Harmonic analysis applied to PDE: dispersive inequalities and Strichartz estimates

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1. Restriction theorems

The Fourier transform
\[ \hat{f}(\xi) = \int_{\mathbb{R}^n} dx \, f(x) e^{-i(x,\xi)} \]
of \( f \in L^1(\mathbb{R}^n) \) is a continuous function, which may be restricted to submanifolds \( S \) such as spheres. On the other hand, the Fourier transform of \( f \in L^2(\mathbb{R}^n) \) is a genuine \( L^2 \) function and it makes no sense to restrict it to a sphere or any other subset of measure zero. Stein’s restriction problem is concerned with a priori inequalities
\[ \| \hat{f} \|_{L^q(S)} \lesssim \| f \|_{L^p(\mathbb{R}^n)} \]
Such results hold for \( q = 2 \) and \( p' \) close to 1 under curvature conditions on \( S \). More information can be found for instance in Stein’s book [11].

**Theorem 1.1** (Stein–Tomas [14]). Assume that \( 1 \leq p' \leq \frac{2n+1}{n-1} \) i.e. \( 2 \frac{n+1}{n-1} \leq p \leq \infty \). Then
\[ \| \hat{f} \|_{L^2(S^{n-1})} \lesssim \| f \|_{L^{p'}(\mathbb{R}^n)} \quad \forall \, f \in \mathcal{S}(\mathbb{R}^n). \]
We give three successive proofs, each one improving upon the previous one.

The first proof relies on the \( T^*T \) method. Consider the operator
\[ Tf = \hat{f} \big|_{S^{n-1}}, \]
its formal adjoint
\[ T^*g(x) = \int_{S^{n-1}} d\sigma(\xi) \, g(\xi) e^{i(x,\xi)}, \]
where \( \sigma \) denotes the surface measure on \( S^{n-1} \) induced by the Lebesgue measure on \( \mathbb{R}^n \), and their composition
\[ \mathcal{T}f(x) = T^*Tf(x) = f \ast \hat{\sigma}. \]
Then the following a priori estimates are equivalent:
\[ \| Tf \|_{L^2} \lesssim \| f \|_{L^{p'}} \iff \| T^*g \|_{L^p} \lesssim \| g \|_{L^2} \iff \| \mathcal{T}f \|_{L^p} \lesssim \| f \|_{L^{p'}}. \]
In order to prove the third one, we use the estimate
\[ |\hat{\sigma}(x)| \lesssim (1 + \|x\|)^{-\frac{n-1}{2}}. \]
Such decay at infinity can be obtained by general oscillatory integral methods. In our particular case, it can be also obtained by expressing the Fourier transform
\[ \hat{\sigma}(x) = (2\pi)^{\frac{n}{2}} \|x\|^{-\frac{n+1}{2}} J_{\frac{n-1}{2}}(\|x\|) \]
in terms of Bessel functions and by using their behavior at infinity. Hence \( \hat{\sigma} \) belongs to \( L^q(\mathbb{R}^n) \) if \( q > \frac{2n}{n-1} \) and to the Lorentz space \( L^{q,\infty}(\mathbb{R}^n) \) in the limit case \( q = \frac{2n}{n-1} \). By using Young’s inequality, we deduce the \( L^{p'} \to L^p \) boundedness of \( \mathcal{T} \) when \( p \geq 2q = \frac{4n}{n-1} \), which is larger than \( 2 \frac{n+1}{n-1}. \)

The second proof relies on a dyadic decomposition. Let \( \chi : \mathbb{R} \to [0,1] \) be a smooth cut–off function such that
and set, for every \( j \in \mathbb{N} \) and \( x \in \mathbb{R}^n \),

\[
\chi_j(x) = \begin{cases} 
\chi(\|x\|) & \text{if } j = 0, \\
\chi(2^{-j} \|x\|) - \chi(2^{1-j} \|x\|) & \text{if } j \in \mathbb{N}^*. 
\end{cases}
\]

Then \( \chi_j \) is supported in

\[
\begin{cases} 
\text{the ball } B(0, 2) & \text{if } j = 0 \\
\text{the dyadic annulus } \Omega_j = B(0, 2^{j+1}) \setminus B(0, 2^{j-1}) & \text{if } j \in \mathbb{N}^*. 
\end{cases}
\]

and

\[
\sum_{j \in \mathbb{N}} \chi_j = 1.
\]

Let us split up \( \sigma = \sum_{j \in \mathbb{N}} \hat{\chi}_j \sigma \) and \( T = \sum_{j \in \mathbb{N}} T_j \) accordingly. On the one hand,

\[
(1) \quad \|T_j\|_{L^1 \to L^\infty} = \|k_j\|_{L^1} \lesssim 2^{-n-j}.
\]

On the other hand,

\[
(2) \quad \|T_j\|_{L^2 \to L^2} = \|\hat{k}_j\|_{L^\infty}. 
\]

We claim that

\[
\|\hat{k}_j\|_{L^\infty} \lesssim 2^j.
\]

This estimate is obvious for \( \hat{k}_0 = \hat{\chi}_0 \star \sigma \). Let us prove it for \( \hat{k}_j = \hat{\chi}_j \star \sigma \) with \( j \in \mathbb{N}^* \). As \( \hat{\chi}_j \) is a rescaled Schwartz function, we have

\[
|\hat{\chi}_j(\xi)| \lesssim 2^{nj}(1 + 2^j \|\xi\|)^{-n} \quad \forall \ j \in \mathbb{N}^*, \forall \ \xi \in \mathbb{R}^n.
\]

Hence,

\[
|\hat{k}_j(\xi)| \lesssim \int_{\mathbb{R}^n} d\sigma(\eta) 2^{nj}(1 + 2^j \|\xi - \eta\|)^{-n}
\]

\[
\lesssim \int_{B(\xi, 2^{-j})} d\sigma(\eta) 2^{nj} + \sum_{k \geq -j} \int_{B(\xi, 2^{k+1}) \setminus B(\xi, 2^k)} d\sigma(\eta) 2^{-nk}
\]

\[
\lesssim 2^{nj} \sigma(B(\xi, 2^{-j})) + \sum_{k \geq -j} 2^{-nk} \sigma(B(\xi, 2^k))
\]

\[
\lesssim 2^j + \sum_{k \geq -j} 2^{-k} \lesssim 2^j.
\]

Here we have used the uniform estimate

\[
\sigma(B(\xi, r)) \lesssim r^{n-1} \quad \forall \ \xi \in \mathbb{R}^n, \forall \ r > 0,
\]

which matters actually for \( r \) small and for \( \|\xi\| \) close to 1. Thus

\[
(2) \quad \|T_j\|_{L^2 \to L^2} \lesssim 2^j \quad \forall \ j \in \mathbb{N}.
\]

By interpolation between (1) and (2), we obtain

\[
(3) \quad \|T_j\|_{L^p \to L^p} \lesssim 2^{-\left(\frac{n}{p} - \frac{n}{p'}\right)j} \quad \forall \ j \in \mathbb{N}.
\]

By adding up (3) over \( j \in \mathbb{N} \), we get the \( L^{p'} \to L^p \) boundedness of \( T \) when \( p > 2 \frac{n+1}{n-1} \). □

The **third proof** relies on Stein’s interpolation for an analytic family of operators. Let us embed \( \sigma \) into the holomorphic family of tempered distributions

\[
\sigma_z = 2^{1-z} \frac{T(z)}{T(1)} (1 - \|\xi\|^2)^{z-1}.
\]

Specifically, \( \sigma_z \) is well-defined in the half-space \( \text{Re } z > 0 \), it extends analytically to \( \mathbb{C} \) by means of the functional relation

\[
\sigma_z = \Delta \sigma_{z+2} + (2z + n) \sigma_{z+1},
\]

and \( \sigma_0 = \sigma \). Moreover,
\[ \hat{\sigma}_z(x) = (2\pi)^\frac{n}{2} \|x\|^{-\frac{n}{2} + 1} J_{\frac{n}{2} - 1}(\|x\|). \]

Consider the analytic family of operators \( \Sigma_z f = f \ast \hat{\sigma}_z \) in the strip \(-\frac{n-1}{2} \leq \text{Re } z \leq 1). On the one hand, if \( \text{Re } z = -\frac{n-1}{2} \), then \( \Sigma_z \) is bounded from \( L^1(\mathbb{R}^n) \) to \( L^\infty(\mathbb{R}^n) \) and
\[ \|\Sigma_z\|_{L^1 \rightarrow L^\infty} = \|\hat{\sigma}_z\|_{L^\infty} \]
grows at most exponentially in \( \text{Im } z \). On the other hand, if \( \text{Re } z = 1 \), then \( \Sigma_z \) is bounded from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) and
\[ \|\Sigma_z\|_{L^2 \rightarrow L^2} = (2\pi)^n \|\sigma_z\|_{L^\infty} \]
grows at most exponentially in \( \text{Im } z \). In conclusion, Stein’s interpolation theorem yields the \( L^{p'} \rightarrow L^p \) boundedness of \( \Sigma_0 = T \) for the endpoint \( p = \frac{n+1}{n-1} \). □

**Remark 1.2** (Knapp). Theorem 1.1 doesn’t hold for \( p' > 2\frac{n+1}{n+3} \) i.e. \( p < 2\frac{n+1}{n-1} \). For small \( \varepsilon > 0 \), consider indeed the function

\[ f_\varepsilon(x) = (4\pi)^n \sin \frac{\sqrt{x_1}}{\varepsilon} \ldots \sin \frac{\sqrt{x_n}}{\varepsilon} e^{ix_n}, \]

whose Fourier transform is the characteristic function of the set

\[ \left[ -\sqrt{\varepsilon}, +\sqrt{\varepsilon} \right]^{n-1} \times [1 - \frac{x_1}{\varepsilon}, 1 + \frac{x_1}{\varepsilon}] \].

Then \( \|f_\varepsilon\|_{L^{p'}} \sim \varepsilon^{\frac{n+1}{p'}} \) while \( \|T f_\varepsilon\|_{L^p} \sim \varepsilon^{\frac{n}{p}} \). By letting \( \varepsilon \to 0 \), we see that \( p \geq 2\frac{n+1}{n-1} \) is a necessary condition for Theorem 1.1 to hold.

**Theorem 1.3** (Strichartz [12]). Consider the paraboloid

\[ S = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n | \tau = -\|\xi\|^2 \} \]
in \( \mathbb{R} \times \mathbb{R}^n \) and assume that \( p' = 2\frac{n+2}{n+3} \) i.e. \( p = 2\frac{n}{n+2} \). Then

\[ \|\hat{u}\|_{L^2(S)} \lesssim \|u\|_{L^{p'}(\mathbb{R} \times \mathbb{R}^n)} \quad \forall \ u \in S(\mathbb{R} \times \mathbb{R}^n). \]

**Proof.** Let us prove Theorem 1.3 by adapting the third proof above. We have now

\[ Tu = \hat{u}\big|_S, \quad T^* v(t, x) = \int_{\mathbb{R}^n} d\xi \ v(-\|\xi\|^2, \xi) e^{i(\tau - \|\xi\|^2 + (x, \xi))} \]

and

\[ T u = T^* T u = u \ast \hat{\sigma}, \]

where

\[ \int_{\mathbb{R} \times \mathbb{R}^n} \ d\tau \ v(\tau, \xi) \big| \tau + \|\xi\|^2 \big|^\frac{n}{2} \ e^{-|\tau|^2/2} \]

and

\[ \hat{\sigma}(t, x) = \int_{\mathbb{R} \times \mathbb{R}^n} \ d\xi \ e^{-i(t\|\xi\|^2 - (x, \xi))} = \left( \frac{i\pi}{2} \right)^\frac{n}{2} \ e^{-i|\tau|^2/4} \]

Consider the analytic family of tempered distributions

\[ \sigma_z(\tau, \xi) = \frac{1}{(1-z)^{\frac{n}{2}}} \big( \tau + \|\xi\|^2 \big)^{\frac{n}{2} - 1} \]

and the associated operators \( \Sigma_z u = u \ast \hat{\sigma}_z \). The \((n+1)\)–dimensional Fourier transform

\[ \hat{\sigma}_z(t, x) = \pi \frac{n}{2} \ e^{-i\frac{\pi}{2}(t-i0)^{-z}} e^{i\text{sign}(t) \frac{n}{2} \ |t-i0|^{-z}}, \]

in the sense of distributions, is computed by combining the \(1\)–dimensional Fourier transform

\[ \frac{1}{(1-z)^{\frac{n}{2}}} \int_{\mathbb{R}} d\tau \tau^{\frac{n}{2} - 1} e^{-i\tau t} = e^{-i\frac{\pi}{2} t} (t-i0)^{-z}, \]

which yields

\[ \frac{1}{(1-z)^{\frac{n}{2}}} \int_{\mathbb{R}} d\tau \big( \tau + \|\xi\|^2 \big)^{\frac{n}{2} - 1} e^{-i\tau t} = e^{i\frac{\pi}{2} t} (t-i0)^{-z} e^{i\|\xi\|^2}, \]

with the \(n\)–dimensional Fourier transform

\[ \int_{\mathbb{R}^n} d\xi \ e^{i\|\xi\|^2} e^{-i(x, \xi)} = \pi \frac{n}{2} \ e^{i\text{sign}(t) \frac{n}{2} |t|^{-\frac{n}{2}} e^{i\|\xi\|^2}.} \]

On the one hand, if \( \text{Re } z = -\frac{n}{2} \), then \( \Sigma_z \) is bounded from \( L^1(\mathbb{R} \times \mathbb{R}^n) \) to \( L^\infty(\mathbb{R} \times \mathbb{R}^n) \) and
The wave equation grows at most exponentially in \( \text{Im } z \). On the other hand, if \( \text{Re } z = 1 \), then \( \Sigma_z \) is bounded from \( L^2(\mathbb{R} \times \mathbb{R}^n) \) to \( L^2(\mathbb{R} \times \mathbb{R}^n) \) and
\[
\| \Sigma_z \|_{L^2 \to L^2} = (2\pi)^{n+2} \| \sigma_z \|_{L^\infty}
\]
grows at most exponentially in \( \text{Im } z \). In conclusion, Stein’s interpolation theorem yields the \( L^{p'} \to L^p \) boundedness of \( \Sigma_0 = \mathcal{T} \) for \( p' = \frac{2n+2}{n+1} \) i.e. \( p = \frac{2n+2}{n+1} \).

**Remark 1.4.** Theorem 1.3 doesn’t hold for \( p' = \frac{2n+2}{n+1} \) i.e. \( p \neq \frac{2n+2}{n+1} \). This is easily proved by rescaling. Specifically, let
\[
(\delta_s u)(t, x) = u(\frac{s^2 t}{s}, sx), \quad \forall s > 0, \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n.
\]
Then \( \| \delta_s u \|_{L^{p'}} = s^{-\frac{n+2}{n+1}} \| u \|_{L^{p'}} \) while \( \| T(\delta_s u) \|_{L^2} = s^{-\frac{2}{n+2}} \| Tu \|_{L^2} \), as
\[
\| \delta_s u(\tau, \xi) \| = s^{-n-2} \| \hat{u}(s^{-\frac{2}{n+2}}, s^{-1} \xi) \|.
\]
By letting \( s \to 0 \) or \( s \to +\infty \), we see that \( p = \frac{2n+2}{n+1} \) is a necessary condition for Theorem 1.3 to hold.

Let us next apply Theorem 1.3 to the Schrödinger equation
\[
\begin{aligned}
i \partial_t u(t, x) &= -\Delta_x u(t, x), \\
u(t, x) &= f(x),
\end{aligned}
\]
(4)

Via the Fourier transform, (4) becomes
\[
\begin{aligned}
i \partial_t \hat{u}(t, \xi) &= ||\xi||^2 \hat{u}(t, \xi), \\
\hat{u}(t, \xi) &= \hat{f}(\xi),
\end{aligned}
\]
whose solution is given by
\[
\hat{u}(t, \xi) = e^{-it||\xi||^2} \hat{f}(\xi).
\]
By applying the inverse Fourier transform, we get
\[
u(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \hat{f}(\xi) e^{i(-t||\xi||^2 + (x, \xi))},
\]
which boils down to the expression of the operator \( T^* \).

**Corollary 1.5.** Let \( p = \frac{2n+2}{n+1} \). Then the following a priori inequality holds for solutions to (4):
\[
\| u \|_{L^p(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| f \|_{L^2(\mathbb{R}^n)}.
\]

**Proof.** This result is a restatement of the \( L^2 \to L^p \) boundedness of the operator \( T^* \). Specifically,
\[
\| u \|_{L^p(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| \hat{f} \|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \| f \|_{L^2(\mathbb{R}^n)}.
\]

**Remark 1.6.** In his seminal work [12], Strichartz studies restriction theorems for general quadratic hypersurfaces and applies some of them to related PDE. Another important example is the cone
\[
S = \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid \tau^2 = ||\xi||^2 \}
\]
and the wave equation
\[
\begin{aligned}
\partial^2_t u(t, x) &= \Delta_x u(t, x), \\
u(t, x) &= f(x), \quad \partial_{\tau} u|_{\tau = 0} = g(x).
\end{aligned}
\]
(5)

In this case, \( p' = \frac{2n+1}{n+3} \) i.e. \( p = \frac{2n+1}{n+1} \), the restriction theorem reads
\[
\| \hat{u} \|_S L^2(S) \lesssim \| u \|_{L^{p'}(\mathbb{R} \times \mathbb{R}^n)} \quad \forall u \in S(\mathbb{R} \times \mathbb{R}^n),
\]
and the following a priori inequality holds for solutions to (5):
\[
\| u \|_{L^p(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| f \|_{H^{1/2}(\mathbb{R}^n)} + \| g \|_{H^{-1/2}(\mathbb{R}^n)}.
\]
where the initial data belong to homogeneous Sobolev spaces $\dot{H}^{\frac{\pm}{2}}(\mathbb{R}^n) = (-\Delta)^{\frac{\pm}{4}}L^2(\mathbb{R}^n)$.

2. Schrödinger equation on $\mathbb{R}^n$ (Linear Case)

In this section we consider the linear Schrödinger equation

\[
\begin{cases}
i \partial_t u(t, x) + \Delta x u(t, x) = F(t, x) \\
u(t, x) = f(x)
\end{cases}
\]

on $\mathbb{R} \times \mathbb{R}^n$ and we prove two fundamental inequalities in Theorem 2.1 and Theorem 2.3, the latter improving upon Corollary 1.5. Our main references are [6] and [8].

Consider first the homogeneous Schrödinger equation

\[
\begin{cases}
i \partial_t u(t, x) + \Delta x u(t, x) = 0 \\
u(t, x) = f(x)
\end{cases}
\]

whose solution is given by

\[
u(t, x) = e^{it\Delta}f(x) = f \ast s_t(x),
\]

where

\[
s_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} e^{-\frac{|\xi|^2}{4t}} d\xi = (4\pi it)^{-\frac{n}{2}} e^{i \frac{|x|^2}{4t}},
\]

is the heat kernel with imaginary time.

**Theorem 2.1.** Let $1 \leq q' \leq 2$ i.e. $2 \leq q \leq \infty$. Then the following dispersive estimate holds, for every $t \in \mathbb{R}^*$:

\[
\| e^{it\Delta} \|_{L^{q'} \to L^q} \lesssim |t|^{-(\frac{n}{2} - \frac{1}{q})}
\]

**Proof.** This result is obtained by interpolation between the elementary estimate

\[
\| e^{it\Delta} \|_{L^1 \to L^\infty} = \| s_t \|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}
\]

and the $L^2$ conservation

\[
\| e^{it\Delta} f \|_{L^2} = \| f \|_{L^2}.
\]

Let us turn to the inhomogeneous equation (6), whose solution is given by Duhamel’s formula:

\[
u(t, x) = e^{it\Delta}f(x) - i \int_0^t ds \ e^{i(t-s)\Delta x} F(s, x).
\]

**Figure 1.** Admissibility for $\mathbb{R}^n$
Definition 2.2. A couple \((p, q)\) of indices is called admissible if
\[ 2 \leq p \leq \infty, \ 2 \leq q < \infty \quad \text{and} \quad \frac{1}{p} = \frac{n}{2} - \frac{1}{q}. \]

Theorem 2.3. Let \((p, q)\) and \((\tilde{p}, \tilde{q})\) be two admissible couples of indices. Then the following Strichartz estimate holds for solutions (9) to (6):
\[ \|u\|_{L^p_t(L^q_x(\mathbb{R}^n))} \lesssim \|f\|_{L^2_x(\mathbb{R}^n)} + \|F\|_{L^p_t(L^{\tilde{q}'}_x(\mathbb{R}^n))} \]

We shall prove Theorem 2.3 in several steps.

**Step 1:** \(TT^*\) method revisited.
Consider the operator
\[ Tf(t, x) = e^{i\Delta_x}f(x), \]
its formal adjoint
\[ T^*F(x) = \int_{-\infty}^{+\infty} ds \ e^{-is\Delta_x}F(s, x), \]
their composition
\[ TF(t, x) = TT^*F(t, x) = \int_{-\infty}^{+\infty} ds \ e^{i(t-s)\Delta_x}F(s, x), \]
and its truncated version
\[ \mathcal{T}F(t, x) = \int_{-\infty}^{t} ds \ e^{i(t-s)\Delta_x}F(s, x). \]

Then the following a priori estimates are equivalent:
\[ \|Tf\|_{L^p_t(L^q_x(\mathbb{R}^n))} \lesssim \|f\|_{L^2_x(\mathbb{R}^n)}, \]
\[ \|T^*F\|_{L^2_x(\mathbb{R}^n)} \lesssim \|F\|_{L^p_t(L^{\tilde{q}'}_x(\mathbb{R}^n))}, \]
\[ \|TF\|_{L^p_t(L^q_x(\mathbb{R}^n))} \lesssim \|F\|_{L^{\tilde{p}}_t(L^q_x(\mathbb{R}^n))}. \]

**Step 2:** Proof of (11), (12), (13) when \((p, q)\) is admissible and \(p > 2\).
On the one hand, according to Theorem 2.1,
\[ \|TF(t, x)\|_{L^q_x} \lesssim \int_{-\infty}^{+\infty} ds \ |t-s|^{-\alpha} \|F(s, x)\|_{L^q_x}, \]
where \(\alpha = n\left(\frac{1}{2} - \frac{1}{q}\right)\). On the other hand, according to the Hardy–Littlewood–Sobolev inequality, the convolution kernel \(|t-s|^{-\alpha}\) defines a bounded operator from \(L^p(\mathbb{R})\) to \(L^p(\mathbb{R})\), provided that \(0 \leq \alpha < 1\) and \(\frac{1}{p} = \frac{n}{2}\). These two results yield (13), hence (11) and (12), under the assumptions above.

**Step 3:** Decoupling indices.
Let \((p, q)\) and \((\tilde{p}, \tilde{q})\) be admissible couples with \(p > 2\) and \(\tilde{p} > 2\). By combining the \(L^p(\mathbb{R}) \to L^2(\mathbb{R}^n)\) boundedness of \(T^*\) with the \(L^2(\mathbb{R}^n) \to L^p(\mathbb{R}, L^{\tilde{q}'}(\mathbb{R}^n))\) boundedness of \(T\), we obtain the \(L^p(\mathbb{R}) \to L^2(\mathbb{R}^n)\) boundedness of \(T^*\). Moreover, as \(\tilde{p}' < p\), the same result holds true for the truncated operator \(\mathcal{T}\), according to the Christ–Kiselev lemma [4].

**Step 4:** Endpoint estimates when \(n \geq 3\) and \((p, q) = (\tilde{p}, \tilde{q}) = (2, 2\frac{n}{n-2})\).
The following arguments are due to Keel and Tao [8]. Instead of \(\mathcal{T}\) and \(\mathcal{T}\), consider the bilinear (actually Hermitian) form
\[ \mathcal{B}(F, G) = \iint_{\mathbb{R}^2} ds \ dt \int_{\mathbb{R}^n} dx \ e^{-is\Delta_x}F(s, x) e^{-it\Delta_x}G(t, x) \]
and its truncated version
\[ \mathcal{B}(F, G) = \iint_{s < t} ds \ dt \int_{\mathbb{R}^n} dx \ e^{-is\Delta_x}F(s, x) e^{-it\Delta_x}G(t, x). \]
In order to estimate (14), let us split up dyadically \( \tilde{\mathcal{B}} = \sum_{j \in \mathbb{Z}} \tilde{B}_j \), where
\[
\tilde{B}_j(F, G) = \int_{2^j \leq t - s \leq 2^{j+1}} ds \, dt \int_{\mathbb{R}^n} dx \, e^{-is\Delta_x} F(s, x) \, e^{-it\Delta_x} G(t, x),
\]
and let us further split up
\[
F(s, x) = \sum_{k = -\infty}^{+\infty} \mathbb{1}_{|k|2^j, (k+1)2^j)}(s) \, F(s, x) \quad \text{and} \quad G(t, x) = \sum_{\ell = -\infty}^{+\infty} \mathbb{1}_{(\ell2^j, (\ell+1)2^j)}(t) \, G(t, x).
\]
Notice the orthogonality
\[
\Vert F \Vert_{L^{p'} L^{q'}} = \left\{ \sum_{k = -\infty}^{+\infty} \Vert F_k^{(j)} \Vert_{L^{p'} L^{q'}} \right\}^{1/p'}, \quad \Vert G \Vert_{L^{p'} L^{q'}} = \left\{ \sum_{\ell = -\infty}^{+\infty} \Vert G_\ell^{(j)} \Vert_{L^{p'} L^{q'}} \right\}^{1/p'}
\]
and the almost orthogonality
\[
\tilde{B}_j(F, G) = \sum_{k, \ell \in \mathbb{Z}} \tilde{B}_j(F_k^{(j)}, G_\ell^{(j)}).
\]
We claim that, for all indices corresponding to the region pictured in Figure 2, we have
\[
|\tilde{B}_j(F_k^{(j)}, G_\ell^{(j)})| \lesssim 2^{\kappa(\tilde{q}, q)j} \Vert F_k^{(j)} \Vert_{L^{2}L^{q'}} \Vert G_\ell^{(j)} \Vert_{L^{2}L^{q'}} \quad \forall \, j, k, \ell \in \mathbb{Z},
\]
where \( \kappa(\tilde{q}, q) = \frac{n}{2} \left( \frac{1}{\tilde{q}} + \frac{1}{q} \right) - \frac{n-2}{2} \).

\[
\text{Figure 2. Interpolation region}
\]
This estimate is obtained by interpolation between the following three cases:
\[
\begin{cases}
(a) \, 2 \leq \tilde{q} < 2 \frac{n}{n-2} \text{ and } q = 2, \\
(b) \, \tilde{q} = 2 \text{ and } 2 \leq q < 2 \frac{n}{n-2}, \\
(c) \, 2 < \tilde{q} = q < \infty.
\end{cases}
\]
\circ Case (a): Let us estimate
\[
|\tilde{B}_j(F_k^{(j)}, G_\ell^{(j)})| \lesssim \sup_{t \in \mathbb{R}} \left\| \int_{t-2^j}^{t-2^{j+1}} ds \, e^{-is\Delta_x} F_k^{(j)}(s, x) \right\|_{L^2_s} \int_{-\infty}^{+\infty} dt \left\| e^{-it\Delta_x} G_\ell^{(j)}(t, x) \right\|_{L^2_t}.
\]
On the one hand, as \( T^* \) is bounded from \( L^{\tilde{p}'}(\mathbb{R}, L^{\tilde{q}'}(\mathbb{R}^n)) \) into \( L^2(\mathbb{R}^n) \), with \( \frac{1}{p'} = \frac{\tilde{q}}{2} \left( \frac{1}{2} - \frac{1}{q} \right) \),
\[
\left\| \int_{\mathbb{R}} ds \mathbb{1}_{(t-2^j+1,t+2^j)}(s) \, e^{-is\Delta_x} F_k^{(j)}(s, x) \right\|_{L^2} \lesssim \left\| \mathbb{1}_{(t-2^j+1,t+2^j)}(s) \, F_k^{(j)}(s, x) \right\|_{L^{\tilde{p}'}_s L^{\tilde{q}'}_x}.
\]
On the other hand,
\[
\left\| e^{-it\Delta_x} G_\ell^{(j)}(t, x) \right\|_{L^2_t} \lesssim \left\| G_\ell^{(j)}(t, x) \right\|_{L^2_t}.
\]
By combining these estimates and by using Hölder’s inequality in time, we conclude that
\[
|\tilde{B}_j(F_k^{(j)}, G_\ell^{(j)})| \lesssim 2^{\kappa(\tilde{q}, q)j} \Vert F_k^{(j)} \Vert_{L^{2}L^{q'}} \Vert G_\ell^{(j)} \Vert_{L^{2}L^{q'}}.
\]
Step 5: Endpoint estimates when $n \geq 3$ and either $(p, q) = (\tilde{p}, \tilde{q}) = (2, 2 \frac{n}{n-2})$ or $(\tilde{p}, \tilde{q}) = (2, 2 \frac{n}{n-2})$.

This case is simpler and relies on the same arguments.

3. Schrödinger equation on $\mathbb{R}^n$ (nonlinear case)

In this section, we consider the semilinear Schrödinger equation

\[
\begin{cases}
  i \partial_t u(t, x) + \Delta_x u(t, x) = F(u(t, x)), \\
  u(t, x) = f(x),
\end{cases}
\]

with power-like nonlinearities $F$. By this we mean that there exist constants $C > 0$ and $\gamma > 1$ such that

\[
\begin{cases}
  |F(u)| \leq C|u|^{\gamma}, \\
  |F(u) - F(\tilde{u})| \leq C \{|u|^{\gamma-1} + |\tilde{u}|^{\gamma-1}\} |u - \tilde{u}|.
\end{cases}
\]

Typical examples are

\[
F(u) = \text{const.} \times \begin{cases}
  |u|^\gamma, \\
  u |u|^{\gamma-1}.
\end{cases}
\]
Following the strategy developed by Kato, Ginibre–Velo [6] and Keel–Tao [8], we apply the Strichartz estimates in Theorem 2.3 to the well–posedness of (18), which means roughly existence and uniqueness of solutions in some suitable function space.

**Theorem 3.1.** (a) Critical case: Assume that \( \gamma = 1 + \frac{4}{n} \).

Then (18) is globally well–posed in \( L^2(\mathbb{R}^n) \) for small initial data \( f \).

(b) Subcritical case: Assume that \( 1 < \gamma < 1 + \frac{4}{n} \).

Then (18) is locally well–posed in \( L^2(\mathbb{R}^n) \) for arbitrary initial data \( f \).

(c) Subcritical case: Assume again that \( 1 < \gamma < 1 + \frac{4}{n} \)

and assume that \( F(u) \) is a real multiple of \( u |u|^{\gamma-1} \).

Then (18) is globally well–posed in \( L^2(\mathbb{R}^n) \) for arbitrary initial data \( f \).

**Proof of (a).** Define \( u = \Phi(v) \) as the solution to the Cauchy problem

\[
\begin{align*}
&i \partial_t u(t, x) + \Delta_x u(t, x) = F(v(t, x)), \\
&u(t, x) = f(x),
\end{align*}
\]

which is given by Duhamel’s formula (9):

\[
u(t, x) = e^{it\Delta} f(x) - i \int_0^t ds e^{i(t-s)\Delta} F(s, x).
\]

According to Theorem 2.3, the following Strichartz estimate holds

\[
\|u\|_{L^\infty_t L^2(\mathbb{R}^n)} + \|u\|_{L^p_t L^q(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)} + C \|F(v)\|_{L^p(\mathbb{R}^n)}
\]

for all couples \((p, q)\) and \((\tilde{p}, \tilde{q})\) satisfying the admissibility conditions

\[
\begin{align*}
2 \leq p &\leq \infty, \quad 2 \leq q < \infty, \quad \frac{1}{p} = \frac{n}{2} \left( \frac{1}{\tilde{p}} - \frac{1}{q} \right), \\
2 \leq \tilde{p} &\leq \infty, \quad 2 \leq \tilde{q} < \infty, \quad \frac{1}{\tilde{p}} = \frac{n}{2} \left( \frac{1}{\tilde{q}} - \frac{1}{q} \right).
\end{align*}
\]

Moreover

\[
\|F(v)\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)} \leq C \|v\|_\gamma \|F(v)\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)} \leq C \|v\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)}
\]

by our nonlinear assumption (19). Thus

\[
\|u\|_{L^\infty_t L^2(\mathbb{R}^n)} + \|u\|_{L^p_t L^q(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)} + C \|v\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)}.
\]

In order to remain within the same function space, we require that

\[
\begin{align*}
p = \gamma p' &\iff \frac{n}{p} + \frac{1}{p'} = 1, \\
q = \gamma q' &\iff \frac{n}{q} + \frac{1}{q'} = 1.
\end{align*}
\]

All conditions (22) and (24) can be fulfilled provided that \( \gamma = 1 + \frac{4}{n} \). In this case, one may consider for instance the Strichartz point, which is given by

\[
p = q = \tilde{p} = \tilde{q} = \gamma + 1 = 2 + \frac{4}{n}.
\]

For such a choice, \( \Phi \) maps \( L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)) \cap L^p(\mathbb{R}, L^q(\mathbb{R}^n)) \) into itself, and actually \( X = C(\mathbb{R}, L^2(\mathbb{R}^n)) \cap L^p(\mathbb{R}, L^q(\mathbb{R}^n)) \) into itself. As \( X \) is a Banach space for the norm

\[
\|u\|_X = \|u\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))} + \|u\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))},
\]

it remains for us to show that \( \Phi \) is a contraction in the ball

\[
X_\varepsilon = \{ u \in X \mid \|u\|_X \leq \varepsilon \},
\]

provided that \( \varepsilon > 0 \) and \( \|f\|_{L^2} \) are sufficiently small. Let \( v, \tilde{v} \in X \) and \( u = \Phi(v), \tilde{u} = \Phi(\tilde{v}) \).

Arguing as above and using in addition Hölder’s inequality, we estimate

\[
\|u - \tilde{u}\|_X \leq C \|F(v) - F(\tilde{v})\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)} 
\leq C \|\{ |v|^{\gamma-1} + |\tilde{v}|^{\gamma-1}\} |v - \tilde{v}|\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)} 
\leq C \left\{ \|v\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)}^{\gamma-1} + \|\tilde{v}\|_{L^q_t L^\tilde{q}(\mathbb{R}^n)}^{\gamma-1} \right\} \|v - \tilde{v}\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))},
\]

hence
(25) \[ \|u - \tilde{u}\|_X \leq C \left\{ \|v\|^{-1}_X + \|\tilde{v}\|^{-1}_X \right\} \|v - \tilde{v}\|_X. \]

If we assume \(\|v\|_X \leq \varepsilon, \|\tilde{v}\|_X \leq \varepsilon\) and \(\|f\| \leq \delta\), then (23) and (25) yield
\[ \|u\|_X \leq C \delta + C \varepsilon^\gamma, \quad \|\tilde{u}\|_X \leq C \delta + C \varepsilon^\gamma \quad \text{and} \quad \|u - \tilde{u}\|_X \leq 2 C \varepsilon^{\gamma-1} \|v - \tilde{v}\|_X. \]

Thus
\[ \|u\|_X \leq \varepsilon, \quad \|\tilde{u}\|_X \leq \varepsilon \quad \text{and} \quad \|u - \tilde{u}\|_X \leq \frac{1}{2} \|v - \tilde{v}\|_X, \]

if \(C \varepsilon^{\gamma-1} \leq \frac{1}{4}\) and \(C \delta \leq \frac{3}{4} \varepsilon\). We conclude by applying the fixed point theorem in the complete metric space \(X_\varepsilon\).

**Proof of (b).** In the subcritical case \(\gamma < 1 + \frac{4}{n}\), the same arguments yield the local well-posedness of (18) in \(L^2(\mathbb{R}^n)\) for arbitrary initial data \(f\). Specifically, we restrict to a small time interval \(I = [-T, +T]\) and we proceed as above, except that we apply in addition Hölder’s inequality in time. This way, we get the Strichartz estimate
\[ \|u\|_X \leq C \|f\| \|L^2 \| + C T^\lambda \|v\|_X^\gamma, \]

where \(X = C(I, L^2(\mathbb{R}^n)) \cap L^p(I, L^q(\mathbb{R}^n))\) and \(\lambda = 1 - \frac{2}{p} - \frac{1}{q} > 0\), and the related estimate
\[ \|u - \tilde{u}\|_X \leq C T^\lambda \left\{ \|v\|^{-1}_X + \|\tilde{v}\|^{-1}_X \right\} \|v - \tilde{v}\|_X. \]

As a consequence, we deduce that \(\Phi\) is a contraction in the ball
\[ X_M = \{ u \in X \mid \|u\|_X \leq M \}, \]

provided \(M > 0\) is large enough and \(T > 0\) small enough, more precisely \(\frac{3}{4} M \geq C \|f\| \|L^2\|\) and \(C T^\lambda M^{\gamma-1} \leq \frac{1}{4}\). We conclude as before. Notice that the size of \(T\) depends only on the \(L^2\) norm of the initial data \(f\).

**Proof of (c).** Assume moreover that \(F(u) = c u |u|^\gamma - 1\) with \(c \in \mathbb{R}\). Then the expression
\[ i \int_{\mathbb{R}^n} dx \partial_t u(t, x) \overline{u(t, x)} = - \int_{\mathbb{R}^n} dx \Delta x u(t, x) \overline{u(t, x)} + c \int_{\mathbb{R}^n} dx |u(t, x)|^{\gamma+1} \]

is real, hence
\[ \partial_t \int_{\mathbb{R}^n} dx |u(t, x)|^2 = 2 \text{ Re } \int_{\mathbb{R}^n} dx \partial_t u(t, x) \overline{u(t, x)} \]

vanishes and we have \(L^2\) conservation:
\[ (28) \quad \int_{\mathbb{R}^n} dx |u(t, x)|^2 = \int_{\mathbb{R}^n} dx |f(x)|^2. \]

As the time interval in part (b) depends only on (28), we may iterate and deduce global existence from local existence, for arbitrary initial data \(f \in L^2\). We refer to [3] for more information about conservation properties of the Schrödinger equation.

**Remark 3.2.** Similar results hold if \(1 + \frac{4}{n}\) is replaced by \(1 + \frac{4}{n-2\sigma}\) and \(L^2(\mathbb{R}^n)\) by \(H^\sigma(\mathbb{R}^n)\), with \(0 < \sigma < \frac{1}{2}\).

**Remark 3.3.** The semilinear wave equation
\[ (29) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta x u(t, x) = F(u(t, x)), \\ u(t, x) = f(x), \ \partial_t|_{t=0} u(t, x) = g(x), \end{cases} \]

with power-like nonlinearities \(F\), can be studied in a similar way but its analysis is more involved. We refer to [5, 6, 8, 9, 13] for more information.
4. Schrödinger equation on hyperbolic spaces \( \mathbb{H}^n \)

This section consists in an introduction to my joint work [1] with Vittoria Pierfelice, where we have investigated the semilinear Schrödinger equation (18) on real hyperbolic spaces \( \mathbb{H}^n \). Among other references, let us mention on the one hand the earlier works [2, 10] and on the other hand [7], which is mainly devoted to scattering theory in \( H^1(\mathbb{H}^n) \). As might be expected, dispersion properties are better in negative curvature. Consequently, Strichartz estimates hold for a wider range and one obtains stronger well-posedness results.

Fourier analysis on \( \mathbb{H}^n \) yields the following explicit expression for the Schrödinger kernel (i.e. the heat kernel with imaginary time).

**Lemma 4.1.** For every \( t \in \mathbb{R}^* \) and \( r \geq 0 \), we have

\[
s_t(r) = \text{const.} \, (it)^{-\frac{n}{2}} e^{-i \frac{(\varphi(t))^2}{2} t} \left( -\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{\frac{n-1}{2}} e^{\frac{r^2}{4t}}.
\]

Here \((it)^{-\frac{1}{2}} = e^{-i \text{sgn}(t) \frac{\varphi}{2}} |t|^{-\frac{1}{2}} \) and, in the even dimensional case, the fractional derivative reads

\[
(-\frac{1}{\sinh r} \frac{\partial}{\partial r})^{\frac{n-1}{2}} e^{\frac{r^2}{4t}} = \frac{1}{\pi} \int_{|r|}^{+\infty} \frac{\sinh s ds}{\sqrt{\cosh s - \cosh r}} \left( -\frac{1}{\sinh s} \frac{\partial}{\partial s} \right)^{\frac{n-1}{2}} e^{\frac{s^2}{4t}}.
\]

We deduce first the following sharp kernel estimates.

**Corollary 4.2.** For every \( t \in \mathbb{R}^* \) and \( r \geq 0 \), we have

\[
|s_t(r)| \lesssim \begin{cases} 
|t|^{-\frac{n}{2}} (1+r)^{\frac{n-1}{2}} e^{-\frac{n-1}{4} r} & \text{if } |t| \leq 1+r, \\
|t|^{-\frac{n}{2}} (1+r)^{-\frac{n+1}{2} r} & \text{if } |t| \geq 1+ r.
\end{cases}
\]

**Remark 4.3.** Notice that

\[(1+r)^{\frac{n-1}{2}} e^{-\frac{n-1}{4} r} \asymp j(r)^{-\frac{n}{4}},\]

where \(j\) is the jacobian of the exponential map, and

\[(1+r)^{-\frac{n+1}{2} r} \asymp \varphi_0(r)\]

where \(\varphi_0\) is the radial ground state.

We deduce next the following dispersive estimates.

**Theorem 4.4.** For every \( 2 < q \leq \infty \) and \( t \in \mathbb{R}^* \), we have

\[
\|e^{it\Delta}\|_{L^q(\mathbb{H}^n) \to L^q(\mathbb{H}^n)} \lesssim \begin{cases} 
|t|^{-n\left(\frac{1}{2} - \frac{1}{q}\right)} & \text{if } 0 < |t| < 1, \\
|t|^{-\frac{n}{2}} & \text{if } |t| \geq 1.
\end{cases}
\]

**Hint of proof.** The first estimate is obtained by interpolation. The second one is obtained by applying the following version of the Kunze–Stein phenomenon:

\[
L^q(\mathbb{H}^n) * L^{\hat{q}}_{\text{rad}}(\mathbb{H}^n) \subset L^q(\mathbb{H}^n)
\]

if \( 2 < q, \hat{q} < \infty \) satisfy \( \frac{2}{\hat{q}} < \hat{q} < q \).

Consider the linear Schrödinger equation

\[
\begin{cases}
& i \partial_t u(t, x) + \Delta_x u(t, x) = F(t, x) \\
& u(t, x) = f(x)
\end{cases}
\]

on \( \mathbb{R} \times \mathbb{H}^n \).

**Theorem 4.5.** The Strichartz estimate

\[
\|u\|_{L^{q/2}(\mathbb{R}, L^q(\mathbb{H}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))}
\]

holds for (6) and for all admissible couples \((p, q), (\hat{p}, \hat{q})\) corresponding to the triangle pictured in Figure 3.

Figure 3. Admissibility for $\mathbb{H}^n$

Consider eventually the semilinear Schrödinger equation

\[
\begin{cases}
  i \partial_t u(t, x) + \Delta_x u(t, x) = F(u(t, x)), \\
  u(t, x) = f(x),
\end{cases}
\]

on $\mathbb{R} \times \mathbb{H}^n$, with power–like nonlinearities $F$ as in (19).

**Theorem 4.6.** Assume that $1 < \gamma \leq 1 + \frac{4}{n}$.
Then (18) is globally well–posed in $L^2(\mathbb{R}^n)$ for small initial data $f$.

**Remark 4.7.** Parts (b) and (c) in Theorem 3.1 hold also on $\mathbb{H}^n$.

**Remark 4.8.** All these results extend straightforwardly to the so–called rank one case, which include all hyperbolic spaces, as well as Damek–Ricci spaces.

**References**


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