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Spectral properties of differential operators

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SPECTRAL PROPERTIES OF DIFFERENTIAL OPERATORS - METHOD OF WAVE EQUATION IN HARMONIC ANALYSIS AND PDE'S

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1. Overview

We will discuss here the notion of the finite speed propagation of solutions of wave equations, which is also called sometimes the weak Huygens principle. Finite speed propagation property is very common phenomenon in our every day experience. One can mention such apparent examples as the speed of sound and light or water waves. Here we describe efficient and elegant way of obtaining rigorous proof of finite propagation speed in a very wide variety of self-adjoint second order differential operators.

We start with a brief discussion of the standard Laplace operator and the corresponding wave and heat equation. Then we recall notion of spectral theory and discuss equivalence of so called Davies-Gaffney property and finite speed propagation. This equivalence provides very efficient tool to obtain rigorous proofs of the finite speed property in the wide range of situation. Next we discuss application of finite speed property in harmonic analysis and PDEs including heat kernels theory, Riesz transform, spectral multipliers and Schrödinger propagator.

The main point which I would like to make in these lecture is that thanks to its simple geometric interpretation the notion of finite speed propagation can be use in very efficient and intuitive wave to study various aspects of harmonic analysis and PDEs. In many cases this simple geometric picture behind finite speed propagation allow us to design the efficient framework for calculation before working our all necessary and sometimes tedious details. In other words it is very useful tool to "understand" sometimes quite complicated calculations.

2. Heat Equation

We consider an infinite homogeneous wire. Suppose that at time $t \geq 0$ and point $x \in \mathbf{R}$ the temperature is equal to $T(t,x)$. Then the amount of heat energy contained in the interval $[a,b]$ is equal to

$$\int_a^b T(t,x)dx.$$

and the ratio of change of this quantity is equal to

$$\partial_t \int_a^b T(t,x)dx = \int_a^b \partial_t T(t,x)dx.$$

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Now the heat energy flow through point a and b should be proportional to $\partial_x T(t,a)$ and $\partial_x T(t,b)$ respectively so

$$\partial_t \int_a^b T(t,x)dx = \int_a^b \partial_t T(t,x)dx = \partial_x T(t,b) - \partial_x T(t,a) = \int_a^b \partial_x^2 T(t,x)dx.$$

However, the above equality holds for all intervals $[a,b]$ so it follows that

$$\partial_t T(t,x) = \partial_x^2 T(t,x).$$

This equation is called the heat equation and our argument above shows that this equation governs the heat propagation in a infinite homogeneous wire. Note that in a more physical discussion we should consider a couple of constants such as heat capacity and heat conductivity. To simplify the discussion we will just assume that all constants are equal to 1. Now we are going to use the Divergence Theorem to argue that similar equations should govern the heat propagation in two, three (or more) dimensional objects. For example we consider the heat propagation in the homogeneous space $\mathbf{R}^3$. Let V be any domain (subset of $\mathbf{R}^3$) and ∂V boundary surface of V. The amount of heat energy enclosed by V is equal to

$$\int_V T(t,x,y,z)dV,$$

where $T(t,x,y,z)$ is the temperature at point (x,y,z) at time t. Next one can expect that the heat energy flow through the surface ∂V is proportional to

$$\int_{\partial V} \mathbf{n}T(t,x,y,z)dS = \int_{\partial V} \nabla T(t,x,y,z)\cdot \mathbf{n}dS$$

Hence by the Divergence Theorem

$$\int_{\partial V} \nabla T(t,x,y,z)\cdot \mathbf{n}dS = \int_V div(\nabla T(t,x,y,z))dV.$$

A simple calculation shows that

$$div(\nabla f)(x,y,z) = \partial_x^2 f(x,y,z) + \partial_y^2 f(x,y,z) + \partial_z^2 f(x,y,z)$$

Often the following notation is used

$$\Delta f(x,y,z) = \partial_x^2 f(x,y,z) + \partial_y^2 f(x,y,z) + \partial_z^2 f(x,y,z)$$

and Δ is called the Laplace operator. In other words

$$\begin{aligned}\int_{\partial V} \nabla T(t,x,y,z)\cdot \mathbf{n}dS &= \int_V \Big(\partial_x^2 T(t,x,y,z) + \partial_y^2 T(t,x,y,z) + \partial_z^2 T(t,x,y,z)\Big)dV \\ &= \int_V \Delta T(t,x,y,z)dV\end{aligned}$$

The same argument as in the one dimensional case shows that

$$\begin{aligned}\int_V \Delta T(t,x,y,z)dV = \int_{\partial V} \nabla T(t,x,y,z)\cdot \mathbf{n}dS &= \partial_t \int_V T(t,x,y,z)dV \\ &= \int_V \partial_t T(t,x,y,z)dV.\end{aligned}$$

However, the above equality holds for all subsets $V \subset \mathbf{R}^3$ so the temperature distribution $T(t,x,y,z)$ satisfies the following equation

$$\partial_t T(t,x,y,z) = \Delta T(t,x,y,z).$$

This equation governs heat propagation in a homogeneous three dimensional medium. It is also an important and significant example of a Partial Differential Equation (PDE). PDEs are any equations which involve partial derivatives.

2.1. **Initial conditions.** An natural physical problem related to our discussion of heat propagation is as follows: Suppose that one knows that initial temperature at time 0 is described by function $T(0,x,y,z) = f(x,y,z)$. Find the temperature at time $t > 0$ at (x,y,z) that is $T(t,x,y,z)$.

Mathematically we can formulate the problem in the following way: Find the function $T(t,x,y,z)$ such that

$$\partial_t T(t,x,y,z) = \Delta T(t,x,y,z)$$

and $T(0,x,y,z) = f(x,y,z)$. The function f is called an initial condition for the considered heat equation.

2.2. **Boundary conditions.** It would be more natural and more practical to consider finite interval $[a,b]$ or finite subset $\Omega \subset \mathbf{R}^3$ as our medium. That is we would like to consider the temperature function $T \colon \mathbf{R}_+ \times [a,b] \to \mathbf{R}$ or $T \colon \mathbf{R}_+ \times V \to \mathbf{R}$. *How should we describe the heat propagation at points a and b or at ∂V?* There are some different choices, which correspond to different physical possibilities. Vector Calculus can help us to understand the situation. Suppose that you wrap your interval $[a,b]$ or your solid V in a perfect thermal isolator. Then for an interval this should mean that

$$\partial_x T(t,a) = 0 \quad \text{and} \quad \partial_x T(t,b) = 0 \tag{1}$$

that is the heat energy does not flow through points a and b.

For V it should mean that heat energy does not flow through ∂V so

$$\nabla T(t,x,y,z) \cdot \mathbf{n} dS = 0$$

for all $(x,y,z) \in \partial V$. That is

$$\nabla T(t,x,y,z) \cdot \mathbf{n} = \mathbf{n} T(t,x,y,z) = 0 \tag{2}$$

for all $(x,y,z) \in \partial V$. Conditions (1) and (2) are called Neumann boundary conditions.

One can also think about different boundary conditions like for example

$$T(t,a) = 0 \quad \text{and} \quad T(t,b) = 0$$

or for V

$$T(t,x,y,z) = 0$$

for all $(x,y,z) \in \partial V$. These conditions are called Dirichlet boundary conditions.

3. Energy and Finite speed propagation for the wave equation

Now we are going to discuss the wave equation. This equation has the following form

$$\partial_t^2 F = \Delta F.$$

For example the expression

$$\partial_t^2 F(t,x) = \partial_x^2 F(t,x)$$

is the one dimensional wave equation;

$$\partial_t^2 F(t,x,y) = \Delta F(t,x,y) = \partial_x^2 F(t,x,y) + \partial_x^2 F(t,x,y) \tag{3}$$

is the two dimensional version. We discuss the wave equation in $\mathbf{R}^3$ below. Wave equation can be used to model such physical phenomena as vibration of a string or thread, propagation of acoustic sound, or electromagnetic waves.

3.1. **Initial conditions for wave equation.** When one considers the heat equations it is natural to expect that it is enough to know the initial temperature $T(0,x)$ to determine the heat distribution at any future time. To determine propagation of the solution of the wave equation one has to know the initial position $F(0,x)$ as well as the values of $F_t(0,x) = \partial_t F(0,x)$. An illuminating comparison can be made with the behaviour of system in Newton mechanic, where we have to know not only the initial positions but also the velocity of all elements of the considered system.

3.2. **Finite speed of propagation of solutions of the wave equation.** Next we will discuss a fundamental property of the wave equation - the finite speed propagation of solutions. In our discussion we will use the Divergence Theorem again.

Theorem 3.1. *Suppose that function $F\colon \mathbf{R}^n \to \mathbf{R}$ satisfies the wave equation. Then finite speed propagation property holds for F. That is if at point $t = t_o$ and $y \in \mathbf{R}^n$ one has*

$$F(t_o,x) = 0 \quad and \quad F_t(t_o,x) = 0 \quad \forall_{x \in \in B(r,y) \subset \mathbf{R}^n}$$

then for all $t \geq t_o$

$$F(t,x) = 0 \quad and \quad F_t(t,x) = 0 \quad \forall_{x \in B(r+t-t_o,y)},$$

where $B(r,y)$ is the ball or radius r defined by $B(r,y) = \{x \in \mathbf{R}^n\colon \sum_i (x_i - y_i)^2 \leq r^2\}$.

The above theorem is a special case of more general results which we discuss later. Here we would like to explain main point of the classical proof based on energy estimates and for simplicity we consider here only the straight line and the Euclidean space, $n = 1$ or $n = 3$. Suppose that the function $F\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ ($n = 1$ or $n = 3$) is the solution of the wave equation

$$\partial_t^2 F(t,x) = \Delta_x F(t,x) = div \nabla F(t,x).$$

Consider the following quantity

$$E(t) = \int_{\mathbf{R}^n} (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) dx.$$

We will show that $E(t)$ does not depend on t. To do this it is enough to show that $\partial_t E(t) = 0$ for all $t \in \mathbf{R}$.

$$\partial_t E(t) = \partial_t \int_{\mathbf{R}^n} (|F_t(t,x)|^2 + |\nabla F(t,x)|^2)dx$$
$$= \int_{\mathbf{R}^n} \partial_t(|F_t(t,x)|^2 + |\nabla F(t,x)|^2)dx$$
$$= \int_{\mathbf{R}^n} 2(F_t(t,x)F_{tt}(t,x) + \nabla F_t(t,x) \cdot \nabla F(t,x))dx$$
$$= \int_{\mathbf{R}^n} 2F_t(t,x)(F_{tt}(t,x) - \Delta F(t,x)) = 0.$$

To verify the above calculation note that by product rule

$$\partial_t |F_t(t,x)|^2 = 2F_t(t,x)F_{tt}(t,x) \quad \text{and} \quad \partial_t |\nabla F(t,x)|^2 = 2\nabla F_t(t,x) \cdot \nabla F(t,x)$$

and that integration by parts gives

$$\int_{\mathbf{R}^n} \nabla F(x) \cdot \nabla G(x)dx = -\int_{\mathbf{R}^n} G(x)\Delta F(x)dx.$$

Thus E does not depend on time and this quantity has a natural interpretation as the energy of the system. One can think about the function

$$|F_t(t,x)|^2 + |\nabla F(t,x)|^2$$

as a density (distribution in the space) of the energy at time t.

Proof of the finite propagation property in the one and three dimensional cases. We will use the idea of the energy corresponding to the wave equation to prove finite speed propagation property of the wave equation. First we consider the case $n = 1$ and for every $r \in R$ and $s > 0$ we set

$$E_{r,s}(t) = \int_{r-s}^{r+s} |F_t(t,x)|^2 + |\nabla F(t,x)|^2$$

Next we calculate $\partial_t E_{r,t}(t)$. We set $u = r + t$ and $v = r - t$. In a similar way as before we conclude that

$$\partial_t E_{r,t}(t) = \partial_t \int_v^u (|F_t(t,x)|^2 + |\nabla F(t,x)|^2)dx \tag{4}$$
$$= \int_v^u \partial_t(|F_t(t,x)|^2 + |\nabla F(t,x)|^2)dx$$
$$+|F_t(t,v)|^2 + |\nabla F(t,v)|^2 + |F_t(t,u)|^2 + |\nabla F(t,u)|^2.$$

The last four terms come from the change in the domain of integration. Indeed recall that by Fundamental Theorem of Analysis

$$\frac{d}{dx} \int_a^x f(s)ds = f(x).$$

We discuss the one-dimensional case so we write F' instead of ∇F. Integrating by parts we get (recall that $u = r + t$ and $v = r - t$)

$$\begin{aligned}(5)\qquad &\int_v^u \partial_t(|F_t(t,x)|^2 + |F'(t,x)|^2)dx \\ &= \int_v^u 2(F_t(t,x)F_{tt}(t,x) + F_t'(t,x)F'(t,x)) \\ &= \int_v^u 2(F_t(t,x)F_{tt}(t,x) - F_t(t,x)F''(t,x)) \\ &\quad +2F_t(t,u)F'(t,u) - 2F_t(t,v)F'(t,v) \\ &= 2F_t(t,u)F'(t,u) - 2F_t(t,v)F'(t,v) \geq \\ &- \left(|F_t(t,v)|^2 + |F'(t,v)|^2 + |F_t(t,u)|^2 + |F'(t,u)|^2\right)\end{aligned}$$

To verify the last estimates we use the inequality between arithmetic and geometric means (a simple version of the Schwarz inequality)

$$p^2 + q^2 \geq 2pq.$$

By (4) and (5) we get $\partial_t E_{r,t}(t) \geq 0$ that is the function $e(t) = E_{r,t}(t)$ is an increasing function of t. This property has a very useful consequence. Namely we note $E(t) = E$ is constant so $E - E_{r,t}(t)$ is a decreasing function. However

$$E - E_{r,t}(t) \geq 0 \quad \text{and} \quad E - E_{\frac{a+b}{2},\frac{b-a}{2}}(t_o) = 0.$$

because we assume that $F(t_o, x) = F_t(t_o, x) = 0$ for all x such that $x \leq a$ or $x \geq b$. Hence

$$E - E_{\frac{a+b}{2},\frac{b-a}{2}+|t-t-o|}(t) = 0.$$

Thus if at time $t = t_o$, $F(t_o, x) = F_t(t_o, x) = 0$ for all x such that $x \leq a$ or $x \geq b$ for some $a < b$ then $F(t, x) = F_t(t, x) = 0$ for all $t \geq t_o$ and all $x \leq a - |t - t_0|$ and for all $x \geq b + |t - t_0|$. This means that the solution of the wave equation propagates only with unit speed. This observation is called the finite speed of propagation property of the solution of the wave equation.

Next, we are going to extend our discussion of the energy corresponding to the wave equation and finite speed propagation for the wave equation to three dimensional case. We start by defining

$$E_\Omega(t) = \int_\Omega (|F_t(t,x)|^2 + |\nabla F(t,x)|^2)dx,$$

where Ω is a connected domain in $\mathbf{R}^3$. We note that

$$div(F\nabla G) = \nabla F \cdot \nabla G + F\Delta G$$

and by Divergence theorem

$$\int_{\partial\Omega} F\nabla G \cdot \mathbf{n} d\sigma = \int_\Omega div(F\nabla G)dx = \int_\Omega (\nabla F \cdot \nabla G + F\Delta G)dx$$

where $d\sigma$ is the surface area of the boundary $\partial\Omega$ of Ω. Hence

$$\partial_t E_\Omega(t) = \partial_t \int_\Omega (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) dx$$
$$= \int_\Omega \partial_t (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) dx$$
$$= \int_\Omega 2F_t(t,x)(F_{tt}(t,x) + \nabla F_t(t,x) \cdot \nabla F(t,x)) dx$$
$$= \int_\Omega 2F_t(t,x)(F_{tt}(t,x) - \Delta F(t,x)) + \int_{\partial\Omega} 2F_t \nabla F \cdot \mathbf{n} d\sigma = \int_{\partial\Omega} 2F_t \nabla F \cdot \mathbf{n} d\sigma.$$

It means that the vector field $2F_t\nabla F$ describes the flow of the energy in the system. Next we denote by $B(x,t)$ the ball of radius t and center at x and we calculate

$$\partial_t E_{B(x,t)}(t) = \partial_t \int_{B(x,t)} (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) dx$$
$$= \int_{B(x,t)} 2(F_t(t,x)F_{tt}(t,x) + \nabla F_t(t,x) \cdot \nabla F(t,x)) dx$$
$$+ \int_{\partial B(x,t)} (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) d\sigma.$$

The second term comes from the change in the domain of integration. Hence

$$\partial_t E_{B(x,t)}(t) = \partial_t \int_{\partial B(x,t)} 2F_t(t,x) \nabla F(t,x) \cdot \mathbf{n} d\sigma(x).$$
$$+ \int_{\partial B(x,t)} (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) d\sigma(x)$$
$$\geq - \int_{\partial B(x,t)} (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) d\sigma(x)$$
$$+ \int_{\partial B(x,t)} (|F_t(t,x)|^2 + |\nabla F(t,x)|^2) d\sigma(x) \geq 0.$$

The above calculation shows that $e(t) = E_{B(x,t)}(t)$ is an increasing function and this, in turn, implies the finite speed of propagation of the solutions of the wave equation.

Remark It is not difficult to check that the above proof without any changes work for all dimensions $n = 2, 3, 4, \ldots$.

4. Spectral theory

There is a significant need to generalize notion of the standard Laplace operator and the corresponding wave and heat equations. For example one would like to consider different geometries of underling ambient space, to introduce non-homogeneous heat capacity or to consider Neumann and Dirichlet boundary conditions mention above. We would also would like to be able to consider Schrödinger operators with some potentials which significance comes from their interpretation in quantum mechanics.

All mentioned above operators are examples of self-adjoint operators. We next discuss such operators in more systematic way.

Recall that, if L is a non-negative self-adjoint operator on $L^2(X, d\mu)$, one can construct the spectral decomposition $E_L(\lambda)$ of the operator L. For any bounded Borel function $m\colon [0,\infty) \to \mathbb{C}$, one then defines the operator $m(L)\colon L^2(M, d\mu) \to L^2(X, d\mu)$ by the formula

$$m(L) = \int_0^\infty m(\lambda)\, \mathrm{d}E_L(\lambda).$$

Now, for $z \in \mathbb{C}_+$ and $m_z(\lambda) = \exp(-z\lambda)$, one sets $m_z(L) = \exp(-zL)$, $z \in \mathbb{C}_+$. By spectral theory the family $\exp(-zL) = \{\exp(-zL)\colon z \in \mathbb{C}_+\}$, (which is called semigroup of operators generated by L) is a family of contraction on $L^2(X)$. That is $\|\exp(-zL)\|_{2\to 2} \le 1$ for all $z \in \mathbb{C}_+$. Now the solution of the heat equation

$$\partial_t F(t,x) = -L_x F(t,x)$$

with initial condition

$$F(x,0) = f(x)$$

can be written down using the notion of semigroup $\exp(-zL)$ as

$$F(x,t) = \exp(-tL) f(x).$$

Indeed at least formally

$$\frac{d}{dt}\exp(-tL)f(x) = -L\exp(-tL)f(x)$$

and $\exp(0L) = Id$. Similarly the solution of wave equation

$$\partial_t^2 F(t,x) = -L_x F(t,x)$$

with initial condition

$$F(x,0) = f(x) \quad \text{and} \quad F_t(x,0) = g(x)$$

can expressed in terms of spectral multipliers defined above as

$$F(x,t) = \cos(-t\sqrt{L})f(x) + \frac{\sin(t\sqrt{L})}{\sqrt{L}}g(x).$$

There is a simple relation between the wave and heat equation which we going to use extensively later and which can be in simple manner stated in the following way

$$\exp(-sL) = \int_0^\infty \cos(t\sqrt{L})\frac{2}{\sqrt{\pi s}}e^{-\frac{t^2}{4s}}\, \mathrm{d}t.$$

Indeed in virtue of spectral theorem it is enough to show that the above relation holds if L in the the above equality is replace by λ^2 for any $\lambda \in \mathbf{R}$. Then it just becomes a standard exercise in Fourier Transform.

5. Examples

The range of possible examples of operator L which we would like to discuss includes the following operators

(a) The standard Laplace operator $\Delta = \sum_i \partial_i^2$.
(b) Δ_Ω the Laplace operator with Dirichlet or Neumann boundary condition for some subset $\Omega \subset \mathbf{R}^n$.

(c) Uniformly elliptic second order operator in divergence form. For any $f \in C_c^\infty(\mathbf{R}^n)$ we define

$$Lf(x) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j f).$$

Uniform ellipticity assumption means that

$$C|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \le c|\xi|^2$$

for all $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$.

(d) Laplace-Beltrami operator on complete Riemannian manifolds.

(e) Hodge Laplacian operator on complete Riemannian manifolds and other operators acting on forms.

(f) Schrödinger operators

$$\Delta + V$$

where $V\colon \mathbf{R}^n \to R$ is a real potential function.

(g) The operators corresponding to local or strongly local Dirichlet quadratic forms and in particularly MMD, measurable metric Dirichlet spaces.

6. Heat and integral kernels

Let (X, d, μ) be a metric measure space, that is μ is a Borel measure with respect to the topology defined by the metric d. Next let $B(x, r) = \{y \in X,\ d(x, y) < r\}$ be the open ball with center $x \in X$ and radius $r > 0$. For $1 \le p \le +\infty$, we denote the norm of a function $f \in L^p(X, d\mu)$ by $\|f\|_p$, by $\langle .,. \rangle$ the scalar product in $L^2(M, d\mu)$, and if T is a bounded linear operator from $L^p(X, d\mu)$ to $L^q(X, d\mu)$, $1 \le p, q \le +\infty$, we write $\|T\|_{p\to q}$ for the operator norm of T.

If there is a locally integrable function $K_T\colon X \times X \to \mathbf{C}$ such that

$$\langle Tf_1, f_2 \rangle = \int_X Tf_1\, \overline{f_2}\, \mathrm{d}\mu = \int_X K_T(x, y)\, f_1(y)\, \overline{f_2(x)}\, \mathrm{d}\mu(y)\, \mathrm{d}\mu(x)$$

for all f_1 and f_2 in $C_c(X)$, then we say that T is a *kernel operator* with kernel K_T. It is well known that if T is bounded from $L^1(X)$ to $L^q(X)$, where $q > 1$, then T is a kernel operator, and

$$\|T\|_{L^1\to L^q} = \sup_{y\in X} \|K_T(\cdot, y)\|_{L^q};$$

vice versa, if T is a kernel operator and the right hand side of the above inequality is finite, then T is bounded from $L^1(X)$ to $L^q(X)$, even if $q = 1$.

One of especially important instant of this notion is a heat kernel, that is the kernels corresponding to the semigroup operators $\exp(-zL)$ which are often denote by p_t. More precisely

$$e^{-tL}f(x) = \int_X p_t(x, y)f(y)d\mu(y),\ \ f \in L^2(X, \mu),\ \ \mu - a.e.\ x \in M.$$

In the Euclidean space $\mathbb{R}^n$, p_t is given by the classical Gauss-Weierstrass kernel:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right),\ t > 0,\ x, y \in \mathbb{R}^n.$$

On general non-compact manifolds, where of course no such formula is available, the subject of upper estimates of the heat kernel has led to an intense activity in the last few decades (see for instance [6], [7], [13], [14] for references and background).

7. Theorems of Phragmén-Lindelöf type

Our discussion of finite propagation speed for the solutions of the wave equation requires some rather simple observation related to the Phragmén-Lindelöf theorem. Let us start with stating the Phragmén-Lindelöf theorem for sectors.

Theorem 7.1. *Let S be the open region in $\mathbb{C}$ bounded by two rays meeting at an angle π/α, for some $\alpha > 1/2$. Suppose that F is analytic on S, continuous on $\bar{S}$, and satisfies $|F(z)| \le C\exp(c|z|^\beta)$ for some $\beta \in [0,\alpha)$ and for all $z \in S$. Then the condition $|F(z)| \le B$ on the two bounding rays implies $|F(z)| \le B$ for all $z \in S$.*

For the proof see [10, Theorem 7.5, p.214, vol.II] or [17, Lemma 4.2, p.108]. Proposition 7.2 is a simple consequence of Theorem 7.1.

Proposition 7.2. *Suppose that F is an analytic function on $\mathbb{C}_+$. Assume that, for given numbers $A, B, \gamma > 0$, $a \ge 0$,*

$$|F(z)| \le B, \quad \forall\, z \in \mathbb{C}_+, \qquad \textit{and} \tag{6}$$

$$|F(t)| \le Ae^{at}e^{-\frac{\gamma}{t}}, \quad \forall\, t \in \mathbb{R}_+. \tag{7}$$

Then

$$|F(z)| \le B\exp\left(-\mathrm{Re}\frac{\gamma}{z}\right), \quad \forall\, z \in \mathbb{C}_+. \tag{8}$$

Proof. Consider the function

$$u(\zeta) = F\left(\frac{\gamma}{\zeta}\right), \tag{9}$$

which is also defined on $\mathbb{C}_+$. By (6),

$$|u(\zeta)e^{\zeta}| \le B\exp|\zeta|, \quad \forall\, \zeta \in \mathbb{C}_+.$$

Again by (6) we have, for any $\varepsilon > 0$,

$$\sup_{\mathrm{Re}\zeta=\varepsilon} |u(\zeta)e^{\zeta}| \le Be^{\varepsilon}. \tag{10}$$

By (7),

$$\sup_{\zeta\in[\varepsilon,\infty)} |u(\zeta)e^{\zeta}| \le Ae^{a\gamma/\varepsilon}. \tag{11}$$

Hence, by Phragmén-Lindelöf theorem with angle $\pi/2$ and $\beta = 1$, applied to

$$S_\varepsilon^+ = \{z \in \mathbb{C}\colon\ \mathrm{Re}z > \varepsilon \quad \text{and} \quad \mathrm{Im}z > 0\}$$

and

$$S_\varepsilon^- = \{z \in \mathbb{C}\colon\ \mathrm{Re}z > \varepsilon \quad \text{and} \quad \mathrm{Im}z < 0\},$$

one obtains

$$\sup_{\mathrm{Re}\zeta\ge\varepsilon} |u(\zeta)e^{\zeta}| \le \max\{Ae^{a\gamma/\varepsilon}, Be^{\varepsilon}\}, \quad \forall\, \varepsilon > 0.$$

Now by the Phragmén-Lindelöf theorem with angle π and $\beta = 0$,

$$\sup_{\mathrm{Re}\zeta \geq \varepsilon} |u(\zeta)e^{\zeta}| \leq Be^{\varepsilon}, \quad \forall\, \varepsilon > 0. \tag{12}$$

Letting $\varepsilon \to 0$ we obtain

$$\sup_{\mathrm{Re}\zeta > 0} |u(\zeta)e^{\zeta}| \leq B.$$

This proves (8) by putting $\zeta = \frac{\gamma}{z}$. □

Note that the estimate (8) does not depend on constants A, a in (7).

8. Finite speed propagation of solutions of wave eqaution

Set

$$\mathcal{D}_\rho = \{(x,\, y) \in X \times X : d(x,\, y) \leq \rho\}.$$

Given an operator T from $L^p(X)$ to $L^q(X)$, we write

$$\operatorname{supp} K_T \subseteq \mathcal{D}_\rho \tag{13}$$

if $\langle Tf_1, f_2\rangle = 0$ for all open sets $U_i \subset M$, $\;f_i \in L^2(U_i, d\mu)$, $i = 1, 2$, where $r = d(U_1, U_2)$. Note that if T is an integral operator with a kernel K_T, then (13) coincides with the standard meaning of $\operatorname{supp} K_T \subseteq \mathcal{D}_\rho$, that is $K_T(x,\, y) = 0$ for all $(x,\, y) \notin \mathcal{D}_\rho$.

Given a non-negative self-adjoint operator L on $L^2(X)$. We say that L satisfies the finite speed propagation property if

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \mathcal{D}_t \quad \forall t > 0\,. \tag{14}$$

For $U_1, U_2 \subset M$ open subsets of M, let $d(U_1, U_2) = \inf_{x\in U_1, y\in U_2} d(x, y)$. We say that the family $\{\exp(-zL)\colon\; z \in \mathbb{C}_+\}$ satisfies the Davies-Gaffney estimate if

$$|\langle \exp(-tL)f_1, f_2\rangle| \leq \exp\left(-\frac{r^2}{4t}\right)\|f_1\|_2\|f_2\|_2 \tag{15}$$

for all $t > 0$, $U_i \subset M$, $\;f_i \in L^2(U_i, d\mu)$, $i = 1, 2$ and $r = d(U_1, U_2)$. Note that we only assume that (15) holds for positive real t.

Theorem 8.1. *Let L be a self-adjoint positive definite operator acting on $L^2(X)$. Then conditions* (14) *and* (15) *are equivalent.*

Remark. The connection of the heat and the wave equation has a long history (see [1, 11], see also [16] and the third proof of [6, Theorem 3.2, p. 157]). For the origin of the L^2 Gaussian estimates (15) so-called the Davies or the Davies-Gaffney estimates see [3].

Proof. Suppose that $\operatorname{supp} f_n \subseteq U_n$ for $n = 1, 2$, and that $0 \leq r < d(U_1, U_2)$. Put

$$u(z) =< \exp(-L/(4z))f_1, f_2 > .$$

L is a self-adjoint positive definite operator so u is an analytic function on the complex half-plane $\mathrm{Re}\, z > 0$, continuous and bounded on the set $\{z \in \mathbf{C}\colon \mathrm{Re}\, z \geq 0,\; z \neq 0\}$, and

$$\sup_{\,\mathrm{Re}\ z=0} |e^{r^2 z}u(z)| \leq \|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)}.$$

By (15)

$$\sup_{\,z\in\mathbf{R}_+} |e^{r^2 z}u(z)| \leq C\|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)}.$$

Hence, by Phragmén-Lindelöf theorem for an angle (see [10, Theorem 7.5, p.214, vol. II] or [17, Lemma 4.2, p.107])

$$|e^{r^2z}u(z)| \le \|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)}$$

and

$$|u(z)| \le \exp(-r^2\mathrm{Re}\, z)\|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)} \tag{16}$$

for all z such that $\mathrm{Re}\, z > 0$. Next we note that

$$< \exp(-sL)f_1, f_2 >= \int_0^\infty < \cos(t\sqrt{L})f_1, f_2 > \frac{2}{\sqrt{\pi s}}e^{-\frac{t^2}{4s}}\, \mathrm{d}t. \tag{17}$$

By change of variable $t := \sqrt{t}$ in integral (17) and putting $s := 1/(4s)$ we get

$$s^{-1/2} < \exp\Big(-\frac{L}{4s}\Big)f_1, f_2 >= 2\int_0^\infty (\pi t)^{-1/2} < \cos(\sqrt{t}\sqrt{L})f_1, f_2 > e^{-st}\, \mathrm{d}t, \tag{18}$$

so the function $v(z) = z^{-1/2}u(z)$ is a Fourier-Laplace transform of the function $w(t) = (\sqrt{\pi t})^{-1} < \cos(\sqrt{t}\sqrt{L})f_1, f_2 >$. Now by (16) and the Paley-Wiener Theorem (Theorem 7.4.3 [8])

$$\mathrm{supp}\; w \subseteq [r^2, \infty). \tag{19}$$

This proves that (15) implies (14). Now if (14) holds, then by (17)

$$| < \exp(-sL)f_1, f_2 > | \le \int_0^\infty | < \cos(t\sqrt{L})f_1, f_2 > |\frac{2}{\sqrt{\pi s}}e^{-\frac{t^2}{4s}}\, \mathrm{d}t$$
$$= \int_r^\infty | < \cos(t\sqrt{L})f_1, f_2 > |\frac{2}{\sqrt{\pi s}}e^{-\frac{t^2}{4s}}\, \mathrm{d}t \le \|f_1\|_{L^2(X,\mu)}\|f_2\|_{L^2(X,\mu)}\int_r^\infty \frac{2}{\sqrt{\pi s}}e^{-\frac{t^2}{4s}}\, \mathrm{d}t$$
$$\le e^{-\frac{r^2}{4s}}\|f_1\|_{L^2(X,\mu)}\|f_2\|_{L^2(X,\mu)}.$$

□

The following lemma is a very simple but useful consequence of (14).

Lemma 8.2. *Assume that* L *satisfies* (14) *and that* $\widehat{F}$ *is the Fourier transform of an even bounded Borel function* F *with* $\mathrm{supp}\, \widehat{F} \subseteq [-r, r]$. *Then* $\mathrm{supp}\, K_{F(\sqrt{L})} \subseteq \mathcal{D}_r$.

Proof. Since F is even, by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{F}(t)\cos(t\sqrt{L})\; dt.$$

But $\mathrm{supp}\, \widehat{F} \subseteq [-r, r]$ and Lemma 8.2 follows from (14). □

9. Davies-Gaffney estimates

A slightly different form of Davies-Gaffney estimate is mostly considered in the literature (see for instance [3] or [6]): in our notation, it reads

$$|\langle \exp(-tL)\chi_{U_1}, \chi_{U_2}\rangle| \le \exp\left(-\frac{r^2}{4t}\right)\sqrt{\mu(U_1)\mu(U_2)} \tag{20}$$

where χ_U denotes the characteristic function of the set U. Of course, (20) follows from (15) by taking $f_1 = \chi_{U_1}$ and $f_2 = \chi_{U_2}$. Conversely, assume (20) and let $f_i = \sum_j c^i_j \chi_{A^i_j}$, where $A^i_j \subset U_i$. Then

$$\begin{aligned} < \exp(-tL)f_1, f_2 > \;&\leq\; \sum_j \sum_\ell |c^1_j c^2_\ell| (\mu(A^1_j)\mu(A^2_\ell))^{1/2} \exp\left(-\frac{d^2(A^1_j, A^2_\ell)}{4t}\right) \\ &\leq\; \sum_j \sum_\ell |c^1_j c^2_l| (\mu(A^1_j)\mu(A^2_\ell))^{1/2} \exp\left(-\frac{d^2(U_1, U_2)}{4t}\right). \end{aligned}$$

By spectral theorem $< \exp(-zL)f_1, f_2 > \leq \|f_1\|_2 \|f_2\|_2$. Proposition 7.2 then yields (15) for such f_1, f_2, and one concludes by density.

One may wonder what is the justification of the constant 4 in (15); we shall see in Theorem 8.1 below that in the case where $\exp(-zL)$ is a semigroup e^{-zL}, 4 is the good normalisation between the operator L and the distance d, namely it translates the fact that the associated wave equation has propagation speed 1.

Examples. As we already said, condition (15) holds for all kinds of self-adjoint, elliptic, second order like operators. Condition (15) is well-known to hold for Laplace-Beltrami operators on all complete Riemannian manifolds. More precisely, Condition (20) is proved for such operators in [3] and [6]. See also the remark after Theorem 9.3. In the more general setting of Laplace type operators acting on vector bundles, condition (15) is proved in [16]. Another important class of semigroups satisfying condition (15) are semigroups generated by Schrödinger operators with real potential and magnetic field (see for example [15], as well as Theorem 9.3 below).

Estimates (15) also hold in the setting of local Dirichlet forms (see for example [5, Theorem 2.8], and also [18], [19]). In this case the metric measure spaces under consideration are possibly not equipped with any differential structure. However, the semigroups associated with these Dirichlet forms do satisfy in general Davies-Gaffney estimates with respect to an intrinsic distance.

9.1. **Self-improving properties of Davies-Gaffney estimates.** It is enough to test (15) on balls only. Then we observe that any additional multiplicative constant or even additional exponential factor in (15) can be replaced by the 1.

Lemma 9.1. *Suppose that (M, d, μ) is a separable metric space and that the analytic family $\{\exp(-zL)\colon z \in \mathbb{C}_+\}$ of bounded operators on $L^2(M, d\mu)$ satisfies condition* (15) *restricted to all balls $U_i = B(x_i, r_i)$, $i = 1, 2$, for all $x_1, x_2 \in M$, $r_1, r_2 > 0$. Then it satisfies condition* (15) *for all open subsets U_1, U_2.*

Proof. Let U_1 and U_2 be arbitrary open subsets of M; set $r = d(U_1, U_2)$. Let $f = \sum_{i=1}^k f_i$, where for all $1 \leq i \leq k$, $f_i \in L^2(B(x_i, r_i), d\mu)$, $B(x_i, r_i) \subset U_1$, and $f_{i_1}(x) f_{i_2}(x) = 0$ for all $x \in M$, $1 \leq i_1 < i_2 \leq k$. Similarly let $g = \sum_{j=1}^\ell g_j$ where $g_j \in L^2(B(y_j, s_j), d\mu)$, $B(y_j, s_j) \subset U_2$ for all $1 \leq j \leq \ell$, and $g_{j_1}(x) g_{j_2}(x) = 0$ for all $x \in M$, $1 \leq j_1 < j_2 \leq \ell$. Note that $d(B(x_i, r_i), B(y_j, s_j)) \geq r$. Now if condition (15) holds for balls then

$$\begin{aligned}
|\langle \exp(-tL)f,g\rangle| &= |\langle \exp(-tL)\sum_{i=1}^k f_i, \sum_{j=1}^\ell g_j\rangle| \\
&= \sum_{i=1}^k\sum_{j=1}^\ell |\langle \exp(-tL)f_i,g_j\rangle| \\
&\le \sum_{i=1}^k\sum_{j=1}^\ell e^{-\frac{r^2}{4t}}\|f_i\|_2\|g_j\|_2 \\
&\le e^{-\frac{r^2}{4t}}\left(\sum_{i=1}^k\|f_i\|_2\right)\left(\sum_{j=1}^\ell\|g_j\|_2\right) \\
&\le e^{-\frac{r^2}{4t}}\sqrt{k\ell}\left(\sum_{i=1}^k\|f_i\|_2^2\right)^{1/2}\left(\sum_{j=1}^\ell\|g_j\|_2^2\right)^{1/2} \\
&= e^{-\frac{r^2}{4t}}\sqrt{k\ell}\|f\|_2\|g\|_2.
\end{aligned}$$

Now if we put $F(z)=\langle \exp(-zL)f,g\rangle$ then Proposition 7.2 shows that the term Ckl in the above inequality can be replaced by 1. This means that (15) holds for f and g. Now to finish the proof of the lemma, it is enough to note that, since M is separable, the space of all possible finite linear combinations of functions f such that $\text{supp} f\subset B(x,r)\subset U$ is dense in $L^2(U,d\mu)$. Moreover, if $f=\sum_{i=1}^k f_i$ and $f_i\in L^2(B(x_i,r_i),d\mu)$ for all $1\le i\le k$ then there exist functions $\tilde f_i\in L^2(B(x_i,r_i),d\mu)$ such that $f=\sum_{i=1}^k\tilde f_i$ and in addition, for all $1\le i_1<i_2\le k$, $\tilde f_{i_1}(x)\tilde f_{i_2}(x)=0$ for all $x\in M$. □

Lemma 9.2. *Suppose that, for some $C\ge 1$ and some $a>0$,*

$$|\langle \exp(-tL)f_1,f_2\rangle|\le Ce^{at}e^{-\frac{r^2}{4t}}\|f_1\|_2\|f_2\|_2, \quad \forall t>0, \tag{21}$$

whenever $f_i\in L^2(M,d\mu)$, $\text{supp} f_i\subseteq B(x_i,r_i)$, $i=1,2$, and $r=d(B(x_1,r_1),B(x_2,r_2))$. Then the family $\{\exp(-zL)\colon z\in\mathbb{C}_+\}$ satisfies condition (15).

Proof. Lemma 9.2 is a straightforward consequence of Proposition 7.2 and Lemma 9.1. □

Let us give an application of Lemma 9.2 by giving yet another example where Davies-Gaffney estimates hold, namely Schrödinger semigroups with real potential. Suppose that Δ is the non-negative Laplace-Beltrami operator on a Riemannian manifold M with Riemannian measure μ and geodesic distance d, and consider the operator $\Delta+\mathcal{V}$ acting on $C_c^\infty(M)$, where $\mathcal{V}\in L^1_{\text{loc}}(M,d\mu)$. If we assume that $\Delta+\mathcal{V}\ge 0$ then we can define the Friedrichs extension of $\Delta+\mathcal{V}$, which with some abuse of notation we also denote by $\Delta+\mathcal{V}$ (see for example [2, Theorem 1.2.8].

Theorem 9.3. *Suppose that Δ is the Laplace-Beltrami operator on a Riemannian manifold M, that $\mathcal{V}\in L^1_{\text{loc}}(M,d\mu)$ and that $\Delta+\mathcal{V}\ge 0$ as a quadratic form. Then the semigroup $\{\exp(-zL)=\exp(-z(\Delta+\mathcal{V}))\colon z\in\mathbb{C}_+\}$ satisfies condition* (15).

Proof. We start our proof with the additional assumption $\mathcal{V} \geq 0$. For $f \in L^2(M, d\mu)$, $t > 0$, $x \in M$, we put $f_t(x) = f(t,x) = \exp(-t(\Delta + \mathcal{V}))f(x)$. Let $\kappa > 0$, and a function $\xi \in C^\infty(M)$, both to be chosen later, such that $|\nabla \xi| \leq \kappa$, where ∇ is the Riemannian gradient on M. Next, as in [4, 6, 16], we consider the integral

$$E(t) = \int_M |f(t,x)|^2 e^{\xi(x)}\, d\mu(x).$$

Then

$$\begin{aligned}
\frac{E'(t)}{2} &= \mathrm{Re} \int_M \partial_t f(t,x)\overline{f(t,x)} e^{\xi(x)}\, d\mu(x) = -\mathrm{Re}\int_M ((\Delta + \mathcal{V})f_t)\ \overline{f_t} e^{\xi}\, d\mu \\
&= -\mathrm{Re}\int_M \left(\nabla f_t \cdot \nabla (f_t e^{\xi}) + |f_t|^2 \mathcal{V} e^{\xi}\right)\, d\mu \\
&= -\mathrm{Re}\int_M \left(|\nabla f_t|^2 + \nabla f_t \cdot f_t \nabla \xi + |f_t|^2 \mathcal{V}\right) e^{\xi}\, d\mu \\
&\leq \int_M \left(-|\nabla f_t|^2 + |\nabla f_t||\nabla \xi||f_t|\right) e^{\xi}\, d\mu \\
&\leq \frac{1}{4}\int_M |f_t|^2 |\nabla \xi|^2 e^{\xi}\, d\mu \leq \frac{\kappa^2 E(t)}{4}
\end{aligned}$$

(note that the non-negativity of $\mathcal{V}$ is used in the first inequality). Hence $E(t) \leq \exp(\kappa^2 t/2)E(0)$.

Consider now two disjoints open sets U_1 and U_2 in M. Choose $\xi = \kappa d(.,U_1)$. One has $|\nabla \xi| \leq \kappa$, $\xi \equiv 0$ on U_1, and, for any $g \in L^2{}_{\mathrm{loc}}(M, d\mu)$,

$$\int_{U_2} |g|^2 e^{\xi}\, d\mu \geq e^{\kappa r} \int_{U_2} |g|^2\, d\mu,$$

where $r = d(U_1, U_2)$. Hence if $\mathrm{supp}\ f \subseteq U_1$ then, taking $g = f_t$,

$$\int_{U_2} |f_t|^2\, d\mu \leq e^{-\kappa r} E(t) \leq \exp\left(\frac{\kappa^2 t}{2} - \kappa r\right) E(0) = \exp\left(\frac{\kappa^2 t}{2} - \kappa r\right)\int_{U_1} |f|^2\, d\mu.$$

Choosing finally $\kappa = r/t$ we obtain

$$\int_{U_2} |\exp(-tL)f|^2\, d\mu \leq \exp\left(-\frac{r^2}{2t}\right)\int_{U_1} |f|^2\, d\mu,$$

that is, for all $f \in L^2(U_1, d\mu)$,

$$\sup_{g\in L^2(U_2,d\mu),\ \|g\|_2 = 1} |\langle \exp(-tL)f, g\rangle|^2 = \int_{U_2} |\Psi(t)f|^2\, d\mu \leq \exp\left(-\frac{r^2}{2t}\right)\|f\|_2^2,$$

which yields (15). Next, we consider a potential $\mathcal{V} \in L^1{}_{\mathrm{loc}}(M)$ such that $\Delta + \mathcal{V} \geq 0$. We put $\mathcal{V}_a(x) = \max\{\mathcal{V}(x), -a\}$ and $L_a = \Delta + \mathcal{V}_a$. When a goes to ∞ then L_a converges to $L = \Delta + \mathcal{V}$ in the strong resolvent sense (see [9, Theorem VIII.3.3, p.454] or [12, Theorem S.16 p.373]). Hence by [12, Theorem VIII.20, p.286] or by [9, Theorem VIII.3.11, p.459 and Theorem IX.2.16, p.504], $\exp(-tL_a)f$ converges to $\exp(-tL)f = \exp(-t(\Delta + \mathcal{V}))f$ for any $f \in L^2(M)$. Therefore it is enough to prove (15) for $\exp(-tL_a)f$, with a given in $\mathbb{R}$. Finally we note that $\mathcal{V}_a + a \geq 0$, thus it follows from the first part of the proof that

$$\exp(-t(\Delta + \mathcal{V}_a + a)) = e^{-at}\exp(-tL_a)$$

satisfies condition (15). But this implies that the semigroup $\exp(-tL_a)$ satisfies condition (21) and Theorem 9.3 follows from Lemma 9.2. □

Remark : Note that the case $\mathcal{V}=0$ is allowed in Theorem 9.3, in other words it yields a proof of (15) for the Laplace-Beltrami operator on complete Riemannian manifolds.

10. Davies-Gaffney estimates for Hodge Laplacian

Suppose that M is a complete Riemannian manifold and μ is an absolutely continuous measure with a smooth density not equal to zero at any point of M. By $\Lambda^k T^*M$ we denote the bundle of k-forms on M. For fixed $\beta,\beta_*\in L^2(\Lambda^1 T^*M)$ and $\gamma\in L^2(\Lambda^k T^*M)$ We define the operator L ($L=L_{\beta,\beta_*,\gamma}$) acting on $L^2(\Lambda^k T^*M)$ by the formula

$$\langle L\omega,\omega\rangle=\int_M |d_k\omega+\omega\wedge\beta|^2+|d_{n-k}*\omega+*\omega\wedge\beta_*|^2+|*\omega\wedge\gamma|^2\,\mathrm{d}\mu(x), \tag{22}$$

where ω is a smooth compactly supported k-form and $*$ is the Hodge star operator. With some abuse of notation we also denote by L its Friedrichs extension. Note that for example the Hodge Laplacian and Schrödinger operators with electromagnetic fields can be defined by (22)).

Theorem 10.1. *The operator L defined by (22) satisfies (14) and (15).*

Proof. We put $\omega_t(x)=\omega(t,x)=\exp(-tL)\omega$. Then we fix some function $\xi\in C^\infty(M)$ such that $|d\xi|\le\kappa$ and we consider the integral

$$E(t)=\int_M(\omega(t,x),\omega(t,x))e^{\xi(x)}\,\mathrm{d}\mu(x).$$

Next we note that for every k-form η and 1-form ζ we have $|\zeta\wedge\eta|^2+|\zeta\wedge*\eta|^2=|\eta|^2|\zeta|^2$ and

$$\begin{aligned}
\frac{E'(t)}{2}&=\mathrm{Re}\int_M(\partial_t\omega(t,x),\omega(t,x))e^{\xi}\,\mathrm{d}\mu(x)=-\mathrm{Re}\int_M(L\omega_t,\omega_te^{\xi})\,\mathrm{d}\mu\\
&=-\mathrm{Re}\int_M\left[(d_k\omega_t+\omega_t\wedge\beta,d_k(\omega_te^{\xi})+(\omega_te^{\xi})\wedge\beta)+(*\omega_t\wedge\gamma,*\omega_t\wedge\gamma)e^{\xi}\right]\mathrm{d}\mu\\
&\quad-\mathrm{Re}\int_M(d_{n-k}*\omega_t+*\omega_t\wedge\beta_*,d_{n-k}(*\omega_te^{\xi})+(*\omega_te^{\xi})\wedge\beta_*)\,\mathrm{d}\mu\\
&=-\int_M\left[|d_k\omega_t+\omega_t\wedge\beta|^2e^{\xi}+|d_{n-k}*\omega_t+*\omega_t\wedge\beta|^2e^{\xi}+|*\omega_t\wedge\gamma|^2e^{\xi}\right]\mathrm{d}\mu\\
&\quad-\mathrm{Re}\int_M(d_k\omega_t+\omega_t\wedge\beta,d_0\xi\wedge\omega_t)e^{\xi}\,\mathrm{d}\mu-\int_M(d_{n-k}*\omega_t+*\omega_t\wedge\beta_*,d_0\xi\wedge*\omega_t)e^{\xi}\,\mathrm{d}\mu\\
&\le-\int_M\left[|d_k\omega_t+\omega_t\wedge\beta|^2e^{\xi}+|d_{n-k}*\omega_t+*\omega_t\wedge\beta_*|^2e^{\xi}+|*\omega_t\wedge\gamma|^2e^{\xi}\right]\mathrm{d}\mu\\
&\quad+\int_M\left[|d_k\omega_t+\omega_t\wedge\beta|^2e^{\xi}+|d_{n-k}*\omega_t+*\omega_t\wedge\beta_*|^2e^{\xi}+|*\omega_t\wedge\gamma|^2e^{\xi}\right]\mathrm{d}\mu\\
&\quad+\frac14\int_M\left[|d_0\xi\wedge\omega_t|^2e^{\xi}+|d_0\xi\wedge*\omega_t|^2e^{\xi}\right]\mathrm{d}\mu=\frac14\int_M|\omega_t|^2|d_0\xi|^2e^{\xi}\,\mathrm{d}\mu\le\frac{\kappa^2E(t)}{4}\,.
\end{aligned}$$

Hence $E(t) \le \exp(\kappa^2 t/2)E(0)$. Now we say that $\xi \in \Theta_\kappa \subseteq C^\infty(M)$ if $\xi(x) = 0$ for $x \in B(x_1, r_1)$ and $|d\xi| \le \kappa$. Next assume that $0 \le r < \rho(x_1, x_2) - (r_1 + r_2)$. Then

$$\sup_{\xi\in\Theta_\kappa} \int_{B(x_2,r_2)} |\omega|^2 e^\xi \,\mathrm{d}\mu \ge e^{r\kappa} \int_{B(x_2,r_2)} |\omega|^2 \,\mathrm{d}\mu.$$

Hence if supp $\omega_0 \subseteq B(x_1, r_1)$ then

$$\int_{B(x_2,r_2)} |\omega_t|^2 \,\mathrm{d}\mu \le \exp\left(\frac{\kappa^2 t}{2} - r\kappa\right) \int_M |\omega_0|^2 \,\mathrm{d}\mu$$

Putting $\kappa = r/t$ we get

$$\int_{B(x_2,r_2)} |\omega_t|^2 \,\mathrm{d}\mu \le \exp\left(\frac{-r^2}{2t}\right) \int_M |\omega_0|^2 \,\mathrm{d}\mu = \exp\left(\frac{-r^2}{2t}\right) \int_{B(x_1,r_1)} |\omega_0|^2 \,\mathrm{d}\mu \tag{23}$$

Now (15) is a straightforward consequence of (23). □

11. Gaussian bounds

In this section we assume that X satisfies the doubling property, i.e. there exists a constant C such that

$$\mu(B(x, 2r)) \le C\mu(B(x, r)) \tag{24}$$

uniformly for all $x \in X$ and for all $r > 0$. Note also that (24) implies that there exist positive constants C and D such that

$$\mu(B(x, \gamma r)) \le C(1 + \gamma)^D \mu(B(x, r)) \quad \forall \gamma > 0, x \in X, r > 0. \tag{25}$$

In the sequel the value D always refers to the constant in (25).

Before we discuss our main result in this section we discuss the following very simply but useful technical results.

Theorem 11.1. *Let X be a measurable metric space with the doubling condition and let L be a self-adjoint positive definite operator. The following conditions are equivalent:*

$$\|K_{\exp(-tL)}(x, \cdot)\|^2_{L^2(X)} \le C\mu(B(x, t^{1/2}))^{-1} \quad \forall t > 0, x \in X; \tag{26}$$

$$\|K_{(I+tL)^{-m/4}}(x, \cdot)\|^2_{L^2(X)} \le C_m \mu(B(x, t^{1/2}))^{-1} \quad \forall t > 0, x \in X \tag{27}$$

for any $m > D$, where D is the constant from condition (25) .

Proof. Note that

$$(I + (tL))^{-m/4} = \frac{1}{\Gamma(m/4)} \int_0^\infty e^{-s}\, s^{m/4-1} \exp(-s(tL)) \,\mathrm{d}s.$$

Hence by (25)

$$\begin{aligned}
\|K_{(I+tL)^{-m/4}}(x, \cdot)\|_{L^2(X)} &\le \frac{1}{\Gamma(m/4)} \int_0^\infty e^{-s}\, s^{m/4-1} \|K_{\exp(-tsL)}(x, \cdot)\|_{L^2(X)} \,\mathrm{d}s \\
&\le \frac{1}{\Gamma(m/4)} \int_0^\infty e^{-s}\, s^{m/4-1} \mu(B(x, (st)^{1/2}))^{-1/2} \,\mathrm{d}s \\
&\le \frac{1}{\Gamma(m/4)} \mu(B(x, t^{1/2}))^{-1/2} \int_0^\infty e^{-s}\, s^{m/4-1} (1 + 1/s)^{D/4} \,\mathrm{d}s \\
&= C\mu(B(x, t^{1/2}))^{-1/2}.
\end{aligned}$$

To prove that (27) implies (26) we note that

$$\begin{aligned}
&\|K_{\exp(-tL)}(x,\,\cdot\,)\|_{L^2(X)} \le \|\exp(-tL)(1+tL)^m\|_{L^2\to L^2}\|K_{(I+tL)^{-m}}(x,\,\cdot\,)\|_{L^2(X)} \\
(28)\qquad &\le \sup_{\lambda\in\mathbf{R}^+} e^{-t\lambda}(1+t\lambda)^m\|K_{(I+tL)^{-m}}(x,\,\cdot\,)\|_{L^2(X)} \le C\|K_{(I+tL)^{-m}}(x,\,\cdot\,)\|_{L^2(X)}.
\end{aligned}$$

□

Theorem 11.2. *Suppose that for some number $N \in \mathbf{N}$ and points $x, y \in X$ there exist functions $V_x, V_y\colon \mathbf{R}^+ \mapsto \mathbf{R}$ such that*

$$(29)\qquad \|K_{(I+t^2L)^{-N/4}}(z,\,\cdot\,)\|_{L^2(X)} \le V_z(t) \quad \forall t>0,\ z=x,y.$$

Then, there exists a constant C_N such that for all $t < d(x,y)^2$

$$(30)\qquad K_{\exp(-tL)}(x,y) \le C_N V_x\Big(\frac{t}{d(x,y)}\Big)V_y\Big(\frac{t}{d(x,y)}\Big)\frac{\exp\Big(\frac{-d(x,y)^2}{4t}\Big)}{d(x,y)t^{-1/2}}.$$

Thus if L satisfies (26) or (27), then

$$(31)|K_{\exp(-tL)}(x,y)| \le C\mu\Big(B\Big(x,\frac{t}{d(x,y)}\Big)\Big)^{-\frac12}\mu\Big(B\Big(y,\frac{t}{d(x,y)}\Big)\Big)^{-\frac12}\frac{\exp\Big(\frac{-d(x,y)^2}{4t}\Big)}{d(x,y)t^{-1/2}}$$

for all $t < d(x,y)^2$.

Proof. For $s > 1$, we define the family of functions ϕ_s by the formula

$$\phi_s(x) = \psi(s(|x|-s)),$$

where $\psi \in C^\infty(\mathbf{R})$ and

$$\psi(x) = \begin{cases} 0 & \text{if } x \le -1 \\ 1 & \text{if } x \ge -1/2\,. \end{cases}$$

Finally we define functions F_s and R_s by the following formula

$$F_s(x) = \frac{1}{\sqrt{4\pi}}\exp\big(\frac{-x^2}{4}\big) - R_s(x) = \phi_s(x)\frac{1}{\sqrt{4\pi}}\exp\big(\frac{-x^2}{4}\big)$$

so that $\widehat{F_s}(\lambda) + \widehat{R_s}(\lambda) = \exp(-\lambda^2)$ and

$$(32)\qquad \widehat{F_s}(\sqrt{tL}) + \widehat{R_s}(\sqrt{tL}) = \exp(-tL).$$

Integration by parts N times yields

$$\int \phi_s(x)e^{-\frac{x^2}{4}}e^{-ix\lambda} = \int \underbrace{\Big(\frac{1}{x/2+i\lambda}\Big(\dots\Big(\frac{1}{x/2+i\lambda}\phi_s(x)\Big)'\dots\Big)'\Big)'}_{N} e^{-\frac{x^2}{4}-i\lambda x}dx.$$

Hence for any natural number N and $s > 1$

$$(33)\qquad |\widehat{F_s}(\lambda)| \le C'_N\frac{1}{s(1+\lambda^2/s^2)^{N/2}}e^{-\frac{s^2}{4}},$$

where C'_N is a constant depending only on N. Next we note that supp $R_s \subseteq [-s+\frac{1}{2s}, s-\frac{1}{2s}]$, so if we put $s_{xy} = d(x,y)t^{-1/2}$, then Lemma 8.2 $K_{\widehat{R_{s_{xy}}}(\sqrt{tL})}(x,y) = 0$.

Hence by (32) [1]

$$K_{\exp(-tL)}(x,y) = K_{\widehat{F_{s_{xy}}}(\sqrt{tL})}(x,y). \tag{34}$$

Now let $J_{s_{xy}}$ be a function such that $J_{s_{xy}}(\lambda)^2 = \widehat{F_{s_{xy}}}(t^{1/2}\lambda)$. By (33)

$$\sup_{\lambda\ge 0}\left|J_{s_{xy}}(\lambda)\left(1+\frac{\lambda^2t^2}{d(x,y)^2}\right)^{N/4}\right| \le C\frac{\exp\left(-\frac{d(x,y)^2}{8t}\right)}{\sqrt{d(x,y)t^{-1/2}}}.$$

Note that

$$\|K_{F_1F_2(L)}(x,\,\cdot\,)\|_{L^2(X)} \le \|F_1\|_{L^\infty}\|K_{F_2(L)}(x,\,\cdot\,)\|_{L^2(X)}. \tag{35}$$

Now by (35)

$$\begin{aligned}\||K_{J_{s_{xy}}(\sqrt{L})}(x,\,\cdot\,)|\|_{L^2(X)} &\le C\frac{\exp\left(-\frac{d(x,y)^2}{8t}\right)}{\sqrt{d(x,y)t^{-1/2}}}\left\|\left|K_{\left(I+\frac{t^2L}{d(x,y)^2}\right)^{-N/4}}(x,\,\cdot\,)\right|\right\|_{L^2(X)}\\ &\le CV_x\left(\frac{t}{d(x,y)}\right)\frac{\exp\left(-\frac{d(x,y)^2}{8t}\right)}{\sqrt{d(x,y)t^{-1/2}}}.\end{aligned} \tag{36}$$

Note that Note also that

$$K_{F_1F_2(L)}(x,y)| \le \|K_{F_1(L)}(x,\,\cdot\,)\|_{L^2(X)}\|K_{F_2(L)}(\,\cdot\,,y)\|_{L^2(X)}. \tag{37}$$

Finally by (37)

$$|K_{\exp(-tL)}(x,y)| = |K_{\widehat{F_{s_{xy}}}(\sqrt{tL})}(x,y)| \le \|K_{J_{s_{xy}}(\sqrt{L})}(x,\,\cdot\,)\|_{L^2(X)}\|K_{J_{s_{xy}}(\sqrt{L})}(y,\,\cdot\,)\|_{L^2(X)} \tag{38}$$

and (30) follows from (36) and (38). □

12. Spectral multipliers

We make the following assumptions about $\mathbf{L}$ and (X,d,μ):

- The space X is separable and has dimension n in the sense of the volume growth of balls: that is, there exist constants $0<c_1<c_2<\infty$ such that

$$c_1\rho^n \le \mu(B(x,\rho)) \le c_2\rho^n \tag{39}$$

  for every $x\in X$ and $\rho>0$;
- $\cos(t\sqrt{\mathbf{L}})$ satisfies finite speed propagation in the sense that

$$\operatorname{supp}\cos(t\sqrt{\mathbf{L}}) \subset \mathcal{D}_t := \{(z_1,z_2)\subset X\times X \mid d(z_1,z_2)\le |t|\}. \tag{40}$$

  The meaning of this statement is that $\langle f_1, \cos(t\sqrt{\mathbf{L}})f_2\rangle = 0$ whenever $\operatorname{supp} f_1 \in B(z_1,\rho_1)$, $\operatorname{supp} f_2\in B(z_2,\rho_2)$ and $|t|+\rho_1+\rho_2\le d(z_1,z_2)$.
- $\mathbf{L}$ satisfies restriction estimates, which come in a strong and a weak form. We say that $\mathbf{L}$ satisfies *L^p to $L^{p'}$ restriction estimates for all energies* if the spectral measure $dE_{\sqrt{\mathbf{L}}}(\lambda)$ maps $L^p(X)$ to $L^{p'}(X)$ for some p satisfying $1\le p<2$ and all $\lambda>0$, with an operator norm estimate

$$\left\|dE_{\sqrt{\mathbf{L}}}(\lambda)\right\|_{L^p(X)\to L^{p'}(X)} \le C\lambda^{n(1/p-1/p')-1}, \text{ for all } \lambda>0. \tag{41}$$

[1](34) shows that the remainder $\widehat{R_{s_{xy}}}(\sqrt{tL})$ does not contribute to the value of the heat kernel $K_{\exp(-tL)}(x,y)$. Subtracting the remainder from the heat propagator is the main idea of the proof.

Lemma 12.1. *Suppose that (x,d,μ) satisfies (39) and S is an bounded linear operator from $L^p(X)\to L^q(X)$ such that*

$$\operatorname{supp} S\subset \mathcal{D}_\rho$$

for some $\rho>0$. Then for any any $1\le p<q\le\infty$ there exists a constant $C=C_{p,q}$ such that

$$\|S\|_{p\to p}\le C\rho^{n(1/p-1/q)}\|S\|_{p\to q}.$$

Proof. We fix $\rho>0$. Then first we choose a sequence $x_n\in M$ such that $d(x_i,x_j)>\rho/10$ for $i\neq j$ and $\sup_{x\in X}\inf_i d(x,x_i)\le \rho/10$. Such sequence exists because M is separable. Second we define $\widetilde{B_i}$ by the formula

$$\widetilde{B_i}=\bar B\left(x_i,\frac{\rho}{10}\right)-\left(\cup_{j<i}\bar B\left(x_j,\frac{\rho}{10}\right)\right), \tag{42}$$

where $\bar B\,(x,\rho)=\{y\in M\colon d(z,z')\le\rho\}$. Third we put $\chi_i=\chi_{\widetilde{B}_i}$, where $\chi_{\widetilde{B}_i}$ is the characteristic function of set $\widetilde{B}_i$. Fourth we define the operator M_{χ_i} by the formula $M_{\chi_i}g=\chi_i g$.

Note that for $i\neq j$ $B(x_i,\frac{\rho}{20})\cap B(x_i,\frac{\rho}{20})=\emptyset$. Hence

$$K=\sup_i \#\{j;\ d(x_i,x_j)\le 2\rho\}\le \sup_x\frac{|\bar B(x,2\rho)|}{\left|B\left(x,\frac{\rho}{20}\right)\right|}<\frac{40^n c_2}{c_1}<\infty.$$

It is not difficult to see that

$$\mathcal{D}_\rho\subset\cup_{\{i,j;\,d(x_i,x_j)<2\rho\}}\widetilde{B}_i\times\widetilde{B}_j\subset\mathcal{D}_{4\rho}$$

so

$$Sf=\sum_{d(x_i,x_j)<2\rho}M_{\chi_i}SM_{\chi_j}f.$$

Hence by Hölder inequality

$$\begin{aligned}
\|Sf\|_p^p=\Big\|\sum_{d(x_i,x_j)<2\rho}M_{\chi_i}SM_{\chi_j}f\Big\|_{L^p}^p&=\sum_i\Big\|\sum_{j;\,d(x_i,x_j)<2\rho}M_{\chi_i}SM_{\chi_j}f\Big\|_p^p\\
&\le\sum_i|\widetilde{B}_i|^{p(1/p-1/q)}\Big\|\sum_{j;\,d(x_i,x_j)<2\rho}M_{\chi_i}SM_{\chi_j}f\Big\|_q^p\\
&\le C\rho^{np(1/p-/q)}\sum_i\Big\|\sum_{j;\,d(x_i,x_j)<2\rho}M_{\chi_i}SM_{\chi_j}f\Big\|_q^p\\
&\le CK^{p-1}\rho^{np(1/p-1/q)}\sum_i\sum_{j;\,d(x_i,x_j)<2\rho}\big\|M_{\chi_i}SM_{\chi_j}f\big\|_q^p\\
&\le CK^p\rho^{np(1/p-1/q)}\sum_j\big\|SM_{\chi_j}f\big\|_q^p\\
&\le CK^p\rho^{np(1/p-1/q)}\|S\|_{p\to q}^p\sum_j\big\|M_{\chi_j}f\big\|_p^p\\
&=CK^p\rho^{np(1/p-1/q)}\|S\|_{p\to q}^p\|f\|_p^p
\end{aligned}$$

This finishes the proof of Lemma 12.1. $\square$

Theorem 12.2. *Suppose that* (X, d, μ) *and* $\mathbf{L}$ *satisfy* (39) *and* (40), *and that* $\mathbf{L}$ *satisfies* L^p *to* $L^{p'}$ *restriction estimates for all energies,* (41), *for some* p *with* $1 \le p < 2$. *Let* $s > n(1/p - 1/2)$ *be a Sobolev exponent. Then there exists* C *depending only on* n, p, s, *and the constant in* (41) *such that, for every even* $F \in H^s(\mathbf{R})$ *supported in* $[-1,1]$, $F(\sqrt{\mathbf{L}})$ *maps* $L^p(X) \to L^p(X)$, *and*

$$\sup_{\alpha>0} \big\|F(\alpha\sqrt{\mathbf{L}})\big\|_{p\to p} \le C\|F\|_{H^s}. \tag{43}$$

Proof of Theorem 12.2. We first assume that $\mathbf{L}$ satisfies L^p to $L^{p'}$ restriction estimates for all energies. We take $\eta \in C_c^\infty(-4,4)$ even and such that

$$\sum_{l\in\mathbb{Z}} \eta\Big(\frac{t}{2^l}\Big) = 1 \quad \text{for all } t \neq 0.$$

Then we set $\phi(t) = \sum_{l\le 0} \eta(2^{-l}t)$,

$$F_0(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(t)\hat{F}(t)\cos(t\lambda)\, dt$$

and

$$F_l(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \eta\Big(\frac{t}{2^l}\Big)\hat{F}(t)\cos(t\lambda)\, dt. \tag{44}$$

Note that by virtue of the Fourier inversion formula

$$F(\lambda) = \sum_{l\ge 0} F_l(\lambda)$$

and by Lemma 8.2

$$\operatorname{supp} F_l(\alpha\sqrt{\mathbf{L}}) \subset \mathcal{D}_{2^{l+2}\alpha}.$$

Now by Lemma 12.1,

$$\big\|F(\alpha\sqrt{\mathbf{L}})\big\|_{p\to p} \le \sum_{l\ge 0}\big\|F_l(\alpha\sqrt{\mathbf{L}})\big\|_{p\to p} \le C\sum_{l\ge 0}(2^l\alpha)^{n(1/p-1/2)}\big\|F_l(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2}. \tag{45}$$

Unfortunately, F_l is no longer compactly supported. To remedy this we choose a function $\psi \in C_c^\infty(-4,4)$ such that $\psi(\lambda) = 1$ for $\lambda \in (-2,2)$ and note that

$$\big\|F_l(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2} \le \big\|(\psi F_l)(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2} + \big\|((1-\psi)F_l)(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2}.$$

To estimate the norm $\|\psi F_l(\alpha\sqrt{\mathbf{L}})\|_{p\to 2}$ we use our restriction estimates (41). Using a T^*T argument and the fact that $\operatorname{supp}\psi \subset [-4,4]$, we note that

$$\begin{aligned}\big\|\psi F_l(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2}^2 = \big\||\psi F_l|^2(\alpha\sqrt{\mathbf{L}})\big\|_{p\to p'} &\le \int_0^{4/\alpha} |\psi F_l(\alpha\lambda)|^2 \big\|dE_{\sqrt{\mathbf{L}}}(\lambda)\big\|_{p\to p'}\, d\lambda \\ &\le \frac{C}{\alpha}\int_0^4 |\psi F_l(\lambda)|^2 \big\|dE_{\sqrt{\mathbf{L}}}(\lambda/\alpha)\big\|_{p\to p'}\, d\lambda. \end{aligned} \tag{46}$$

It follows from the above calculation and (41) that

$$\alpha^{n(1/p-1/2)}\big\|\psi F_l(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2} \le C\|\psi F_l\|_2, \tag{47}$$

for all $\alpha > 0$. As a consequence, we obtain

$$\sum_{l\ge 0} 2^{ln(1/p-1/2)}\alpha^{n(1/p-1/2)}\big\|\psi F_l(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2} \le \sum_{l\ge 0} 2^{ln(1/p-1/2)}\|\psi F_l\|_2$$

for all $\alpha > 0$. Now let us recall that by definition of Besov space

$$\sum_{l\geq 0} 2^{ln(1/p-1/2)}\|\psi F_l\|_2 \leq \sum_{l\geq 0} 2^{ln(1/p-1/2)}\|F_l\|_2 = \|F\|_{B^{n(1/p-1/2)}_{1,2}}.$$

See, e.g., [20, Chap. I and II] for more details. We also recall that if $s > s'$ then $H^s \subset B^{s'}_{1,2}$ and $\|F\|_{B^{n(1/p-1/2)}_{1,2}} \leq C_s\|F\|_{H^s}$ for all $s > n(1/p-1/2)$, see again [20]. Therefore, we have shown that

$$\sum_{l\geq 0} 2^{ln(1/p-1/2)}\alpha^{n(1/p-1/2)}\big\|\psi F_l(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2} \leq C\|F\|_{H^s}. \tag{48}$$

Now it follows from standard Gaussian bounds that

$$\big\|E_{\sqrt{L}}[0,\lambda]\big\|_{L^p(X)\to L^{p'}(X)} \leq C\lambda^{n(1/p-1/p')}, \quad \lambda \geq \lambda_0 \tag{49}$$

with a uniform C. (Here $E_{\sqrt{L}}[0,\lambda]$ is the same as $\chi_{[0,\lambda]}(\sqrt{L})$.)

Next we obtain bounds for the part of estimate (45) corresponding to the term $\|(1-\psi)F_l(\alpha\sqrt{\mathbf{L}})\|_{p\to 2}$. This only requires the spectral projection estimates (49). We write

$$\begin{aligned}|(1-\psi)F_l|^2(\alpha\sqrt{\mathbf{L}}) &= \int_0^\infty \big|(1-\psi)(\alpha\lambda)F_l(\alpha\lambda)\big|^2 dE_{\sqrt{\mathbf{L}}}(\lambda)\\ &= -\int_0^\infty \Big(\frac{d}{d\lambda}\big|(1-\psi)(\alpha\lambda)F_l(\alpha\lambda)\big|^2\Big)E_{\sqrt{\mathbf{L}}}(\lambda)\,d\lambda\\ &= -\int_0^\infty \Big(\frac{d}{d\lambda}\big|(1-\psi)(\lambda)F_l(\lambda)\big|^2\Big)E_{\sqrt{\mathbf{L}}}(\lambda/\alpha)\,d\lambda.\end{aligned}$$

Hence, using (49),

$$\big\|(1-\psi)F_l(\alpha\sqrt{\mathbf{L}})\big\|^2_{p\to 2} \leq C\int_0^\infty \Big(\frac{d}{d\lambda}|(1-\psi)(\lambda)F_l(\lambda)|^2\Big)\Big(\frac{\lambda}{\alpha}\Big)^{n(1/p-1/p')}\,d\lambda. \tag{50}$$

We write

$$F_l(\lambda) = \frac{1}{2\pi}\int e^{it(\lambda-\lambda')}\eta\Big(\frac{t}{2^l}\Big)F(\lambda')\,d\lambda'\,dt,$$

use the identity

$$e^{it(\lambda-\lambda')} = i^{-N}(\lambda-\lambda')^{-N}(d/dt)^N e^{it(\lambda-\lambda')},$$

and integrate by parts N times. Note that if $\lambda \in \operatorname{supp} 1-\psi$ and $\lambda' \in \operatorname{supp} F$ then $\lambda \geq 2$ and $\lambda' \leq 1$, and hence $\lambda - \lambda' \geq \lambda/2$. It follows that

$$\big|((1-\psi)F_l)(\lambda)\big| \leq C\lambda^{-N}2^{-N(l-1)}\|F\|_2$$

with C independent of N. Similarly,

$$\Big|\frac{d}{d\lambda}((1-\psi)F_l)(\lambda)\Big| \leq C\lambda^{-N}2^{-N(l-1)}2^l\|F\|_2.$$

Using this in (50) with N sufficiently large and $l \geq 2$, we obtain

$$(2^l\alpha)^{n(1/p-1/2)}\big\|((1-\psi)F_l)(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2} \leq C2^{-l}\|F\|_2.$$

Therefore, we have

$$\sum_l (2^l\alpha)^{n(1/p-1/2)}\big\|((1-\psi)F_l)(\alpha\sqrt{\mathbf{L}})\big\|_{p\to 2} \leq C\|F\|_2 \leq C\|F\|_{H^s}. \tag{51}$$

Equations (45), (48) and (51) prove (43).

□

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