# Moufang Polygons

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## Contents



## 1 Buildings

Let $\Pi$ be a Coxeter diagram with vertex set $I$ and let $W$ be the corresponding Coxeter group. Thus $W$ is a group generated by the set $I$ subject to relations which can be read off from the labels on the edges of $\Pi$. Let $\Delta$ be a building of type $\Pi$. For these notes, it suffices to consider $\Delta$ as a graph whose vertices are the chambers, where two chambers are joined by an edge whenever they lie in a common panel. If the type of a panel $P$ (viewed as a simplex) is $I\backslash\{i\}$, we give each edge joining two chambers of $P$ the "color" $i$.

The pair $(W, I)$ is called a *Coxeter system.* Let $\Sigma_\Pi$ be the corresponding Cayley graph. Thus $\Sigma_\Pi$ is the graph with vertex set $W$, where two vertices $x, y \in W$ are joined by an edge whenever $y^{-1}x \in I$. We endow $\Sigma_\Pi$ with its natural edge-coloring whereby an edge $\{x, y\}$ has "color" $i \in I$ if and only if $x = yi$ in $W$. The *apartments* of $\Delta$ are subgraphs isomorphic to this edge-colored Cayley graph.

For each edge $\{x, y\}$ of $\Sigma_\Pi$, the set of vertices nearer to $x$ than to $y$ and the set of vertices nearer to $y$ than to $x$ form a partition of the vertex set of $\Sigma_\Pi$ (i.e.

of $W$). Sets of this form are called *roots.* A root of an apartment $\Sigma$ of $\Delta$ is the image in $\Sigma$ of a root under some isomorphism from $\Sigma_\Pi$ to $\Sigma$. A root of $\Delta$ is a root of one of its apartments.

For each subset $J$ of $I$, let $\Pi_J$ denote the subgraph of $\Pi$ spanned by $J$. The $J$*- residues* of $\Delta$ (for some $J \subset I$) are the connected components of the graph obtained from $\Delta$ by deleting all the edges of $\Delta$ whose color is not in $J$ (but without deleting any chambers). For a given $J$-residue $R$, the set $J$ is precisely the set of colors appearing on the edges of $R$; this set is called the *type* of $R$ and its cardinality is called the *rank* of $R$. A *residue* is a $J$-residue for some $J \subset I$. The *panels* of $\Delta$ are the residues of rank one. Every $J$-residue is a building of type $\Pi_J$ . In particular, the panels are complete graphs. A $J$-residue is irreducible if the diagram $\Pi_J$ is connected.

# 2 Spherical Buildings

The classification of spherical buildings (in rank at least three) was first given in [8]. In this section we follow more closely the revised proof given in Chapter 40 of [13].

From now on, we suppose that $\Delta$ is *spherical* (i.e. that its apartments are finite), *thick* (i.e. that every panel contains more than two chambers) and *irreducible* (i.e. that the Coxeter diagram $\Pi$ is connected).

**Definition 1.** *For each root $\alpha$ of $\Delta$, let $U_\alpha$ denote the intersection of the stabilizers in the group Aut($\Delta$) of all the chambers of $\Delta$ that are contained in a panel containing two chambers of $\alpha$. The group $U_\alpha$ is called the root group belonging to $\alpha$. It acts trivially on $\alpha$ and faithfully on every panel in the wall of $\alpha$ (i.e. on every panel containing exactly one chamber of $\alpha$).*

**Definition 2.** *The building $\Delta$ is Moufang if for every root $\alpha$, the root group $U_\alpha$ acts transitively on the set of apartments containing $\alpha$. If $\Delta$ is Moufang and $P$ is a panel in the wall of a root $\alpha$, then the root group acts sharply transitive on the set $P\backslash\alpha$.*

Now let $C$ be a chamber of $\Delta$ and let $E_2(C)$ be the subgraph of $\Delta$ spanned by all the irreducible rank two residues of $\Delta$ that contain $C$. (There is exactly one such residue for each edge of $\Pi$ and the intersection of two of these residues is a panel containing $C$ if the corresponding edges are adjacent, $\{C\}$ otherwise.) The subgraph $E_2(C)$ is called the *foundation* of $\Delta$.

**Theorem 3.** *The building $\Delta$ is uniquely determined by $E_2(C)$.*

In other words, if $\Delta'$ is a second building of type $\Pi$ and $\pi$ is an isomorphism from $E_2(C)$ to $E_2(C')$, where $C'$ is a chamber of $\Delta'$, then there is an isomorphism from $\Delta$ to $\Delta'$ that extends $\pi$. A map $\pi$ from $E_2(C)$ to $E_2(C')$ sending $C$ to $C'$ is an isomorphism if and only if for each edge $J\{i,j\}$ of $\Pi$, the restriction of $\pi$ to the $J$-residue of $\Delta$ containing $C$ is an isomorphism from this residue to the $J$-residue of $\Delta'$ containing $C'$.

Applying Theorem 3 with $\Delta' = \Delta$, it is possible to deduce that the automorphism group of $\Delta$ is large. More precisely, we have the following:

**Theorem 4.** *If the rank of $\Delta$ (i.e. the cardinality of $I$) is at least three, then $\Delta$ is Moufang and all the irreducible residues of rank at least two of $\Delta$ are also Moufang.*

# 3 Generalized polygons

An irreducible spherical building of rank two is the same thing as a generalized polygon. A generalized polygon is simply a bipartite graph whose diameter is half the length of a shortest circuit. To avoid certain trivialities, we assume as well that every vertex has at least three (but possibly infinitely many) neighbors and that the diameter is at least three. (The diameter is not allowed to be infinite). A generalized $n$-gon is a generalized polygon of diameter $n$.

Let $\Gamma$ be a generalized $n$-gon for some $n \geq 3$ and let $G = \mathrm{Aut}(G)$. A circuit of length $2n$ in $\Gamma$ is called an apartment. A root of $\Gamma$ is an undirected path of length $n$. For each vertex $x$ of $\Gamma$, let $\Gamma_x$ denote the set of neighbors of $x$. For each root $\alpha = (x_0, x_1, ..., x_n)$ of $\Gamma$, we denote by $U_\alpha$ the pointwise stabilizer in $G$ of the set $\Gamma_{x_1} \cup ... \cup \Gamma_{x_{n-1}}$ . The group $U_\alpha$ is called the root group associated with $\alpha$.

# 4 Moufang polygons

**Definition 5.** *A generalized n-gon satisfies the Moufang property if for each root $\alpha$ of $\Gamma$, the root group $U_\alpha$ acts transitively on the set of apartments containing $\alpha$.*

A Moufang $n$-gon is a generalized $n$-gon satisfying the Moufang property.

A generalized 3-gon (or triangle) is the same thing as the incidence graph of a projective plane. The notion of a Moufang generalized $n$-gon generalizes the notion of a Moufang projective plane first introduced in [4].

We now assume that $\Gamma$ is a Moufang $n$-gon for some $n \geq 3$. We choose an apartment $\Sigma$ and label the vertices of $\Sigma$ by the integers modulo $2n$ so that $i$ is adjacent to $i+1$ and different from $i+2$ for all $i$. Let $U_i$ be the root group corresponding to the root $(i, i+1, ..., i+n)$ for all $i$ and let $U_+$ denote the subgroup of $G$ generated by the subgroups $U_1, U_2, ..., U_n$. The $(n+1)$-tuple $(U_+, U_1, U_2, ..., U_n)$ is called the root group sequence associated with $\Gamma$; it is unique up to conjugation in $G$ and up to the re-numbering $U_i \mapsto U_{n+1-i}$ of the root groups $U_1, ..., U_n$. In [12], we show:

**Theorem 6.** *A Moufang n-gon is uniquely determined by the associated root group sequence $(U_+, U_1, U_2, ..., U_n)$.*

As an example, we consider the case $n = 3$. Let $A$ be an alternative division ring. This is a ring satisfying all the axioms of a skew-field except the law of associativity for multiplication, but with a strengthened law of inverses: For each non-zero $u \in A$, there exists an element $u'$ such that $u' \cdot uv = v$ and $vu \cdot u' = v$ for all $v \in A$. Now let $U_1, U_2, U_3$ be three groups parameterized by the additive group of $A$. By this we mean that we can choose isomorphisms $u \mapsto x_i(u)$ from the additive group of $A$ to the (multiplicative) group $U_i$ for all $i \in [1, 3]$. We now impose the relations $[U_1, U_2] = [U_2, U_3] = 1$ and $[x_1(u), x_3(v)] = x_2(uv)$ for all $u$, $v \in A$. These equations determine the structure of a group $U_+ = \langle U_1, U_2, U_3 \rangle$. It turns out that $(U_+, U_1, U_2, U_3)$ is the root group sequence associated with a generalized triangle which we denote by $\mathcal{T}(A)$. If $A$ is a skew-field, $\mathcal{T}(A)$ is the incidence graph of the projective plane associated with a 3-dimensional right vector space over $A$. Moufang showed in [4] that every Moufang projective plane is parameterized by an alternative division ring. This result can be reformulated as follows: Every Moufang triangle is isomorphic to $\mathcal{T}(A)$ for some alternative division ring $A$.

Alternative division rings were classified by Bruck and Kleinfeld [2, 3]: An alternative division ring is either a skew-field (possibly commutative) or a kind of 8-dimensional nonassociative algebra over a commutative field $K$ called a Cayley-Dickson division algebra.

This case is typical. For each $n$, we find an algebraic system (in some general sense) with which we can parameterize the groups $U_1, ..., U_n$ and give formulas for all the commutators in $[U_i, U_j]$ for all distinct $i$, $j$ in $[1, n]$ expressed in terms of the parameters. These formulas determine the root group sequence $(U_+, U_1, ..., U_n)$ and thus $\Gamma$ In each case, there then remains the problem of classifying the relevant algebraic systems. That this strategy has any chance of success rests on the following result [10, 14]:

**Theorem 7.** *Moufang $n$-gons exist only for $n = 3$, 4, 6 and 8.*

In [13] is proved that *every Moufang polygon is isomorphic to one of the nine families of Moufang polygons*:

1. Moufang Triangles
2. Moufang Quadrangles of type involutory
3. Moufang Quadrangles of type quadratic form
4. Moufang Quadrangles of type indifferent
5. Moufang Quadrangles of type pseudo-quadratic form
6. Moufang Quadrangles of type $E_6$, $E_7$ and $E_8$
7. Moufang Quadrangles of type $F_4$
8. Moufang hexagons
9. Moufang octagons

We indicate the conclusions in each case beginning with the largest value of $n$. Moufang octagons are parameterized (see [11]) by pairs $(K, \sigma)$ where $K$ is a commutative field of characteristic two and $\sigma$ is an endomorphism of $K$ such that $\sigma^2$ is the Frobenius map of $K$, i.e. $(x^\sigma)^\sigma = x^2$ for all $x \in K$.

Moufang hexagons are parameterized (see [9]) by certain triples $(J, F, \#)$, where $F$ is a commutative field, $J$ a vector space over $F$ and $\#$ a map from $J$ to itself satisfying a certain list of properties. These triples are closely related to certain Jordan algebras which have been closely studied by Albert, Jacobson and several of Jacobsons students. They were classified by Petersson and Racine [6, 7]. We give two examples: Let $J$ be a commutative field containing $F$ and suppose either that $J^3 \subseteq F$ or that $[J : F] = 3$ and $J/F$ is separable. We set $x^\# = x^2$ for all $x \in J$ in the first case and $x^\# = x/N(x)$ for all $x \in J*$ in the second, where $N$ is the norm of the extension $J/F$. In all the other cases, the dimension of $J$ over $F$ is 9 or 27.

**Example 1.** The hexagons $H(J, F, \#)$.
Let

$$(J, F, N, \#, T, \times, 1)$$

be one of the hexagonal system defined in Chapter 15 in [13]. The functions $T$, $N$ and $\times$ and the element 1 are all uniquely determined by $J$, $F$ and the adjoin map $\#$. Let $U_1$, $U_3$ and $U_5$ be groups parametrized by $J$ and let $U_2$, $U_4$ and $U_6$ be groups defined by the relations

$$\begin{aligned}
[x_1(a), x_3(b)] &= x_2(T(a, b)) \\
[x_3(a), x_5(b)] &= x_4(T(a, b)) \\
[x_1(a), x_5(b)] &= x_2(-T(a^\#, b))x_3(a \times b)x_4(T(a, b^\#)) \\
[x_2(t), x_6(u)] &= x_4(tu) \\
[x_1(a), x_6(t)] &= x_2(-tN(a))X_3(ta^\#)x_4(t^2N(a))x_5(-ta)
\end{aligned}$$

for all $a, b \in J$ and $t, u \in F$.

There are three distinct classes of Moufang quadrangles: classical, indifferent and exceptional. The classical quadrangles are parameterized by pseudo-quadratic forms (see 8.2 of [8]). The indifferent quadrangles are parameterized by algebraic systems involving certain purely inseparable field extensions in characteristic two. The exceptional quadrangles (of which there are four families) are parameterized by pairs of vector spaces and several maps connecting these vector spaces and the fields over which they are defined. The parameter systems for the first three families involve the even Clifford algebra of a certain type of quadratic form. The parameter systems for the fourth family is still more exotic; these quadrangles (like the indifferent quadrangles and the Moufang octagons) exist only in characteristic two. See [5].

**Example 2.** The quadrangles $\mathcal{Q}_{\mathcal{T}}(K, K_0, \sigma)$ of involutory type.
Let $(K, K_0, \sigma)$ be an involutory set and let $U_1$ and $U_3$ be groups parametrized by the group $K_0$ and let $U_2$ and $U_4$ be groups parametrized by the additive group of $K$. Let $\mathcal{Q}_{\mathcal{T}}(K, K_0, \sigma)$ denote the graph defined by the relations

$$[x_2(a), x_4(b)^{-1}] = x_3(a^\sigma b + b^\sigma a) \text{ and}$$

$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(a^\sigma ta)$$

for all $t \in K_0$ and $a$, $b \in K$.

**Example 3.** The quadrangles $\mathcal{Q}_{\mathcal{T}}(K, L_0, q)$ of quadratic form type.
Let $(K, L_0, q)$ be an anisotropic quadratic space as defines in (12.2)-(12.4) in [13] with $L_0 \neq 0$ and let$f$ denote the bilinear form associated with $q$. Let $U_1$ and $U_3$ be groups parametrized by the additive group of $K$ and let $U_2$ and $U_4$ be groups parametrized by $L_0$. Let $\mathcal{Q}_{\mathcal{T}}(K, L_0, q)$ denote the graph defined by the relations

$$[x_2(a), x_4(b)^{-1}] = x_3(f(a, b)) \text{ and}$$
$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(tq(a))$$

for all $t \in K$ and $a$, $b \in L_0$.

The Moufang triangles $\mathcal{T}(A)$ for $A$ a field or a skew-field and the classical quadrangles are the spherical buildings associated with certain classical groups. The remaining Moufang triangles (those parameterized by a Cayley-Dickson division algebra), the remaining quadrangles (except those defined only in characteristic two) and all the Moufang hexagons (except those defined over a purely inseparable field extension in characteristic three) are the spherical buildings associated with $k$-forms of absolutely simple algebraic groups of $k$-rank two. All other Moufang polygons, namely those which are defined only in characteristic two or three, are related to groups of mixed type as defined in (10.3.2) of [8].

**Example 4.** The triangles $\mathcal{T}(A)$.
Lat $A$ be an alternative division ring and let $U_1$, $U_2$, $U_3$ be three groups all parametrized by additive group of $A$. Let $\mathcal{T}(A)$ denote the graph defined by

the relations

$$[x_1(t), x_3(u)] = x_2(tu)$$

for all $t, u \in A$ according to the introduced conventions.

# 5 Root Systems

Let $V$ be an $n$-dimensional Euclidean space and let $\Phi$ be an irreducible reduced root system spanning $V$ (as defined in Chapter 6 of [1]; see especially §4). Thus $\Phi = X_n$ for some $X \in \{A; B; C; D; E; F; G\}$. For each $\alpha \in \Phi$, let

$$H_\alpha = \{v \in V | \alpha \cdot v = 0\}$$

and

$$s_\alpha(v) = v - 2(v \cdot \alpha)\alpha = (\alpha \cdot \alpha)$$

for all $v \in V$ . A *Weyl chamber* of $\Phi$ is the closure of a connected component of the space $V$ with all the hyperplanes of the form $H_\alpha$ for $\alpha \in \Phi$ removed. We define two Weyl chambers to be adjacent if their intersection spans a subspace of dimension $n - 1$. This defines a graph $\Theta$ on the Weyl chambers that is isomorphic to the Cayley graph associated with the spherical Coxeter diagram called $X_n$. For each $\alpha \in \Phi$, the set of all Weyl chambers contained in

$$\{v \in V | \alpha \cdot \geq 0\}$$

is a root of $\Phi$ (as defined in Section 1) and every root of $\Theta$ is of this form (for a unique $\alpha \in \Phi$). We can thus identify $\Phi$ with the set of roots of $\Theta$.

**Notation 1.** *Let $\alpha$, $\beta$ be two roots of $\Phi$ such that $\beta \neq \pm\alpha$. Let $(\alpha, \beta)$ denote the set of elements*

$$\alpha_1, ..., \alpha_s$$

*of $\Phi$ of the form $p\alpha + q\beta$ for positive real numbers $p$ and $q$. We order the vectors in this set so that the angle between $\alpha$ and $\alpha_i$ increases as $i$ increases. This ordered set is called the interval from $\alpha$ to $\beta$.*

**Remark 1.** *Suppose that $\Delta$ is a building of type $X_n$ with $n \geq 3$, so $\Delta$ is one of the spherical buildings described in the previous section. Let $\Sigma$ be an apartment* *of* $\Delta$*, let* $\alpha$*,* $\beta$ *be linearly independent elements of* $\Phi$ *and let* $\alpha_1, ..., \alpha_s$ *be as in 3.1. If we identify* $\Sigma$ *with the graph* $\Theta$ *and thus* $\Phi$ *with the set of roots of* $\Sigma$*, then*

$$[U_\alpha, U_\beta] \subset \prod_{i=1}^{s} U\alpha_i$$

*unless the interval* $(\alpha, \beta)$ *is empty, in which case* $[U_\alpha, U_\beta] = 1$*.*

Let $\tilde{X}_n$ denote the corresponding extended Dynkin diagram with the arrows on the multiple bonds deleted. The diagrams $\tilde{X}_n$ which arise in this way are precisely the connected affine Coxeter diagrams; they can be viewed, for example, in Theorem 4 in Chapter VI, §4, of [1].

For each $\alpha \in \Phi$ and each integer $k$, let

$$H_{\alpha,k} = \{v \in V | \alpha \cdot v = k\}$$

and

$$s_{\alpha,k}(v) = s_\alpha(v) + 2k\alpha = (\alpha \cdot \alpha)$$

for all $v \in$ V . The affine hyperplane $H_{\alpha,k}$ is thus the fixed point set of $s_{\alpha,k}$. An *alcove* of $\Phi$ is the closure of a connected component of the space V with all the affine hyperplanes of the form $H_{\alpha,k}$ removed. Let $\Gamma$ be the graph whose vertices are the alcoves such that two alcoves are adjacent whenever their intersection is of dimension $n-1$ (i.e. the vectors $u-v$ for all $u$, $v$ in the intersection span a subspace of dimension $n-1$). The graph $\Gamma$ is isomorphic to the Cayley graph of the Coxeter system associated with the Coxeter diagram $\tilde{X}_n$, and the corresponding Coxeter group is isomorphic to the group generated by all the maps of the form $s_{\alpha,k}$.

For each $\alpha \in \Phi$ and each integer $k$, let $K_{\alpha,k}$ denote the set of alcoves in the set

$$\{v \in V | v \cdot \alpha \geq k\}.$$

Each $K_{\alpha,k}$ is a root of $\Gamma$ and every root of .. is of this form (for a unique pair $\alpha$, $k$). We can think of the affine hyperplane $H_{\alpha,k}$ as the wall of $K_{\alpha,k}$. Two walls $H_{\alpha,k}$ and $H_{\beta,l}$ of $\Gamma$ are *parallel* if $\alpha = \beta$ and adjacent if, in *addition*, $|k-l| = 1$.

A point $v$ in $V$ is called *special* if $v \cdot \alpha$ is an integer for all $\alpha \in \Phi$. A *sector* is a translation of a Weyl chamber by a special point, i.e. a set of the form

$C + v$, where $C$ is a Weyl chamber and $v$ is a special point. We think of a sector $S$ as the subgraph of $\Gamma$ spanned by all the alcoves in $S$. Two sectors $S_1$ and $S_2$ are adjacent if there are two Weyl chambers $C_1$ and $C_2$ whose intersection spans a subspace of dimension $n-1$ and two special points $v_1$ and $v_2$ such that $S_1 = C_1 + v_1$ and $S_2 = C_2 + v_2$.

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