pl. Grunwaldzki 2/4
50-384 Wrocław
tel. +48 71 375 74 01, +48 71 328 07 17
fax +48 71 375 74 29
e-mail: dyrekcja@math.uni.wroc.pl
www.math.uni.wroc.pl

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Harmonic analysis applied to PDE: dispersive inequalities and Strichartz estimates

Jean-Philipp Anker

(Université d' Orléans)

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HARMONIC ANALYSIS APPLIED TO PDE : DISPERSIVE INEQUALITIES AND STRICHARTZ ESTIMATES

JEAN–PHILIPPE ANKER
(UNIVERSITY OF ORLEANS, FRANCE)

1. Restriction theorems

The Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} dx\, f(x)\, e^{-i\langle x,\xi\rangle}$$

of $f \in L^1(\mathbb{R}^n)$ is a continuous function, which may be restricted to submanifolds S such as spheres. On the other hand, the Fourier transform of $f \in L^2(\mathbb{R}^n)$ is a genuine L^2 function and it makes no sense to restrict it to a sphere or any other subset of measure zero. Stein's restriction problem is concerned with a priori inequalities

$$\| \widehat{f}\,\big|_S \|_{L^q(S)} \lesssim \| f \|_{L^{p'}(\mathbb{R}^n)}\,.$$

Such results hold for $q = 2$ and p' close to 1 under curvature conditions on S. More information can be found for instance in Stein's book [11].

Theorem 1.1 (Stein–Tomas [14])**.** *Assume that* $1 \le p' \le 2\frac{n+1}{n+3}$ *i.e.* $2\frac{n+1}{n-1} \le p \le \infty$. *Then*

$$\| \widehat{f}\,\big|_{\mathbb{S}^{n-1}} \|_{L^2(\mathbb{S}^{n-1})} \lesssim \| f \|_{L^{p'}(\mathbb{R}^n)} \qquad \forall\, f \in \mathcal{S}(\mathbb{R}^n).$$

We give three successive proofs, each one improving upon the previous one.

The **first proof** relies on the T^*T method. Consider the operator

$$Tf = \widehat{f}\,\big|_{\mathbb{S}^{n-1}}\,,$$

its formal adjoint

$$T^*g(x) = \int_{\mathbb{S}^{n-1}} d\sigma(\xi)\, g(\xi)\, e^{i\langle x,\xi\rangle}\,,$$

where σ denotes the surface measure on $\mathbb{S}^{n-1}$ induced by the Lebesgue measure on $\mathbb{R}^n$, and their composition

$$\mathcal{T}f(x) = T^*Tf(x) = f * \widehat{\sigma}.$$

Then the following a priori estimates are equivalent :

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^{p'}} \iff \|T^*g\|_{L^p} \lesssim \|g\|_{L^2} \iff \|\mathcal{T}f\|_{L^p} \lesssim \|f\|_{L^{p'}}.$$

In order to prove the third one, we use the estimate

$$|\,\widehat{\sigma}(x)| \lesssim \big(1 + \|x\|\big)^{-\frac{n-1}{2}}.$$

Such decay at infinity can be obtained by general oscillatory integral methods. In our particular case, it can be also obtained by expressing the Fourier transform

$$\widehat{\sigma}(x) = (2\pi)^{\frac{n}{2}}\, \|x\|^{-\frac{n}{2}+1} J_{\frac{n}{2}-1}(\|x\|)$$

in terms of Bessel functions and by using their behavior at infinity. Hence $\widehat{\sigma}$ belongs to $L^q(\mathbb{R}^n)$ if $q > \frac{2n}{n-1}$ and to the Lorentz space $L^{q,\infty}(\mathbb{R}^n)$ in the limit case $q = \frac{2n}{n-1}$. By using Young's inequality, we deduce the $L^{p'} \to L^p$ boundedness of $\mathcal{T}$ when $p \ge 2q = \frac{4n}{n-1}$, which is larger than $2\frac{n+1}{n-1}$. □

The **second proof** relies on a dyadic decomposition. Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cut–off function such that

$$\chi=\begin{cases}1 & \text{on } (-\infty,1]\\ 0 & \text{on } [2,+\infty)\end{cases}$$

and set, for every $j\in\mathbb{N}$ and $x\in\mathbb{R}^n$,

$$\chi_j(x)=\begin{cases}\chi(\|x\|) & \text{if } j=0,\\ \chi(2^{-j}\|x\|)-\chi(2^{1-j}\|x\|) & \text{if } j\in\mathbb{N}^*.\end{cases}$$

Then χ_j is supported in

$$\begin{cases}\text{the ball } \overline{B}(0,2) & \text{if } j=0\\ \text{the dyadic annulus } \Omega_j=\overline{B}(0,2^{j+1})\smallsetminus B(0,2^{j-1}) & \text{if } j\in\mathbb{N}^*\end{cases}$$

and

$$\sum\nolimits_{j\in\mathbb{N}}\chi_j\equiv 1\,.$$

Let us split up $\widehat{\sigma}=\sum_{j\in\mathbb{N}}\overbrace{\chi_j\widehat{\sigma}}^{k_j}$ and $\mathcal{T}=\sum_{j\in\mathbb{N}}\mathcal{T}_j$ accordingly. On the one hand,

$$\|\mathcal{T}_j\|_{L^1\to L^\infty}=\|k_j\|_{L^1}\lesssim 2^{-\frac{n-1}{2}j}\,. \tag{1}$$

On the other hand,

$$\|\mathcal{T}_j\|_{L^2\to L^2}=\|\widehat{k_j}\|_{L^\infty}\,.$$

We claim that

$$\|\widehat{k_j}\|_{L^\infty}\lesssim 2^j.$$

This estimate is obvious for $\widehat{k_0}=\widehat{\chi_0}*\sigma$. Let us prove it for $\widehat{k_j}=\widehat{\chi_j}*\sigma$ with $j\in\mathbb{N}^*$. As $\widehat{\chi_j}$ is a rescaled Schwartz function, we have

$$|\widehat{\chi_j}(\xi)|\lesssim 2^{nj}(1+2^j\|\xi\|)^{-n}\qquad\forall\,j\in\mathbb{N}^*,\ \forall\,\xi\in\mathbb{R}^n.$$

Hence,

$$\begin{aligned}|\widehat{k_j}(\xi)|&\lesssim\int_{\mathbb{R}^n}d\sigma(\eta)\,2^{nj}(1+2^j\|\xi-\eta\|)^{-n}\\ &\le\int_{B(\xi,2^{-j})}d\sigma(\eta)\,2^{nj}+\sum\nolimits_{k\ge -j}\int_{B(\xi,2^{k+1})\smallsetminus B(\xi,2^k)}d\sigma(\eta)\,2^{-nk}\\ &\le 2^{nj}\sigma(B(\xi,2^{-j}))+\sum\nolimits_{k\ge -j}2^{-nk}\,\sigma(B(\xi,2^k))\\ &\lesssim 2^j+\sum\nolimits_{k\ge -j}2^{-k}\lesssim 2^j.\end{aligned}$$

Here we have used the uniform estimate

$$\sigma(B(\xi,r))\lesssim r^{n-1}\qquad\forall\,\xi\in\mathbb{R}^n,\ \forall\,r>0\,,$$

which matters actually for r small and for $\|\xi\|$ close to 1. Thus

$$\|\mathcal{T}_j\|_{L^2\to L^2}\lesssim 2^j\qquad\forall\,j\in\mathbb{N}. \tag{2}$$

By interpolation between (1) and (2), we obtain

$$\|\mathcal{T}_j\|_{L^{p'}\to L^p}\lesssim 2^{-(\frac{n-1}{2}-\frac{n+1}{p})j}\qquad\forall\,j\in\mathbb{N}. \tag{3}$$

By adding up (3) over $j\in\mathbb{N}$, we get the $L^{p'}\to L^p$ boundedness of $\mathcal{T}$ when $p>2\frac{n+1}{n-1}$. $\square$

The **third proof** relies on Stein's interpolation for an analytic family of operators. Let us embed σ into the holomorphic family of tempered distributions

$$\sigma_z=\tfrac{2^{1-z}}{\Gamma(z)}\,(1-\|\xi\|^2)_+^{z-1}.$$

Specifically, σ_z is well–defined in the half-space $\operatorname{Re}z>0$, it extends analytically to $\mathbb{C}$ by means of the functional relation

$$\sigma_z=\Delta\,\sigma_{z+2}+(2z+n)\,\sigma_{z+1}\,,$$

and $\sigma_0=\sigma$. Moreover,

$$\widehat{\sigma_z}(x) = (2\pi)^{\frac{n}{2}}\,\|x\|^{-z-\frac{n}{2}+1}\,J_{z+\frac{n}{2}-1}(\|x\|).$$

Consider the analytic family of operators $\Sigma_z f = f * \widehat{\sigma_z}$ in the strip $-\frac{n-1}{2} \le \operatorname{Re} z \le 1$. On the one hand, if $\operatorname{Re} z = -\frac{n-1}{2}$, then Σ_z is bounded from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^1\to L^\infty} = \|\widehat{\sigma_z}\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im} z$. On the other hand, if $\operatorname{Re} z = 1$, then Σ_z is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^2\to L^2} = (2\pi)^n\,\|\sigma_z\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im} z$. In conclusion, Stein's interpolation theorem yields the $L^{p'}\to L^p$ boundedness of $\Sigma_0 = \mathcal{T}$ for the endpoint $p = 2\frac{n+1}{n-1}$. □

Remark 1.2 (Knapp)**.** *Theorem 1.1 doesn't hold for* $p' > 2\frac{n+1}{n+3}$ *i.e.* $p < 2\frac{n+1}{n-1}$. *For small* $\varepsilon > 0$, *consider indeed the function*

$$f_\varepsilon(x) = (4\pi)^n\,\frac{\sin\sqrt{\varepsilon}\,x_1}{x_1}\,\dots\,\frac{\sin\sqrt{\varepsilon}\,x_{n-1}}{x_{n-1}}\,\frac{\sin\frac{\varepsilon}{2}x_n}{x_n}\,e^{i\,x_n}\,,$$

whose Fourier transform is the characteristic function of the set

$$[-\sqrt{\varepsilon},+\sqrt{\varepsilon}\,]^{n-1}\times[1-\tfrac{\varepsilon}{2},1+\tfrac{\varepsilon}{2}]\,.$$

Then $\|f_\varepsilon\|_{L^{p'}} \asymp \varepsilon^{\frac{n+1}{2p}}$ *while* $\|Tf_\varepsilon\|_{L^2} \asymp \varepsilon^{\frac{n-1}{4}}$. *By letting* $\varepsilon\to 0$, *we see that* $p \ge 2\frac{n+1}{n-1}$ *is a necessary condition for Theorem 1.1 to hold.*

Theorem 1.3 (Strichartz [12])**.** *Consider the paraboloid*

$$S = \{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n \mid \tau = -\|\xi\|^2\}$$

in $\mathbb{R}\times\mathbb{R}^n$ *and assume that* $p' = 2\frac{n+2}{n+4}$ *i.e.* $p = 2\frac{n+2}{n}$. *Then*

$$\big\|\,\widehat{u}\,\big|_S\big\|_{L^2(S)} \lesssim \|u\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^n)} \qquad \forall\, u\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^n).$$

Proof. Let us prove Theorem 1.3 by adapting the third proof above. We have now

$$Tu = \widehat{u}\,\big|_S\,,\quad T^*v(t,x) = \int_{\mathbb{R}^n} d\xi\,v(-\|\xi\|^2,\xi)\,e^{\,i(-t\|\xi\|^2+\langle x,\xi\rangle)}$$

and

$$\mathcal{T}u = T^*Tu = u*\widehat{\sigma}\,,$$

where

$$\int_{\mathbb{R}\times\mathbb{R}^n} d\sigma(\tau,\xi)\;v(\tau,\xi) \;=\; \int_{\mathbb{R}^n} d\xi\,v(-\|\xi\|^2,\xi)$$

and

$$\widehat{\sigma}(t,x) = \int_{\mathbb{R}\times\mathbb{R}^n} d\xi\,e^{\,i(t\|\xi\|^2-\langle x,\xi\rangle)} \;=\; \underbrace{\big(i\tfrac{\pi}{t}\big)^{\frac{n}{2}}}_{\pi^{\frac{n}{2}}\,e^{\,i\,\operatorname{sign}(t)\,n\frac{\pi}{4}}\,|t|^{-\frac{n}{2}}}\;e^{-i\frac{|x|^2}{4t}}$$

Consider the analytic family of tempered distributions

$$\sigma_z(\tau,\xi) = \tfrac{1}{\Gamma(z)}\,(\tau+\|\xi\|^2)_+^{z-1}$$

and the associated operators $\Sigma_z u = u*\widehat{\sigma_z}$. The $(n{+}1)$–dimensional Fourier transform

$$\widehat{\sigma_z}(t,x) = \pi^{\frac{n}{2}}\,e^{-i\frac{\pi}{2}z}\,e^{\,i\,\operatorname{sign}(t)\,n\frac{\pi}{4}}\,(t-i\,0)^{-z}\,|t|^{-\frac{n}{2}}\,e^{\,i\frac{|x|^2}{4t}}\,,$$

in the sense of distributions, is computed by combining the 1–dimensional Fourier transform

$$\tfrac{1}{\Gamma(z)}\int_{\mathbb{R}} d\tau\,\tau_+^{z-1}\,e^{-i\,t\,\tau} \;=\; e^{-i\frac{\pi}{2}z}\,(t-i\,0)^{-z}\,,$$

which yields

$$\tfrac{1}{\Gamma(z)}\int_{\mathbb{R}} d\tau\,(\tau+\|\xi\|^2)_+^{z-1}\,e^{-i\,t\,\tau} \;=\; e^{\,i\frac{\pi}{2}z}\,(t-i\,0)^{-z}\,e^{\,i\,t\,\|\xi\|^2}\,,$$

with the n–dimensional Fourier transform

$$\int_{\mathbb{R}^n} d\xi\;e^{\,i\,t\,\|\xi\|^2}\,e^{-i\langle x,\xi\rangle} \;=\; \pi^{\frac{n}{2}}\,e^{\,i\,\operatorname{sign}(t)\,n\frac{\pi}{4}}\,|t|^{-\frac{n}{2}}\,e^{\,i\frac{|x|^2}{4t}}\,.$$

On the one hand, if $\operatorname{Re} z = -\frac{n}{2}$, then Σ_z is bounded from $L^1(\mathbb{R}\times\mathbb{R}^n)$ to $L^\infty(\mathbb{R}\times\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^1\to L^\infty} = \|\widehat{\sigma_z}\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im} z$. On the other hand, if $\operatorname{Re} z = 1$, then Σ_z is bounded from $L^2(\mathbb{R}\times\mathbb{R}^n)$ to $L^2(\mathbb{R}\times\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^2\to L^2} = (2\pi)^{n+1}\,\|\sigma_z\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im} z$. In conclusion, Stein's interpolation theorem yields the $L^{p'}\to L^p$ boundedness of $\Sigma_0 = \mathcal{T}$ for $p' = 2\frac{n+2}{n+4}$ i.e. $p = 2\frac{n+2}{n}$. □

Remark 1.4. *Theorem 1.3 doesn't hold for* $p' \neq 2\frac{n+2}{n+4}$ *i.e.* $p \neq 2\frac{n+2}{n}$. *This is easily proved by rescaling. Specifically, let*

$$(\delta_s u)(t,x) = u(s^2 t, sx)\,, \quad \forall\, s>0,\, \forall\,(t,x)\in\mathbb{R}\times\mathbb{R}^n.$$

Then $\|\delta_s u\|_{L^{p'}} = s^{-\frac{n+2}{p'}}\|u\|_{L^{p'}}$ *while* $\|T(\delta_s u)\|_{L^2} = s^{-\frac{n}{2}-2}\|Tu\|_{L^2}$, *as*

$$\widehat{\delta_s u}(\tau,\xi) = s^{-n-2}\,\widehat{u}(s^{-2}\tau, s^{-1}\xi)\,.$$

By letting $s\to 0$ *or* $s\to+\infty$, *we see that* $p = 2\frac{n+2}{n+4}$ *is a necessary condition for Theorem 1.3 to hold.*

Let us next apply Theorem 1.3 to the Schrödinger equation

$$\text{(4)}\qquad \begin{cases} i\,\partial_t u(t,x) = -\,\Delta_x u(t,x), \\ u(t,x) = f(x), \end{cases}$$

Via the Fourier transform, (4) becomes

$$\begin{cases} i\,\partial_t \widehat{u}(t,\xi) = \|\xi\|^2\,\widehat{u}(t,\xi), \\ \widehat{u}(t,\xi) = \widehat{f}(\xi), \end{cases}$$

whose solution is given by

$$\widehat{u}(t,\xi) = e^{-it\|\xi\|^2}\,\widehat{f}(\xi)\,.$$

By applying the inverse Fourier transform, we get

$$u(t,x) = (2\pi)^{-n}\int_{\mathbb{R}^n} d\xi\,\widehat{f}(\xi)\,e^{\,i(-t\|\xi\|^2+\langle x,\xi\rangle)}\,,$$

which boils down to the expression of the operator T^*.

Corollary 1.5. *Let* $p = 2\frac{n+2}{n+4}$. *Then the following a priori inequality holds for solutions to* (4):

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\,.$$

Proof. This result is a restatement of the $L^2\to L^p$ boundedness of the operator T^*. Specifically,

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|\widehat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}}\,\|f\|_{L^2(\mathbb{R}^n)}\,.$$ □

Remark 1.6. *In his seminal work* [12], *Strichartz studies restriction theorems for general quadratic hypersurfaces and applies some of them to related PDE. Another important example is the cone*

$$S = \{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n \mid \tau^2 = \|\xi\|^2\}$$

and the wave equation

$$\text{(5)}\qquad \begin{cases} \partial_t^2 u(t,x) = \Delta_x u(t,x), \\ u(t,x) = f(x),\ \partial_t|_{t=0}\,u(t,x) = g(x). \end{cases}$$

In this case, $p' = 2\frac{n+1}{n+3}$ *i.e.* $p = 2\frac{n+1}{n-1}$, *the restriction theorem reads*

$$\big\|\,\widehat{u}\,\big|_S\big\|_{L^2(S)} \lesssim \|u\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^n)} \qquad \forall\, u\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^n),$$

and the following a priori inequality holds for solutions to (5):

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{\dot H^{1/2}(\mathbb{R}^n)} + \|g\|_{\dot H^{-1/2}(\mathbb{R}^n)}\,,$$

where the initial data belong to homogeneous Sobolev spaces $\dot{H}^{\pm\frac{1}{2}}(\mathbb{R}^n) = (-\Delta)^{\mp\frac{1}{2}} L^2(\mathbb{R}^n)$.

2. Schrödinger equation on $\mathbb{R}^n$ (linear case)

In this section we consider the linear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x) \\ u(t,x) = f(x) \end{cases} \tag{6}$$

on $\mathbb{R}\times\mathbb{R}^n$ and we prove two fundamental inequalities in Theorem 2.1 and Theorem 2.3, the latter improving upon Corollary 1.5. Our main references are [6] and [8].
Consider first the homogeneous Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = 0, \\ u(t,x) = f(x), \end{cases} \tag{4}$$

whose solution is given by

$$u(t,x) = e^{it\Delta} f(x) = f * s_t(x)\,, \tag{7}$$

where

$$s_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi\; e^{-it\|\xi\|^2}\, e^{i\langle x,\xi\rangle} = \overbrace{(4\pi i t)^{-\frac{n}{2}}}^{e^{-i\,\mathrm{sign}(t)\,n\,\frac{\pi}{4}}\,(4\pi|t|)^{-\frac{n}{2}}}\, e^{i\frac{|x|^2}{4t}}\,, \tag{8}$$

is the heat kernel with imaginary time.

Theorem 2.1. *Let* $1 \le q' \le 2$ *i.e.* $2 \le q \le \infty$. *Then the following dispersive estimate holds, for every* $t \in \mathbb{R}^*$:

$$\|\, e^{it\Delta} \|_{L^{q'} \to L^q} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{q})}$$

Proof. This result is obtained by interpolation between the elementary estimate

$$\|\, e^{it\Delta} \|_{L^1 \to L^\infty} = \|\, s_t \|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}$$

and the L^2 conservation

$$\|\, e^{it\Delta} f \|_{L^2} = \|\, f \|_{L^2}\,. \qquad \square$$

Let us turn to the inhomogeneous equation (6), whose solution is given by Duhamel's formula :

$$u(t,x) = \overbrace{e^{it\Delta_x} f(x)}^{\text{homogeneous}} - \overbrace{i \int_0^t ds\; e^{i(t-s)\Delta_x} F(s,x)}^{\text{inhomogeneous}}\,. \tag{9}$$

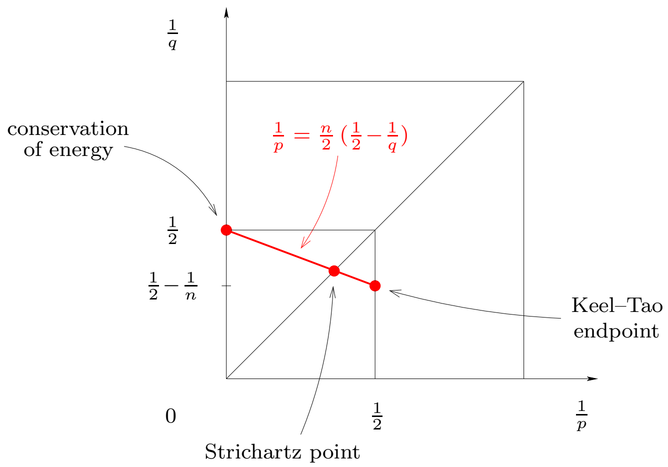

Figure 1. Admissibility for $\mathbb{R}^n$

Definition 2.2. *A couple* (p,q) *of indices is called admissible if*

$$2\le p\le\infty\,,\;2\le q<\infty\quad\textit{and}\quad \tfrac{1}{p}=\tfrac{n}{2}\left(\tfrac{1}{2}-\tfrac{1}{q}\right).$$

Theorem 2.3. *Let* (p,q) *and* $(\tilde p,\tilde q)$ *be two admissible couples of indices. Then the following Strichartz estimate holds for solutions* (9) *to* (6):

$$\|u\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\lesssim\|f\|_{L^2(\mathbb{R}^n)}+\|F\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}$$

We shall prove Theorem 2.3 in several steps.

Step 1: TT^* method revisited.
Consider the operator

$$Tf(t,x)=e^{it\Delta_x}f(x)\,,$$

its formal adjoint

$$T^*F(x)=\int_{-\infty}^{+\infty}ds\;e^{-is\Delta_x}F(s,x)\,,$$

their composition

$$\mathcal{T}F(t,x)=TT^*F(t,x)=\int_{-\infty}^{+\infty}ds\;e^{i(t-s)\Delta_x}F(s,x)\,,$$

and its truncated version

$$\widetilde{\mathcal{T}}F(t,x)=\int_{-\infty}^{t}ds\;e^{i(t-s)\Delta_x}F(s,x)\,. \tag{10}$$

Then the following a priori estimates are equivalent:

$$\|Tf\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\lesssim\|f\|_{L^2(\mathbb{R}^n)}\,, \tag{11}$$

$$\|T^*F\|_{L^2(\mathbb{R}^n)}\lesssim\|F\|_{L^{p'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}\,, \tag{12}$$

$$\|\mathcal{T}F\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\lesssim\|F\|_{L^{p'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}\,. \tag{13}$$

Step 2: Proof of (11), (12), (13) when (p,q) is admissible and $p>2$.
On the one hand, according to Theorem 2.1,

$$\|\,\mathcal{T}F(t,x)\|_{L^q_x}\lesssim\int_{-\infty}^{+\infty}ds\;|t-s|^{-\alpha}\;\|\,F(s,x)\|_{L^{q'}_x}$$

where $\alpha=n\left(\frac{1}{2}-\frac{1}{q}\right)$. On the other hand, according to the Hardy–Littlewood–Sobolev inequality, the convolution kernel $|t-s|^{-\alpha}$ defines a bounded operator from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$, provided that $0\le\alpha<1$ and $\frac{1}{p}=\frac{\alpha}{2}$. These two results yield (13), hence (11) and (12), under the assumptions above.

Step 3: Decoupling indices.
Let (p,q) and $(\tilde p,\tilde q)$ be admissible couples with $p>2$ and $\tilde p>2$. By combining the $L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))\to L^2(\mathbb{R}^n)$ boundedness of T^* with the $L^2(\mathbb{R}^n)\to L^p(\mathbb{R},L^q(\mathbb{R}^n))$ boundedness of T, we obtain the $L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))\to L^p(\mathbb{R},L^q(\mathbb{R}^n))$ boundedness of $\mathcal{T}$. Moreover, as $\tilde p'<p$, the same result holds true for the truncated operator $\widetilde{\mathcal{T}}$, according to the Christ–Kiselev lemma [4].

Step 4: Endpoint estimates when $n\ge3$ and $(p,q)=(\tilde p,\tilde q)=(2,2\frac{n}{n-2})$.
The following arguments are due to Keel and Tao [8]. Instead of $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, consider the bilinear (actually Hermitian) form

$$\mathcal{B}(F,G)=\iint_{\mathbb{R}^2}ds\,dt\int_{\mathbb{R}^n}dx\;e^{-is\Delta_x}F(s,x)\,\overline{e^{-it\Delta_x}G(t,x)}$$

and its truncated version

$$\widetilde{\mathcal{B}}(F,G)=\iint_{s<t}ds\,dt\int_{\mathbb{R}^n}dx\;e^{-is\Delta_x}F(s,x)\,\overline{e^{-it\Delta_x}G(t,x)}\,. \tag{14}$$

In order to estimate (14), let us split up dyadically $\widetilde{\mathcal{B}}=\sum_{j\in\mathbb{Z}}\widetilde{\mathcal{B}}_j$, where

$$\widetilde{\mathcal{B}}_j(F,G)=\iint_{2^j\le t-s<2^{j+1}} ds\,dt\int_{\mathbb{R}^n}dx\;e^{-is\Delta_x}F(s,x)\;\overline{e^{-it\Delta_x}G(t,x)}\,,$$

and let us further split up

$$F(s,x)=\sum_{k=-\infty}^{+\infty}\underbrace{\mathbb{1}_{[k2^j,(k+1)2^j)}(s)\,F(s,x)}_{F_k^{(j)}(s,x)}\quad\text{and}\quad G(t,x)=\sum_{\ell=-\infty}^{+\infty}\underbrace{\mathbb{1}_{[\ell 2^j,(\ell+1)2^j)}(t)\,G(t,x)}_{G_\ell^{(j)}(t,x)}.$$

Notice the orthogonality

$$(15)\quad \|F\|_{L^{\tilde p'}L^{\tilde q'}}=\Big\{\sum\nolimits_{k=-\infty}^{+\infty}\|F_k^{(j)}\|_{L^{\tilde p'}L^{\tilde q'}}^{\tilde p'}\Big\}^{1/\tilde p'},\quad \|G\|_{L^2L^{q'}}=\Big\{\sum\nolimits_{\ell=-\infty}^{+\infty}\|G_\ell^{(j)}\|_{L^{p'}L^{q'}}^{p'}\Big\}^{1/p'}$$

and the almost orthogonality

$$(16)\qquad \widetilde{\mathcal{B}}_j(F,G)=\sum\nolimits_{\substack{k,\ell\in\mathbb{Z}\\ 1\le\ell-k\le2}}\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})\,.$$

We claim that, for all indices corresponding to the region pictured in Figure 2, we have

$$(17)\qquad |\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})|\lesssim 2^{\kappa(\tilde q,q)j}\,\|F_k^{(j)}\|_{L^2L^{\tilde q'}}\|G_\ell^{(j)}\|_{L^2L^{q'}}\quad\forall\,j,k,\ell\in\mathbb{Z},$$

where $\kappa(\tilde q,q)=\frac n2\big(\frac1{\tilde q}+\frac1q\big)-\frac{n-2}2$.

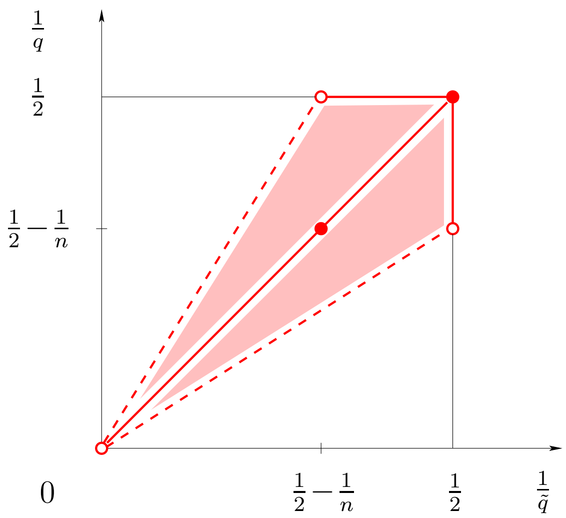

FIGURE 2. Interpolation region

This estimate is obtained by interpolation between the following three cases:

$$\begin{cases}\text{(a) } 2\le\tilde q<2\frac{n}{n-2}\ \text{and}\ q=2,\\ \text{(b) } \tilde q=2\ \text{and}\ 2\le q<2\frac{n}{n-2},\\ \text{(c) } 2<\tilde q=q<\infty.\end{cases}$$

∘ *Case* (a): Let us estimate

$$|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})|\lesssim\sup_{t\in\mathbb{R}}\Big\|\int_{t-2^{j+1}}^{t-2^j}ds\;e^{-is\Delta_x}F_k^{(j)}(s,x)\Big\|_{L_x^2}\int_{-\infty}^{+\infty}dt\,\big\|e^{-it\Delta_x}G_\ell^{(j)}(t,x)\big\|_{L_x^2}.$$

On the one hand, as T^* is bounded from $L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))$ into $L^2(\mathbb{R}^n)$, with $\frac1{\tilde p}=\frac n2\big(\frac12-\frac1{\tilde q}\big)$,

$$\Big\|\int_{\mathbb{R}}ds\;\mathbb{1}_{(t-2^{j+1},t-2^j]}(s)\;e^{-is\Delta_x}F_k^{(j)}(s,x)\Big\|_{L_x^2}\lesssim\big\|\mathbb{1}_{(t-2^{j+1},t-2^j]}(s)\,F_k^{(j)}(s,x)\big\|_{L_s^{\tilde p'}L_x^{\tilde q'}}$$

On the other hand,

$$\big\|e^{-it\Delta_x}G_\ell^{(j)}(t,x)\big\|_{L_x^2}\lesssim\big\|G_\ell^{(j)}(t,x)\big\|_{L_x^2}\,.$$

By combining these estimates and by using Hölder's inequality in time, we conclude that

$$|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})|\lesssim 2^{\frac1{\tilde p}j}\,\big\|F_k^{(j)}\big\|_{L^2L^{\tilde q'}}\big\|G_\ell^{(j)}\big\|_{L^2L^2}$$

with $\frac{1}{\tilde{p}'} = \kappa(\tilde{q},2)$.

○ *Case* (b) is handled similarly.

○ *Case* (c): Let us estimate this time

$$|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})| \lesssim \iint_{2^j \le t-s < 2^{j+1}} ds\,dt\, \big\| e^{\,i(t-s)\Delta_x} F_k^{(j)}(s,x)\big\|_{L_x^q} \big\| G_\ell^{(j)}(t,x)\big\|_{L_x^{q'}}.$$

According to the dispersive estimate in Theorem 2.1,

$$\big\| e^{\,i(t-s)\Delta_x} F_k^{(j)}(s,x)\big\|_{L_x^q} \lesssim |t-s|^{-n(\frac12-\frac1q)} \big\| F_k^{(j)}(s,x)\big\|_{L_x^{q'}}.$$

By using again Hölder's inequality in time, we conclude that

$$\begin{aligned}|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})| &\lesssim 2^{\,n(\frac12-\frac1q)j} \big\|F_k^{(j)}\big\|_{L^1L^{q'}} \big\|G_\ell^{(j)}\big\|_{L^1L^{q'}}\\ &\lesssim 2^{\,\kappa(q,q)\,j} \big\|F_k^{(j)}\big\|_{L^2L^{q'}} \big\|G_\ell^{(j)}\big\|_{L^2L^{q'}}.\end{aligned}$$

By adding up (17) over $k,\ell\in\mathbb{Z}$ and by using the orthogonality properties (15) and (16), we deduce that

$$\sup\nolimits_{j\in\mathbb{Z}} \overbrace{2^{-\kappa(\tilde{q},q)\,j}}^{w_{\tilde{q},q}(j)} |\widetilde{\mathcal{B}}_j(F,G)| \lesssim \|F\|_{L^2L^{\tilde{q}'}} \|G\|_{L^2L^{q'}}.$$

In other words, $\{\widetilde{\mathcal{B}}_j\}_{j\in\mathbb{Z}}$ defines a bounded bilinear (actually Hermitian) map from the product $L^2(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^n))\times L^2(\mathbb{R},L^{q'}(\mathbb{R}^n))$ into the weighted space $\ell^\infty(\mathbb{Z},w_{\tilde{q},q})$. This result can be improved by real interpolation. Specifically, by moving $(\tilde{q},q)$ and varying $w_{\tilde{q},q}$, one gets a bounded map into $\ell^1(\mathbb{Z},w_{\tilde{q},q})$ instead of $\ell^\infty(\mathbb{Z},w_{\tilde{q},q})$. When $\tilde{q}=q=2\frac{n}{n-2}$, we have in particular $\kappa(\tilde{q},q)=0$ i.e. $w_{\tilde{q},q}=1$. Hence

$$|\,\widetilde{\mathcal{B}}(F,G)| \le \sum\nolimits_{j\in\mathbb{Z}} |\,\widetilde{\mathcal{B}}_j(F,G)| \lesssim \|F\|_{L^2L^{\tilde{q}'}} \|G\|_{L^2L^{q'}}.$$

We conclude with two elementary observations. On the one hand, the truncated form

$$\mathcal{B}(F,G)-\widetilde{\mathcal{B}}(F,G) = \iint_{s>t} ds\,dt \int_{\mathbb{R}^n} dx\; e^{-is\Delta_x}F(s,x)\,\overline{e^{-it\Delta_x}G(t,x)}\,.$$

and hence $\mathcal{B}(F,G)$ are estimated in the same way. On the other hand, the truncated operator defined by

$$\int_0^t ds\; e^{\,i(t-s)\Delta_x}F(s,x)$$

is deduced from (10) by multiplying $F(s,x)$ by $1\!\!1_{\mathbb{R}_+}(s)$.

Step 5: Endpoint estimates when $n\ge3$ and either $(p,q)=(\tilde{p},\tilde{q})=(2,2\frac{n}{n-2})$ or $(\tilde{p},\tilde{q})=(2,2\frac{n}{n-2})$.

This case is simpler and relies on the same arguments. □

3. Schrödinger equation on $\mathbb{R}^n$ (nonlinear case)

In this section, we consider the semilinear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x)+\Delta_x u(t,x)=F(u(t,x)),\\ u(t,x)=f(x),\end{cases} \tag{18}$$

with power–like nonlinearities F. By this we mean that there exist constants $C>0$ and $\gamma>1$ such that

$$\begin{cases} |F(u)|\le C\,|u|^\gamma\,,\\ |F(u)-F(\tilde{u})|\le C\,\{|u|^{\gamma-1}+|\tilde{u}|^{\gamma-1}\}\,|u-\tilde{u}|\,.\end{cases} \tag{19}$$

Typical examples are

$$F(u)=\text{const.}\times\begin{cases}|u|^\gamma,\\ u\,|u|^{\gamma-1}.\end{cases}$$

Following the strategy developed by Kato, Ginibre–Velo [6] and Keel–Tao [8], we apply the Strichartz estimates in Theorem 2.3 to the well–posedness of (18), which means roughly existence and uniqueness of solutions in some suitable function space.

Theorem 3.1. (a) Critical case: *Assume that* $\gamma = 1+\frac{4}{n}$.
Then (18) *is globally well–posed in* $L^2(\mathbb{R}^n)$ *for small initial data* f.
(b) Subcritical case: *Assume that* $1<\gamma<1+\frac{4}{n}$.
Then (18) *is locally well–posed in* $L^2(\mathbb{R}^n)$ *for arbitrary initial data* f.
(c) Subcritical case: *Assume again that* $1<\gamma<1+\frac{4}{n}$
and assume that $F(u)$ *is a real multiple of* $u\,|u|^{\gamma-1}$.
Then (18) *is globally well–posed in* $L^2(\mathbb{R}^n)$ *for arbitrary initial data* f.

Proof of (a). Define $u=\Phi(v)$ as the solution to the Cauchy problem

$$\begin{cases} i\,\partial_t u(t,x)+\Delta_x u(t,x)=F(v(t,x))\,, \\ u(t,x)=f(x), \end{cases} \tag{20}$$

which is given by Duhamel's formula (9):

$$u(t,x)=e^{\,it\Delta_x}f(x)-i\int_0^t ds\; e^{\,i(t-s)\Delta_x}F(s,x).$$

According to Theorem 2.3, the following Strichartz estimate holds

$$\|u\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\le C\,\|f\|_{L^2(\mathbb{R}^n)}+C\,\|F(v)\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))} \tag{21}$$

for all couples (p,q) and $(\tilde p,\tilde q)$ satisfying the admissibility conditions

$$\begin{cases} 2\le p\le\infty\,,\; 2\le q<\infty\,,\; \frac{1}{p}=\frac{n}{2}\,(\frac12-\frac1q)\,, \\ 2\le \tilde p\le\infty\,,\; 2\le \tilde q<\infty\,,\; \frac{1}{\tilde p}=\frac{n}{2}\,(\frac12-\frac1{\tilde q})\,. \end{cases} \tag{22}$$

Moreover

$$\|F(v)\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\le C\,\|\,|v|^\gamma\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\le C\,\|v\|^{\gamma}_{L^{\gamma\tilde p'}(\mathbb{R},L^{\gamma\tilde q'}(\mathbb{R}^n))}\,.$$

by our nonlinear assumption (19). Thus

$$\|u\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\le C\,\|f\|_{L^2(\mathbb{R}^n)}+C\,\|v\|^{\gamma}_{L^{\gamma\tilde p'}(\mathbb{R},L^{\gamma\tilde q'}(\mathbb{R}^n))}\,. \tag{23}$$

In order to remain within the same function space, we require that

$$\begin{cases} p=\gamma\,\tilde p' \iff \frac{\gamma}{p}+\frac{1}{\tilde p}=1\,, \\ q=\gamma\,\tilde q' \iff \frac{\gamma}{q}+\frac{1}{\tilde q}=1\,. \end{cases} \tag{24}$$

All conditions (22) and (24) can be fulfilled provided that $\gamma=1+\frac{4}{n}$. In this case, one may consider for instance the Strichartz point, which is given by

$$p=q=\tilde p=\tilde q=\gamma+1=2+\tfrac{4}{n}\,.$$

For such a choice, Φ maps $L^\infty(\mathbb{R},L^2(\mathbb{R}^n))\cap L^p(\mathbb{R},L^q(\mathbb{R}^n))$ into itself, and actually $X=C(\mathbb{R},L^2(\mathbb{R}^n))\cap L^p(\mathbb{R},L^q(\mathbb{R}^n))$ into itself. As X is a Banach space for the norm

$$\|u\|_X=\|u\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\,,$$

it remains for us to show that Φ is a contraction in the ball

$$X_\varepsilon=\{\,u\in X\mid \|u\|_X\le\varepsilon\,\}\,,$$

provided that $\varepsilon>0$ and $\|f\|_{L^2}$ are sufficiently small. Let $v,\tilde v\in X$ and $u=\Phi(v)$, $\tilde u=\Phi(\tilde v)$. Arguying as above and using in addition Hölder's inequality, we estimate

$$\begin{aligned}\|u-\tilde u\|_X&\le C\,\|F(v)-F(\tilde v)\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\\ &\le C\,\|\{|v|^{\gamma-1}+|\tilde v|^{\gamma-1}\}\,|v-\tilde v|\,\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\\ &\le C\,\{\|v\|^{\gamma-1}_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}+\|\tilde v|^{\gamma-1}_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\}\,\|v-\tilde v\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\,,\end{aligned}$$

hence

$$\|u-\tilde u\|_X \le C\,\big\{\|v\|_X^{\gamma-1}+\|\tilde v\|_X^{\gamma-1}\big\}\,\|v-\tilde v\|_X\,. \tag{25}$$

If we assume $\|v\|_X\le\varepsilon$, $\|\tilde v\|_X\le\varepsilon$ and $\|f\|_{L^2}\le\delta$, then (23) and (25) yield

$$\|u\|_X \le C\,\delta+C\,\varepsilon^\gamma\,,\quad \|\tilde u\|_X \le C\,\delta+C\,\varepsilon^\gamma \quad\text{and}\quad \|u-\tilde u\|_X\le 2\,C\,\varepsilon^{\gamma-1}\,\|v-\tilde v\|_X\,.$$

Thus

$$\|u\|_X\le\varepsilon\,,\quad \|\tilde u\|_X\le\varepsilon \quad\text{and}\quad \|u-\tilde u\|_X\le\tfrac12\,\|v-\tilde v\|_X\,,$$

if $C\,\varepsilon^{\gamma-1}\le\frac14$ and $C\,\delta\le\frac34\,\varepsilon$. We conclude by applying the fixed point theorem in the complete metric space X_ε. □

Proof of (b). In the subcritical case $\gamma<1+\frac4n$, the same arguments yield the local well–posedness of (18) in $L^2(\mathbb{R}^n)$ for arbitrary initial data f. Specifically, we restrict to a small time interval $I=[-T,+T]$ and we proceed as above, except that we apply in addition Hölder's inequality in time. This way, we get the Strichartz estimate

$$\|u\|_X\le C\,\|f\|_{L^2}+C\,T^\lambda\,\|v\|_X^\gamma\,, \tag{26}$$

where $X=C(I,L^2(\mathbb{R}^n))\cap L^p(I,L^q(\mathbb{R}^n))$ and $\lambda=1-\frac{\gamma}{p}-\frac{1}{\tilde p}>0$, and the related estimate

$$\|u-\tilde u\|_X\le C\,T^\lambda\,\big\{\|v\|_X^{\gamma-1}+\|v\|_X^{\gamma-1}\big\}\,\|v-\tilde v\|_X\,. \tag{27}$$

As a consequence, we deduce that Φ is a contraction in the ball

$$X_M=\{\,u\in X\mid \|u\|_X\le M\,\}\,,$$

provided $M>0$ is large enough and $T>0$ small enough, more precisely $\frac34\,M\ge C\,\|f\|_{L^2}$ and $C\,T^\lambda M^{\gamma-1}\le\frac14$. We conclude as before. Notice that the size of T depends only on the L^2 norm of the initial data f.

Proof of (c). Assume moreover that $F(u)=c\,u\,|u|^{\gamma-1}$ with $c\in\mathbb{R}$. Then the expression

$$i\int_{\mathbb{R}^n}dx\;\partial_t u(t,x)\,\overline{u(t,x)}=-\int_{\mathbb{R}^n}dx\;\Delta_x u(t,x)\,\overline{u(t,x)}+c\int_{\mathbb{R}^n}dx\;|u(t,x)|^{\gamma+1}$$

is real, hence

$$\partial_t\int_{\mathbb{R}^n}dx\,|u(t,x)|^2=2\,\mathrm{Re}\int_{\mathbb{R}^n}dx\;\partial_t u(t,x)\overline{u(t,x)}$$

vanishes and we have L^2 conservation:

$$\int_{\mathbb{R}^n}dx\,|u(t,x)|^2=\int_{\mathbb{R}^n}dx\,|f(x)|^2\,. \tag{28}$$

As the time interval in part (b) depends only on (28), we may iterate and deduce global existence from local existence, for arbitrary initial data $f\in L^2$. We refer to [3] for more information about conservation properties of the Schödinger equation. □

Remark 3.2. *Similar results hold if* $1+\frac4n$ *is replaced by* $1+\frac{4}{n-2\sigma}$ *and* $L^2(\mathbb{R}^n)$ *by* $H^\sigma(\mathbb{R}^n)$, *with* $0<\sigma<\frac n2$.

Remark 3.3. *The semilinear wave equation*

$$\begin{cases}\partial_t^2 u(t,x)-\Delta_x u(t,x)=F(u(t,x)),\\ u(t,x)=f(x),\ \partial_t|_{t=0}\,u(t,x)=g(x),\end{cases} \tag{29}$$

with power–like nonlinearities F, can be studied in a similar way but its analysis is more involved. We refer to [5, 6, 8, 9, 13] for more information.

4. Schrödinger equation on hyperbolic spaces $\mathbb{H}^n$

This section consists in an introduction to my joint work [1] with Vittoria Pierfelice, where we have investigated the semilinear Schrödinger equation (18) on real hyperbolic spaces $\mathbb{H}^n$. Among other references, let us mention on the one hand the earlier works [2, 10] and on the other hand [7], which is mainly devoted to scattering theory in $H^1(\mathbb{H}^n)$. As might be expected, dispersion properties are better in negative curvature. Consequently, Strichartz estimates hold for a wider range and one obtains stronger well–posedness results.

Fourier analysis on $\mathbb{H}^n$ yields the following explicit expression for the Schrödinger kernel (i.e. the heat kernel with imaginary time).

Lemma 4.1. *For every $t\in\mathbb{R}^*$ and $r\geq 0$, we have*

$$s_t(r) = \text{const.}\,(it)^{-\frac{1}{2}}\,e^{-i(\frac{n-1}{2})^2 t}\,\Big(-\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\Big)^{\frac{n-1}{2}} e^{\frac{i}{4}\frac{r^2}{t}}\,.$$

Here $(it)^{-\frac{1}{2}} = e^{-i\,\mathrm{sign}(t)\frac{\pi}{4}}\,|t|^{-\frac{1}{2}}$ and, in the even dimensional case, the fractional derivative reads

$$\Big(-\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\Big)^{\frac{n-1}{2}} e^{\frac{i}{4}\frac{r^2}{t}} = \tfrac{1}{\sqrt{\pi}}\int_{|r|}^{+\infty}\tfrac{\sinh s\,ds}{\sqrt{\cosh s-\cosh r}}\,\Big(-\tfrac{1}{\sinh s}\tfrac{\partial}{\partial s}\Big)^{\frac{n}{2}} e^{\frac{i}{4}\frac{s^2}{t}}\,.$$

We deduce first the following sharp kernel estimates.

Corollary 4.2. *For every $t\in\mathbb{R}^*$ and $r\geq 0$, we have*

$$|s_t(r)| \lesssim \begin{cases} |t|^{-\frac{n}{2}}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{n-1}{2}r} & \text{if } |t|\leq 1+r,\\ |t|^{-\frac{3}{2}}\,(1+r)\,e^{-\frac{n-1}{2}r} & \text{if } |t|\geq 1+r.\end{cases}$$

Remark 4.3. *Notice that*

$$(1+r)^{\frac{n-1}{2}}\,e^{-\frac{n-1}{2}r} \asymp j(r)^{-\frac{1}{2}},$$

where j is the jacobian of the exponential map, and

$$(1+r)\,e^{-\frac{n-1}{2}r} \asymp \varphi_0(r)$$

where φ_0 is the radial ground state.

We deduce next the following dispersive estimates.

Theorem 4.4. *For every $2<q\leq\infty$ and $t\in\mathbb{R}^*$, we have*

$$\|e^{it\Delta}\|_{L^{q'}(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} \lesssim \begin{cases} |t|^{-n(\frac{1}{2}-\frac{1}{q})} & \text{if } 0<|t|<1,\\ |t|^{-\frac{3}{2}} & \text{if } |t|\geq 1.\end{cases}$$

Hint of proof. The first estimate is obtained by interpolation. The second one is obtained by applying the following version of the Kunze–Stein phenomenon:

$$L^{q'}(\mathbb{H}^n) * L^{\tilde{q}}_{\text{rad}}(\mathbb{H}^n) \subset L^q(\mathbb{H}^n)$$

if $2<q,\tilde{q}<\infty$ satisfy $\frac{q}{2}<\tilde{q}<q$. □

Consider the linear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x)\\ u(t,x) = f(x)\end{cases} \tag{6}$$

on $\mathbb{R}\times\mathbb{H}^n$.

Theorem 4.5. *The Strichartz estimate*

$$\|u\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}$$

holds for (6) and for all admissible couples (p,q), $(\tilde{p},\tilde{q})$ corresponding to the triangle pictured in Figure 3.

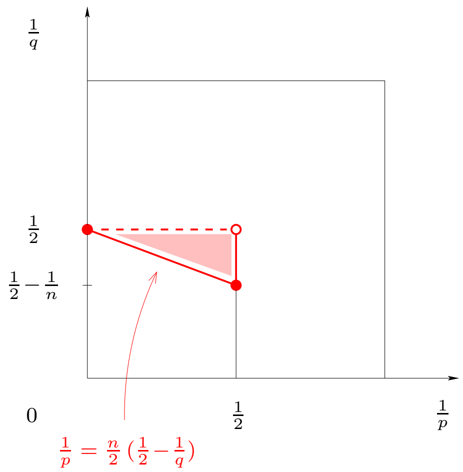

FIGURE 3. Admissibility for $\mathbb{H}^n$

Consider eventually the semilinear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x)+\Delta_x u(t,x)=F(u(t,x)), \\ u(t,x)=f(x), \end{cases} \tag{18}$$

on $\mathbb{R}\times\mathbb{H}^n$, with power–like nonlinearities F as in (19).

Theorem 4.6. *Assume that* $1<\gamma\le 1+\frac{4}{n}$.
Then (18) *is globally well–posed in* $L^2(\mathbb{R}^n)$ *for small initial data* f.

Remark 4.7. *Parts* (b) *and* (c) *in Theorem 3.1 hold also on* $\mathbb{H}^n$.

Remark 4.8. *All these results extend straightforwardly to the so–called rank one case, which include all hyperbolic spaces, as well as Damek–Ricci spaces.*

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Université d'Orléans & CNRS, Fédération Denis Poisson (FR 2964) & Laboratoire MAPMO (UMR 6628), Bâtiment de mathématiques – Route de Chartres, B.P. 6759 – 45067 Orléans cedex 2 – France
E-mail address: anker@univ-orleans.fr