theta}{\theta}, \tag{50}$$

derive Navier-Stokes equations by principle of maximization of entropy production

$$\max_{(50)+\mathbf{T}^d, m, \mathbf{q}} \tilde{\xi}(\mathbf{T}^d, m, \mathbf{q}),$$

where $\tilde{\xi}$ *is given by*

$$\tilde{\xi} = \frac{1}{2\mu}|\mathbf{T}^d|^2 + \frac{3}{2\mu+3\lambda}(m+p)^2 + \frac{1}{K}\frac{|\mathbf{q}|^2}{\theta}.$$

Homework 2. *Consider the following model*

$$\mathbf{T} = -p\mathbf{I} + \nu_\infty + \frac{\nu_0 - \nu_\infty}{(1+\Gamma^2|\mathbf{D}|^2)^{\frac{1-n}{2}}}\mathbf{D},$$

where $\nu_0, \nu_\infty, \Gamma > 0$*. Explain which non-Newtonian behaviour exhibits fluid described by that model for* $n \in \mathbb{R}$*.*

Homework 3. *Show that constitutive relation for Bingham fluid*

$$|\mathbf{S}| \le \tau^* \Leftrightarrow \mathbf{D} = \mathbf{0},$$
$$|\mathbf{S}| > \tau^* \Leftrightarrow \mathbf{S} = \frac{\tau^*\mathbf{D}}{|\mathbf{D}|} + 2\mu(|\mathbf{D}|^2)\mathbf{D}$$

is equivalent to

$$2\mu(|\mathbf{D}|^2)\left(\tau^* + (|\mathbf{S}|-\tau^*)^+\right)\mathbf{D} = (|\mathbf{S}|-\tau^*)^+\mathbf{S},$$

where $\tau^* > 0$, $\mu(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *and* $x^+ = \max\{0, x\}$*.*

Homework 4. *Use the principle of maximization of entropy production* $\xi = \tilde{\xi}(\mathbf{T}^d, \mathbf{D}^d)$ *for* $\mathbf{T}^d \in A$*, where*

$$A := \left\{\mathbf{T}^d \in \mathbb{R}^{3\times3}_{\mathrm{sym}}, \xi = \mathbf{T}^d : \mathbf{D}^d\right\}.$$

In two cases for $\tilde{\xi}$ *given by:*

1. $\tilde{\xi} = |\mathbf{T}^d|^{r/(r-1)}$,
2. $\tilde{\xi} = |\mathbf{D}^d|^{2-r}|\mathbf{T}^d|^2$.

Homework 5. *Stokes fluid is given by constitutive relation*

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + \alpha\mathbf{D}^2,$$

where α *and* μ *are constant. Explain which non-Newtonian behaviour exhibits Stokes fluid.*

Homework 6. *Consider Korteweg's model in unidirectional velocity field* $\mathbf{v} = (u(y), 0, 0)$*. Show that model exhibits normal stress differences*

$$T_{11} - T_{22} \neq 0,$$
$$T_{22} - T_{33} \neq 0.$$

Bibliography

[1] C. Barus. Isotherms, isopiestics and isometrics relative to viscosity. *Amer. J. Sci.*, 45:87–96, 1893.

[2] Santanu Basu and U.S. Shivhare. Rheological, textural, micro-structural and sensory properties of mango jam. *J. Food Eng.*, 100(2):357–365, 2010.

[3] B. Bernstein, E. A. Kearsley, and L. J. Zapas. A study of stress relaxation with finite strain. *Trans. Soc. Rheol.*, 7(1):391–410, 1963.

[4] C. E. Bingham. *Fluidity and plasticity.* McGraw–Hill, New York, 1922.

[5] H Blatter. Velocity and stress-fields in grounded glaciers – a simple algorithm for including deviatoric stress gradients. *J. Glaciol.*, 41(138):333–344, 1995.

[6] P. W. Bridgman. The effect of pressure on the viscosity of forty-four pure liquids. *Proc. Am. Acad. Art. Sci.*, 61(3/12):57–99, FEB-NOV 1926.

[7] J. M. Burgers. Mechanical considerations – model systems – phenomenological theories of relaxation and viscosity. In *First report on viscosity and plasticity*, chapter 1, pages 5–67. Nordemann Publishing, New York, 1939.

[8] Pierre J. Carreau. Rheological equations from molecular network theories. *J. Rheol.*, 16(1):99–127, 1972.

[9] I-Jen Chen and D. C. Bogue. Time-dependent stress in polymer melts and review of viscoelastic theory. *Trans. Soc. Rheol.*, 16(1):59–78, 1972.

[10] William O. Criminale, J. L. Ericksen, and G. L. Filbey. Steady shear flow of non-Newtonian fluids. *Arch. Rat. Mech. Anal.*, 1:410–417, 1957.

[11] Malcolm M. Cross. Rheology of non-newtonian fluids: A new flow equation for pseudoplastic systems. *J. Colloid Sci.*, 20(5):417–437, 1965.

[12] A. de Waele. Viscometry and plastometry. *J. Oil Colour Chem. Assoc.*, 6:33–69, 1923.

[13] J.E. Dunn and J. Serrin. On the thermomechanics of interstitial working. *Archive for Rational Mechanics and Analysis*, 88:95–133, 1985.

[14] Henry Eyring. Viscosity, plasticity, and diffusion as examples of absolute reaction rates. *J. Chem. Phys.*, 4(4):283–291, 1936.

[15] H. Giesekus. A simple constitutive equation for polymer fluids based on the concept of deformation-dependent tensorial mobility. *J. Non-Newton. Fluid Mech.*, 11(1-2):69–109, 1982.

[16] J. W. Glen. The creep of polycrystalline ice. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 228(1175):519–538, 1955.

[17] Winslow H. Herschel and Ronald Bulkley. Konsistenzmessungen von Gummi-Benzollösungen. *Colloid Polym. Sci.*, 39(4):291–300, August 1926.

[18] K. L. Johnson and J. L. Tevaarwerk. Shear behaviour of elastohydrodynamic oil films. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 356(1685):215–236, 1977.

[19] M. W. Johnson and D. Segalman. A model for viscoelastic fluid behavior which allows non-affine deformation. *J. Non-Newton. Fluid Mech.*, 2(3):255–270, 1977.

[20] Michael M. Khonsari and E. Richard Booser. *Applied Tribology: Bearing Design and Lubrication*. John Wiley & Sons Ltd, Chichester, second edition, 2008.

[21] Seikichi Matsuhisa and R. Byron Bird. Analytical and numerical solutions for laminar flow of the non-Newtonian Ellis fluid. *AIChE J.*, 11(4):588–595, 1965.

[22] J. Clerk Maxwell. On the dynamical theory of gases. *Philos. Trans. R. Soc.*, 157:49–88, 1867.

[23] J. G. Oldroyd. On the formulation of rheological equations of state. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 200(1063):523–541, 1950.

[24] Wolfgang Ostwald. Über die Geschwindigkeitsfunktion der Viskosität disperser Systeme. I. *Colloid Polym. Sci.*, 36:99–117, 1925.

[25] Erin C. Pettit and Edwin D. Waddington. Ice flow at low deviatoric stress. *J. Glaciol.*, 49(166):359–369, 2003.

[26] N. Phan Thien. Non-linear network viscoelastic model. *J. Rheol.*, 22(3):259–283, 1978.

[27] N. Phan Thien and Roger I. Tanner. A new constitutive equation derived from network theory. *J. Non-Newton. Fluid Mech.*, 2(4):353–365, 1977.

[28] K. R. Rajagopal and A. R. Srinivasa. A thermodynamic frame work for rate type fluid models. *J. Non-Newton. Fluid Mech.*, 88(3):207–227, 2000.

[29] Francis Ree, Taikyue Ree, and Henry Eyring. Relaxation theory of transport problems in condensed systems. *Ind. Eng. Chem.*, 50(7):1036–1040, 1958.

[30] M. Reiner. A mathematical theory of dilatancy. *Am. J. Math.*, 67(3):350–362, 1945.

[31] R. S. Rivlin and J. L. Ericksen. Stress-deformation relations for isotropic materials. *J. Ration. Mech. Anal.*, 4:323–425, 1955.

[32] R. S. Rivlin and K. N. Sawyers. Nonlinear continuum mechanics of viscoelastic fluids. *Annu. Rev. Fluid Mech.*, 3:117–146, 1971.

[33] T. A. Savvas, N. C. Markatos, and C. D. Papaspyrides. On the flow of non-newtonian polymer solutions. *Appl. Math. Modelling*, 18(1):14–22, 1994.

[34] Gilbert R. Seely. Non-newtonian viscosity of polybutadiene solutions. *AIChE J.*, 10(1):56–60, 1964.

[35] A. W. Sisko. The flow of lubricating greases. *Ind. Eng. Chem.*, 50(12):1789–1792, 1958.

[36] Kenji Yasuda. *Investigation of the analogies between viscometric and linear viscoelastic properties of polystyrene fluids*. PhD thesis, Massachusetts Institute of Technology. Dept. of Chemical Engineering., 1979.