Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny
pl. Grunwaldzki 2/4
50-384 Wrocław
tel. +48 71 375 74 01, +48 71 328 07 17
fax +48 71 375 74 29
e-mail: dyrekcja@math.uni.wroc.pl
www.math.uni.wroc.pl

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Orthogonal decompositions for generalized stochastic processes with independent values

Eugene Lytvynov

(Swansea University, UK)

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Chapter 1

Preliminaries

1.1 What is a white noise?

Consider a Brownian motion $(B(t))_{t\geq 0}$. A (Gaussian) white noise is, informally, the time derivative of Brownian motion:

$$\omega(t) = \frac{d}{dt}\, B(t).$$

From here, for any $0 \leq s < t$, the increment of the Brownian motion on the time interval $[s, t]$ is given by

$$B(t) - B(s) = \int_s^t \omega(u)\, du = \int_{\mathbb{R}} \chi_{[s,t]}(u)\omega(u)\, du.$$

We also note that, since the derivative $\frac{d}{dt}B(t)$ only depends on increments of Brownian motion near the point t and since the increments of Brownian motion on disjoint intervals are independent, for any choice of $t_1 < t_2 < \cdots < t_n$, the random variables $\omega(t_1),\ldots, \omega(t_n)$ are independent.

However, as well known, although the trajectories of Brownian motion are continuous, they are nowhere differentiable with probability one. Hence, the only way to rigorously treat the derivative of a Brownian motion is to look at it as a generalized function. Thus, we naturally arrive at a notion of a probability measure μ on a space of generalized functions on $\mathbb{R}$, say $\mathcal{D}'$. A probability measure on a space of generalized functions is called a generalized stochastic process.

For any test function $\varphi \in \mathcal{D}$, the mapping

$$\mathcal{D}' \ni \omega \mapsto \langle \omega, \varphi\rangle \in \mathbb{R}$$

is then a random variable. Note that, informally, we may write

$$\langle \omega, \varphi \rangle = \int_{\mathbb{R}} \omega(t)\varphi(t)\, dt.$$

One says that a generalized stochastic process has independent values, if for any test functions $\varphi_1, \dots, \varphi_n \in \mathcal{D}(\mathbb{R})$ with disjoint support, the random variables

$$\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_n \rangle$$

are independent under μ.

It is intuitively clear that a generalized stochastic process on $\mathcal{D}'(\mathbb{R})$ with independent values may appear as a time derivative of a Brownian motion, a Poisson process, a Lévy process, or even more generally, of a stochastic process with independent increments. Furthermore, we will below discuss generalized stochastic processes with independent values over a multi-dimensional space $\mathbb{R}^n$, or even over a Riemannian manifolds X.

Our aim is the study of L^2-spaces $L^2(\mathcal{D}'(\mathbb{R}^n), \mu)$ where μ is a (rather general) generalized stochastic process.

In sections below, we will recall some known results on orthogonal polynomials and spaces of test and generalized functions. We will also discuss a version of Berezansky's projection spectral theorem.

1.2 Orthogonal polynomials

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \sigma)$ be a probability space. We assume that the probability measure σ has all moments finite, i.e.

$$\int_{\mathbb{R}} |x|^n \sigma(dx) < \infty, \quad \forall n \in \mathbb{N}.$$

Therefore the integrals $\int_{\mathbb{R}} x^n \sigma(dx)$ are well defined. The numbers

$$m_n = \int_{\mathbb{R}} x^n \sigma(dx), \quad n \in \mathbb{N},$$

are called the moments of σ. Additionally, we will assume that the support of the measure σ contains an infinite number of points.

If we take a sequence of monomials $(x^n)_{n=0}^\infty$, then according to the Gram–Schmidt procedure they may be orthogonalized. Thus, we get a system of monic orthogonal polynomials:

$$P_n(x) = x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + \cdots + \alpha_0. \tag{1.1}$$

('Monic' means that the leading coefficient, i.e., the coefficient by x^n, is 1.)

Theorem 1.1. *Assume that the support of σ is infinite. Then, there exist $a_n > 0$, $n = 1, 2, \dots,$ and $b_n \in \mathbb{R}$, $n = 0, 1, 2, \dots,$ such that*

$$\begin{array}{rcl} xP_n(x) &=& P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \quad n \geq 1, \\ xP_0(x) &=& P_1(x) + b_0. \end{array} \tag{1.2}$$

Furthermore, for any $a_n > 0$, $n = 1, 2, \dots,$ and $b_n \in \mathbb{R}$, $n = 0, 1, 2, \dots,$ there exists a probability measure σ with finite moments and infinite support such that the corresponding polynomials $(P_n(x))_{n=0}^\infty$ defined by (1.2) form a system of monic orthogonal polynomials for measure σ.

We note that, generally speaking, there may exist different probability measures which have the same moments. That is, the measure σ in the second part of the above theorem is, generally speaking, not unique. However, there exist sufficient conditions which guarantee that the measure σ is unique. The following theorem is an example of such a condition.

Theorem 1.2. *Assume that σ is a probability measure on $\mathbb{R}$ which has finite moments. Then the following three conditions are equivalent:*

(i) There exists $c > 0$ such that, for all $n \in \mathbb{N}$,

$$m_n \leq c^n n! \tag{1.3}$$

(ii) There exists $\epsilon > 0$, such that

$$\int_{\mathbb{R}} e^{\epsilon|x|}\sigma(dx) < \infty.$$

(iii) There exists $\epsilon > 0$ such that the Laplace transform of σ,

$$(-\epsilon, \epsilon) \ni y \mapsto \int_{\mathbb{R}} e^{yx}\sigma(dx) \in \mathbb{R}$$

is well defined and hence it can be extended to an analytic function

$$\{z \in \mathbb{C} \mid |z| < \epsilon\} \ni z \mapsto \int_{\mathbb{R}} e^{zx} \sigma(dx) \in \mathbb{C}.$$

If either (i), or (ii), or (iii) holds, then the measure σ is a unique probability measure on $\mathbb{R}$ which has moments m_n, so that there is a one-to-one correspondence between σ and the system of orthogonal polynomials.

Remark 1.3. Let σ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which satisfies either condition (i), or (ii), or (iii) of Theorem 1.2. Then, if we know the Fourier transform of σ, $\int_{\mathbb{R}} e^{itx}\,\sigma(dx)$, for t from a neighborhood of zero in $\mathbb{R}$, then we can evaluate the moments m_n of σ by differentiating the Fourier transform at zero, and so we can recover the measure σ. Hence the measure σ is uniquely identified by its Fourier transform in a neighborhood of zero.

1.3 Rigged Hilbert spaces

Let H_0 be a real Hilbert space with scalar product $(\cdot,\cdot)_{H_0}$ and norm $\|\cdot\|_{H_0}$. We suppose that

$$H_+ \subseteq H_0, \tag{1.4}$$

where H_+ is a dense subset of H_0. We also suppose that H_+ is a Hilbert space with respect to another scalar product $(\cdot,\cdot)_{H_+}$ and the norm $\|\cdot\|_{H_+}$ in H_+ is such that

$$\|\varphi\|_{H_0} \le \|\varphi\|_{H_+}, \quad \varphi \in H_+. \tag{1.5}$$

(The more general case when $\|\cdot\|_{H_0} \le c\|\cdot\|_{H_+}$ for some constant $c < \infty$ can be reduced to (1.5) by introducing an equivalent norm in H_+.) The elements of the set H_+ play the role of test functions.

Each element $f \in H_0$ generates a linear continuous functional $\langle f, \cdot\rangle$ on H_+ according to the formula

$$\langle f, \varphi\rangle := (f, \varphi)_{H_0}. \tag{1.6}$$

We introduce a new norm $\|\cdot\|_{H_-}$ in H_0 by taking the norm of f as the norm of the functional $\langle f, \cdot\rangle$ corresponding to it:

$$\|f\|_{H_-} := \|\langle f, \cdot\rangle\| = \sup\left\{\frac{|(f,\varphi)_{H_0}|}{\|\varphi\|_{H_+}} \;\middle|\; \varphi \in H_+,\ \varphi \ne 0\right\}. \tag{1.7}$$

We complete H_0 in the norm (1.7) and obtain a linear normed space H_-, which is called the space with negative norm and its elements play the role of generalized functions. Thus we have constructed the chain

$$H_+ \subseteq H_0 \subseteq H_- \tag{1.8}$$

of spaces with positive, zero and negative norms. (The elements of H_-, H_0, H_+ will be called generalized functions, ordinary functions and test functions, respectively). A rigging of the space H_0 by the spaces H_+ and H_- is given by (1.8).

Each element $\xi \in H_-$ is evidently a linear continuous functional on H_+ so that

$$H_- \subseteq (H_+)', \tag{1.9}$$

where $(H_+)'$ denotes the dual space of H_+. We will write $(\xi, \varphi)_{H_0}$, or $\langle \xi, \varphi \rangle$ for the action of the functional ξ on an element $\varphi \in H_+$. It is obvious that

$$|(\xi, \varphi)_{H_0}| \leq \|\xi\|_{H_-} \|\varphi\|_{H_+}, \quad \xi \in H_-, \varphi \in H_+ \tag{1.10}$$

which is a generalization of the Cauchy–Schwarz inequality.

Initially, we have defined H_- as a Banach space. However, one may prove that H_- is a Hilbert space, i.e., the norm $\|\cdot\|_{H_-}$ in H_- is generated by some scalar product $(\cdot, \cdot)_{H_-}$. Furthermore, $H_- = (H_+)'$, i.e. H_- can be thought of as the dual space of H_+.

A rigging (1.8) is called quasi-nuclear if the inclusion operator $O : H_+ \to H_0$ is quasi-nuclear (or of the Hilbert–Schmidt class), that is for one (and hence any) orthonormal basis $(e_n)_{n=1}^{\infty}$ of H_+, we have

$$\sum_{n=1}^{\infty} \|e_n\|_{H_0}^2 < \infty.$$

In this case, we will say that the space H_+ is imbedded into H_0 quasi-nuclearly. The corresponding rigging (or the chain (1.8)) will also be called quasi-nuclear.

Example 1.4. Let $H_0 = \ell_2 = \ell_2(\mathbb{R})$ be the Hilbert space of all square summable real sequences $x = (x_k)_{k=0}^{\infty}$ with scalar product

$$(x, y)_{H_0} = \sum_{k=0}^{\infty} x_k y_k.$$

More generally, for each sequence $\tau = (\tau_k)_{k=0}^{\infty}$, $\tau_k > 0$, we define the corresponding Hilbert space

$$H_\tau = \ell_2(\tau) = \{(x_k)_{k=0}^{\infty} \mid x_k \in \mathbb{R},\ \sum_{k=0}^{\infty} x_k^2 \tau_k =: \|x\|_{H_\tau}^2 < \infty\},$$
$$(x, y)_{H_\tau} = \sum_{k=0}^{\infty} x_k y_k \tau_k. \tag{1.11}$$

Evidently, $\ell_2 = \ell_2(\tau)$ with $\tau_k = 1$, $k \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Denote by T the set of all sequences $\tau = (\tau_k)_{k=0}^{\infty}$ with $\tau_k \geq 1$, $k \in \mathbb{Z}_+$. Clearly, for any $\tau, \tau' \in T$ such that $\tau'_k \geq \tau_k$, $k \in \mathbb{Z}_+$, $H_{\tau'} \subset H_\tau$ and $\|\cdot\|_{H'_\tau} \geq \|\cdot\|_{H_\tau}$. Denote by $\mathbb{R}_0^\infty$ the set of all finite real sequences, i.e., all real sequences $(x_k)_{k=0}^{\infty}$ such that $x_k = 0$, $k \geq K$, for some $K \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$. For each $\tau \in T$, $\mathbb{R}_0^\infty \subset \ell_2(\tau)$ with dense inclusion. Therefore, if τ and τ' are as above, $\ell_2(\tau')$ is dense in $\ell_2(\tau)$.

For every $\tau = (\tau_k)_{k=0}^{\infty} \in T$, one can take $\tau' = (2^k \tau_k)_{k=0}^{\infty}$ so that the imbedding $O_{\tau',\tau} : H_{\tau'} \to H_\tau$ is quasi-nuclear. Indeed, let $(e_n)_{n=0}^{\infty}$ be the natural orthonormal basis in ℓ_2, that is

$$e_n = (x_k)_{k=0}^{\infty}, \quad x_k = 0 \text{ for } k \neq n,\ x_n = 1. \tag{1.12}$$

Then the vectors $(\tau_n^{-\frac{1}{2}} e_n)_{n=0}^{\infty}$ form a basis in H_τ and, therefore, for the Hilbert–Schmidt norm of the imbedding operator $O_{\tau',\tau}$, we have

$$\|O_{\tau',\tau}\|_{HS}^2 = \sum_{k=0}^{\infty} \|(\tau'_k)^{-\frac{1}{2}} e_k\|_{H_\tau}^2 = \sum_{k=0}^{\infty} 2^{-k} < \infty.$$

For every $\tau \in T$, the Hilbert space $H_{\tau^{-1}} = \ell_2(\tau^{-1})$, $\tau^{-1} := (\tau_k^{-1})_{k=0}^{\infty}$, is dual to $H_\tau = \ell_2(\tau)$ with respect to $H_0 = \ell_2$. The scalar product in $H_0 = \ell_2$ defines a natural pairing of the elements of $\ell_2(\tau^{-1})$ and $\ell_2(\tau)$, namely,

$$(\xi, \varphi)_{H_0} = \sum_{k=0}^{\infty} \xi_k \varphi_k, \quad \xi \in \ell_2(\tau^{-1}),\ \varphi \in \ell_2(\tau).$$

Example 1.5. Given $l \in \mathbb{Z}_+$ and a weight function $p : \mathbb{R}^d \to \mathbb{R}$, $p(x) \geq 1$, $x \in \mathbb{R}^d$, $p \in C(\mathbb{R}^d)$ (the space of all continious functions on $\mathbb{R}^d$), the Sobolev space $W_2^l(\mathbb{R}^d, p(x)\,dx)$ is defined as the completion of $C_0^\infty(\mathbb{R}^d)$ (the space of all infinitely differentiable functions on $\mathbb{R}^d$ with bounded support) with respect to the scalar product

$$(\varphi,\psi)_{W_2^l(\mathbb{R}^d,\,p(x)\,dx)} = \sum_{|\alpha|\le l}(D^\alpha\varphi, D^\alpha\psi)_{L^2(\mathbb{R}^d,\,p(x)\,dx)}, \quad \varphi,\psi\in C_0^\infty(\mathbb{R}^d). \tag{1.13}$$

The summation in (1.13) is over all indices $\alpha=(\alpha_1,\dots,\alpha_d)$, $\alpha_1,\dots,\alpha_d\in\mathbb{Z}_+$, $|\alpha|=\alpha_1+\dots+\alpha_d\le l$, and D^α denotes the corresponding partial derivative.

We set $H_0=L^2(\mathbb{R}^d,dx)=W_2^0(\mathbb{R}^d,dx)$ and $H_+=W_2^l(\mathbb{R}^d,p(x)\,dx)$, $l\in\mathbb{N}$. Clearly, H_+ is densely and continuously embedded into H_0. (Note that each $W_2^l(\mathbb{R}^d,p(x)\,dx)$ contains the set $C_0^\infty(\mathbb{R}^d)$, which is dense in every Sobolev space, by its construction.) Then H_- is called a Sobolev space with negative index $-l$ and is denoted by $W_2^{-l}(\mathbb{R}^d,p^{-1}(x)\,dx)$.

If $l>d/2$, then the Sobolev space $W_2^l(\mathbb{R}^d,p(x)\,dx)$ consists of continuous functions, i.e., $W_2^l(\mathbb{R}^d,p(x)\,dx)\subset C(\mathbb{R}^d)$. Furthermore, if $m\in\mathbb{Z}_+$, $l>d/2$, and if $1\le p_1(x)\le p_2(x)$ with

$$\int_{\mathbb{R}^d}\frac{p_1^2(x)}{p_2^2(x)}\,dx<\infty,$$

then the inclusion $W_2^{m+l}(\mathbb{R}^d,p_2(x)\,dx)\subset W_2^m(\mathbb{R}^d,p_1(x)\,dx)$ is quasi-nuclear.

1.4 Rigging of a Hilbert space by a nuclear space

Let Φ be a linear topological space that is topologically (i.e., densely and continuously) imbedded into a Hilbert space H_0. Just as in Section 1.3, each element $f\in H_0$ generates the linear continuous functional on Φ according to the formula

$$l_f(\varphi)=(f,\varphi)_{H_0}, \quad \varphi\in\Phi.$$

Let Φ' denote the dual space of Φ (i.e., the space of all linear continuous functionals on Φ). Identifying f with l_f, we obtain the imbedding of H_0 into the space Φ'. Hence, we have constructed the chain

$$\Phi\subseteq H_0\subseteq\Phi'. \tag{1.14}$$

We also say that (1.14) is a rigging of H_0 by the spaces Φ and Φ', and Φ' is the dual space of H_0 with respect to zero space H_0.

In what follows, we will only consider the case where Φ is a nuclear space. So, let us define such a space. Let $(H_\tau)_{\tau\in T}$ be a family of Hilbert spaces. We assume that the set $\Phi := \bigcap_{\tau\in T} H_\tau$ is dense in each H_τ, and that the family $(H_\tau)_{\tau\in T}$ is directed by imbedding, i.e.,

$$\forall\tau',\tau''\in T\ \exists\tau'''\in T:\ H_{\tau'''}\subset H_{\tau'},\ H_{\tau'''}\subset H_{\tau''}, \tag{1.15}$$

where the imbeddings are continuous. On Φ, we introduce a projective limit topology with respect to the family of Hilbert spaces $(H_\tau)_{\tau\in T}$ and natural imbeddings $O_\tau:\Phi\to H_\tau$. By definition, this means that we consider the weakest topology on Φ for which all the mappings O_τ, $\tau\in T$, are continuous. One can show that the collection of all possible open balls

$$\mathcal{U}(\varphi;\tau;\epsilon)=\{\psi\in\Phi\mid\|\varphi-\psi\|_{H_\tau}<\epsilon\},\quad \varphi\in\Phi,\ \tau\in T,\ \epsilon>0. \tag{1.16}$$

may be taken as a system of base neighbourhoods of this topology. This space Φ constructed as above is called the projective limit of the family $(H_\tau)_{\tau\in T}$ and is denoted by

$$\Phi=\operatorname*{proj\,lim}_{\tau\in T} H_\tau. \tag{1.17}$$

If, additionally, for each $\tau\in T$, there exists $\tau'\in T$ such that $H_{\tau'}\subset H_\tau$ and the inclusion operator $O_{\tau',\tau}, H_{\tau'}\to H_\tau$ is quasi-nuclear, then the space Φ is called a nuclear space.

Let $\Phi=\operatorname{proj\,lim}_{\tau\in T}H_\tau$ be a nuclear space, and let H_0 be a Hilbert space. Assume that, for each $\tau\in T$, $H_\tau\subset H_0$ with continuous imbedding, and that Φ (and therefore each H_τ) is a dense subset of H_0. We can now construct the riggings

$$\begin{aligned} H_\tau \subseteq\ & H_0 \ \subseteq H_{-\tau},\quad \tau\in T,\\ \Phi\subseteq\ & H_0\ \subseteq\Phi'. \end{aligned}$$

Notice that if $H_{\tau'}\subseteq H_\tau$, we have

$$H_{-\tau}\subseteq H_{-\tau'}. \tag{1.18}$$

Theorem 1.6 (Schwartz)**.** *We have*

$$\Phi'=\bigcup_{\tau\in T}H_\tau'=\bigcup_{\tau\in T}H_{-\tau}.$$

This equality should be understood as follows: for each $l\in\Phi'$ there exists $\tau\in T$ such that l may be extended by continuity from Φ to a linear continuous functional on H_τ, and vice versa if $l\in H_{-\tau}$ for some $\tau\in T$, then $l\restriction\Phi\in\Phi'$.

Since $\Phi' = \bigcup_{\tau\in T} H_{-\tau}$, one can introduce on Φ' the topology of the inductive limit of Hilbert spaces $(H_{-\tau})_{\tau\in T}$ (however, we will not use it actually). One writes $\Phi' = \operatorname{ind\,lim}_{\tau\in T} H_{-\tau}$. Thus, we have

$$\Phi = \operatorname*{proj\,lim}_{\tau\in T} H_\tau \subseteq H_0 \subseteq \operatorname*{ind\,lim}_{\tau\in T} H_{-\tau} = \Phi', \tag{1.19}$$

which is called a Gel'fand (standard) triple. The dual space Φ' is often called a co-nuclear space.

Example 1.7. Recall Example 1.4. Clearly, $\bigcap_{\tau\in T} \ell_2(\tau) = \mathbb{R}_0^\infty$. Indeed, the inclusion $\mathbb{R}_0^\infty \subset \bigcap_{\tau\in T} \ell_2(\tau)$ is evident, whereas for any sequence of real numbers $(x_k)_{k=0}^\infty$ which has an infinite number of non-zero elements, one can always find $\tau\in T$ such that $(x_k)_{k=0}^\infty \notin \ell_2(\tau)$.

Setting $\Phi = \mathbb{R}_0^\infty = \operatorname{proj\,lim}_{\tau\in T} \ell_2(\tau)$, we get a nuclear space. In fact, convergence in this space means uniform finiteness of all sequences and coordinate-wise convergence. That is, a sequence $(x^{(n)})_{n=1}^\infty$ converges to x in $\mathbb{R}_0^\infty$ if and only if there exists $N\in\mathbb{Z}_+$ such that $x_k^{(n)} = 0$, $x_k = 0$ for all $k \ge N$ and all $n\in\mathbb{N}$, and $x_k^{(n)} \to x_k$ as $n\to\infty$ for each $k \in \{0,1,2,\dots,N-1\}$.

By Theorem 1.6 and Example 1.4,

$$\Phi' = \operatorname*{ind\,lim}_{\tau\in T} \ell_2(\tau^{-1}).$$

Note that

$$\{\tau^{-1} \mid \tau\in T\} = \{(\tau_k)_{k=0}^\infty \mid 0<\tau_k<1,\ k\in\mathbb{N}\}.$$

Denote by $\mathbb{R}^\infty = \mathbb{R}\times\mathbb{R}\times\cdots$. Then, clearly, $\Phi'\subset\mathbb{R}^\infty$, since each $\ell_2(\tau^{-1}) \subset \mathbb{R}^\infty$. On the other hand, for each sequence $(x_k)_{k=0}^\infty \in \mathbb{R}^\infty$ one can always find $\tau\in T$ such that $(x_k)_{k=0}^\infty \in \ell_2(\tau^{-1})$. Therefore, $\Phi' = \mathbb{R}^\infty$. Thus, we get the Gel'fand triple

$$\mathbb{R}_0^\infty \subset \ell_2 \subset \mathbb{R}^\infty. \tag{1.20}$$

In fact for each $\xi = (\xi_k)_{k=0}^\infty \in \mathbb{R}^\infty$ and $x = (x_k)_{k=0}^\infty \in \mathbb{R}_0^\infty$, the dual pairing between ξ and x with respect to the zero space ℓ_2 is given by

$$\langle \xi, x\rangle = (\xi, x)_{H_0} = \sum_{k=0}^\infty \xi_k x_k. \tag{1.21}$$

(Note that since $x\in\mathbb{R}_0^\infty$, the series in (1.21) contains only a finite number of non-zero terms, and hence it is well-defined).

Example 1.8. Recall Example 1.5. Denote $\mathcal{D} = \mathcal{D}(\mathbb{R}^d) := C_0^\infty(\mathbb{R}^d)$. As mentioned in Example 1.5, $\mathcal{D}$ is a dense subset of each Sobolev space $W_2^l(\mathbb{R}^d, p(x)dx)$, $l \in \mathbb{Z}_+$, $p(x) \geq 1$. Therefore

$$\mathcal{D} \subset \bigcap_{l \in \mathbb{Z}_+,\, p(x) \geq 1} W_2^l(\mathbb{R}^d, p(x)dx).$$

In fact, one can show that

$$\mathcal{D} = \bigcap_{l \in \mathbb{Z}_+,\, p(x) \geq 1} W_2^l(\mathbb{R}^d, p(x)dx).$$

Furthermore, using the fact stated in the end of Example 1.5, one sees $\mathcal{D}$ is a nuclear space. The convergence in $\mathcal{D}$ may be described as follows: If $(f_n)_{n=1}^\infty \subset \mathcal{D}, \quad f \in \mathcal{D}$, then $f_n \to f$ on $\mathcal{D}$ if and only if

$$\bigcup_{n \in \mathbb{N}} \mathrm{supp}(f_n)$$

is a bounded set in $\mathbb{R}^d$ (i.e., the functions f_n are uniformly finite), and for each index $(\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, 2, \ldots, d$,

$$(D^\alpha f_n)(x) \to (D^\alpha f)(x) \quad \text{as } n \to \infty$$

uniformly on $\mathbb{R}^d$. Here $D^\alpha f$ denotes the corresponding partial derivative of f. By Theorem 1.6,

$$\mathcal{D}' = \operatorname*{ind\,lim}_{l \in \mathbb{Z}_+,\, p(x) \geq 0} W_2^{-l}(\mathbb{R}^d, p^{-1}(x)dx).$$

Example 1.9. Let σ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which satisfies condition (i) (or equivalently condition (ii)) of Theorem 1.2 and has an infinite support. Recall Example 1.7. Denote by $\mathcal{P}$ the set of all polynomials on $\mathbb{R}$. Recall the vectors e_n in ℓ_2 defined by (1.12). They form an orthonormal basis in ℓ_2. Consider the linear mapping I defined by

$$Ie_n = x^n,$$

and extended by linearity to the linear span of the e_n vectors, i.e., to $\mathbb{R}_0^\infty$. Thus, I is a bijective mapping between $\mathbb{R}_0^\infty$ and $\mathcal{P}$. Through it we define a nuclear space topology on $\mathcal{P}$. Note that $p_k \to p$ in $\mathcal{P}$ as $k \to \infty$, means that there exists $N \in \mathbb{N}$ such that

$$p_k = \sum_{i=0}^{N} a_{ik} x^i, \quad p = \sum_{i=0}^{N} a_i x^i,$$

and for each $i = 0, 1, \dots, N$, $a_{ik} \to a_i$ as $k \to \infty$.

As easily seen, the nuclear space $\mathcal{P}$ is densely and continuously embedded into $L^2(\mathbb{R}, \sigma)$.

1.5 Probability measures on co-nuclear spaces, Minlos theorem

Let

$$\Phi \subseteq H_0 \subseteq \Phi'$$

be a Gel'fand triple. We first need a σ-algebra on Φ'. For each $\varphi \in \Phi$, we define a mapping as follows

$$\Phi' \ni \omega \mapsto \langle \omega, \varphi \rangle \in \mathbb{R}. \tag{1.22}$$

Then $\mathcal{C}(\Phi')$ is the minimal σ-algebra on Φ' with respect to which all mappings (1.22) are measurable. In particular, if $\varphi_1, \varphi_2, \dots, \varphi_n \in \Phi, n \in \mathbb{N}$ and $g := \mathbb{R}^n \to \mathbb{R}$ measurable, then

$$F(\omega) = g(\langle \omega, \varphi_1 \rangle, \langle \omega, \varphi_2 \rangle, \dots, \langle \omega, \varphi_n \rangle) \tag{1.23}$$

is a measurable function on Φ'. A function of the form (1.23) is called a cylinder function, and $\mathcal{C}(\Phi')$ is called the cylinder σ-algebra on Φ'.

Now, if μ is a probability measure on $(\Phi', \mathcal{C}(\Phi'))$, then we call μ a generalized stochastic process.

Let μ be a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then the Fourier transform of μ is defined by

$$F_\mu(x) = \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} \mu(dy), \quad x \in \mathbb{R}^n,$$

where $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$. The classical Bochner theorem states that a function $F : \mathbb{R}^n \to \mathbb{C}$ is the Fourier transform of a unique probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, i.e., $F = F_\mu$, if and only if $F(0) = 1$ and F is positive definite, i.e., for all $c_1, \ldots, c_n \in \mathbb{C}$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}^d$, we have

$$\sum_{j,k=1}^{n} c_j \overline{c_k} F(x_j - x_k) \geq 0.$$

Let now μ be a probability measure on $(\Phi', \mathcal{C})$. Analogously to the finite-dimensional case, we define the Fourier transform of μ by

$$F_\mu(\varphi) = \int_{\Phi'} e^{i\langle \varphi, \omega \rangle} \mu(d\omega), \quad \varphi \in \Phi.$$

The following theorem is an infinite-dimensional generalization of the Bochner theorem.

Theorem 1.10 (Minlos)**.** *Suppose $F : \Phi \to \mathbb{C}$. Then F is the Fourier transform of a unique probability measure μ on $(\Phi', \mathcal{C}(\Phi'))$ if and only if*

- *$F(0) = 1$;*
- *F is positive definite, i.e., for all $c_1, \ldots, c_n \in \mathbb{C}$, $n \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_n \in \Phi$ we have*

$$\sum_{j,k=1}^{n} c_j \overline{c_k} F(\varphi_j - \varphi_k) \geq 0;$$

- *F is continuous on Φ, i.e., F_μ is continuous on each H_τ, $\tau \in T$.*

Remark: The third condition of the Minlos theorem is a new condition compared with the Bochner theorem (in the finite-dimensional case, one automatically gets the continuity of the Fourier transform).

1.6 Projection spectral theorem

As we see from the previous section, one way of defining a probability measure on a co-nuclear space is through the Minlos theorem. Another possible way of construction of such a measure is given through the projection spectral theorem, which we will discuss below.

Let us first recall the spectral theorem in the case of one self-adjoint operator. Let H be a real, separable Hilbert space and let $(A, D(A))$ be a self-adjoint operator. Let $\Omega \in H$ and assume that Ω is cyclic for A, i.e., $\Omega \in D(A^n)$, $n \in \mathbb{N}$, and the set $\{\Omega, A\Omega, A^2\Omega, \dots\}$ is a total set in H. Then, the spectral theorem states that there exists a unique probability measure μ on $\mathbb{R}$ such that the linear mapping I given through

$$I\Omega = 1, \quad IA^n\Omega = x^n, \quad n \in \mathbb{N},$$

extends by continuity to a unitary operator

$$I : H \to L^2(\mathbb{R}, \mu).$$

Under I, the operator A goes over into the operator of multiplication by x, given by

$$D\{x\cdot\} = \{f \in L^2(\mathbb{R}, \mu) : xf(x) \in L^2(\mathbb{R}, \mu)\}$$

and $(x \cdot f)(x) = xf(x)$, $f \in D(x\cdot)$. Thus, $IAI^{-1} = x\cdot$. In fact, the measure μ is given by

$$\mu(\alpha) = (E(\alpha)\Omega, \Omega)_H, \quad \alpha \in \mathcal{B}(\mathbb{R}),$$

where $E(\cdot)$ is the resolution of the identity of A. The measure μ is called the spectral measure of A (at Ω).

The following theorem generalizes the above result to the case of a family of commuting self-adjoint operators indexed by elements of a nuclear space.

Theorem 1.11. (Projection spectral theorem for a family of commuting, self-adjoint operators) *Assume that we have two Gel'fand triples $\Phi' \supset H \supset \Phi$ and $\Psi' \supset \mathcal{F} \supset \Psi$, where H and $\mathcal{F}$ are separable Hilbert spaces. Also assume that we have a family $(A(\varphi))_{\varphi\in\Phi}$ of Hermitian operators in $\mathcal{F}$ such that*

1. *$D(A(\varphi)) = \Psi$, $\varphi \in \Phi$;*
2. *$A(\varphi)\Psi \subset \Psi$ for each $\varphi \in \Phi$, and furthermore $A(\varphi) : \Psi \to \Psi$ is continuous;*
3. *$A(\varphi_1)A(\varphi_2)f = A(\varphi_2)A(\varphi_1)f$, $f \in \Psi$, (i.e., the $A(\varphi)$'s algebraically commute on Ψ);*
4. *for all $f, g \in \Psi$, the mapping*

   $$\Phi \ni \varphi \mapsto (A(\varphi)f, g)_{\mathcal{F}} \in \mathbb{R}$$

   is continuous;

5. *There exists a vector* Ω *in* $\mathcal{F}$ *which is cyclic for* $(A(\varphi))_{\varphi\in\Phi}$*, i.e., the set*

$$\{\Omega,\ A(\varphi_1)\cdots A(\varphi_k)\Omega \mid \varphi_1,\dots,\varphi_k\in\Phi,\ k\in\mathbb{N}\}$$

is total in $\mathcal{F}$*;*

6. *for any* $f\in\Psi$ *and* $\varphi\in\Phi$*, the vector* f *is analytic for the operator* $A(\varphi)$*, that is, there exists* $t>0$ *such that*

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,\|A^n(\varphi)f\|_{\mathcal{F}}<\infty.$$

Then, each operator $A(\varphi)$*,* $\varphi\in\Phi$*, is essentially self-adjoint and we denote its closure by* $(\tilde{A}(\varphi),D(\tilde{A}(\varphi)))$*. These operators commute in the sense of their resolutions of the identity. Furthermore, there exists a unique probability measure* μ *on* $(\Phi',\mathcal{C}(\Phi'))$ *such that the linear operator* $I:\mathcal{F}\to L^2(\Phi',\mu)$ *given through* $I\Omega=1$ *and*

$$\begin{aligned}I(\tilde{A}(\varphi_1)\cdots\tilde{A}(\varphi_n)\Omega)&=I(A(\varphi_1)\cdots A(\varphi_n)\Omega)\\&=\langle\varphi_1,\omega\rangle\cdots\langle\varphi_n,\omega\rangle\in L^2(\Phi',\mu)\end{aligned}$$

is unitary. Under the action of I*, each operator* $(\tilde{A}(\varphi),D(\tilde{A}(\varphi)))$*,* $\varphi\in\Phi$ *goes over into the operator of multiplication by* $\langle\omega,\varphi\rangle$ *in* $L^2(\Phi',\mu)$*, given by*

$$D(\langle\varphi,\omega\rangle\cdot)=\{F\in L^2(\Phi',\mu):\langle\varphi,\omega\rangle F(\omega)\in L^2(\Phi',\mu)\}$$

and for each $F\in D(\langle\varphi,\omega\rangle\cdot)$*,*

$$(\langle\varphi,\omega\rangle\cdot F)(\omega)=\langle\varphi,\omega\rangle F(\omega).$$

Remark 1.12. In fact, under the conditions of Theorem 1.11, for μ-a.e. $\omega\in\Phi'$, there exists $P(\omega)\in\Psi'$ such that, for all $\varphi\in\Phi$,

$$\langle P(\omega),A(\varphi)f\rangle=\langle\omega,\varphi\rangle\langle P(\omega),f\rangle.$$

Thus, $P(\omega)$ is a generalized eigenvector of the operator family $(A(\varphi))_{\varphi\in\Phi}$ which belongs to the eigenfuction $\Phi\ni\varphi\mapsto\langle\omega,\varphi\rangle$. Furthermore, for each $f\in\Psi$,

$$(If)(\omega)=\langle P(\omega),f\rangle.$$

Chapter 2

Symmetric Fock space

2.1 Symmetric tensor power of a Gel'fand triple

Let us assume that we have a Gel'fand triple

$$\Phi = \operatorname*{proj\,lim}_{\tau\in T} H_\tau \subset H_0 \subset \operatorname*{ind\,lim}_{\tau\in T} H_{-\tau} = \Phi'.$$

Let $n \in \mathbb{N}$, $n \geq 2$. For each $\tau \in T$, we can evidently construct the n-th tensor power of H_τ, i.e., $H_\tau^{\otimes n}$. As easily seen, the Hilbert space $H_\tau^{\otimes n}$ is topologically imbedded into $H_0^{\otimes n}$. One can prove that the dual space of $H_\tau^{\otimes n}$ with respect to the zero space $H_0^{\otimes n}$ is $H_{-\tau}^{\otimes n}$, so that we get the following rigging of $H_0^{\otimes n}$:

$$H_\tau^{\otimes n} \subset H_0^{\otimes n} \subset H_{-\tau}^{\otimes n}.$$

Furthermore, a straightforward calculation shows that, if the imbedding $H_\tau \subset H_0$ is quasi-nuclear, then so is the imbedding $H_\tau^{\otimes n} \subset H_0^{\otimes n}$. Finally, one can show that the intersection of all $H_\tau^{\otimes n}$, $\tau \in T$, is dense in each $H_\tau^{\otimes n}$. Therefore, $\bigcap_{\tau\in T} H_\tau^{\otimes n}$ may be considered as a nuclear space, which is usually denoted by $\Phi^{\otimes n}$. Furthermore, the dual of $\Phi^{\otimes n}$ with respect to centre space $H_0^{\otimes n}$ is denoted by $\Phi'^{\otimes n}$, and is equal to $\operatorname{ind\,lim}_{\tau\in T} H_{-\tau}^{\otimes n}$. Thus, for each $n \geq 2$, we get the Gel'fand triple

$$\Phi^{\otimes n} = \operatorname*{proj\,lim}_{\tau\in T} H_\tau^{\otimes n} \subset H_0^{\otimes n} \subset \operatorname*{ind\,lim}_{\tau\in T} H_{-\tau}^{\otimes n} = \Phi'^{\otimes n}. \tag{2.1}$$

Next, we define on $H_0^{\otimes n}$ the symmetrization operator Sym_n by

$$\mathrm{Sym}_n\, f_1 \otimes \cdots \otimes f_n := \frac{1}{n!} \sum_{\sigma\in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},$$

where S_n is the set of all permutations of $\{1, \dots, n\}$. As easily seen, Sym_n extends to an orthogonal projection in $H_0^{\otimes n}$.

Denote $H_0^{\odot n} := \mathrm{Sym}_n H_0^{\otimes n}$, where $\mathrm{Sym}_n H_0^{\otimes n}$ is the subspace of $H_0^{\otimes n}$ onto which Sym_n projects. Also, for $f_1, \dots, f_n \in H_0$, we denote

$$f_1 \odot \cdots \odot f_n = \mathrm{Sym}_n(f_1 \otimes \cdots \otimes f_n).$$

Note that, for each $\sigma \in S_n$,

$$f_1 \odot \cdots \odot f_n = f_{\sigma(1)} \odot \cdots \odot f_{\sigma(n)}.$$

The space $H_0^{\odot n}$ is called the n-th symmetric tensor power of H_0, and for $f_1, \dots, f_n \in H_0$, $f_1 \odot \cdots \odot f_n$ is called symmetric tensor product of $f_1, \dots, f_n$. Clearly, for each $f \in H_0$, $f^{\otimes n} = f^{\odot n}$.

In the case where $H_0 = L^2(R, \mu)$, so that $H_0^{\otimes n} = L^2(R^n, \mu^{\otimes n})$, the symmetric tensor power $H_0^{\odot n}$ is the subspace of $L^2(R^n, \mu^{\otimes n})$ which consists of all functions $f \in L^2(R^n, \mu^{\otimes n})$ which remain invariant under the permutation of their variables, i.e., for each $\sigma \in S_n$,

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \mu^{\otimes n}\text{-a.e.}$$

In the case of Gel'fand triple (2.1), we have that $\bigcap_{\tau \in T} H_\tau^{\odot n}$ is dense in $H_0^{\odot n}$, and if $H_{\tau'} \subset H_\tau$, then $H_{\tau'}^{\odot n} \subset H_\tau^{\odot n}$ topologically, and if the former inclusion was quasi-nuclear, then so is the latter inclusion. Thus, $\mathrm{proj}\lim_{\tau \in T} H_\tau^{\odot n}$ is a nuclear space, which is denote by $\Phi^{\odot n}$. Next, one can easily shows that for each $\tau \in T$ the dual space of $H_\tau^{\odot n}$ with respect to zero space $H_0^{\odot n}$ is $H_{-\tau}^{\odot n}$, so that we get the Gel'fand triple

$$\Phi^{\odot n} = \operatorname*{proj\,lim}_{\tau \in T} H_\tau^{\odot n} \subset H_0^{\odot n} \subset \operatorname*{ind\,lim}_{\tau \in T} H_{-\tau}^{\odot n} = \Phi'^{\odot n}.$$

2.2 Symmetric Fock space

The symmetric Fock space is the Hilbert space made of the direct sum of symmetric tensor powers of a single-particle Hilbert space.

Below, we will need the following notation: If H is a Hilbert space and $a > 0$, then aH is the Hilbert space which coincides with H as a set and the inner product in aH is given by

$$(f, g)_{aH} = a(f, g)_H.$$

Let H be a separable, real Hilbert space. We define the symmetric Fock space over H as

$$\mathcal{F}(H) := \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(H),$$

where $\bigoplus_{n=0}^{\infty}$ denotes orthogonal sum of Hilbert spaces, for each $n \in \mathbb{N}$

$$\mathcal{F}^{(n)}(H) := H^{\odot n} n!, \tag{2.2}$$

and $\mathcal{F}_0(H) := \mathbb{R}$. That is, $\mathcal{F}(H)$ consists of all sequences $f = (f^{(0)}, f^{(1)}, \dots)$ where $f^{(0)} \in \mathbb{R}$, $f^{(1)} \in H$, $f^{(2)} \in H^{\odot 2}$, $f^{(3)} \in H^{\odot 3}, \dots$,

$$\|f\|^2_{\mathcal{F}(H)} = |f_0|^2 + \sum_{n=1}^{\infty} \|f^{(n)}\|^2_{\mathcal{F}^{(n)}(H)} = |f_0|^2 + \sum_{n=1}^{\infty} \|f^{(n)}\|^2_{H^{\odot n}} n! < \infty,$$

and for any $f \in \mathcal{F}(H)$ as above and $g = (g^{(0)}, g^{(1)}, \dots) \in \mathcal{F}(H)$

$$(f, g)_{\mathcal{F}(H)} = f^{(0)} g^{(0)} + \sum_{n=1}^{\infty} (f^{(n)}, g^{(n)})_{H^{\odot n}} n!\,.$$

The subspaces $\mathcal{F}^{(n)}(H)$ are often called the n-particle subspaces, and the vector $\Omega = (1, 0, 0, \cdots) \in \mathcal{F}^{(0)}(H)$ is called the vacuum.

In view of the definition of the Fock space, we will treat any $H^{\odot n}$ as a subspace of $\mathcal{F}(H)$, so that any vector $f^{(n)} \in H^{\odot n}$ will be identified with the vector

$$(0, \dots, 0, f^{(n)}, 0, 0, \dots)$$

in $\mathcal{F}(H)$.

2.3 Rigging of a Fock space

Let H be a real, separable Hilbert space. For any sequence $q = (q_n)_{n=0}^{\infty}$, $q_n \geq 1$, we define the weighted Fock space $\mathcal{F}(H, q)$ as follows:

$$\mathcal{F}(H, q) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H) q_n.$$

(Recall the notation introduced in the beginning of Section 2.2). In particular, if $q_n = 1$ for all $n \geq 0$, $\mathcal{F}(H, q) = \mathcal{F}(H)$.

Let $H_+ \subset H_0$ quasi-nuclearly. Fix any sequence q as above. As we discussed in Section 2.1, for each $n \geq 2$, the inclusion $H_+^{\odot n} \subset H_0^{\odot n}$ is also quasi-nuclear. Therefore, for $q = (q_n)_{n=0}^\infty$ as above, one can find another sequence $q' = (q'_n)_{n=0}^\infty$, $q'_n \geq q_n$, such that

$$\mathcal{F}(H_+, q') \subset \mathcal{F}(H_0, q)$$

quasi-niclearly. Indeed, denoted by $\|O_n\|_{HS}$ the Hilbert-Schmidt norm of the inclusion operator $O_n : H_+^{\odot n} \to H_0^{\odot n}$. We know that $\|O_n\|_{HS} < \infty$. Fix any $q' = (q'_n)_{n=0}^\infty$, $q'_n \geq q_n$. Then, clearly, the imbedding operator

$$O : \mathcal{F}(H_+, q) \to \mathcal{F}(H_0, q)$$

is continuous. Then the Hilbert-Schmidt norm of O is equal to

$$\|O\|_{HS}^2 = \sum_{n=0}^\infty \|O_n\|_{HS}^2 \frac{q_n}{q'_n}$$

which is finite for sufficiently quickly growing $(q'_n)_{n=0}^\infty$.

Let us take any Gel'fand triple

$$\Phi = \operatorname*{proj\,lim}_{\tau \in T} H_\tau \subset H_0 \subset \operatorname*{ind\,lim}_{\tau \in T} H_{-\tau} = \Phi'.$$

It follows from the above that

$$\operatorname*{proj\,lim}_{\tau \in T,\ q = (q_n)_{n=0}^\infty,\ q_n \geq 1} \mathcal{F}(H_\tau, p) \tag{2.3}$$

is a nuclear space. Indeed, fix any τ and p as above. Choose first $\tau' \in T$ such that $H'_\tau \subset H_\tau$ quasi-nuclearly and then choose $q' = (q'_n)_{n=0}^\infty$, $q'_n \geq q_n$, so that $\mathcal{F}(H_{\tau'}, q') \subset \mathcal{F}(H_\tau, q)$ quasi-nuclearly. In fact, the space (2.3) consists of all finite sequences $(f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$ such that $f^{(i)} \in \Phi^{\odot i}$, $i = 0, 1, \dots, n$, $n \in \mathbb{N}$. We denote this space by $\mathcal{F}_{\text{fin}}(\Phi)$. The convergence in this space means uniform finiteness and coordinate-wise convergence in each $\Phi^{\odot n}$.

Thus, we get the Gel'fand triple

$$\mathcal{F}_{\text{fin}}(\Phi) \subset \mathcal{F}(H_0) \subset \mathcal{F}^*_{\text{fin}}(\Phi),$$

where $\mathcal{F}_{\mathrm{fin}}^*(\Phi)$ is the dual space of $\mathcal{F}_{\mathrm{fin}}(\Phi)$ with respect to the zero space $\mathcal{F}(H_0)$. This space consists of all sequences $F = (F^{(0)}, F^{(1)}, \dots)$, where $F^{(n)} \in \Phi'^{\odot n}$, and the dual pairing between F and

$$f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$$

is given by

$$(F, f)_{\mathcal{F}(H_0)} = \sum_{i=0}^{n} (F^{(i)}, f^{(i)})_{H_0^{\odot n}} n!.$$

2.4 Creation, annhilation and neutral operators

Let us introduce some linear operators in the Fock space which will be heavily used in our research.

Let H be a real, separable Hilbert space. Denote by $\mathcal{F}_{\mathrm{fin}}(H)$ the subspace of the Fock space $\mathcal{F}(H)$ consisting of all vectors of the form

$$f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots), \quad f^{(i)} \in \mathcal{F}^{(i)}(H).$$

We can endow $\mathcal{F}_{\mathrm{fin}}(H)$ with a topology such that convergence in $\mathcal{F}_{\mathrm{fin}}(H)$ means uniform finiteness and coordinate-wise convergence in each $\mathcal{F}^{(i)}(H)$ (this topology is similar to that of $\mathbb{R}_0^\infty$).

Let $f \in H$. We define a creation operator $a^+(f)$ as a linear continuous operator on $\mathcal{F}_{\mathrm{fin}}(H)$ given through

$$a^+(f) g^{(n)} = f \odot g^{(n)}, \quad g^{(n)} \in \mathcal{F}^{(n)}(H), \quad n \in \mathbb{Z}_+.$$

Note that, since

$$\mathrm{Sym}_{n+1}(\mathbf{1} \otimes \mathrm{Sym}_n) = \mathrm{Sym}_{n+1},$$

we have, for each $u^{(n)} \in H^{\otimes n}$,

$$a^+(f)\, \mathrm{Sym}_n\, u^{(n)} = \mathrm{Sym}_{n+1}(f \otimes u^{(n)}). \tag{2.4}$$

Next, we define an annihilation operator $a^-(f)$ as a linear continuous operator on $\mathcal{F}_{\mathrm{fin}}(H)$ given through

$$a^-(f) g^{\odot n} = n (f, g)_H g^{\odot (n-1)}, \quad n \in \mathbb{N},$$

$$a^-(f)\Omega = 0. \tag{2.5}$$

Assume that $H = L^2(R,\mu)$, so that $H^{\odot n} = L^2_{\mathrm{sym}}(R^n, \mu^{\otimes n})$. Then, (2.5) implies that for each $g^{(n)} \in H^{\odot n}$,

$$(a^-(f)g^{(n)})(x_1, x_2, \dots, x_{n-1}) = n\int_R f(x)g^{(n)}(x, x_1, x_2, \dots, x_{n-1})\mu(dx).$$

Assume again that $H = L^2(R,\mu)$, and let f be a bounded, measurable function on R. We define the neutral (also called preservation) operator $a^0(f)$ as a linear continuous operator on $\mathcal{F}_{\mathrm{fin}}(H)$ given through

$$\begin{aligned} a^0(f)\Omega &= 0, \\ a^0(f)g^{\odot n} &= n(fg)\odot g^{\otimes(n-1)} \in \mathcal{F}^{(n)}(H), \end{aligned} \tag{2.6}$$

where fg denotes the point-wise product of function f and g. (Note that since f is bounded $fg \in L^2(R,\mu)$.) Note also that, for each $g^{(n)} \in H^{\odot n}$,

$$(a^0(f)g^{(n)})(x_1, \dots, x_n) = (f(x_1) + \dots + f(x_n))g^{(n)}(x_1, \dots, x_n).$$

Remark 2.1. In what follows, we will also use a neutral operator $a^0(f)$ for functions f which are not necessarily bounded. In that case, the domain of $a^0(f)$ must be reduced in order to allow the vector fg in (2.6) to be an element of $L^2(R,\mu)$. For example, if $f \in L^2(R,\mu)$, the function $g \in L^2(R,\mu)$ could be bounded.

Direct calculations show that $a^-(f)$ is the restriction to $\mathcal{F}_{\mathrm{fin}}(H)$ of the adjoint operator of $a^+(f)$ in $\mathcal{F}(H)$:

$$(a^+(f)F, G)_{\mathcal{F}(H)} = (F, a^-(f)G)_{\mathcal{F}(H)}, \quad F, G \in \mathcal{F}_{\mathrm{fin}}(H).$$

On the other hand, the neutral operator $a^0(f)$ is symmetric in $\mathcal{F}(H)$,

$$(a^0(f)F, G)_{\mathcal{F}(H)} = (F, a^0(f)G)_{\mathcal{F}(H)}, \quad F, G \in \mathcal{F}_{\mathrm{fin}}(H).$$

Remark 2.2. Note that formulas (2.5) and (2.6) imply that, for $f, g_1, \dots, g_n \in H$

$$\begin{aligned} a^-(f)g_1\odot g_2\odot\dots\odot g_n =&(f,g_1)_H g_2\odot\dots\odot g_n + (f,g_2)_H g_1\odot g_3\odot\dots\odot g_n \\ &+ \dots + (f,g_n)_H g_1\odot\dots\odot g_{n-1}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} a^0(f)g_1\odot g_2\odot\dots\odot g_n =&(fg_1)\odot g_2\odot\dots\odot g_n + g_1\odot(fg_2)\odot g_3\odot\dots\odot g_n \\ &+ \dots + g_1\odot g_2\odot\dots\odot g_{n-1}\odot(fg_n). \end{aligned} \tag{2.8}$$

Chapter 3

Generalized stochastic processes with independent values through the projection spectral theorem

We will now discuss how a generalized stochastic process with independent values can be constructed by using the projection spectral theorem (Theorem 1.11).

Assume that for each $x \in \mathbb{R}^d$, $\sigma(x, ds)$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with infinite support. We also assume that for each $\Delta \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{R}^d \ni x \mapsto \sigma(x, \Delta) \tag{3.1}$$

is a measurable mapping. (Note that, if $d = 1$, $\sigma(x, ds)$ is just a Markov kernel on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.) Hence, we can define a σ-finite measure $dx\, \sigma(x, ds)$ on $(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}))$.

Let $\mathcal{B}_0(\mathbb{R}^d)$ denote the collection of all sets $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ which are bounded. We will additionally assume that, for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, there exists for $C_\Lambda > 0$ such that

$$\int_{\mathbb{R}} |s|^n \sigma(x, ds) \le C_\Lambda^n n! \quad \text{for all } x \in \Lambda,\ n \in \mathbb{N}, \tag{3.2}$$

In particular, for each fixed $x \in \mathbb{R}^d$, the measure $\sigma(x, ds)$ satisfies conditions (i), or equivalently condition (ii), of Theorem 1.2.

We fix the Hilbert space

$$H_0 = L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds)). \tag{3.3}$$

Recall the nuclear space $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ from Example 1.8. Recall the nuclear space $\mathcal{P}$ from Example 1.9. We construct the nuclear space

$$\mathscr{S} = \mathcal{D} \otimes \mathcal{P}. \tag{3.4}$$

This space consists of all functions of the form

$$f(x, s) = \sum_{k=0}^{n} s^k a_k(x),$$

where $n \in \mathbb{N}$ and $a_0(x), a_1(x), \cdots, a_n(x) \in \mathcal{D}$.

It is not hard to show that the following lemma holds.

Lemma 3.1. *The space $\mathscr{S}$ is topologically, i.e., densely and continuously, embedded into*

$$H_0 = L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds)).$$

Thus, we get a Gel'fand triple

$$\mathscr{S} \subset H_0 \subset \mathscr{S}',$$

where $\mathscr{S}'$ is the dual space of $\mathscr{S}$ with respect to the zero space H_0.

Thus, by Section 2.3, we get the Gel'fand triple

$$\mathcal{F}_{\mathrm{fin}}(\mathscr{S}) \subset \mathcal{F}(H_0) \subset \mathcal{F}^*_{\mathrm{fin}}(\mathscr{S}).$$

Our aim is to construct a special probability measure on $\mathcal{D}'$ using the projection spectral theorem for a family of commuting self-adjoint operators (Theorem 1.11).

So, we set $\Psi := \mathcal{F}_{\mathrm{fin}}(\mathscr{S})$, $\mathcal{F} := \mathcal{F}(H_0)$, $\Psi^* := \mathcal{F}^*_{\mathrm{fin}}(\mathscr{S})$, $\Phi = \mathcal{D}$, $H = L^2(\mathbb{R}^d, dx)$, $\Phi' = \mathcal{D}'$.

For each $\varphi \in \mathcal{D}$, we define

$$A(\varphi) := a^+(\varphi \otimes 1) + a^-(\varphi \otimes 1) + a^0(\varphi(x)s), \tag{3.5}$$

where $(\varphi \otimes 1)(x, s) := \varphi(x)$. We set $D(A(\varphi)) = \Psi$, and as easily seen $A(\varphi)\Psi \subset \Psi$.

Theorem 3.2. *The operators $(A(\varphi))_{\varphi\in\Phi}$ and the Gel'fand triples $\Phi'\supset H\supset\Phi$ and $\Psi'\supset\mathcal{F}\supset\Psi$ satisfy the conditions of Theorem 1.11, so that the statement of this theorem holds for these operators and Gel'fand triples. Thus, each operator $A(\varphi)$ is essentially selfadjoint on $\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$, and we denote its closure by $(\tilde{A}(\varphi),D(\tilde{A}(\varphi)))$. These operators commute in the sense of their resolution of identity. Furthermore, there exists a unique probability measure μ on $\mathcal{D}'$ such that the linear operator*

$$I:\mathcal{F}(H_0)\to L^2(\mathcal{D}',\mu)$$

given through $I\Omega=1$ and

$$\begin{aligned}I(\tilde{A}(\varphi_1)\cdots\tilde{A}(\varphi_n)\Omega)&=I(A(\varphi_1)\cdots A(\varphi_n)\Omega)\\&=\langle\varphi_1,\omega\rangle\cdots\langle\varphi_n,\omega\rangle\in L^2(\mathcal{D}',\mu)\end{aligned}$$

is unitary. Under the action of I, each operator $(\tilde{A}(\varphi),D(\tilde{A}(\varphi)))$, $\varphi\in\mathcal{D}$ goes over into the operator of multiplication by $\langle\omega,\varphi\rangle$ in $L^2(\mathcal{D}',\mu)$.

Proof. We check the conditions of Theorem 1.11.

1. This condition is clearly satisfied.

2. We already know that $A(\varphi)\mathcal{F}_{\mathrm{fin}}(\mathscr{S})\subset\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$, $\varphi\in\Phi$. For each $\varphi\in\mathcal{D}$, the linear operator

$$\mathcal{D}\ni f\mapsto\varphi f\in\mathcal{D}$$

is continuous (see Example 1.8 for the description of convergence in $\mathcal{D}$) and the linear operator

$$\mathcal{P}\ni f\mapsto sf(s)\in\mathcal{P}$$

is also continuous (see Example 1.9 for the description of convergence in $\mathcal{P}$). This implies that for each $n\in\mathbb{N}$ the linear operator

$$a^0(\varphi(x)s):(\mathcal{D}\otimes\mathcal{P})^{\odot n}\to(\mathcal{D}\otimes\mathcal{P})^{\odot n}$$

is continuous. Therefore,

$$a^0(\varphi\otimes s):\mathcal{F}_{\mathrm{fin}}(\mathscr{S})\to\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$$

is continuous.

Next the continuity of

$$a^+(\varphi\otimes 1):\mathcal{F}_{\mathrm{fin}}(\mathscr{S})\to\mathcal{F}_{\mathrm{fin}}(\mathscr{S})\quad\text{and}\quad a^-(\varphi\otimes 1):\mathcal{F}_{\mathrm{fin}}(\mathscr{S})\to\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$$

easily follows from their definition. Thus, the operator

$$A(\varphi) : \Psi \mapsto \Psi$$

is continuous.

3. For any linear operators A, B, we denote by $[A, B]$ the commutator of A, B: $[A, B] := AB - BA$.

Let $\mathcal{H} = L^2(R, \nu)$ be an L^2-space and let $\mathcal{F}(\mathcal{H})$ be the corresponding Fock space. Then, for any $f_1, f_2 \in \mathcal{H}$,

$$[a^+(f_1),\ a^+(f_2)] = 0. \tag{3.6}$$

Indeed, for each $g^{\odot n} \in \mathcal{F}(\mathcal{H})$,

$$\begin{aligned} a^+(f_1)a^+(f_2)g^{\odot n} &= f_1 \odot f_2 \odot g^{\odot n} \\ &= f_2 \odot f_1 \odot g^{\odot n} \\ &= a^+(f_2)a^+(f_1)g^{\odot n}. \end{aligned}$$

Taking the adjoint operators in (3.6), we get

$$[a^-(f_1),\ a^-(f_2)] = 0. \tag{3.7}$$

Next

$$[a^-(f_1),\ a^+(f_2)] = (f_1, f_2)_{\mathcal{H}}\ 1. \tag{3.8}$$

Indeed by (2.7),

$$\begin{aligned} a^-(f_1)a^+(f_2)g^{\otimes n} &= a^-(f_1)\ f_2 \odot g^{\otimes n} \\ &= (f_1, f_2)_{\mathcal{H}} g^{\otimes n} + n(f_1, g)_{\mathcal{H}}\ f_2 \odot g^{\otimes(n-1)}, \end{aligned}$$

and

$$\begin{aligned} a^+(f_2)a^-(f_1)g^{\otimes n} &= a^+(f_2)(f_1, g)_{\mathcal{H}}\ n\ g^{\otimes(n-1)} \\ &= n(f_1, g)_{\mathcal{H}}\ f_2 \odot g^{\otimes(n-1)}, \end{aligned}$$

so that

$$\big(a^-(f_1)a^+(f_2) - a^+(f_2)a^-(f_1)\big)g^{\otimes n} = (f_1, f_2)_{\mathcal{H}} g^{\otimes n},$$

which proves (3.8).

Next

$$[a^0(f_1), a^0(f_2)] = 0.$$

Indeed, by (2.8),

$$\begin{aligned} a^0(f_1)a^0(f_2)g^{\otimes n} &= a^0(f_1)\ n\ (f_2 g) \odot g^{(n-1)}, \\ &= n(f_1 f_2 g) \odot g^{(n-1)} + n(n-1)(f_1 g) \odot (f_2 g) \odot g^{(n-2)}, \\ &= a^0(f_2)a^0(f_1)\ g^{\otimes n}. \end{aligned}$$

Next

$$[a^0(f_1), a^+(f_2)] = a^+(f_1 f_2). \tag{3.9}$$

Indeed by (2.8)

$$\begin{aligned} a^0(f_1)a^+(f_2)g^{\otimes n} &= a^0(f_1) f_2 \odot g^{\otimes n} \\ &= (f_1 f_2) \odot g^{\otimes n} + n f_2 \odot (f_1 g) \odot g^{\otimes (n-1)} \\ &= a^+(f_1 f_2)g^{\otimes n} + a^+(f_2)a^0(f_1)g^{\otimes n}. \end{aligned}$$

Note that (3.9) means that

$$a^0(f_1)a^+(f_2) - a^+(f_2)a^0(f_1) = a^+(f_1 f_2),$$

and taking the adjoint of these operators we get

$$a^-(f_1)a^0(f_2) - a^0(f_1)a^-(f_2) = a^-(f_1 f_2).$$

Thus

$$[a^-(f_1),\ a^0(f_2)] = a^-(f_1 f_2). \tag{3.10}$$

By (3.6)–(3.10), taking a function λ, we get;

$$\begin{aligned} &[a^+(f_1) + a^-(f_1) + a^0(\lambda f_1), a^+(f_2) + a^-(f_2) + a^0(\lambda f_2)] \\ &= [a^+(f_1), a^+(f_2)] + [a^+(f_1), a^-(f_2)] + [a^+(f_1), a^0(\lambda f_2)] \\ &+ [a^-(f_1), a^+(f_2)] + [a^-(f_1), a^-(f_2)] + [a^-(f_1), a^0(\lambda f_2)] \\ &+ [a^0(\lambda f_1), a^+(f_2)] + [a^0(\lambda f_1), a^-(f_2)] + [a^0(\lambda f_1), a^0(\lambda f_2)] \\ &= -(f_1, f_2)_{\mathcal{H}} 1 - a^+(\lambda f_1 f_2) + (f_1, f_2)_{\mathcal{H}} 1 \\ &+ a^-(\lambda f_1 f_2) + a^+(\lambda f_1 f_2) - a^-(\lambda f_1 f_2) \end{aligned}$$

$$= 0. \tag{3.11}$$

Applying formula (3.11) to our case with function $\lambda(x,s)=s$, we get that

$$[A(\varphi_1), A(\varphi_2)] = 0.$$

4. To prove condition 4, we need to check that for any $f, g \in \mathcal{F}_{\mathrm{fin}}(\mathscr{S})$, the mappings

$$\mathcal{D} \ni \varphi \mapsto (a^+(\varphi\otimes 1)f, g)_{\mathcal{F}(H_0)}, \tag{3.12}$$
$$\mathcal{D} \ni \varphi \mapsto (a^-(\varphi\otimes 1)f, g)_{\mathcal{F}(H_0)}, \tag{3.13}$$
$$\mathcal{D} \ni \varphi \mapsto (a^0(\varphi(x)s)f, g)_{\mathcal{F}(H_0)} \tag{3.14}$$

are continuous. Indeed, for any $f^{(n)} \in (\mathcal{D}\otimes\mathcal{P})^{\odot n}$, $g^{(n+1)} \in (\mathcal{D}\otimes\mathcal{P})^{\odot(n+1)}$,

$$\begin{aligned}
&\big(a^+(\varphi\otimes 1)f^{(n)}, g^{(n+1)}\big)_{H_0^{\odot(n+1)}} = \big((\varphi\otimes 1)\odot f^{(n)}, g^{(n+1)}\big)_{H_0^{\odot(n+1)}}\\
&= \big((\varphi\otimes 1)\otimes f^{(n)}, g^{(n+1)}\big)_{H_0^{\otimes(n+1)}}\\
&= \int_{(\mathbb{R}^d\times\mathbb{R})^{(n+1)}} \varphi(x_1) f^{(n)}(x_2, s_2, \dots, x_{n+1}, s_{n+1}) g^{(n+1)}(x_1, s_1, \dots, x_{n+1}, s_{n+1})\\
&\times dx_1 \cdots dx_{n+1}\, \sigma(x_1, ds_1)\cdots\sigma(x_{n+1}, ds_{n+1}),
\end{aligned}$$

which continuously depends on $\varphi \in \mathcal{D}$, by the dominated convergence theorem. This proves continuity of (3.12). Next, we note that the mapping (3.13) can be written as

$$\big(f, a^+(\varphi\otimes 1)g\big)_{\mathcal{F}} = \big(a^+(\varphi\otimes 1)g, f\big)_{\mathcal{F}},$$

which is continuous in φ by the proved above.

Finally, for any $f^{(n)}, g^{(n)} \in (\mathcal{D}\otimes\mathcal{P})^{\odot n}$ by (2.6),

$$\begin{aligned}
&\big(a^0(\varphi(x)s)f^{(n)}, g^{(n)}\big)_{H_0^{\odot n}}\\
&= \int_{(\mathbb{R}^d\times\mathbb{R})^n} \Big(\sum_{i=1}^n \varphi(x_i)s_i\Big) f^{(n)}(x_1, s_1, \dots, x_n, s_n)\, g^{(n)}(x_1, s_1, \dots, x_n, s_n)\\
&\times dx_1 \cdots dx_n\, \sigma(x_1, ds_1)\cdots\sigma(x_n, ds_n),
\end{aligned}$$

and again by the dominated convergence theorem, (3.14) is continuous in $\varphi \in \mathcal{D}$.

5. We will now prove that the vacuum Ω is cyclic for $(A(\varphi))_{\varphi\in\mathcal{D}}$. We note that this fact is not trivial sense the set $\{\varphi\otimes 1, \varphi\in\mathcal{D}\}$ is not total in H_0. For each $\Delta\in\mathcal{B}_0(\mathbb{R}^d)$, we denoted by χ_Δ the indicator function of Δ. For each $\Delta\in\mathcal{B}_0(\mathbb{R}^d)$, we define

$$A(\Delta)=a^+(\chi_\Delta\otimes 1)+a^-(\chi_\Delta\otimes 1)+a^0(\chi_\Delta(x)s).$$

As domain for $A(\Delta)$, we will choose all finite sequences

$$(f^{(0)},f^{(1)},\dots,f^{(n)},0,0,\dots),\quad n\in\mathbb{N},$$

where for each $i=0,1,2,\dots,n$, $f^{(i)}(x_1,s_1\dots,x_i,s_i)$ is a finite sum of functions of the form $f_1^{(i)}(x_1,\dots,x_i)f_2^{(i)}(s_1,\dots,s_i)$, where $f_1^{(i)}$ is a measurable, bounded, symmetric functions on $(\mathbb{R}^d)^i$ with bounded support and $f_2^{(i)}$ is a symmetric polynomial on $\mathbb{R}^i$.

For each $\Delta\in\mathcal{B}_0(\mathbb{R}^d))$, we can always find a sequence $(\varphi_n)_{n=1}^\infty$, $\varphi_n\in\mathcal{D}$, such that $\varphi_n\to\chi_\Delta(x)$ a.e. and the sequence $(\varphi_n)_{n=1}^\infty$ is uniformly finite (i.e., $\bigcup_{n\in\mathbb{N}}\operatorname{supp}\varphi_n$ is bounded) and uniformly bounded.

From here it follows that it suffices to prove that Ω is cyclic for the family $(A(\Delta))_{\Delta\in\mathcal{B}_0(\mathbb{R}^d)}$ in $\mathcal{F}(H_0)$. Denote by F the linear span of the vectors

$$\{\Omega, A(\Delta_1)\cdots A(\Delta_n)\Omega,\ \Delta_1,\dots,\Delta_n\in\mathcal{B}_0(\mathbb{R}^d),\ n\in\mathbb{N}\}.$$

Thus, we have to prove that the closure of F is $\mathcal{F}(H_0)$.

Lemma 3.3. *For any sets $\Delta_1,\dots,\Delta_i\in\mathcal{B}_0(\mathbb{R}^d)$ which are mutually disjoint, and any $l_1,\dots,l_i\in\mathbb{N}$ such that $l_1+\cdots+l_i=n$, $n\in\mathbb{N}$.*

$$(\chi_{\Delta_1}(x_1)s_1^{l_1-1})\odot\cdots\odot(\chi_{\Delta_i}(x_i)s_i^{l_i-1})\in F.$$

Proof. We first note that each $A(\Delta)$ maps F into itself. We will prove the statement by induction in n. We have

$$A(\Delta)\Omega=\chi_\Delta\otimes 1,$$

which is the statement for $n=1$. Let us assume that the statement holds up to n, and let us prove it for $n+1$. So, let $l_1,\dots,l_i\in\mathbb{N}$, $l_1+\cdots+l_i=n+1$, and let $\Delta_1,\dots,\Delta_i$ be mutually disjoint. We have to consider two cases.

Case 1: $l_1=1$. Then

$$(\chi_{\Delta_1}(x_1)1)\odot(\chi_{\Delta_2}(x_2)s_2^{l_2-1})\odot\cdots\odot(\chi_{\Delta_i}(x_i)s_i^{l_i-1})$$

$$= A(\Delta_1)(\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1})$$

and by the assumption

$$(\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}) \in F,$$

since $l_2 + l_3 + \cdots + l_i = n$.

Case 2: $l_1 > 1$. We get

$$\begin{aligned}
&A(\Delta_1)(\chi_{\Delta_1}(x_1)s_1^{l_1-2}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}) \\
&= (\chi_{\Delta_1}(x_1)1) \odot (\chi_{\Delta_1}(x_1)s_1^{l_1-2}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}) \\
&+ \left(\int_{\Delta_1} dx \int_{\mathbb{R}} \sigma(x, ds_1)s_1^{l_1-2}\right)(\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}) \\
&+ (\chi_{\Delta_1}(x_1)s_1^{l_1-1}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}).
\end{aligned} \tag{3.15}$$

The left hand side of the equation (3.15) belongs to F, since

$$(l_1 - 2) + l_2 + \cdots + l_i = n,$$

the vector

$$\left(\int_{\Delta_1} dx \int_{\mathbb{R}} \sigma(x, ds_1)s_1^{l_1-2}\right)(\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1})$$

belongs to F, since $l_2 + \cdots + l_i \le n - 1$, and

$$(\chi_{\Delta_1}(x_1)1) \odot (\chi_{\Delta_1}(x_1)s_1^{l_1-2}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1})$$

belongs to F, since

$$1 + (l_1 - 1) + l_2 + \cdots + l_i = n + 1$$

and we use Case 1. □

Lemma 3.4. *$\mathcal{F}(H_0)$ coincides with the closed linear span of the functions of the form*

$$(\chi_{\Delta_1}(x_1)s_1^{l_1-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}),$$

where sets $\Delta_1, \ldots, \Delta_i \in \mathcal{B}_0(\mathbb{R}^d)$ are mutually disjoint, and $l_1, \ldots, l_i \in \mathbb{N}$

Proof. Using that, for each $x \in \mathbb{R}^d$, the set of polynomials is dense in $L^2(\mathbb{R}, \sigma(x, ds))$ and and using that the Lebesgue measure dx is nonatomic, the statement easily follows. □

By Lemma 3.3 and Lemma 3.4, we get that the closure of F is $\mathcal{F}(H_0)$.

6. Finally, let us prove that each vector from $\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$ is analytic for each operator $A(\varphi)$, $\varphi\in\mathcal{D}$.

We start with the following lemma.

Lemma 3.5. *Fix any $\Lambda\in\mathcal{B}_0(\mathbb{R}^d)$. Then, for $n,m\in\mathbb{N}$ and any $x_1,\dots,x_n\in\Lambda$,*

$$\int_{\mathbb{R}^n}(|s_1|+|s_2|+\dots+|s_n|)^m\sigma(x_1,ds_1)\cdots\sigma(x_n,ds_n)$$
$$\le C_\Lambda^m\frac{(n+m-1)!}{(n-1)!}.$$

Proof. For any $x_1,x_2,\dots,x_n$, we have, by (3.2) and using an easy combinatoric formula,

$$\begin{aligned}
&\int_{\mathbb{R}^n}(|s_1|+|s_2|+\dots+|s_n|)^m\sigma(x_1,ds_1)\cdots\sigma(x_n,ds_n)\\
&=\sum_{l_1,\dots,l_n\in\mathbb{Z}_+,\,l_1+l_2+\dots+l_n=m}\frac{m!}{l_1!\,l_2!\cdots l_n!}\int_{\mathbb{R}^n}|s_1|^{l_1}|s_2|^{l_2}\cdots|s_n|^{l_n}\\
&\quad\times\sigma(x_1,ds_1)\cdots\sigma(x_n,ds_n) \qquad (3.16)\\
&=\sum_{l_1,\dots,l_n\in\mathbb{Z}_+,\,l_1+l_2+\dots+l_n=m}\frac{m!}{l_1!\,l_2!\cdots l_n!}\int_{\mathbb{R}}|s_1|^{l_1}\sigma(x_1,ds_1)\int_{\mathbb{R}}|s_2|^{l_2}\sigma(x_2,ds_2)\\
&\quad\times\dots\times\int_{\mathbb{R}}|s_n|^{l_n}\sigma(x_n,ds_n)\\
&\le C_\Lambda^m\sum_{l_1,\dots,l_n\in\mathbb{Z}_+,\,l_1+l_2+\dots+l_n=m}\frac{m!}{l_1!\,l_2!\cdots l_n!}l_1!\,l_2!\cdots l_n! \qquad (3.17)\\
&=C_\Lambda^m\,m!\sum_{l_1,\dots,l_n\in\mathbb{Z}_+,l_1+l_2+\dots+l_n=m} \qquad (3.18)
\end{aligned}$$

Note that the number of choices of $l_1,\dots,l_n\in\mathbb{Z}$ such that $l_1+l_2+\dots+l_n=m$ is equal to the number of choices of $l_1,\dots,l_n\in\mathbb{N}$ such that $l_1+l_2+\dots+l_n=m+n$. But the latter number is equal to the number of choices of n numbers out of the set $\{1,\dots,n+m\}$ such that one number is 1. That is we have to choose $n-1$ numbers out of $\{2,\dots,m+n\}$. Thus, we continue (3.18) as follows:

$$\le C_\Lambda^m m!\frac{(n+m-1)!}{m!\,(n-1)!}$$

$$= C_\Lambda^m \frac{(n+m-1)!}{(n-1)!}.$$

□

Remark 3.6. In fact, we will use the following weaker estimate:

$$\int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_n|)^m \sigma(x_1, ds_1) \cdots \sigma(x_n, ds_n) \le C_\Lambda^m (m+n)! \quad (3.19)$$

for all $x_1, x_2, \dots, x_n \in \Lambda$.

It suffices to prove that each vector of the form

$$f^{(m)}(x_1, s_1, x_2, s_2 \dots, x_m, s_m) = g^{(m)}(x_1, x_2, \dots, x_m) s_1^{l_1} \odot \cdots \odot s_m^{l_m},$$

where $g^{(m)} \in \mathcal{D}^{\odot m}$, $l_1, \dots, l_m \ge 0$, $m \in \mathbb{N}$, is analytic for each $A(\varphi)$, $\varphi \in \mathcal{D}$.

Below we will denote by C different positive constants whose explicit values are not essential for us. So, we fix $\varphi \in \mathcal{D}$ and we have to prove that there exists C such that

$$\|A^n(\varphi) f^{(m)}\|_{\mathcal{F}(H_0)} \le C^n n!, \quad n \in \mathbb{N}. \quad (3.20)$$

Since

$$\begin{aligned} \|A^n(\varphi) f^{(m)}\|^2_{\mathcal{F}(H_0)} &= (A^n(\varphi) f^{(m)}, A^n(\varphi) f^{(m)})_{\mathcal{F}(H_0)} \\ &= (A^{2n}(\varphi) f^{(m)}, f^{(m)})_{\mathcal{F}(H_0)}, \end{aligned}$$

(3.20) is equivalent to

$$(A^{2n}(\varphi) f^{(m)}, f^{(m)})_{\mathcal{F}(H_0)} \le C^n (n!)^2. \quad (3.21)$$

Choose $\Delta \in \mathcal{B}_0(\mathbb{R}^d)$ such that

$$|\varphi| \le C\chi_\Delta, \quad |g^{(m)}| \le C\chi_\Delta^{\otimes m}.$$

Then (3.21) would follow from

$$\left(B^{2n}(\Delta)\psi^{(m)}, \psi^{(m)}\right)_{\mathcal{F}} \le C^n (n!)^2, \quad (3.22)$$

where

$$B(\Delta) = a^+(\chi_\Delta(x)1(s)) + a^-(\chi_\Delta(x)1(s)) + a^0(\chi_\Delta(x)|s|)$$

and

$$\psi^{(m)}(x_1, s_1, x_2, s_2, \ldots, x_m, s_m) = \chi_\Delta^{\otimes m}(x_1, x_2, \ldots, x_m)|s_1|^{l_1} \odot \cdots \odot |s_m|^{l_m}.$$

We see that

$$B^{2n}(\Delta) = \big(a^+(\chi_\Delta(x)1(s)) + a^-(\chi_\Delta(x)1(s)) + a^0(\chi_\Delta(x)|s|)\big)^{2n},$$

which is a sum of 3^{2n} terms where every term is a product of $2n$ operators each of which is one of the operators $a^+(\chi_\Delta(x)1(s))$, $a^-(\chi_\Delta(x)1(s))$, $a^0(\chi_\Delta(x)|s|)$. Since we have to estimate $\big(B^{2n}(\Delta)\psi^{(m)}, \psi^{(m)}\big)$, we are interested only in those terms which have the same number of creation and annihilation operators. Denote this number by k. Thus, we consider a term in which we have a product (in arbitrary order) of k creation operators $a^+(\chi_\Delta(x)1(s))$, k annihilation operators $a^-(\chi_\Delta(x)1(s))$ and $(2n-2k) = 2(n-k)$ neutral operators $a^0(\chi_\Delta(x)|s|)$. Denote such a term by D. Without loss of generality, we assume that

$$\mathrm{vol}(\Delta) := \int_\Delta dx$$

is ≥ 1 (otherwise we need to extent the set Δ).

By using (3.19), for any $q, r \in \mathbb{N}$, $f \in L^2(\mathbb{R}^q, \sigma(x_1, ds_1) \otimes \cdots \otimes \sigma(x_q, ds_q))$, $x_1, \ldots, x_q \in \Delta$, by the Cauchy inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_q|)^r f(s_1, s_2, \ldots, s_q)\sigma(x_1, ds_1)\sigma(x_2, ds_2)\cdots\sigma(x_q, ds_q)\\
&\quad \leq \left(\int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_q|)^{2r}\sigma(x_1, ds_1)\sigma(x_2, ds_2)\cdots\sigma(x_q, ds_q)\right)^{\frac{1}{2}}\\
&\qquad \times \|f\|_{L^2(\mathbb{R}^q, \sigma(x_1, ds_1)\otimes\cdots\otimes\sigma(x_q, ds_q))}\\
&\quad \leq C^{2r+q}((2r+q)!)^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^q, \sigma(x_1, ds_1)\otimes\cdots\otimes\sigma(x_q, ds_q))}\\
&\quad \leq C^{r+q}(r+q)!\|f\|_{L^2(\mathbb{R}^q, \sigma(x_1, ds_1)\otimes\cdots\otimes\sigma(x_q, ds_q))},
\end{aligned} \tag{3.23}$$

where we used the inequality:

$$\begin{aligned}
(2l)! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2l-1) \cdot (2l)\\
&\leq 2 \cdot 2 \cdot 4 \cdot 4 \cdots (2l)(2l)\\
&= (2 \cdot 4 \cdot 6 \cdots (2l))^2
\end{aligned}$$

$$= (2^l\, 1 \cdot 2 \cdot 3 \cdots l)^2$$
$$= 2^{2l}(l!)^2, \quad l \in \mathbb{N}.$$

Therefore,

$$(D\psi^{(m)}, \psi^{(m)})_{\mathcal{F}} \le C^n m!(m+1)(m+2)\cdots(m+k)(m+k+2(n-k))! \quad (3.24)$$

where the factor $m!$ comes from the fact that $\psi^{(m)}$ belongs to $\mathcal{F}^{(m)}L^2(\mathbb{R} \times \mathbb{R}, dx\, \sigma(x, ds))$, the factor $(m+1)(m+2)\cdots(m+k)$ comes from the fact that we have k annihilation operators, and the factor $(m+k+2(n-k))!$ comes from the estimate (3.23) and the fact that we have $2(n-k)$ neutral operators.

Hence, by (3.24), recalling that m is fixed

$$\begin{aligned}(D\psi^{(m)}, \psi^{(m)})_{\mathcal{F}} &\le C^n (m+k)!\, ((m+k)+2(n-k))! \\ &\le C^n (2(m+k)+2(n-k))! \\ &= C^n (2m+2n)! \\ &\le C^n (2n)! \\ &\le C^n (n!)^2.\end{aligned}$$

From here the estimate (3.22) follows, and the theorem is proven. $\square$

Let us find the Fourier transform of the spectral measure μ of the family $(A(\varphi))_{\varphi\in\mathcal{D}}$ in $\mathcal{F}(H_0)$ from Theorem 3.2. $\sigma(x, ds)$.

Theorem 3.7. *The spectral measure μ of the family of operators $(A(\varphi))_{\varphi\in\mathcal{D}}$ in $\mathcal{F}(H_0)$, which exists due to Theorem 3.2, has the following Fourier transform:*

$$\begin{aligned}\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\Big[-\frac{1}{2}\int_{\mathbb{R}^d} dx\, \sigma(x,\{0\})\varphi(x)^2 \\ + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds)\frac{1}{s^2}[e^{i\varphi(x)s} - i\varphi(x)s - 1]\Big],\end{aligned} \quad (3.25)$$

where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$.

Proof. We will divide the proof into several steps.

Step 1. Formula (3.25) is evidently equivalent to the following formula:

$$\begin{aligned}\int_{\mathcal{D}'} e^{i\,t\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\Big[-\frac{1}{2}\int_{\mathbb{R}^d} dx\, \sigma(x,\{0\})\varphi(x)^2 t^2 \\ + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds)\frac{1}{s^2}[e^{i\varphi(x)s\,t} - i\varphi(x)s\,t - 1]\Big],\end{aligned} \quad (3.26)$$

where $\varphi \in \mathcal{D}$ and $t \in \mathbb{R}$. Fix any $\varphi \in \mathcal{D}$. Consider the following measurable mapping:

$$\mathbb{R}^d \times \mathbb{R}^* \ni (x, s) \mapsto \varphi(x)s \in \mathbb{R}. \tag{3.27}$$

Let $\xi(dz)$ be the image of the measure $dx\, \sigma(x, ds)\frac{1}{s^2}$ under the mapping (3.27). Let also

$$a := \int_{\mathbb{R}^d} dx\, \sigma(x, \{0\})\varphi(x)^2.$$

Then, the right hand side of the formula (3.26) can be written in the form

$$\exp\left[\frac{1}{2}a(it)^2 + \int_{\mathbb{R}} \xi(dz)[e^{itz} - itz - 1]\right].$$

Since the function $e^{itz} - itz - 1$ vanishes when $z = 0$, we continue

$$= \exp\left[\frac{1}{2}a(it)^2 + \int_{\mathbb{R}^*} \xi(dz)[e^{itz} - itz - 1]\right]. \tag{3.28}$$

Let us check that $\xi(dz)$ is a Lévy measure, i.e., it satisfies

$$\int_{\mathbb{R}^*} (z^2 \wedge 1)\xi(dz) < \infty.$$

So, let us first show that

$$\int_{[-1,1]\setminus\{0\}} z^2 \xi(dz) < \infty.$$

Indeed

$$\begin{aligned}\int_{[-1,1]\setminus\{0\}} z^2 \xi(dz) &\le \int_{\mathbb{R}} z^2 \xi(dz) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds)\frac{1}{s^2}(s\varphi(x))^2 \\ &= \int_{\mathbb{R}^d} dx\, \varphi(x)^2 \int_{\mathbb{R}^*} \sigma(x, ds) < \infty.\end{aligned}$$

Next,

$$\begin{aligned}
\int_{\mathbb{R}} \chi_{\{|z|\geq 1\}}(z)\,\xi(dz) &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\frac{1}{s^2}\chi_{\{|s\varphi(x)|\geq 1\}}(s,x)\\
&= \int_{\mathrm{supp}\ \varphi} dx \int_{\mathbb{R}^*} \sigma(x,ds)\frac{1}{s^2}\chi_{\{|s\varphi(x)|\geq 1\}}(s,x)\\
&\leq \int_{\mathrm{supp}\ \varphi} dx \int_{\mathbb{R}^*} \sigma(x,ds)\frac{1}{s^2}\chi_{\{|s|\geq \frac{1}{\sup_{y\in\mathbb{R}^d}|\varphi(y)|}\}}(s,x)\\
&\leq \int_{\mathrm{supp}\ \varphi} dx \left(\sup_{y\in\mathbb{R}^d}|\varphi(y)|\right)^2 \int_{\mathbb{R}^*} \sigma(x,ds) < +\infty.
\end{aligned}$$

Therefore, the expression in (3.28) is the Fourier transform of an infinitely divisible random variable (see e.g. [4]). In particular, the right hand side of formula (3.26), considered as a function of t, is the Fourier transform of a random variable. We state the Lapace transform of this random variable can be extended to a function of complex variable which is analytic in a neighborhood of zero.

Indeed, we first note that the right hand side of (3.26) can be written as

$$\begin{aligned}
\exp\Bigg[&\frac{(it)^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\varphi(x)^2\\
&+\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\sum_{n=2}^{\infty}\frac{\varphi(x)^n s^{n-2}(it)^n}{n!}\Bigg].
\end{aligned}\tag{3.29}$$

By (3.2),

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\sum_{n=2}^{\infty}\frac{|\varphi(x)|^n|s|^{n-2}|t|^n}{n!} < \infty$$

for

$$|t| < (C_{\mathrm{supp}(\varphi)}\sup_{x\in\mathbb{R}^d}|\varphi(x)|)^{-1} =: C_\varphi.$$

Hence, (3.29) can be extended to an analytic function

$$\begin{aligned}
\{z\in\mathbb{C}\mid |z|<C_\varphi\} \ni z \mapsto \exp\Bigg[&\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\varphi(x)^2\\
&+\sum_{n=2}^{\infty}\frac{z^n}{n!}\int_{\mathbb{R}^d} dx\,\varphi(x)^n\int_{\mathbb{R}^*}\sigma(x,ds)s^{n-2}\Bigg].
\end{aligned}$$

On the other hand, by (3.20),

$$\int_{\mathcal{D}'} |\langle \varphi, \omega\rangle|^n \, \mu(d\omega) \le C^n n!.$$

Hence, by Theorem 1.2 and Remark 1.3, if we show that equality (3.26) holds for all t from a neighborhood of zero in $\mathbb{R}$, then it will follow that equality (3.26) holds for all $t \in \mathbb{R}$, and so (3.25) holds.

Step 2. For each $x \in \mathbb{R}^d$, we define a measure $\nu(x, ds)$ on $\mathbb{R}^*$ by

$$\nu(x, ds) := \frac{1}{s^2}\sigma(x, ds).$$

We also define a measure $\varkappa(x, ds)$ on $\mathbb{R}$ by

$$\varkappa(x, ds) = \sigma(x, \{0\})\delta_0(ds) + \nu(x, ds). \tag{3.30}$$

Here $\delta_0(ds)$ is the Dirac measure at 0. Note that the measure $\sigma(x, \{0\})\delta_0(ds)$ is concentrated at 0, while the measure $\nu(x, ds)$ is concentrated on $\mathbb{R}^*$. Define a Hilbert space

$$\mathcal{H}_0 := L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \varkappa(x, ds)). \tag{3.31}$$

We construct a unitary isomorphism

$$\mathcal{U} : H_0 \to \mathcal{H}_0 \tag{3.32}$$

by

$$(\mathcal{U}f)(x, s) = \begin{cases} f(x, 0) & \text{if } s = 0, \\ f(x, s)s & \text{if } s \neq 0. \end{cases} \tag{3.33}$$

We naturally extend this isomorphism to a unitary operator

$$\mathcal{U} : \mathcal{F}(H_0) \to \mathcal{F}(\mathcal{H}_0). \tag{3.34}$$

We will use the same notation for operators in $\mathcal{F}(H_0)$ and their images under $\mathcal{U}$, i.e. operators in $\mathcal{F}(\mathcal{H}_0)$. An easy calculation shows that an operator $A(\varphi)$ in $\mathcal{F}(\mathcal{H}_0)$ has the form

$$A(\varphi) = a^+(\varphi(x)s) + a^-(\varphi(x)s) + a^0(\varphi(x)s) + a^+(\varphi(x)\chi_{\{0\}}(s)) + a^-(\varphi(x)\chi_{\{0\}}(s)). \tag{3.35}$$

The initial domain of this operator (before closure) is $\mathcal{U}\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$.

Step 3. Fix any $\Lambda_1, \dots, \Lambda_n \in \mathcal{B}_0(\mathbb{R}^d)$, disjoint. Denote $\Lambda = \bigcup_{i=1}^n \Lambda_i$. Fix any $\Delta_1, \dots, \Delta_m$—disjoint, bounded, measurable subsets of $\mathbb{R}^*$, and set $\Delta_0 := \{0\}$. (We consider a metric on $\mathbb{R}^*$ such that the distance from any point in $\mathbb{R}^*$ to 0 (in the limiting sense) is $+\infty$).

Consider the functions

$$e_{ij} = \chi_{\Lambda_i \times \Delta_j}, \quad i = 1, \dots, n, \ j = 0, 1, \dots, m \tag{3.36}$$

in $\mathcal{H}_0$. These functions are evidently orthogonal. Let R be the subspace of $\mathcal{H}_0$ which is the closed linear span of the functions (3.36). Thus, $(e_{ij})_{i=1,\dots,n,\, j=0,1,\dots,m}$ form an orthogonal basis in R. Consider the symmetric Fock space over R, i.e., $\mathcal{F}(R)$. An orthogonal basis of this space is formed by the vectors

$$e_{10}^{\otimes \alpha_{10}} \odot e_{11}^{\otimes \alpha_{11}} \odot \dots \odot e_{nm}^{\otimes \alpha_{nm}} =: e_{(\alpha_{10}, \dots, \alpha_{nm})} \tag{3.37}$$

where $\alpha_{ij} \in \mathbb{Z}_+$. Denote by $\mathcal{F}_{\mathrm{fin}}(R)$ the linear span of the vectors (3.37). Consider operators

$$a_{ij} := \begin{cases} a^+(e_{ij}) + a^-(e_{ij}) + a^0(e_{ij}), & \text{if } j \neq 0, \\ a^+(e_{i0}) + a^-(e_{i0}), & \text{if } j = 0 \end{cases} \tag{3.38}$$

in $\mathcal{F}(R)$ with domain $\mathcal{F}_{\mathrm{fin}}(R)$. We have

$$\begin{aligned} & a^+(e_{ij}) e_{(\alpha_{10}, \dots, \alpha_{nm})} = e_{(\alpha_{10}, \dots, \alpha_{ij}+1, \dots, \alpha_{nm})}, \\ & a^0(e_{ij}) e_{(\alpha_{10}, \dots, \alpha_{nm})} = \alpha_{ij} e_{(\alpha_{10}, \dots, \alpha_{nm})} \quad \text{if } j \neq 0, \\ & a^-(e_{ij}) e_{(\alpha_{10}, \dots, \alpha_{nm})} = \alpha_{ij} \left(\int_{\Lambda_i} dx \int_{\Delta_j} \varkappa(x, ds) \right) e_{(\alpha_{10}, \dots, \alpha_{ij}-1, \dots, \alpha_{nm})}. \end{aligned} \tag{3.39}$$

Denote by $\mathcal{F}_{ij}$ the closed subspace of $\mathcal{F}(R)$ in which the vectors $(e_{ij}^{\otimes \alpha_{ij}})_{\alpha_{ij} \in \mathbb{Z}_+}$ form an orthogonal basis. Consider the tensor product of Hilbert spaces

$$\mathcal{F}_{10} \otimes \mathcal{F}_{11} \otimes \dots \otimes \mathcal{F}_{nm}.$$

By the definition of symmetric tensor product, we may construct a unitary isomorphism

$$S : \mathcal{F}_{10} \otimes \mathcal{F}_{11} \otimes \dots \otimes \mathcal{F}_{nm} \to \mathcal{F}(R)$$

by setting

$$S e_{10}^{\otimes \alpha_{10}} \otimes e_{11}^{\otimes \alpha_{11}} \otimes \dots \otimes e_{nm}^{\otimes \alpha_{nm}} = e_{10}^{\otimes \alpha_{10}} \odot e_{11}^{\otimes \alpha_{11}} \odot \dots \odot e_{nm}^{\otimes \alpha_{nm}}.$$

Then it follows from (3.38) and (3.39) that each operator

$$S^{-1}a_{ij}S$$

has the form

$$\mathbf{1}\otimes\cdots\otimes A_{ij}\otimes\cdots\otimes\mathbf{1},$$

where the operator A_{ij} (staying at the ij-th place) is the operator a_{ij} acting in $\mathcal{F}_{ij}$ with domain which is the linear span of the vectors $(e_{ij}^{\otimes\alpha_{ij}})_{\alpha_{ij}\in\mathbb{Z}_+}$. As easily seen from our previous considerations, each operator A_{ij} is essentially self-adjoint in $\mathcal{F}_{ij}$.

By [7, Chapter 3], we can construct the Fourier transform of the finite family of operators $(A_{ij})_{i=1,\dots,n,\,j=0,1,\dots,m}$ in the Hilbert space $\mathcal{F}_{10}\otimes\mathcal{F}_{11}\otimes\cdots\otimes\mathcal{F}_{nm}$. Its spectral measure, denoted by γ, is the product measure

$$\gamma=\gamma_{10}\otimes\gamma_{11}\otimes\cdots\otimes\gamma_{nm}$$

on $\mathbb{R}^{n(m+1)}$, where γ_{ij} is the spectral measure of the operator A_{ij} in $\mathcal{F}_{ij}$ at the vacuum state $e_{ij}^{\otimes 0}$. By formula (3.39), we have

$$A_{ij}e_{ij}^{\otimes\alpha_{ij}}=e_{ij}^{\otimes(\alpha_{ij}+1)}+\alpha_{ij}e_{ij}^{\otimes\alpha_{ij}}+\alpha_{ij}\left(\int_{\Lambda_i}dx\int_{\Delta_j}\nu(x,ds)\right)e_{ij}^{\otimes(\alpha_{ij}-1)}$$

if $j\neq 0$, and

$$A_{i0}e_{i0}^{\otimes\alpha_{i0}}=e_{i0}^{\otimes(\alpha_{i0}+1)}+\alpha_{i0}\left(\int_{\Lambda_i}dx\,\sigma(x,\{0\})\right)e_{i0}^{\otimes(\alpha_{i0}-1)}.$$

From here we immediately find that γ_{ij} is the centered Poisson measure with parameter $\int_{\Lambda_i}dx\int_{\Delta_j}\nu(x,ds)$ if $j\neq 0$, and the Gaussian measure with mean 0 and variance $\int_{\Lambda_i}dx\,\sigma(x,\{0\})$ if $j=0$. Hence, we have, for $r_{10},\dots r_{nm}\in\mathbb{R}$,

$$
\begin{aligned}
&\int_{\mathbb{R}^{n(m+1)}} e^{i(y_{10}r_{10}+\cdots+y_{nm}r_{nm})} d\gamma_{10}(y_{10})\cdots d\gamma_{nm}(y_{nm}) \\
&= \exp\bigg[-\frac{1}{2}\bigg(\int_{\Lambda_1} dx\,\sigma(x,\{0\})\,r_{10}^2+\cdots+\int_{\Lambda_n} dx\,\sigma(x,\{0\})\,r_{n0}^2\bigg) \\
&\quad + \int_{\Lambda_1} dx \int_{\Delta_1} \nu(x,ds)(e^{ir_{11}}-ir_{11}-1) \\
&\quad + \cdots + \int_{\Lambda_n} dx \int_{\Delta_m} \nu(x,ds)(e^{ir_{nm}}-ir_{nm}-1)\bigg] \\
&= \exp\bigg[-\frac{1}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})(\chi_{\Lambda_1}(x)r_{10}+\cdots+\chi_{\Lambda_n}(x)r_{n0})^2 \\
&\quad + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \nu(x,ds)(e^{if(x,s)}-if(x,s)-1)\bigg],
\end{aligned}
\tag{3.40}
$$

where

$$f(x,s)=\chi_{\Lambda_{11}}(x)\chi_{\Delta_1}(s)r_{11}+\cdots+\chi_{\Lambda_{nm}}(x)\chi_{\Delta_m}(s)r_{nm}.$$

Step 4. We define, for a function $g(x,s)$, an operator

$$B(g):=a^+(g)+a^-(g)+a^0(g(x,s)\chi_{\mathbb{R}^*}(s))$$

in $\mathcal{F}(\mathcal{H}_0)$ (on a proper domain). We now set

$$
\begin{aligned}
g(x,s) :&= \chi_{\Lambda_1}(x)\chi_{\{0\}}(s)r_{10}+\cdots+\chi_{\Lambda_n}(x)\chi_{\{0\}}(s)r_{n0} \\
&\quad + \chi_{\Lambda_1}(x)\chi_{\Delta_1}(s)r_{11}+\cdots+\chi_{\Lambda_n}(x)\chi_{\Delta_m}(s)r_{nm}.
\end{aligned}
\tag{3.41}
$$

By Step 3 and estimate (3.22), for any $z\in\mathbb{C}$ with $|z|$ sufficiently small, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{z^n(B(g)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} &= \exp\bigg[\frac{1}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})z^2 g(x,0)^2 \\
&\quad + \int_{\mathbb{R}^d} dx\int_{\mathbb{R}^*}\nu(x,ds)(e^{z\,g(x,s)}-z\,g(x,s)-1)\bigg].
\end{aligned}
$$

Step 5. Let us fix sets $\Lambda_1,\dots,\Lambda_n$ as above, let $r_1,\dots,r_n\in\mathbb{R}$. Set

$$\psi(x):=\chi_{\Lambda_1}(x)r_1+\cdots+\chi_{\Lambda_n}(x)r_n. \tag{3.42}$$

We now approximate the function

$$f(x,s) := \psi(x)\chi_{\{0\}}(s) + \psi(x)s$$

point-wise by function as in (3.41). Then, at least informally, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n(B(f)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} &= \exp\bigg[\frac{z^2}{2}\int_{\mathbb{R}^d}\sigma(x,\{0\})\psi(x)^2dx \\ &\quad + \int_{\mathbb{R}^d}dx\int_{\mathbb{R}^*}\nu(x,ds)(e^{z\psi(x)s} - z\,\psi(x)s - 1)\bigg] \\ &= \exp\bigg[\frac{z^2}{2}\int_{\mathbb{R}^d}\sigma(x,\{0\})\psi(x)^2dx \\ &\quad + \int_{\mathbb{R}^d}dx\int_{\mathbb{R}^*}\frac{\sigma(x,ds)}{s^2}(e^{z\psi(x)s} - z\,\psi(x)z - 1)\bigg].\end{aligned} \tag{3.43}$$

Let us justify this limit. We can assume that the functions $g_k(x,s)$ of the form (3.41) by which we approximate the function $f(x,s)$ satisfy

$$|g_k(x,s)| \le C\chi_\Lambda(x)(|s| + \chi_{\{0\}}(s)), \tag{3.44}$$

for all $k\in\mathbb{N}$, where $C>0$.

We have, for $x\in\mathbb{R}^d$, $s\in\mathbb{R}^*$, and $z\in\mathbb{C}$,

$$\frac{1}{s^2}\sum_{n=2}^{\infty}\frac{|z|^n|g_k(x,s)|^n}{n!} \le \chi_\Lambda(x)\sum_{n=2}^{\infty}\frac{|z|^nC^n|s|^{n-2}}{n!}\,. \tag{3.45}$$

Hence, by (3.2), (3.45), and the dominated convergence theorem,

$$\begin{aligned}&\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^*}\sigma(x,ds)s^{-2}(e^{z\,g_k(x,s)} - z\,g_k(x,s) - 1) \\ &\qquad \to \int_{\mathbb{R}^d}dx\int_{\mathbb{R}^*}\sigma(x,ds)s^{-2}(e^{z\,\psi(x)s} - z\,\psi(x)s - 1),\end{aligned}$$

as $k\to\infty$, for z from a neighborhood of zero in $\mathbb{C}$. Also by (3.44) and dominated convergence theorem

$$\int_{\mathbb{R}^d}dx\,\sigma(x,\{0\})g_k(x,0)^2 \to \int_{\mathbb{R}^d}dx\,\sigma(x,\{0\})\psi(x)^2.$$

Hence, for z from a neighborhood of zero in $\mathbb{C}$,

$$\begin{aligned}&\exp\Bigg[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})g_k(x,0)^2\\&\quad+\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^*}\sigma(x,ds)\,s^{-2}(e^{zg_k(x,s)}-z\,g_k(x,s)-1)\Bigg]\\&\to\exp\Bigg[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\psi(x)^2\\&\quad+\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^*}\sigma(x,ds)\,s^{-2}(e^{z\psi(x)s}-z\,\psi(x)s-1)\Bigg].\end{aligned}$$

Next, let us show that

$$\sum_{n=0}^{\infty}\frac{z^n(B(g_k)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!}\to\sum_{n=0}^{\infty}\frac{z^n(A(\psi)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!}\tag{3.46}$$

as $k\to\infty$ for z from a neighborhood of zero in $\mathbb{C}$. We first note, by the dominated convergence theorem and (3.44), that for a fixed $n\in\mathbb{N}$,

$$(B(g_k)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}\to(A(\psi)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}$$

as $k\to\infty$. Furthermore, as follows from (3.22), there exists a constant $C>0$ such that

$$|(B(g_k)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}|\le C^n\,n!\tag{3.47}$$

for all $k\in\mathbb{N}$. Therefore, (3.46) holds by the dominated convergence theorem.

Hence, we conclude that, for some $\epsilon>0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n(A(\psi)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!}&=\exp\Bigg[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\psi(x,0)^2\\&\quad+\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^*}\sigma(x,ds)\,s^{-2}(e^{z\psi(x)s}-z\,\psi(x)s-1)\Bigg]\end{aligned}\tag{3.48}$$

for all $z\in\mathbb{C}$, $|z|<\epsilon$. As easily seen from the above considerations, ϵ depends only on $\Lambda=\bigcup_{i=1}^n\Lambda_i$ and on $\sup_{x\in\Lambda}|\psi(x)|=\max_{i=1,\dots,n}|r_i|$.

Step 6. We fix any $\varphi\in\mathcal{D}$. Let Λ be the support of the function φ. We will now approximate φ by functions ψ_k as in (3.42). For each k, we will denote corresponding Λ-sets by $\Lambda_1^{(k)}, \Lambda_2^{(k)}, \dots, \Lambda_{n_k}^{(k)}$, so that $\bigcup_{i=1}^{n_k} \Lambda_{n_i} = \Lambda$. We will also assume that $\sup_{x\in\Lambda} |\psi_k(x)| \le C$ for all $k \in \mathbb{N}$. By the dominated convergence theorem, we get

$$\begin{aligned}
&\exp\left[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\psi_k(x)^2\right.\\
&\qquad\left. + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\, s^{-2}(e^{z\,\psi_k(x)s} - z\,\psi_k(x)s - 1)\right]\\
&\qquad \to \exp\left[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\,\varphi(x)^2\right.\\
&\qquad\left. + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\, s^{-2}(e^{z\,\varphi(x)s} - z\,\varphi(x)s - 1)\right]
\end{aligned}$$

for $z \in \mathbb{C}$ from a neighborhood of zero. So, to prove the theorem, it remains to show that, for $z \in \mathbb{C}$ from a neighborhood of zero,

$$\sum_{n=0}^{\infty} \frac{z^n (A(\psi_k)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} \to \sum_{n=0}^{\infty} \frac{z^n (A(\varphi)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!}\,. \tag{3.49}$$

Similarly to (3.47), we have, for all $k \in \mathbb{R}$,

$$|(A(\psi_k)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)}| \le C^n\, n!$$

Hence, by the dominated convergence theorem, formula (3.49) would follow if we show that, for each $n \in \mathbb{N}$

$$(A(\psi_k)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)} \to (A(\varphi)^n\Omega,\Omega)_{\mathcal{F}(\mathcal{H}_0)} \quad \text{as } k \to \infty, \tag{3.50}$$

which again follows by the dominated convergence theorem. □

Denote by $B_0(\mathbb{R}^d)$ the linear space of all measurable bounded, real-valued functions on $\mathbb{R}^d$. For each $f \in B_0(\mathbb{R}^d)$, we may define a random variable $\langle f, \omega\rangle$ as an $L^2(\mathcal{D}', \mu)$-limit of functions $\langle \varphi_n, \omega\rangle$ with $\varphi_n \in \mathcal{D}$, $n \in \mathbb{N}$, such that $\varphi_n \to f$ in $L^2(\mathbb{R}^d, dx)$. The Fourier transform $\int_{\mathcal{D}'} e^{i\langle f,\omega\rangle}\,\mu(d\omega)$ is clearly given by the right hand side of (3.25) in which φ is replaced by f.

Let $\Lambda_1, \dots, \Lambda_n \in \mathcal{B}_0(\mathbb{R}^d)$, mutually disjoint and let $t_1, \dots, t_n \in \mathbb{R}$. Set $f = t_1\chi_{\Delta_1} + \dots + t_1\chi_{\Delta_1}$. Then, by (3.25),

$$\int_{\mathcal{D}'} \exp\left[i(t_1\langle\chi_{\Lambda_1},\omega\rangle + \dots + t_n\langle\chi_{\Lambda_n},\omega\rangle)\right]\mu(d\omega) = \prod_{i=1}^{n}\int_{\mathcal{D}'} \exp[it_i\langle\chi_{\Lambda_i},\omega\rangle]\,\mu(d\omega).$$

So, the random variables $\langle \chi_{\Lambda_1}, \omega \rangle, \ldots, \langle \chi_{\Lambda_n}, \omega \rangle$ are independent. Thus, the probability measure μ is a *generalized stochastic process with independent values*, see [20].

Chapter 4

Examples: Meixner-type processes with independent values

Let us first recall Meixner's classification of orthogonal polynomials which have generating function of exponential type, see e.g. [15] for further detail.

Assume that functions $f(z)$ and $\Psi(z)$ have a Taylor series representation around zero. Also assume that $f(0) = 1$, $\Psi(0) = 0$, and $\Psi'(0) = 1$. Then, the equation

$$G(x, z) := \exp(x\Psi(z))f(z) = \sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!} z^n \tag{4.1}$$

determines a system of monic polynomials $p^{(n)}(x)$, $n \in \mathbb{Z}_+$. Meixner [35] found all classes of such polynomials which are orthogonal with respect to a probability measure ν on $\mathbb{R}$ and have infinite support. In fact, a given system $(p^{(n)}(x))_{n=0}^{\infty}$ of monic polynomials is orthogonal and has generating function (4.1) if and only if there exists $l \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $k > 0$ and $\eta \geq 0$ such that

$$xp^{(n)}(x) = p^{(n+1)}(x) + (n\lambda + l)p^{(n)}(x) + n(k + \eta(n-1))p^{(n-1)}(x). \tag{4.2}$$

If one only considers the case where the measure of orthogonality, ν, is centered, i.e.,

$$\int_{\mathbb{R}} x \, d\nu(x) = 0,$$

then $l = 0$, so that (4.2) becomes

$$xp^{(n)}(x) = p^{(n+1)}(x) + n\lambda p^{(n)}(x) + n(k + \eta(n-1))p^{(n-1)}(x). \tag{4.3}$$

For fixed parameters λ and η, we define $\alpha, \beta \in \mathbb{C}$ so that

$$\alpha + \beta = -\lambda, \qquad \alpha\beta = \eta, \tag{4.4}$$

or equivalently

$$1 + \lambda t + \eta t^2 = (1 - \alpha t)(1 - \beta t). \tag{4.5}$$

Clearly $\lambda \in \mathbb{R}$ and $\eta \geq 0$ if and only if either $\alpha, \beta \in \mathbb{R}$, α and β being of the same sign, or $\mathrm{Im}(\alpha) \neq 0$ and α and β are complex conjugate.

We have to distinguish the following five cases:

I. (Gaussian case) Now $\alpha = \beta = 0$ (or equivalently $\lambda = \eta = 0$). The orthogonality measure ν is the Gaussian measure:

$$d\nu(x) = (2\pi k)^{-1/2} \exp\left(-\frac{x^2}{2k}\right) dx. \tag{4.6}$$

The Fourier transform of the Gaussian measure ν is given by

$$\int_{\mathbb{R}} \exp(iux) d\nu(x) = \exp\left(-\frac{1}{2}ku^2\right), \qquad u \in \mathbb{R}. \tag{4.7}$$

Furthermore,

$$\Psi(z) = z, \qquad f(z) = \exp(-\frac{1}{2}kz^2),$$

so that

$$G(x, z) = \exp\left(xz - \frac{1}{2}kz^2\right).$$

The $(p^{(n)})_{n=0}^{\infty}$ is a system of Hermite polynomials.

II. (Poisson case) Assume $\alpha \neq 0$, $\beta = 0$ (which corresponds to the choice of $\lambda \neq 0$ and $\eta = 0$). Now ν is a centered Poisson measure:

$$d\nu(x) = \exp\left(-\frac{k}{\alpha^2}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{k}{\alpha^2}\right)^n \delta_{(-\alpha n + \frac{k}{\alpha})}(dx),$$

where δ_a denotes the Dirac measure with mass at a. The Fourier transform of ν is given by

$$\int_{\mathbb{R}} \exp(iux)d\nu(x) = \exp\left(\frac{k}{\alpha^2}(e^{-i\alpha u} - 1 + i\alpha u)\right).$$

Furthermore, in a neighborhood of zero,

$$\begin{aligned}
\Psi(z) &= -\frac{1}{\alpha}\log(1-\alpha z),\\
f(z) &= \exp\left(k(\frac{1}{\alpha^2}\log(1-\alpha z) + \frac{z}{\alpha})\right),
\end{aligned}$$

so that

$$G(x,z) = \exp\left(-\frac{x}{\alpha}\log(1-\alpha z) + k\left(\frac{1}{\alpha^2}\log(1-\alpha z) + \frac{z}{\alpha}\right)\right).$$

The $(p^{(n)})_{n=0}^{\infty}$ is a system of Charlier polynomials.

III. (Gamma case) Assume $\alpha = \beta \neq 0$ (which corresponds to $\lambda = -2\alpha \in \mathbb{R}$, $\eta = \alpha^2 > 0$). ν is a centered gamma measure:

$$\begin{aligned}
d\nu(x) &= \chi_{(-\infty,-k/\alpha)}(x)\left(-x + \frac{k}{\alpha}\right)^{-1+k/\alpha^2} e^{x/\alpha}, \qquad \alpha > 0,\\
d\nu(x) &= \chi_{(-k/\alpha,+\infty)}(x)\left(x + \frac{k}{\alpha}\right)^{-1+k/\alpha^2} e^{x/\alpha}, \qquad \alpha < 0.
\end{aligned}$$

The Fourier transform of the gamma measure ν (in a neighborhood of zero) is given by

$$\int_{\mathbb{R}} \exp(iux)d\nu(x) = \exp\left(k\left(\frac{iu}{\alpha} - \frac{1}{\alpha^2}\log(1+\alpha iu)\right)\right).$$

Next,

$$\begin{aligned}
\Psi(z) &= \frac{z}{1-\alpha z},\\
f(z) &= \exp\left(-k(\frac{1}{\alpha^2}\log(1-\alpha z) + \frac{z}{(1-\alpha z)\alpha})\right),
\end{aligned}$$

so that

$$G(x,z)=\exp\left(\frac{xz}{1-\alpha z}-k\left(\frac{1}{\alpha^2}\log(1-\alpha z)+\frac{z}{(1-\alpha z)\alpha}\right)\right).$$

The $(p^{(n)})_{n=0}^\infty$ is a system of Laguerre polynomials.

IV. (Pascal case) Now $\alpha\neq\beta\neq 0$, $\alpha,\beta\in\mathbb{R}$ (which corresponds to $\eta>0$ and $\lambda^2-4\eta\geq 0$). Then ν is a centered Pascal measure (negative binomial distribution):

$$d\nu(x)=\sum_{n=0}^\infty\left(-\frac{\beta}{\alpha}\right)^n\left(-\frac{k}{\alpha\beta}\right)_n\delta_{(-\frac{k}{\alpha-(\alpha-\beta)n})}(dx),$$

where $(\varkappa)_0:=1$, $(\varkappa)_n:=\varkappa(\varkappa+1)\cdots(\varkappa+n-1)$, $n\in\mathbb{N}$, $x\in\mathbb{R}$, and we assumed that $|\alpha|>|\beta|$. The Fourier transform of ν (in a neighborhood of zero) is given by

$$\int_{\mathbb{R}}\exp(iux)d\nu(x)=\exp\left(-\frac{k}{\alpha\beta}\log\frac{\alpha e^{-i\beta u}-\beta e^{-i\alpha u}}{\alpha-\beta}\right).$$

Furthermore

$$\Psi(z)=\frac{1}{\alpha-\beta}\log\left(\frac{1-\beta z}{1-\alpha z}\right),$$

$$f(z)=\exp\left(-\frac{k}{\alpha-\beta}\log\left(\frac{(1-\beta z)^{\frac{1}{\beta}}}{(1-\alpha z)^{\frac{1}{\alpha}}}\right)\right),$$

so that

$$G(x,z)=\exp\left(\frac{x}{\alpha-\beta}\log\left(\frac{1-\beta z}{1-\alpha z}\right)-\frac{k}{\alpha-\beta}\log\left(\frac{(1-\beta z)^{\frac{1}{\beta}}}{(1-\alpha z)^{\frac{1}{\alpha}}}\right)\right).$$

The $(p^{(n)})_{n=0}^\infty$ is a system of Meixner polynomials of the first kind.

V. (Meixner case). Now $\mathrm{Im}(\alpha)\neq 0$, $\alpha=\overline{\beta}$ (which corresponds to $\eta>0$ and $\lambda^2-4\eta<0$). The measure ν is a centered Meixner measure

$$d\nu(x)=\left(\Gamma\left(\frac{k}{2\alpha\beta}\right)\right)^{-2}\left|\Gamma\left(\frac{k}{2\alpha\beta}+\frac{i}{2}\left(x-\frac{(\alpha+\beta)k}{2\alpha\beta}\right)\right)\right|^2$$
$$\times\exp\left[-\left(x-\frac{(\alpha+\beta)k}{2\alpha\beta}\right)\arctan\left(\frac{i(\alpha+\beta)}{|\alpha+\beta|}\right)\right],$$

where we assumed that $\mathrm{Im}(\alpha) > \mathrm{Im}(\beta)$. The formulas for the Fourier transform of ν, the functions $\Psi(z)$, $\Psi^{-1}(z)$, and $G(x,z)$ have the same form as in the case IV, but with complex conjugate α, β. The $(p^{(n)}(x))_{n=0}^{\infty}$ is a system of Meixner polynomials of the second kind, or the Meixner–Pollaczek polynomials.

In fact, all the above formulas for the Fourier transform and the generating function can be written down in a common form if one uses infinite sums involving α and β, see [39].

For each measure of orthogonality of polynomials from the Meixner class, ν, the Laplace transform of ν,

$$\int_{\mathbb{R}} e^{ux}\nu(dx)$$

is well-defined in a neighborhood of zero in $\mathbb{R}$. The function

$$\mathfrak{C}(u) = \log\left(\int_{\mathbb{R}} e^{ux}\nu(dx)\right),$$

also defined in a neighborhood of zero in $\mathbb{R}$, is called the cumulant transform of ν. We denote by $\mathfrak{C}_{\lambda,\eta}(u)$ the cumulant transform of the measure ν corresponding to parameters $k=1$, λ, and η.

For any $\lambda \in \mathbb{R}$ and $\eta > 0$, let $\sigma_{\lambda,\eta}(ds)$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is the measure of orthogonality of monic polynomials $(q^{(n)}(s))_{n\geq 0}$ which satisfy the recurrence relation:

$$sq^{(n)}(s) = q^{(n+1)}(s) + \lambda(n+1)q^{(n)}(s) + \eta n(n+1)q^{(n-1)}(s). \tag{4.8}$$

Note that the probability measure $\sigma_{\lambda,\eta}$ belongs to the Meixner class. In the case where $\eta = 0$, we set $\sigma_{\lambda,\eta} = \delta_\lambda$ — the Dirac measure with mass at λ.

We have [41], see also [35]:

Proposition 4.1. *For any $k > 0$, $\lambda \in \mathbb{R}$ and $\eta \geq 0$, let $\nu_{k,\lambda,\eta}$ be the measure of orthogonality of monic polynomials satisfying* (4.3). *Then the measure $\nu_{k,\lambda,\eta}$ is infinitely divisible and $k\, s^{-2}\sigma_{\lambda,\eta}(ds)$ is its Lévy measure:*

$$\int_{\mathbb{R}} \exp(iux)\, \nu_{k,\lambda,\eta}(dx) = \exp\left[k\int_{\mathbb{R}} \sigma_{\lambda,\eta}(ds)s^{-2}(e^{ius} - 1 - ius)\right].$$

(Note that, for $s = 0$, the function $s^{-2}(e^{ius} - 1 - ius)$ is assumed to take the value $-\frac{s^2}{2}$.) In particular,

$$\mathfrak{C}_{\lambda,\eta}(u) = \int_{\mathbb{R}} \sigma_{\lambda,\eta}(ds)s^{-2}(e^{us} - 1 - us). \tag{4.9}$$

Note that, in the case where, for all $x \in \mathbb{R}^d$, the measure $\sigma(x, ds)$ has support at only one point, we can trivially modify (rather simplify) our considerations in Chapter 3 and obtain a corresponding generalized stochastic process with independent values. Consider a measurable, locally bounded function $\lambda : \mathbb{R}^d \to \mathbb{R}$ and a function $\eta(x) = 0$ for all $x \in \mathbb{R}^d$. We then choose

$$\sigma(x, ds) := \sigma_{\lambda(x),\eta(x)}(x, ds) = \sigma_{\lambda(x),0}(x, ds) = \delta_{\lambda(x)}(ds),$$

and construct a corresponding generalized stochastic process μ. Its Fourier transform is

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\left[\int_{\mathbb{R}^d} dx\, \frac{1}{\lambda^2}(e^{i\lambda\varphi(x)} - 1 - i\lambda\varphi(x))\right].$$

In particular, for $\lambda = 0$, we get the Gaussian white noise measure:

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\left[-\int_{\mathbb{R}^d} dx\, \frac{\varphi(x)^2}{2}\right].$$

For $\lambda = 1$, we get the centered Poisson measure:

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\left[\int_{\mathbb{R}^d} dx\, (e^{i\varphi(x)} - 1 - i\varphi(x))\right].$$

We define the measure μ' to be the pushforward of the measure μ under the mapping

$$\mathcal{D} \ni \omega \mapsto \omega + 1 \in \mathcal{D}.$$

Note that

$$\langle\omega + 1, \varphi\rangle = \langle\omega, \varphi\rangle + \int_{\mathbb{R}^d} \varphi(x)\, dx.$$

Then the measure μ' is the Poisson measure and it has the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\left[\int_{\mathbb{R}^d} dx\, (e^{i\varphi(x)} - 1)\right].$$

The Poisson measure is concentrated on Radon measures of the form

$$\omega = \sum_{i=1}^{\infty} \delta_{x_i}, \tag{4.10}$$

where the set $\{x_i\}_{i=1}^{\infty}$ is infinite, but locally finite. The Poisson measure is an example of a point process, i.e., a probability measure which is concentrated on measures (4.10).

From now on, we fix any measurable, locally bounded functions $\lambda : \mathbb{R}^d \to \mathbb{R}$ and $\eta : \mathbb{R}^d \to (0, \infty)$. For each $x \in \mathbb{R}^d$, let $\sigma(x, ds)$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\sigma(x, ds) := \sigma_{\lambda(x),\eta(x)}(ds),$$

see Proposition 4.1. It can be easily shown that the measures $\sigma(x, ds)$ satisfy condition (3.2), and so the results of Chapter 3 are applicable in this case and we obtain a corresponding generalized stochastic process with independent values. The Fourier transform of this measure μ on $\mathcal{D}'$ is given by

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}^d} dx \int_{\mathbb{R}} \sigma_{\lambda(x),\eta(x)}(x, ds)\frac{1}{s^2}[e^{i\varphi(x)s} - 1 - i\varphi(x)s]\Big], \tag{4.11}$$

compare with [31, 32] and [39].

Let us choose $\lambda(x) \equiv 2$, $\eta(x) \equiv 1$. We have

$$\sigma_{2,1}(x, ds) = \chi_{(0,\infty)}(s)se^{-s}\, ds.$$

Hence

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\left[\int_{\mathbb{R}^d} dx \int_0^{\infty} ds\, \frac{e^{-s}}{s}\, [e^{i\varphi(x)s} - 1 - i\varphi(x)s]\right].$$

In fact, for φ from a neighborhood of zero, we get

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\left[-\int_{\mathbb{R}^d} \big(\log(1 - i\varphi(x)) + i\varphi(x)\big)\, dx\right].$$

This is the centered gamma measure. We define the measure μ' to be the pushforward of the measure μ under the mapping

$$\mathcal{D} \ni \omega \mapsto \omega + 1 \in \mathcal{D}.$$

Then the measure μ' is the gamma measure and it has the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu'(d\omega) = \exp\left[\int_{\mathbb{R}^d} dx \int_0^{\infty} ds\, \frac{e^{-s}}{s}\, [e^{i\varphi(x)s} - 1]\right]$$

$$= \exp\left[-\int_{\mathbb{R}^d} \log(1 - i\varphi(x))\, dx\right].$$

The gamma measure is a random discrete measure. More precisely, the gamma measure is concentrated on the set of generalized functions $\omega \in \mathcal{D}'$ of the form

$$\omega = \sum_{i=1}^{\infty} s_i \delta_{x_i},$$

where the set $\{x_i\}_{i=1}^{\infty}$ is dense in $\mathbb{R}^d$, and for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$,

$$\sum_{i:\, x_i \in \Lambda} s_i < \infty,$$

the latter meaning that ω is a Radon measure, i.e., it is a measure taking a finite value on each bounded measurable set.

In the general case, as follows from the proof of Theorem 3.7, the Laplace transform of the measure μ,

$$\varphi \mapsto \int_{\mathcal{D}'} e^{\langle \varphi, \omega \rangle} \mu(d\omega)$$

is well-defined in a neighborhood of zero in $\mathcal{D}$. Hence, we can define its cumulant transform

$$\mathfrak{C}(\varphi) := \log\left(\int_{\mathcal{D}'} e^{\langle \varphi, \omega \rangle} \mu(d\omega)\right),$$

which is also defined in a neighborhood of zero in $\mathcal{D}$.

Theorem 4.2. *We have, for φ from a neighborhood of zero in $\mathcal{D}$,*

$$\mathfrak{C}(\varphi) = \int_{\mathbb{R}^d} \mathfrak{C}_{\lambda(x), \eta(x)}(\varphi(x))\, dx. \tag{4.12}$$

Proof. We will only sketch the proof of this theorem. By approximation, it suffices to prove the following statement.

Fix any $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ and any constant $\varepsilon > 0$. Then there exists a constant $C > 0$ for which the following holds. Let $\Lambda, \dots, \Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ be mutually disjoint and $\bigcup_{j=1}^{n} \Lambda_j = \Lambda$. Let functions λ and η take on constant values on each set Λ_j, $j = 1, \ldots, n$, and let the functions $|\lambda|$ and η be bounded by ε on Λ. Let a function φ be given by

$$\varphi(x) = \sum_{j=1}^{n} r_j \chi_{\Lambda_j}(x), \tag{4.13}$$

where

$$\max_{j=1,\ldots,n} |r_j| \le C.$$

Then formula (4.12) holds for this function φ.

Indeed, denote by λ_j and η_j the value of λ and η on Λ_j. By (4.9), (4.11), and (4.13),

$$\begin{aligned}
\mathfrak{C}(\varphi) &= \sum_{j=1}^{n} \int_{\Lambda_j} dx \int_{\mathbb{R}} \sigma(x, ds) s^{-2}(e^{r_j s} - r_j s - 1) \\
&= \sum_{j=1}^{n} \left(\int_{\Lambda_j} dx \right) \int_{\mathbb{R}} \sigma_{\lambda_j,\eta_j}(ds)(e^{r_j s} - r_j s - 1) \\
&= \sum_{j=1}^{n} \left(\int_{\Lambda_j} dx \right) \mathfrak{C}_{\lambda_j,\eta_j}(r_j) \\
&= \int_{\mathbb{R}^d} \sum_{j=1}^{n} \chi_{\Lambda_j}(x) \mathfrak{C}_{\lambda(x),\eta(x)}(\varphi(x))\, dx \\
&= \int_{\mathbb{R}^d} \mathfrak{C}_{\lambda(x),\eta(x)}(\varphi(x))\, dx,
\end{aligned}$$

where we used that $\mathfrak{C}_{\lambda(x),\eta(x)}(0) = 0$. From here the statement follows.

$\square$

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