Harmonic analysis on Gelfand pairs
(a short course)

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Introduction

In these notes I give an overview of the basic theory of Gelfand pairs, adding to the classical material (Chapters 1 and 2) recent developments which clarify what is the concrete structure of Gelfand pairs of Lie groups.

This material is already contained in several books, each containing only part of it, and looked from different points of view. The main references are:

- S. Helgason, *Groups and Geometric Analysis*, Acad. Press, 1984 (for Ch. 2, see Ch. IV there)
- J. Wolf, *Harmonic Analysis on Commutative Spaces*, AMS, 2007 (for Ch. 3, see Ch. 13 there)

Another interesting book, at an introductory level but nonetheless containing more advanced material:


Asterisks are used to indicate statements, examples or other comments requiring a proof that is left for exercise: a single asterisk (*) means that the exercise is accessible without specific preparation in harmonic analysis, a double asterisk (**) that the exercise requires some specialized knowledge.
CHAPTER 1

GENERAL THEORY

1. Group actions and homogeneous spaces

Let $G$ be a group and $M$ a set. An action of $G$ on $M$ is a map $\varphi : G \to M^M$ such that:

- $\varphi(e) = i_M$;
- $\varphi(gh) = \varphi(g)\varphi(h)$ for every $g, h \in G$.

In particular, $\varphi(g)$ is bijective for every $g$ and $\varphi(g)^{-1} = \varphi(g^{-1})$.

An action of $G$ on $M$ can be equivalently defined as a map $\psi : G \times M \to M$ such that

- $\psi(e, x) = x$ for every $x \in M$;
- $\psi(g, \psi(h, m)) = \psi(gh, m)$ for every $g, h \in G$ and $x \in M$.

Examples (*).

(1.a) Let $G = \mathfrak{S}_n$ the symmetric group on $n$ elements, and $M = \{1, 2, \ldots, n\}$. Setting $\psi(\sigma, k) = \sigma(k)$ for $\sigma \in \mathfrak{S}_n$ and $x \in M$, this gives an action of $G$ on $M$.

(1.b) Let $G = \text{GL}_n(\mathbb{R})$ the group of $n \times n$ real invertible matrices, and $M = \mathbb{R}^n$. Then $\psi(g, x) = gx$ is an action.

(1.c) Let $S^2 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. The conformal action of the group $\text{GL}_2(\mathbb{C})$ on $S^2$ is defined as follows: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, then

$$\psi(g, z) = \frac{az + b}{cz + d}.$$ 

We write $g \cdot x$ for $\psi(g, x)$. The defining condition of an action take the form

- $e \cdot x = x$;
- $g \cdot (h \cdot x) = (gh) \cdot x$.

We are interested in topologic actions, in which $G$ is a locally compact group, $M$ a locally compact Hausdorff space, and the action $\psi : G \times M \to M$ is continuous.

Later, we will restrict our attention to differentiable actions, assuming that $G$ is a Lie group, $M$ a differentiable manifold, and the action $\psi : G \times M \to M$ is smooth.
Definitions will be given for topologic actions, even in cases where they also make sense for generic ones.

**Definition 1.1.** Given $x \in M$, the $G$-orbit of $x$ is the subset $O_x = \{g \cdot x : g \in G\}$ of $M$.

The stabilizer of $x \in M$ is the subgroup $G_x = \{g : g \cdot x = x\}$ of $G$.

The action of $G$ on $M$ is called effective if $\bigcap_{x \in M} G_x = \{e\}$.

The action is called transitive if, given $x, y \in M$, there exists $g \in G$ such that $g \cdot x = y$.

The action is called simply transitive if, given $x, y \in M$, there exists a unique $g \in G$ such that $g \cdot x = y$.

$G_x$ is closed for every $x$. If an action is not effective, then $G_0 = \bigcap_{x \in M} G_x$ is a closed normal subgroup.

The action naturally projects to $G/G_0$ and is effective.

The following properties are easy to prove.

**Lemma 1.2 (\textasteriskcentered).**

1. For every $g \in G$, the map $x \mapsto g \cdot x$ is a homeomorphism of $M$.

2. The $G$-orbits in $M$ are the equivalence classes for the equivalence relation

   $x \sim y \iff \exists g \in G : y = g \cdot x$.

3. The action is transitive if and only if $M$ consists of a unique orbit.

4. If $x, y \in M$ belong to the same orbit, their stabilizers are conjugate subgroups of $G$; precisely, if $y = g \cdot x$, then $G_y = gG_x g^{-1}$.

**Definition 1.3.** A homogeneous space of a locally compact group $G$ is a pair $(M, \psi)$, where $M$ is a Hausdorff locally compact space and $\psi$ is a transitive action of $G$ on $M$.

Given two homogeneous spaces $(M, \psi)$ and $(M', \psi')$ of the same group $G$, a continuous map $F : M \rightarrow M'$ is called $G$-equivariant if $\psi'(g, F(x)) = F(\psi(g, x))$ for every $g \in G$ and $x \in M$.

Two homogeneous spaces of $G$ are called equivalent if there exists a $G$-equivariant homeomorphism of $M$ onto $M'$.

The following theorem shows that, under mild topological assumptions on $G$, all its homogeneous spaces are, up to equivalence, the quotients $G/H$, where $H$ is a closed subgroup of $G$.

**Theorem 1.4 (\textasteriskcentered).** Let $H$ be a closed subgroup (not necessarily normal) of $G$. Then the quotient space $G/H$ is Hausdorff and locally compact. The action of $G$ on $G/H$ given by

$$(g, g' H) \mapsto g \cdot (g' H) \triangleq gg' H$$

makes $G/H$ a homogeneous space of $G$.

Let $M$ be a homogeneous space of $G$. For fixed $x_0 \in M$, the map of $G$ on $M$

$$g \mapsto g \cdot x_0$$
induces a $G$-equivariant continuous map $F_{x_0}$ of $G/G_{x_0}$ on $M$.

If $G$ is second countable, $F_{x_0}$ is a homeomorphism.

The last statement is not true in general. Consider for instance $G = \mathbb{R}_d$, the real line with its discrete topology, $M = \mathbb{R}$ with the euclidean topology, and the action $\psi(g, x) = g + x$.

Examples.

(1.d) (*) Let $G = U_n$ and $M = S^{2n-1} \subset \mathbb{C}^n$, with the natural action given by $g \cdot z = gz$. Then $M$ is a homogeneous space.

Fix $e_1 = (1, 0, \ldots, 0) \in S^{2n-1}$. Then $G_{e_1}$ consists of the matrices

$$g = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix},$$

with $h \in U_{n-1}$ and $S^{2n-1} \sim U_n/G_{e_1}$. With an abuse of language, one says that $S^{2n-1} \sim U_n/U_{n-1}$.

If the action on $S^{2n-1}$ is restricted to the subgroup $SU_n$, it remains transitive. In this case we have $S^{2n-1} \sim SU(n)/SU(n-1)$.

Let $\mathbb{C}P^{n-1}$ be the complex projective space

$$\mathbb{C}P^{n-1} = S^{2n-1}/\sim,$$

where

$$v \sim w \iff \exists \theta \in \mathbb{T} : v = e^{i\theta}w.$$

The actions of $U_n$ and of $SU_n$ both pass to the quotient and $\mathbb{C}P^{n-1}$ becomes a homogeneous space of either group. If $[e_1]$ is the image of $e_1$ in $\mathbb{C}P^{n-1}$, its stabilizer in $U_n$ consists of the matrices

$$g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & h \end{pmatrix},$$

and is the product $U_1 \times U_{n-1}$. Hence, under the action of $U_n$,

$$\mathbb{C}P^{n-1} \sim U_n/(U_1 \times U_{n-1}).$$

In term of the action of $SU_n$, one writes

$$\mathbb{C}P^{n-1} \sim SU_n/S(U_1 \times U_{n-1}).$$

(1.e) (*) Let $M$ be the upper half-plane $\{z = x + iy : y > 0\} \subset \mathbb{C}$. The group $G = SL_2(\mathbb{R})$ acts (transitively but non-effectively) on $M$ by linear fractional transformations: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$g \cdot z = \frac{az + b}{cz + d}.$$
The stabilizer of \( i \in M \) consists of the matrices
\[
g = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},
\]
hence \( M \sim SL_2(\mathbb{R})/SO_2 \).

(1.f) In differential geometry, a Riemannian manifold \( M \) is called homogeneous if, for every pair of points \( x, y \in M \), there exists an isometry of \( M \) onto itself mapping \( x \) into \( y \).

The isometries of a homogeneous manifold form a group under composition. This group \( I(M) \) has a natural structure of a Lie group - hence locally compact and second countable. The action \( I(M) \) on \( M \) given by \( g \cdot x = g(x) \) makes \( M \) a homogeneous space of \( G \).

If \( x_0 \in M \) and \( I_{x_0}(M) \) is its stabilizer, the map
\[
g \in I_{x_0}(M) \mapsto d_{x_0}g
\]
is an isomorphism of topological groups. Since \( d_{x_0}g \) preserves the metric on the tangent space \( T_{x_0}M, I_{x_0}(M) \)
is identified with a locally compact subgroup of \( O(T_{x_0}M) \), hence it is compact.

2. \( G \)-invariant measures and operators

Let \( M \) be a homogeneous space of \( G \). From now on we suppose that

(1) the stabilizer \( G_x \) of \( x \in M \) is compact;

(2) with \( K = G_{x_0} \), for a given \( x_0 \in M \), \( M \) is homeomorphic to \( G/K \).

Lemma 2.1 (*). Let \( M = G/K \), with \( K \) a compact subgroup of \( G \), and let \( \Lambda \) be the canonical projection of \( G \) onto \( G/K \). The map \( \Lambda^* \) assigning to \( f \in C_c(G/K) \) the function
\[
\Lambda^* f = f \circ \Lambda
\]
on \( G \) is an isomorphism of \( C_c(G/K) \) onto the space \( C_c(G; K) \) of the functions in \( C_c(G) \) which are invariant under right translations by elements of \( K \) (i.e., such that \( R_k f = f \) for every \( k \in K \)).

If \( dk \) is the normalized Haar measure on \( K \), the map
\[
f \mapsto f^k(g) = \int_K f(gk) \, dk
\]
(2.1) maps \( C_c(G) \) onto \( C_c(G; K) \) and \( (f^k)^* = f^* \).

A regular Borel measure \( \mu \) on \( M \) is called \( G \)-invariant if \( \mu(g \cdot B) = \mu(B) \) for every Borel set \( B \) in \( M \) and every \( g \in G \). \( G \)-invariance is equivalent to the condition
\[
\int_M f(g \cdot x) \, d\mu(x) = \int_M f(x) \, d\mu(x)
\]
for every $f \in C_c(M)$ and $g \in G$.

**Proposition 2.2** (*). Let $dg$ be a fixed left Haar measure on $G$. The measure $dx$ on $G/K$ such that

$$\int_{G/K} f(x) \, dx = \int_G \Lambda^* f(g) \, dg .$$

is the unique (up to multiplication by a positive constant) positive regular Borel measure on $G/K$ which is $G$-invariant.

By $L^p(K\backslash G)$, $L^p(G/K)$, $L^p(K\backslash G/K)$ we denote the spaces of functions which are, respectively, left-$K$-invariant, right-$K$-invariant, bi-$K$-invariant. Each of the operators

$$f \mapsto \int_K f(kg) \, dk , \quad f \mapsto \int_K f(gk) \, dk , \quad f \mapsto \int_{K \times K} f(k_1gk_2) \, dk_1 \, dk_2$$

is a norm-one projection of $L^p(G)$ onto $L^p(K\backslash G)$, $L^p(G/K)$, $L^p(K\backslash G/K)$ respectively. For $p = 2$, they are the orthogonal projections.

For $f$ defined on $M$ and $g \in G$ we set

$$\tau_g f(x) = f(g^{-1} \cdot x) .$$

(2.2)

In each space $L^p(M) = L^p(M, dx)$, each $\tau_g$ is an isometry.

Consider now an integral operator on $M$, defined on $C_c(M)$,

$$Tf(x) = \int_{G\backslash M} \Phi(x,y) f(y) \, dy ,$$

(2.3)

with $\Phi$ locally integrable on $M \times M$.

We say that $T$ is $G$-invariant if

$$T(\tau_g f) = \tau_g (Tf)$$

for every $f$ and $g \in G$.

**Theorem 2.3** (*). Let $x_0$ be a fixed point in $M$, and set $K = G_{x_0}$. The following properties are equivalent:

1. $T$ is $G$-invariant;
2. $\Phi(g \cdot x, g \cdot y) = \Phi(x, y)$ for every $g \in G$ and $x, y \in M$;
3. there exists a locally integrable, $K$-invariant function $\varphi$ on $M$ such that $\Phi(x, g \cdot x_0) = \varphi(g^{-1} \cdot x)$;
4. there exists $\psi$ locally integrable on $G$ and bi-$K$-invariant, such that

$$\Phi(g \cdot x_0, h \cdot x_0) = \psi(h^{-1} g) .$$

(2.4)

Moreover, if $\Phi$ and $\psi$ are related by (2.4) and $\Lambda(g) = g \cdot x_0$, then

$$\Lambda^*(Tf) = (\Lambda^* f) \ast \psi .$$
The functions $\Phi$, $\varphi$ and $\psi$ are related by the identities

$$\varphi = \Phi(\cdot, x_0), \quad \psi = \varphi \circ \Lambda.$$ 

**Corollary 2.4.** Let $T_1, T_2$ be two operators as in (2.3). If the associated functions $\varphi_1, \varphi_2$ are in $L^1(M)$ (and hence $\psi_1, \psi_2 \in L^1(M)$) then $T_1$ and $T_2$ are bounded on $L^p(M)$ for $1 \leq p \leq \infty$, and the function $\psi$ on $G$ associated to $T_1 T_2$ is $\psi_2 \ast \psi_1$.

**Examples (\*)**

(2.a) Let $G = O(n)$, $M = S^{n-1}$. Taking $x_0 = e_1$, we have $K \sim O(n-1)$, like in Example (1.d). A $K$-invariant function on $S^{n-1}$ is a function depending only on the coordinate $x_1$ (the “latitude”). Such a function is called zonal. Same with $O(n)$ replaced by $SO(n)$.

(2.b) Let $G$ be the Euclidean motion group on $\mathbb{R}^n$, defined as the semidirect product of $\mathbb{R}^n$ (subgroup of translations) and $O(n)$ (subgroup of rotations). Multiplication in $G$ (with notation $(k, v) \in O(n) \times \mathbb{R}^n$) is given by $(k, v)(k', v') = (kk', kv' + v)$.

$G$ acts naturally on $\mathbb{R}^n$ by

$$(k, v) \cdot x = kx + v.$$ 

The action is transitive and the stabilizer of the point 0 is the subgroup $K = O(n)$ of rotations. A $K$-invariant function on $\mathbb{R}^n$ is a function which is constant on spheres centered at the origin, i.e., a radial function.

(2.c) Let $H$ be a compact group. Take $G$ as the direct product $H \times H$, $M = H$ with action given by

$$(h, h') \cdot x = hxh'^{-1}.$$ 

The action is transitive and the stabilizer of the identity element $e \in H$ is the “diagonal” subgroup $K = \{(h, h) : h \in H\}$, isomorphic to $H$. A function $f$ on $H = M$ is $K$-invariant if and only if

$$f((h, h) \cdot x) = f(hxh^{-1}) = f(x)$$ 

for every $h, x \in H$, i.e., a central function.

3. **Gelfand pairs**

Theorem 2.3 indicates that the analysis of algebras of $G$-invariant operators on $M$ can be read as analysis of bi-$K$-invariant functions (or measures, or more general entities) on $G$, and composition of such operators reduces to convolution in $G$ of the corresponding bi-$K$-invariant functions.
Convolution respects lateral $K$-invariance, e.g.,

\[ L^1(K \backslash G) \ast L^1(G) \subset L^1(K \backslash G), \quad L^1(G) \ast L^1(G/K) \subset L^1(G/K), \]

(3.1) \[ L^1(K \backslash G) \ast L^1(G/K) \subset L^1(K \backslash G/K). \]

It follows that $L^1(K \backslash G/K)$ is a Banach subalgebra of $L^1(G)$, closed under involution.

**Definition 3.1.** $(G, K)$ is called a Gelfand pair if $L^1(K \backslash G/K)$ is commutative.

In fact, this condition can be replaced by the requirement that certain other convolution algebras are commutative, like $C_c(K \backslash G/K)$ or $M(K \backslash G/K)$.

The most elementary bi-$K$-invariant measures on $G$ are the *bi-$K$-orbital measures* $\mu_g$ defined by

\[ \int_K f(h) d\mu_g(h) = \int_K \int_K f(kgk') dk dk', \]

where $g \in G$. The support of $\mu_g$ is the double coset $KgK$.

Every $f \in L^1(K \backslash G/K)$ can be expressed as an average of bi-$K$-orbital measures, through the Bochner integral

\[ f = \int_G f(g) \mu_g dg. \]

This provides the argument for proving the following statement.

**Proposition 3.2 (\textsuperscript{\textasteriskcentered}).** $(G, K)$ is Gelfand pair if and only if the bi-$K$-orbital measures on $G$ commute.

In particular, when $(G, K)$ is a Gelfand pair, one has commutativity of double cosets\footnote{It is stated in Wolf’s book that this condition is also sufficient to imply that $(G, K)$ is a Gelfand pair, but the proof is not correct. I do not know if the statement is true anyhow.}:

(3.2) \[ (KgK)(KhK) = (KhK)(KgK), \]

for every $g, h \in G$.

Before looking at examples, we give one necessary and one sufficient condition for $(G, K)$ being a Gelfand pair.

**Lemma 3.3.** If $(G, K)$ is a Gelfand pair, then $G$ is unimodular.

**Proof.** Take $\varphi, \psi \in L^1(K \backslash G/K)$. Denoting by $dh$ a left Haar measure on $G$,

\[ \varphi \ast \psi(\epsilon) = \int_G \varphi(h) \psi(h^{-1}) dh = \psi \ast \varphi(\epsilon) = \int_G \psi(h) \varphi(h^{-1}) dh. \]

On the other hand,

\[ \int_G \varphi(h) \psi(h^{-1}) dh = \int_G \varphi(h^{-1}) \psi(h) \Delta(h)^{-1} dh, \]

where $\Delta$ is the modular function on $G$. 

\[ \int_G \varphi(h) \psi(h^{-1}) dh = \int_G \varphi(h^{-1}) \psi(h) \Delta(h)^{-1} dh, \]
By compactness of $K$, $\Delta = 1$ on $K$, and hence $\Delta$ is bi-$K$-invariant. If $G$ were not unimodular, there would then be a bi-$K$-invariant open set $A$ in $G$ on which $\Delta > m > 1$. Taking $\varphi = \tilde{\psi} = \chi_A$, a contradiction would follow. \hfill \Box

**Proposition 3.4.** Suppose there is an automorphism $\theta$ of $G$ such that, for every $g \in G$, $g^{-1} \in K(\theta g)K$. Then $(G, K)$ is a Gelfand pair.

**Proof.** The modular function of $G$ satisfies the condition $\Delta(\theta g) = \Delta(g)$ for every automorphism $\theta$. Since $\Delta$ is bi-$K$-invariant, the assumption implies that $\Delta(g^{-1}) = \Delta(g)$ for every $g$, hence $G$ is unimodular.

Set $\tilde{f}(g) = f(g^{-1})$ and $f^\theta(g) = f(\theta g)$. For every $f \in L^1(K \backslash G / K),$

$$\tilde{f}(g) = f(\theta(g)) = f^\theta(g).$$

If $f, f' \in L^1(K \backslash G / K)$, we have $\tilde{f} \ast \tilde{f}' = (f' \ast f)^\theta$, while

$$f^\theta \ast f'^\theta(g) = \int_G f(\theta h) f'(\theta(h^{-1}g)) \, dh = c_\theta^{-1} \int_G f(h) f'(h^{-1}g) \, dh = c_\theta^{-1} (f \ast f'^\theta)(g),$$

with $c_\theta > 0$. Hence

$$(f \ast f')^\theta = c_\theta f^\theta \ast f'^\theta = c_\theta \tilde{f} \ast \tilde{f}' = c_\theta (f' \ast f)^\theta = c_\theta (f \ast f')^\theta,$$

i.e., $f \ast f' = c_\theta f' \ast f$. Iterating the same identity, we have $c_\theta^j = 1$, so that $c_\theta = 1$. \hfill \Box

**Examples (\textit{\textasteriskcentered}).**

(3.a) One proves that $(SO_n, SO_{n-1})$ is a Gelfand pair by applying Proposition 3.4 with $\theta$ equal to the identity map. This amounts to proving that, with $\epsilon_1 = (1, 0, \ldots, 0)$, for every $g \in SO(n)$ there exists $k \in SO(n-1)$ such that $g^{-1} \epsilon_1 = kg \epsilon_1.$

(3.b) A similar argument applies to the pair $(SU_n, SU_{n-1})$, only with $\theta(g) = g$.

(3.c) Consider the pair $(SL_2(\mathbb{R}), SO_2)$, whose homogeneous space has been identified with the upper half-plane in $\mathbb{C}$. Proposition 3.4 can be applied again, once the following property is proved: given $g \in SL_2(\mathbb{R})$, there exist $k_1, k_2 \in SO_2$ such that $g^{-1} = k_1 g k_2.$

\textsuperscript{2}This requires the following decomposition theorem for invertible matrices: given $A$ invertible, there exists a diagonal matrix $D$ and two orthogonal matrices $U, V$ such that $A = UDV$. 
4. Representation-theoretical characterization of Gelfand pairs

If \( \pi \) is a unitary representation of \( G \) on a Hilbert space \( H_\pi \), denote by \( H^K_\pi \) the closed subspace of \( H_\pi \) of \( K \)-invariant vectors \( v \), i.e., such that \( \pi(k)v = v \) for every \( k \in K \). One says that \( \pi \) is \textit{class-one (relative to} \( K \)) if \( H^K_\pi \) is nontrivial.

The operator

\[
P^K_\pi = \int_K \pi(k) \, dk
\]

is the orthogonal projection of \( H_\pi \) onto \( H^K_\pi \). A function \( f \in L^1(G) \) is

- right-\( K \)-invariant if and only if, for every irreducible unitary representation \( \pi \) of \( G \), \( \pi(f) = \pi(f)P^K_\pi \);
- left-\( K \)-invariant if and only if, for every irreducible unitary representation \( \pi \) of \( G \), \( \pi(f) = P^K_\pi \pi(f) \);
- bi-\( K \)-invariant if and only if, for every irreducible unitary representation \( \pi \) of \( G \),

\[
\pi(f) = P^K_\pi \pi(f) P^K_\pi.
\]

(4.1)

\textbf{Theorem 4.1.} \((G, K)\) is a Gelfand pair if and only if, for every class-one irreducible unitary representation \( \pi \) of \( G \), \( H^K_\pi \) is one-dimensional.

\textbf{Proof.} Suppose that \( \dim H^K_\pi \leq 1 \) for every class-one irreducible unitary representation \( \pi \). By (4.1), if \( f \in L^1(K \backslash G / K) \), \( \pi(f) = 0 \) if \( \pi \) is not class-one, or \( \pi(f) = c(\pi)P^K_\pi \), because \( P^K_\pi \) is a rank-one projection. In either case, given \( f, g \in L^1(K \backslash G / K) \), \( \pi(f) \) and \( \pi(g) \) commute. Hence \( f * g = g * f \).

Conversely, assume that \((G, K)\) is a Gelfand pair, and let \( \pi \) be a class-one irreducible unitary representation of \( G \). The extension of \( \pi \) to \( L^1(G) \) is also irreducible.

For every \( f \in L^1(K \backslash G / K) \), the operator \( \hat{\pi}(f) \) fixes \( H^K_\pi \) by (4.1), so that the \(*\)-representation

\[
\hat{\pi}(f) = \pi(f)|_{H^K_\pi}
\]

of \( L^1(K \backslash G / K) \) on \( H^K_\pi \) is well defined. We prove that \( \hat{\pi} \) is irreducible.

Let \( H^K_\pi = X_1 \oplus X_2 \), with \( X_1, X_2 \) orthogonal and \( \hat{\pi} \)-invariant. Assuming that \( X_1 \) is nontrivial, let \( H_1 = \text{span} \{ \pi(f)v : f \in L^1(G), v \in X_1 \} \subset H_\pi \). Then \( H_1 \) is invariant under \( \pi(L^1(G)) \). Hence \( H_1 \) is dense in \( H_\pi \).
On the other hand, $H_1$ is orthogonal to $X_2$. Indeed, given $v \in X_1$, $w \in X_2$ and $f \in L^1(G)$, we have
\[
\langle \pi(f)v, w \rangle = \int_G f(g) \langle \pi(g)v, w \rangle \, dg
= \int_{K \times K} \int_G f(g) \langle \pi(g) \pi(k_1)v, \pi(k_2)w \rangle \, dg \, dk_1 \, dk_2
= \int_{K \times K} \left( \int_G f(k_2^{-1} g k_1) \, dk_1 \, dk_2 \right) \langle \pi(g)v, w \rangle \, dg
= \langle \hat{\pi}(f)v, w \rangle
= 0,
\]
where $\hat{f}(g) = \int_{K \times K} f(k_2^{-1} g k_1) \, dk_1 \, dk_2 \in L^1(K \backslash G/K)$.

Hence $X_2$ is trivial. So $\hat{\pi}$ is an irreducible *-representation of a commutative *-algebra, and Schur's lemma implies that $\dim \mathcal{H}^K_{\pi} = 1$. \qed

Example (*).

(4.a) Given a compact group $H$, let $G = H \times H \rtimes K = \{(h, h) : h \in H\}$. By Schur's lemma, an irreducible unitary representation $\pi = \pi_1 \otimes \pi_2$ of $H \times H$ is class-one relative to $K$ if and only if $\pi_2 \sim \pi_1^*$, the contragredient of $\pi_1$. Again by Schur's lemma, the only $K$-invariant elements of $\mathcal{H}_\pi = \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}'$ are the scalar multiples of $\sum_j e_j \otimes e_j'$, where $\{e_j\}$ is an orthonormal basis of $\mathcal{H}_{\pi_1}$ and $\{e_j'\}$ its dual basis of $\mathcal{H}_{\pi_1}'$.

Hence $(G, K)$ is a Gelfand pair.

5. Spherical functions

Let $(G, K)$ be a Gelfand pair. We apply Gelfand theory to the commutative convolution algebra $L^1(K \backslash G/K)$ and determine the multiplicative linear functionals on it (characters). Notice that the dual space is naturally identified with $L^\infty(K \backslash G/K)$.

Theorem 5.1. The following conditions are equivalent, for $\varphi \in L^\infty(G)$:

1. $\varphi$ is bi-$K$-invariant and defines a character of $L^1(K \backslash G/K)$;
2. $\varphi$ is continuous, non-trivial, and satisfies the identity
\[
\int_K \varphi(kg'k) \, dk = \varphi(g) \varphi(g').
\]

Proof. Suppose that $\varphi \in L^\infty(K \backslash G/K)$ defines a multiplicative functional. For $v \in L^1(G)$, set
\[
v^\dagger(g) = \int_{K \times K} v(k_1 g k_2) \, dk_1 \, dk_2.
\]
For $u \in L^1(K\backslash G/K)$,

$$\lambda(u \ast v^t) = \int_G (u \ast v^t)(h) \varphi(h) \, dh$$

$$= \int_{G \times G} u(h) \int_{K \times K} v(kh'k') \varphi(h) \, dk \, dk' \, dh$$

$$= \int_{G \times G} u(h) \int_{K \times K} v(kh'k') \varphi(hh') \, dk \, dk' \, dh$$

$$= \int_{G \times G} u(h) v(h') \int_{K} \varphi(hkh') \, dk \, dh'$$

$$= \int_{G \times G} u(h) v(h') \varphi(hh') \, dh' \, dh.$$

This expression must be equal to

$$\lambda(u) \lambda(v^t) = \int_G u(h) \varphi(h) \, dh \int_G v^t(h) \varphi(h) \, dh$$

$$= \int_G u(h) \varphi(h) \, dh \int_G v(h) \varphi(h) \, dh$$

$$= \int_{G \times G} u(h) v(h') \varphi(h) \varphi(h') \, dh \, dh'.$$

Since $v$ is arbitrary, it must be

$$(\int_G u(h) \varphi(h) \, dh) \varphi(h') = \int_G u(h) \varphi(hh') \, dh = (u \ast \varphi)(h')$$

for almost every $h' \in G$. The right-hand side is a continuous function of $h'$, which implies that $\varphi$ coincides a.e. with a continuous function.

Take now $u_0 \in L^1(G)$. With $u = u_0^2$ in (2.2), we have

$$\varphi(h') \int_G u_0(h) \varphi(h) \, dh = \int_G \int_{K \times K} u_0(k_1k_2) \varphi(hh') \, dk_1 \, dk_2 \, dh$$

$$= \int_G u_0(h) \int_K \varphi(hk^{-1}h') \, dk_2 \, dh,$$

which gives (5.1) since $u_0$ is arbitrary.

Conversely, assume that $\varphi$ is continuous, bounded, nontrivial, and satisfies (5.1). This identity implies that, for every $g, g' \in G$, $k \in K$,

$$\varphi(g) \varphi(kg') = \varphi(gk) \varphi(g') = \varphi(g) \varphi(g').$$

Since $\varphi$ is non-identically zero, it is bi-$K$-invariant.
Given \( u, v \in L^1(K \backslash G/K) \),
\[
\int_G u(g)\varphi(g) \, dg \int_G v(h)\varphi(h) \, dh = \iint_{G \times G} \int_K u(g)v(h)\varphi(gh) \, dk \, dh \, dg \\
= \iint_{G \times G} \int_K u(gk^{-1}) \, dk \, v(h)\varphi(gh) \, dh \, dg \\
= \iint_{G \times G} u(g) \, v(h)\varphi(gh) \, dh \, dg \\
= \iint_{G \times G} u(gh^{-1}) \, v(h)\varphi(g) \, dh \, dg \\
= \int_G (u \ast v)(g)\varphi(g) \, dg.
\]

\[\square\]

**Definition 5.2.** The continuous functions on \( G \) satisfying (5.1) are called the spherical functions for the pair \((G, K)\).

Theorem 3.1 has an analogue for general (i.e., not necessarily bounded) spherical functions: they provide the multiplicative functionals on \( C_c(K \backslash G/K) \).

Notice that (5.1) implies, besides bi-\(K\)-invariance, that \( \varphi(e) = 1 \).

It is also common to call spherical the \(K\)-invariant functions \(\hat{\varphi}\) on \(G/K\) such that \(\Lambda^*\hat{\varphi}\) is spherical on \(G\).

### 6. Spherical transform

The Gelfand spectrum \( \Delta = \Delta(G, K) \) of the Banach algebra \( L^1(K \backslash G/K) \) is therefore naturally identified with the set of bounded spherical functions of \((G, K)\). The following theorem is an extension of a well-known result in Fourier analysis of locally compact abelian groups.

**Theorem 6.1 \(^*\).** The Gelfand topology on \( \Delta \) coincides with the compact-open topology (i.e., the topology of uniform convergence on compact sets) on the set of bounded spherical functions.

**Definition 6.2.** The Gelfand transform of \( f \in L^1(K \backslash G/K) \) is called the spherical transform of \( f \).

We write
\[
\hat{f}(\varphi) = \int_G f(g)\varphi(g^{-1}) \, dg,
\]
when \( \varphi \in \Delta \) (i.e., it is a bounded spherical function). It satisfies the properties:

- linearity in \( f \);
- \( \hat{f} \in C_b(\Delta) \) and \( \|\hat{f}\|_\infty \leq \|f\|_1 \);
- \( \hat{f} \ast g = \hat{f} \hat{g} \);
- \( f \ast \varphi = \hat{f}(\varphi)\varphi \) for every \( f \in L^1(K \backslash G/K) \) and \( \varphi \in \Delta \).
7. Positive definite spherical functions

**Theorem 7.1** (*). Let $(G,K)$ be a Gelfand pair, and let $\pi$ be an irreducible unitary representation of $G$, class-one relative to $K$. Let $v$ be a unit vector in $\mathcal{H}_x^K$. Then the function

\[
\varphi_\pi(g) = \langle v, \pi(g)v \rangle ,
\]

(7.1)

does not depend on the choice $v$ and is a bounded spherical function. For $f \in L^1(K\backslash G/K)$,

\[
\hat{f}(\varphi_\pi) = \langle \pi(f)v, v \rangle ,
\]

and

\[
\pi(f) = \hat{f}(\varphi_\pi) \mathcal{P}_\pi^K .
\]

In general, not all bounded spherical transforms arise in this way. Clearly, the spherical functions $\varphi_\pi$ of (7.1) are positive definite on $G$, a property which is quite special in many examples, e.g. for the pair $(SL_2(\mathbb{R}), SO_2)$.

We denote by $\Delta^+$ the subset of $\Delta$ consisting of the positive definite spherical functions.

**Theorem 7.2** (uniqueness for the spherical transform). Let $f \in L^1(K\backslash G/K)$ be such that $\hat{f} = 0$ on $\Delta^+$. Then $f = 0$.

In particular, $L^1(K\backslash G/K)$ is semisimple.

**Proof.** The hypothesis implies that $\pi(f) = 0$ for every irreducible unitary representation of $G$. The conclusion follows from the Gelfand-Raikov theorem. \hfill \Box

The positive definite spherical functions can be characterized by an extremal property, in the same way as characters of an abelian group.

**Theorem 7.3** (**). Let $\mathcal{P}$ be the cone of positive definite bi-$K$-invariant functions on $G$. The extremal lines of $\mathcal{P}$ are those containing a positive definite spherical function.

Like on abelian groups, this fact leads to a Bochner-type theorem.

**Lemma 7.4.** The space $\{\hat{f}_{\Delta^+} : f \in L^1(K\backslash G/K)\}$ is uniformly dense in $C_0(\Delta^+)$. 

**Proof.** If $\varphi \in \Delta^+$, $\varphi^* = \varphi$, hence

\[
\hat{f}^* = \tilde{\hat{f}}
\]

for $f \in L^1(K\backslash G/K)$. The conclusion follows from the Stone-Weierstrass theorem. \hfill \Box
Theorem 7.5 (Bochner-Godement theorem (**)). Let $F$ be a positive definite bi-$K$-invariant function on $G$. There exists one and only one positive Borel measure $\mu_\psi$ on $\Delta^+$ such that
\[
F(g) = \int_{\Delta^+} \varphi(g) \, d\mu_\psi(\varphi).
\]

8. The fundamental theorems of the spherical transform

As for the Fourier transform on locally compact abelian groups, the Bochner theorem implies the existence of a Plancherel measure, with corresponding inversion and Plancherel formulas.

Theorem 8.1 (inversion formula (**)). There exists a unique positive Borel measure $\sigma$ on $\Delta^+$ such that, if $f \in C_c(K\setminus G/K)$ and $\hat{f} \in L^1(\Delta^+, \sigma)$ then
\[
f(g) = \int_{\Delta^+} \hat{f}(\varphi) \varphi(g) \, d\sigma(\varphi) .
\]

Theorem 8.2 (Plancherel-Godement theorem (**)). If $f \in C_c(K\setminus G/K)$, then $\hat{f} \in L^2(\Delta^+, \sigma)$ and
\[
\int_G |f(g)|^2 \, dg = \int_{\Delta^+} |\hat{f}(\varphi)|^2 \, d\sigma(\varphi).\]

The spherical transform extends by continuity to a unitary operator from $L^2(K\setminus G/K)$ onto $L^2(\Delta^+, \sigma)$. The Plancherel measure $\sigma$ is the only measure on $\Delta^+$ for which (8.2) holds for every $f \in C_c(K\setminus G/K)$.

9. Spherical analysis of $G$-invariant operators

Consider the algebra $A$ of bounded linear operators
\[
T : L^2(K\setminus G/K) \to L^2(K\setminus G/K)
\]
comuting with the action of $G$, i.e.,
\[
T(f \circ g) = (Tf) \circ g
\]
for every $g \in G$. This implies that, for every $u \in L^1(G)$ and $f \in L^2(K\setminus G/K)$,
\[
T(u \ast f) = u \ast (Tf).
\]

Lemma 9.1. There exists a function $m \in L^\infty(\Delta^+)$ such that $\hat{T}f = m\hat{f}$ for every $f \in L^2(K\setminus G/K)$. Moreover $\|m\|_{\infty} = \|T\|_{L(L^2(K\setminus G/K))}$.

Proof. Conjugating $T$ with the spherical transform, we obtain an operator $\hat{T}$, bounded on $L^2(\Delta^+, \sigma)$ and such that
\[
\hat{T}(\varphi h) = v\hat{T}(h),
\]
for every $v \in C_c(\Delta^+)$ and $h \in L^2(\Delta^+, \sigma)$.

Given an open, relatively compact $A \subset \Delta^+$, set $m_A = \hat{T}(\chi_A)$. For $h \in C_c(\Delta^+)$ supported in $A$, we then have

$$\hat{T}(h) = hm_A.$$

By continuity, this identity holds for every $h \in L^2(A)$. Moreover,

$$\|m_A\|_{L^\infty(A)} = \sup_{\|h\|_2 \leq 1} \|\eta m_A\|_2 \leq \|\hat{T}\| = \|T\|.$$ 

If $A, A'$ are two sets as above, on $A \cap A'$ we have

$$m_{A \cap A'} = \hat{T}(\chi_{A \cap A'}) = \chi_{A \cap A'} m_A = m_A,$$

and similarly $m_{A' \cap A} = m_{A'}$. Hence $m_A = m_{A'}$ on $A \cap A'$. This gives rise to a unique function $m$ defined on $\Delta^+$, with $\|m\|_\infty \leq \|\hat{T}\|$, and such that

$$\hat{T}h = hm$$

for every $h$.

The opposite implication with the inequality $\|\hat{T}\| \leq \|m\|_\infty$ is obvious from the Plancherel formula. \hfill \Box

**Theorem 9.2.** The algebra $\mathcal{A}$ is commutative and isomorphic to $L^\infty(\Delta^+)$. The spectrum in $L(L^2(M))$ of an operator $T \in \mathcal{A}$ coincides with its spectrum in $\mathcal{A}$ and with the essential image in $\mathcal{C}$ of its spherical multiplier.


10. The special case $(G, K) = (K \ltimes H, K)$

Let $H$ be a locally compact group and $K$ a compact group of automorphisms of $H$. Denote by $L^1(H)^K$ the space of $K$-invariant functions on $H$ which are integrable with respect to the left Haar measure $dh$ on $H$. Then $L^1(H)^K$ is an algebra under convolution.

We are interested in the situation where $L^1(H)^K$ is commutative. This is a Gelfand pair case: we have two groups acting on $H$:

- $K$, acting by automorphisms,
- $H$ itself, acting by left translations.

Combining the two actions together, we obtain the transformations of $H$ onto itself, depending on pairs $(k, h) \in K \times H$ and given by

$$(k, h) \cdot h' = h k(h'),$$

(a special case is that of Example 2.b with the Euclidean motion group).
Since $K$ normalizes the group of translations, the set of these transformations is closed under composition and is a group, the *semidirect product* $G = K \ltimes H$ with product

$$(k_1, h_1)(k_2, h_2) = (k_1 k_2, h_1 k_1(h_2)) .$$

The stabilizer in $G$ of the identity element $e_H$ of $H$ is $K$ (to be precise, $K \times \{e_H\}$). So $H$ is naturally identified with the homogeneous space $(K \ltimes H) / K$.

**Proposition 10.1** (*). *The map $\Lambda^*$ of Lemma 2.1 is a bijection from $L^1(H)^K$ to $L^1(K \backslash G / K)$ and respects convolution. In particular, $L^1(H)^K$ is commutative if and only if $(K \ltimes H, K)$ is a Gelfand pair.*

Since $H$ has its own group structure, it is natural and convenient to look for a reformulation of definitions and criteria presented above which do not require any specific analysis on $G$, but only on $H$.

For instance, orbital measures (cf. Proposition 3.2) can be defined on $H$ by

$$\int_H f(h) \, d\mu_a(h) = \int_K f(k(h_0)) \, dk ,$$

with the following consequence.

**Corollary 10.2.** $L^1(H)^K$ is commutative if and only if orbital measures on $H$ commute.

Similarly, spherical functions can be defined on $H$ by the functional equation

(10.1)\[ \int_K \varphi(h k h') \, dk = \varphi(h) \varphi(h') , \]

instead of (5.1).

The reformulation of Theorem 4.1 is very useful.

Let $\widehat{H}$ be the dual object of $H$, i.e., the set of equivalence classes $[\pi]$ of irreducible unitary representations $\pi$ of $H$. The group $K$ acts on $\widehat{H}$ in the following way.

Given $k \in K$ and $\pi$ irreducible and unitary,

$$\pi^k(h) = \pi(k^{-1} h)$$

defines an irreducible and unitary representation of $H$, which may or may not be equivalent to $\pi$. So we set

$$k \cdot [\pi] = [\pi^k] .$$

Let $K_\pi$ be the stabilizer of $[\pi]$. For $k \in K_\pi$, there exists a (unique up to a unitary factor) unitary operator $\tau(k)$ on $H_\pi$ which intertwines $\pi$ with $\pi^k$. This defines a projective unitary representation of $K_\pi$ on $H_\pi$. 
Since $K_\pi$ is compact, $\mathcal{H}_\pi$ decomposes as the direct sum of irreducible $\tau$-invariant subspaces, which we write as

$$\mathcal{H}_\pi = \bigoplus_{\sigma} m_\sigma V_\sigma,$$

where $\sigma$ varies over a set of inequivalent irreducible projective representations of $K_\pi$ and $m_\sigma$ is the multiplicity of $\sigma$ in $\tau$.

**Theorem 10.3 (**) $(K \ltimes H, K)$ is a Gelfand pair if and only if, for every irreducible unitary representation $\pi$ of $H$, all multiplicites $m_\sigma$ in (10.2) are 0 or 1.

One says that the action of $K_\pi$ on $\mathcal{H}_\pi$ is multiplicity-free.

The positive definite spherical functions on $H$ can be obtained as follows\(^3\).

Given $\pi$ and one irreducible component $V_\sigma$ from (10.2), let $\{e_1, \ldots, e_d\}$ an orthonormal basis of $V_\sigma$. Then the function of $h \in H$

$$\frac{1}{d} \sum_{j=1}^{d} \langle e_j, \pi(h)e_j \rangle$$

is only $K_\pi$ invariant, but

$$\varphi_{\pi,\sigma}(h) = \int_{K} \frac{1}{d} \sum_{j=1}^{d} \langle e_j, \pi(k(h))e_j \rangle \, dk$$

is $K$-invariant.

**Theorem 10.4 (**) Let $(K \ltimes H, K)$ be a Gelfand pair. The functions $\varphi_{\pi,\sigma}$ are all the positive definite spherical functions of $(K \ltimes H, K)$. One has $\varphi_{\pi,\sigma} = \varphi_{\pi',\sigma'}$ if and only if $[\pi] = k \cdot [\pi']$ for some $k \in K$ and $\sigma$ is conjugate to $\sigma'$.

Let $O_{[\pi]}$ be the $K$-orbit of $[\pi]$ in $\hat{H}$. The subset $\Delta^+$ of the Gelfand spectrum $\Delta$ is then parametrized by the pairs $(O_{[\pi]}, \sigma)$ according to (10.2).

**Corollary 10.5 (**) Assume that $(K \ltimes H, K)$ is a Gelfand pair, and let $f \in L^1(H)$. Then $f$ is $K$-invariant if and only if, for every irreducible unitary representation $\pi$ of $H$, $\pi(f)$ preserves the decomposition (10.2) of $\mathcal{H}_\pi$ and is a scalar multiple of the identity on each $V_\sigma$.

If this is the case, then

$$\pi(f)|_{V_\sigma} = \hat{f}(\varphi_{\pi,\sigma})I.$$

---

\(^3\) A $K$-invariant function $f$ on $H$ is positive definite if and only if the bi-$K$-invariant function $\Lambda^* f$ on $K \ltimes H$ is positive definite.
Examples.

(9.a) (*) Consider the case of Example 2.b, with $H = \mathbb{R}^n$ and $K = O_n$. The irreducible unitary representations of $\mathbb{R}^n$ are one-dimensional and given by the characters

$$\chi_{\xi}(x) = e^{i\xi \cdot x},$$

where $\xi \in \mathbb{R}^n$ and $\cdot$ denotes the scalar product. For $k \in O_n$,

$$\chi_{\xi}^k(x) = e^{i\xi \cdot (k^{-1}x)} = \chi_{k\xi}(x).$$

So the $K$-orbit of $\xi$ in $\mathbb{R}^n$ is the sphere of radius $|\xi|$. Since the representation space is one-dimensional, formula (10.3) trivializes quite a bit, giving the Bessel functions

$$\varphi_r(x) = \int_{O_n} e^{i\xi \cdot (kx)} \, dk,$$

where $|\xi| = r$.

The parameter $r$ provides a topological identification of $\Delta$ with the half-line $[0, +\infty)$. The spherical transform of $f \in L^1(\mathbb{R}^n)^{O_n}$ is

$$\hat{f}(r) = \int_{\mathbb{R}^n} f(x) \varphi_r(-x) \, dx = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \, dx = \mathcal{F}f(\xi),$$

where $|\xi| = r$ and $\mathcal{F}$ denoted the Fourier transform.

(9.b) (*) More generally, take $H = \mathbb{R}^n$ and $K$ a closed subgroup of $O_n$. As in the previous case, the bounded spherical functions are averages of characters under $K$, and so they are in one-to-one correspondence with $K$-orbits in $\mathbb{R}^n$. Topologically, $\Delta \sim \mathbb{R}^n/K$, the orbit space. If $\pi$ denotes the canonical projection of $\mathbb{R}^n$ onto $\mathbb{R}^n/K$, the spherical transform is related to the Fourier transform by the identity

$$\hat{f} \circ \pi = \mathcal{F}f.$$

(9.c) Take now $H$ to be the Heisenberg group $\mathbb{H}_n$, with the exponential coordinates $(z, t) \in \mathbb{C}^n \times \mathbb{R}$. It is well known that $U_n$ acts by automorphisms of $\mathbb{H}_n$.

Fix $K$ a closed subgroup of $U_n$. To decide whether $(K \ltimes \mathbb{H}_n, K)$ is a Gelfand pair, it suffices to consider the infinite dimensional representations of $\mathbb{H}_n$. It is convenient to use the Bargmann form of these representations, all acting on the same space $\mathcal{F}$, called the Fock space, of holomorphic functions on $\mathbb{C}^n$. What is relevant to know is that

- the stabilizer of each element of $\mathbb{H}_n$ is all of $K$ (this is a consequence of the Stone-von Neumann theorem);
- the representation $\tau$ of $K$ on $\mathcal{F}$ is a true representation, and is given by the natural action $F \mapsto F \circ k^{-1}$.  

This says, to begin with, that $\tau$ preserves degrees in $F$ and hence the decomposition

$$F = \bigoplus_{k \in \mathbb{N}} P^k,$$

into spaces of homogeneous polynomials of a given degree. In general, the $P^k$ need not be irreducible, and the decomposition of each $P^k$ into irreducibles may produce multiplicities.

In any case, the conclusion is stated in simple algebraic terms.

**Proposition 10.6 (**)**. $(K \rtimes \mathbb{H}_n, K)$ is a Gelfand pair if and only if the space of polynomials in $n$ complex variables decomposes under the natural action of $K$ without multiplicities.

(9.d) (*) Let $H$ be a compact group, and take $K = H$ acting on itself by inner automorphisms. Forming the semidirect product $H \ltimes H$ (where the first factor acts by inner automorphisms and the second by left translations) we obtain once again the pair of Examples 2.c and 4.a. The theory of central functions can then be rewritten in the setting of this section.
CHAPTER 2

LIE THEORY

1. Smooth transitive actions

Let $G$ be a Lie group, $M$ a connected differentiable manifold with a smooth action

$$\psi : G \times M \rightarrow M.$$ 

We assume that the action is transitive and that the stabilizer $K = G_{x_0}$ of a given point is compact. By Theorem 1.4 of Ch. 1, $M$ is homeomorphic to $G/K$ and the action $\psi$ is conjugate to the canonical action on $G/K$. In particular, and $M$ inherits the real-analytic structure of $G/K$ and $\psi$ is real-analytic.

The new object that one can insert in the theory of Gelfand pairs is the rôle played by the algebra $\mathcal{D}(M)^G$ of $G$-invariant differential operators\footnote{(*) We recall that a differential operator on a smooth manifold $M$ is a linear operator $D : C^\infty(M) \rightarrow C^\infty(M)$ such that, in any set of local coordinates $x = \eta(t)$, with $x$ in an open subset of $M$ and $t \in \Omega \subseteq \mathbb{R}^n$,

$$(Df)(\eta(t)) = \sum_{|\alpha| \leq m} a_\alpha(t) \partial^\alpha (f \circ \eta)(t),$$

with coefficients $a_\alpha \in C^\infty(\Omega)$. This is equivalent to requiring that $D$ is linear and preserves supports: for every $f \in C^\infty(M)$,

$$\text{supp}(Df) \subseteq \text{supp} f.$$} on $M$.

A differential operator $D$ on $M$ is $G$-invariant if, for every $f \in C^\infty(M)$ and $g \in G$,

$$D(f \circ g) = (Df) \circ g.$$ 

By $\mathcal{D}(G)$ we denote the algebra of left-invariant differential operators on $G$. Given $L \in \mathcal{D}(G)$ and $k \in K$, we set

$$\text{Ad}(k)L = R_kLR_k^{-1},$$

where $R$ is the right regular representation on $C^\infty(G)$.

We assume now the identification $M \sim G/K$.

Lemma 1.1 \textit{(*)}. Let $L$ be a left-invariant differential operator on $G$. The following conditions are equivalent:
(1) For every $f \in C^\infty(G/K)$, the function $L(\Lambda^* f)$ (with $\Lambda^*$ defined in Lemma 2.1 of Ch. 1) is right-$K$-invariant;

(2) for every $k \in K$, $\text{Ad}(k)L = L$.

When they are satisfied, the operator $D$ such that

\[ \Lambda^*(Df) = L(\Lambda^* f) \]

is well defined on $C^\infty(G/K)$ and is in $\mathcal{D}(G/K)^G$.

Denote by $\mathcal{D}(G)^{\text{Ad}(K)}$ the algebra of $\text{Ad}(K)$-invariant operators in $\mathcal{D}(G)$.

**Proposition 1.2** (*). The map $L \mapsto D$ defined by (1.1) is an isomorphism of $\mathcal{D}(G)^{\text{Ad}(K)}$ onto $\mathcal{D}(G/K)^G$.

Let $\mathfrak{g}, \mathfrak{t}$ be the Lie algebras of $G$ and $K$ respectively. Since $\mathfrak{t}$ is $\text{Ad}(K)$-invariant in $\mathfrak{g}$ and $K$ is compact, we can select an $\text{Ad}(K)$-invariant complementary subspace of $\mathfrak{t}$ in $\mathfrak{g}$, so to have the decomposition\(^2\)

\[ \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} . \]

We will think of $\mathfrak{p}$ as a model of the tangent space to $G/K$ at the point $eK$ (i.e., to $M$ at the base point $x_0$). In fact, if $\exp : \mathfrak{g} \mapsto G$ is the exponential map, the map

\[ \eta : \mathfrak{p} \ni t \mapsto \Lambda \circ \exp(t) , \]

restricted to a neighborhood of 0 in $\mathfrak{p}$, gives a local coordinate system at the base point in $G/K$.

The map $\eta$ intertwines the action of $K$ on $G/K$ with the adjoint action on $\mathfrak{p}$.

**Proposition 1.3** (*). Given an $\text{Ad}(K)$-invariant polynomial $P$ on $\mathfrak{p}$, the differential operator on $G/K$

\[ D_P f(gK) = P(i^{-1} \partial_t |_{t=0} f(g\eta(t))) , \]

is well defined on $C^\infty(G/K)$ and is $K$-invariant. Conversely, any element of $\mathcal{D}(G/K)^G$ arises in way from a unique $\text{Ad}(K)$-invariant polynomial $P$.

The operator $D_P$ is formally self-adjoint (i.e., $\langle D_P f, h \rangle = \langle f, D_P h \rangle$ for every test functions $f, h$) if and only if $P$ is real valued.

In general, the correspondence $P \mapsto D_P$ of (1.3) is not multiplicative. However, it is multiplicative modulo lower-order terms:

\[ D_{P_1} D_{P_2} = D_{P_1 P_2} + \text{terms of order lower than } \deg P_1 + \deg P_2 . \]

This has the following consequence.

\(^2\)In general $\mathfrak{p}$ is not a Lie subalgebra.
Corollary 1.4 (*). Let \( \{P_1, \ldots, P_k\} \) be a finite system of generators of the algebra \( \mathcal{P}(\mathfrak{p})^{\text{Ad}(K)} \) of \( \text{Ad}(K) \)-invariant polynomials on \( \mathfrak{p} \). Then \( \{D_{P_1}, \ldots, D_{P_k}\} \) generates \( \mathbb{D}(G/K)^G \).

It follows from the Hilbert basis theorem that \( \mathcal{P}(\mathfrak{p})^{\text{Ad}(K)} \) is finitely generated. Hence \( \mathbb{D}(G/K)^G \) is also finitely generated.

Consider the Fischer inner product
\[
\langle P, Q \rangle = Q(\partial)P(0)
\]
on the space of real-valued polynomials on \( \mathfrak{p} \). For each degree \( j \) take an orthonormal basis \( \{Q_{d_1}^j, \ldots, Q_{d_j}^j\} \) of the subspace of \( \text{Ad}(K) \)-invariant polynomials homogeneous of degree \( j \). If \( F \) is \( \text{Ad}(K) \)-invariant and analytic near 0, we have the Taylor expansion
\[
F(t) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{d_j} Q_{d_j}^j(\partial)F(0)Q_{\ell}^j(t).
\]

Conjugating with the map \( \eta \) in (1.2), we have the following Taylor formula for \( K \)-invariant functions at \( x_0 \).

Corollary 1.5. If \( f \) is analytic near \( x_0 \) and \( K \)-invariant, then
\[
f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{d_j} D_{Q_{d_j}^j}f(x_0)Q_{\ell}^j(\eta^{-1}(x)) \tag{1.4}.
\]

2. Gelfand pairs

We adopt the following notation. Given \( f \) integrable on \( G/K \), and \( u \in L^1(G) \), define \( u \ast f \) on \( G/K \) as
\[
u \ast f(x) = \int_G u(h)f(h^{-1} \cdot x) \, dh. \tag{2.1}
\]

From the identity
\[
\Lambda^*(u \ast f) = u \ast (\Lambda^* f), \tag{2.2}
\]
we easily obtain various properties of \( \ast \), e.g.,
\[
(u \ast v) \ast f = u \ast (v \ast f), \quad \|u \ast f\|_p \leq \|u\|_1 \|f\|_p, \quad \text{ecc.}
\]

If \( D \in \mathbb{D}(G/K)^G \) and \( f \in C_c^\infty(G/K) \),
\[
D(u \ast f)(x) = D \int_G u(h)\tau_h f(x) \, dx = \int_G u(h)\tau_h Df(x) \, dx = u \ast (Df)(x). \tag{2.3}
\]

Lemma 2.1. If \((G, K)\) is a Gelfand pair, the algebra \( \mathbb{D}(G/K)^G \) is commutative.
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PROOF. Given \( f, g \in C_c^\infty(G/K)^K \), consider \( \Lambda^* f, \Lambda^* g \in C_c^\infty(K \backslash G/K) \). By (2.2),
\[
\Lambda^* ((\Lambda^* f) \ast g) = (\Lambda^* f) \ast (\Lambda^* g)
\]
\[
= (\Lambda^* g) \ast (\Lambda^* f)
\]
\[
= \Lambda^* ((\Lambda^* g) \ast f),
\]
so that
\[
(\Lambda^* f) \ast g = (\Lambda^* g) \ast f.
\]

If \( D_1, D_2 \in \mathbb{D}(G/K)^K \), also \( D_1 f, D_2 g \) are \( K \)-invariant, hence
\[
D_1 D_2 ((\Lambda^* f) \ast g) = D_1 ((\Lambda^* f) \ast D_2 g) = D_1 (\Lambda^* (D_2 g) \ast f)
\]
\[
= \Lambda^* (D_2 g) \ast D_1 f = \Lambda^* (D_1 f) \ast D_2 g
\]
\[
= D_2 (\Lambda^* (D_1 f) \ast g) = D_2 (\Lambda^* (D_1 f) \ast g)
\]
\[
= D_2 ((\Lambda^* g) \ast D_1 f) = D_2 D_1 ((\Lambda^* g) \ast f)
\]
\[
= D_2 D_1 ((\Lambda^* f) \ast g).
\]

Given \( u \in C_c^\infty(G) \), let
\[
u^\#(h) = \int_{K \times K} u(khk') \, dk \, dk' .
\]

So, for \( g \in C_c^\infty(G/K)^K \),
\[
D_1 D_2 (u^\# \ast g)(x_0) = u^\# \ast (D_1 D_2 g)(x_0)
\]
\[
\int_G \int_{K \times K} u(khk')(D_1 D_2 g)(h^{-1} \cdot x_0) \, dk \, dk' \, dh
\]
\[
\int_G \int_{K \times K} u(h)(D_1 D_2 g)(h'k^{-1} \cdot x_0) \, dk \, dk' \, dh
\]
\[
u \ast (D_1 D_2 g)(x_0),
\]
and similarly for the product \( D_2 D_1 \). Hence
\[
u \ast (D_1 D_2 g)(x_0) = u \ast (D_1 D_2 g)(x_0).
\]

Taking \( u \) as an element of an approximate identity and passing to the limit, we obtain that \( D_1 D_2 g(x_0) = D_2 D_1 g(x_0) \).

Taking now \( g \in C_c^\infty(G/K) \), set \( \bar{g}(x) = \int_K g(k^{-1} \cdot x) \, dk \). Then
\[
D_1 D_2 \bar{g}(x_0) = \int_K (D_1 D_2 \tau_k g)(x_0) \, dk = \int_K \tau_k (D_1 D_2 g)(x_0) \, dk = D_1 D_2 g(x_0) .
\]

This implies that \( D_1 D_2 g(x_0) = D_2 D_1 g(x_0) \) for every \( g \in C_c^\infty(G/K) \). Using the \( G \)-invariance of \( D_1 \) and \( D_2 \), we conclude that \( D_1 D_2 = D_2 D_1 \). \( \square \)
3. Spherical functions as joint eigenfunctions

Assume that $(G, K)$ is a Gelfand pair. We give an equivalent characterization of the spherical functions.

It is convenient to regard spherical functions primarily as functions on $G/K$ rather than on $G$. We will however shift continuously back and forth between $G/K$ and $G$, setting $\Phi = \Lambda^* \varphi$.

**Theorem 3.1.** Let $\varphi$ be continuous, non-identically zero on $G/K$. The following conditions are equivalent:

1. $\varphi$ is a spherical function;
2. $\varphi$ is $C^\infty$, $K$-invariant, $\varphi(eK) = 1$, and is an eigenfunction of every $D \in \mathcal{D}(G/K)^G$.

Moreover, spherical functions are analytic, and each spherical function $\varphi$ is uniquely determined by the function $D \mapsto \lambda_D(\varphi)$ assigning to $D \in \mathcal{D}(G/K)^G$ its eigenvalue on $\varphi$.

**Proof.** Suppose that $\Phi = \Lambda^* \varphi$ satisfies (5.1) in Ch. 1. From Ch. 1, Sect. 5, it follows that $\Phi$ is bi-$K$-invariant and $\Phi(e) = 1$.

We prove first that $\Phi$ is $C^\infty$. Given $f \in C^\infty_c(G)$, let $f^\sharp(g) = \int_K (gk) dk$. Then also $f^\sharp$ and $\Phi \ast f^\sharp$ are $C^\infty$. But

$$
\Phi(g) \int_G \Phi(h^{-1}) f(h) dh = \int_G \int_K \Phi(gkh^{-1}) f(h) dk dh = \int_G \int_K \Phi(gh^{-1}) f(hk) dk dh = \Phi \ast f^\sharp(g).
$$

Taking $f$ such that $\int_G \Phi(h^{-1}) f(h) dh \neq 0$ we conclude that $\Phi$ is $C^\infty$. Hence $\varphi$ is $K$-invariant on $G/K$ and $C^\infty$. Notice that (5.1) in Ch. 1 is equivalent to the condition

$$
(3.1) \quad \int_K \varphi(gk \cdot y) dk = \varphi(g \cdot x_0) \varphi(y),
$$

for $g \in G$ and $y \in G/K$. Applying $D \in \mathcal{D}(G/K)^G$ (in the variable $y$) to the left-hand side, we obtain

$$
D \left( \int_K \tau k^{-1} g^{-1} \varphi dk \right)(y) = \int_K \tau k^{-1} g^{-1} D \varphi(y) dk = \int_K D \varphi(gk \cdot y) dk,
$$

while, applying $D$ to the right-hand side, we obtain $\varphi(g \cdot x_0) D \varphi(y)$. Setting $y = x_0$, $g \cdot x_0 = x$, and pairing the two formulas, we have

$$
D \varphi(x) = \varphi(x) D \varphi(x_0).
$$

Hence $\varphi$ is an eigenfunction of $D$ with eigenvalue $\lambda_D(\varphi) = D \varphi(x_0)$.

Conversely, suppose that $\varphi$ satisfies (2). Introducing coordinates $t = (t_1, \ldots, t_n)$ on $\mathfrak{p}$ according to (1.2), let $|t|$ an $\text{Ad}(K)$-invariant norm on $\mathfrak{p}$. The operator $D_p$, with $P(t) = |t|^2 \hat{\xi}$ in $\mathcal{D}(G/K)^G$ and is elliptic (a Laplacian). Hence its eigenfunctions are analytic.
Given $g \in G$, consider the function

\[ \varphi_g(x) = \int_K \varphi(gk \cdot x) \, dk = \int_K \tau(gk)^{-1} \varphi(x) \, dk. \]  

Then $\varphi_g$ is analytic and $K$-invariant. By Corollary 4.4, it can be expanded as

\[ \varphi_g(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{d_j} D_Q G \varphi_g(0) Q^J_\ell \left( \psi_{x_0}(x) \right), \]  

near $x_0$. For $D \in \mathbb{D}(G/K)^G$, let $\lambda_D = D \varphi(x_0)$ be the eigenvalue of $\varphi$ with respect to $D$. Then

\[ D \varphi_g - D \left( \int_K \tau(gk)^{-1} \varphi \, dk \right) = \int_K \tau(gk)^{-1} D \varphi \, dk = \lambda_D \varphi_g = D \varphi(x_0) \varphi_g. \]

But $\varphi_g(x_0) = \varphi(g \cdot x_0)$, so that

\[ \varphi_g(x) = \varphi(g \cdot x_0) \sum_{j=0}^{\infty} \sum_{\ell=1}^{d_j} D^J_\ell \varphi(x_0) Q^J_\ell \left( \eta^{-1}(x) \right) = \varphi(g \cdot x_0) \varphi(x). \]

Hence (3.1) holds for $x$ near $x_0$. By analytic extension, (3.1) holds for every $x$. From (3.3) we derive that, near $x_0$,

\[ \varphi(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{d_j} \lambda_D(\varphi) Q^J_\ell \left( \psi_{x_0}(x) \right), \]

which proves the last statement. 

We prove now a further characterization of Gelfand pairs when $G$ is a Lie group.

**Theorem 3.2 (⋆).** Let $K$ be a compact subgroup of a Lie group $G$, with $G/K$ connected. If $\mathbb{D}(G/K)$ is commutative, then $(G, K)$ is a Gelfand pair.

The proof is based on the following two lemmas. We relabel as $D_r, Q_r$ the operators $D^J_\ell$ and polynomials $Q^J_\ell$ in (3.3).

**Lemma 3.3.** Let $f$ be analytic and $K$-invariant on $G/K$. For $x = g \cdot x_0$, let

\[ F(x, y) = \int_K f(gk \cdot y) \, dk. \]

Then $F$ is $K$-invariant in each variable, analytic in both variables, and, for $x, y$ near $x_0$,

\[ F(x, y) = \sum_{r,s} D_r D_s f(x_0) \eta^{-1}(x) Q_s(\eta^{-1}(y)). \]  

**Proof.** Analyticity and $K$-invariance of $F$ are evident. For $g \in G$, the function

\[ F_g(y) = F(g \cdot x_0, y) = \int_K \tau(gk)^{-1} f(y) \, dk \]
ise analytic in $V \cdot x_0$ and $K$-invariant. Moreover,
\[ D_sF_g(x_0) = \int_K \tau_{(gk)^{-1}}(D_s f)(x_0) \, dk = D_s f(g \cdot x_0), \]
since $D_s f$ is $K$-invariant. Hence, with $x = g \cdot x_0$,
\[ F(x, y) = F_y(y) = \sum_{s=0}^{\infty} D_s f(x)Q_s(\psi_{x_0}(y)) = \sum_{r,s=0}^{\infty} D_r D_s f(x_0)Q_r(\eta^{-1}(x))Q_s(\eta^{-1}(y)). \]

\textbf{Lemma 3.4.} The space of analytic functions on $G$ vanishing at infinity is dense in $C_0(G)$.

\textbf{Proof.} The fundamental fact is Nelson’s theorem\textsuperscript{3}, stating that the heat kernel $p_t(g)$ of the semigroup $e^{t\Delta}$ generated by a left-invariant Laplacian $\Delta$ on $G$ is analytic in $g$. Since $\{p_t\}_{t>0} \subset L^1(G)$ and is an approximate identity, the convolutions $f \ast p_t$ with $f \in C_c(G)$, are analytic, vanish at infinity and form a dense subset of $C_0(G)$. \hfill \Box

\textbf{Proof of Theorem 3.2.} Given $f \in C_0(G/K)$, $K$-invariant and analytic, take $F(x, y)$ as in Lemma 3.2. If $\mathcal{D}(G/K)^G$ is commutative, it follows from (3.4) that $F(x, y) = F(y, x)$ for all $x, y$, since $G/K$ is connected.

Take now $U = \Lambda^* u, V = \Lambda^* v \in L^1(K\setminus G/K)$. We have
\[ \int_G U \ast V(g)\Lambda^* f(g) \, dg = \int_{G \times G} U(h)V(h^{-1}g)\Lambda^* f(g) \, dh \]
\[ = \int_{G \times G} \int_K U(h)V(k^{-1}h^{-1}g)\Lambda^* f(g) \, dk \, dh \]
\[ = \int_{G \times G} \int_K U(h)V(g)\Lambda^* f(hkg) \, dk \, dh \]
\[ = \int_{G/K \times G/K} u(y)v(x)F(y,x) \, dy \, dx. \]

Then
\[ \int_G U \ast V(g)\Lambda^* f(g) \, dg = \int_G V \ast U(g)\Lambda^* f(g) \, dg. \]

From Lemma 3.4 it follows that $K$-invariant analytic functions are dense in $C_0(G/K)^K$, and this implies that $u \ast v = v \ast u$. \hfill \Box

4. Embedding of spectra in Euclidean spaces

Let $\Xi$ the set of (nonnecessarily bounded) spherical functions. On $\Xi$ we consider two topologies: the compact-open topology, and the topology inherited from $\mathcal{E}(G/K)$ (i.e., $C^\infty$ with the topology of uniform convergence on compact sets with all derivatives).

**Lemma 4.1.** For every $D \in \mathbb{D}(G/K)^G$, the function $\lambda_D$ is continuous on $\Xi$ with the compact-open topology.

On $\Xi$, the $\mathcal{E}$-topology and the compact-open topology coincide.

**Proof.** Given $\varphi_0 \in \Xi$, fix a compact neighborhood $V$ of $x_0$ such that $\text{Re } \varphi_0 > \frac{1}{4}$ on $V$. Given $\varepsilon > 0$, the set $U = \{\varphi \in \Xi : |\varphi(x) - \varphi_0(x)| < \varepsilon \ \forall \ x \in V\}$ is a neighborhood of $\varphi_0$ in the compact-open topology. If $\varepsilon < \frac{1}{4}$, Re $\varphi > \frac{1}{4}$ on $V$ for every $\varphi \in U$.

Fix $F = \Lambda^* f \in C_c^\infty(K \backslash G/K)$ a nonnegative function, with integral 1 and supported on a symmetric neighborhood $V'$ of the identity such that $V'^2 \subset \Lambda^{-1} V$. For every $\varphi \in U$, Re $(F \ast \varphi) > \frac{1}{4}$ on $V'$. By (3.1), $F \ast \varphi = (F \ast \varphi)(\varepsilon) = c_{\varphi} F \ast \varphi$, with $|c_{\varphi}| > \frac{1}{4}$.

Hence, with $\Phi = \Lambda^* \varphi$,

$$\varphi = c_{\varphi}^{-1} F \ast \varphi = c_{\varphi}^{-1} \Lambda^{*-1}(F \ast \Phi) = c_{\varphi}^{-1} \Lambda^{*-1}(\Phi \ast F) = c_{\varphi}^{-1} \Phi \ast f .$$

For $D \in \mathbb{D}(G/K)^G$,

$$D \varphi = c_{\varphi}^{-1} \Phi \ast D f = c_{\varphi}^{-1} \Lambda^*(D f) \ast \varphi ,$$

so that, evaluating at $x_0$,

$$|\lambda_D(\varphi) - \lambda_D(\varphi_0)| < 4\varepsilon \|D f\|_1 .$$

This proves the first statement.

For the second part, we only need to prove that the compact-open topology is finer than the $\mathcal{E}$-topology.

Given $\varphi_0 \in \Xi$, fix a compact set $C$ in $G/K$ and a neighborhood $U_{\varphi_0}(C, n, \varepsilon) = \{\varphi \in \Xi : \|\varphi - \varphi_0\|_{C^n(C)} < \varepsilon\}$ in the $\mathcal{E}$-topology.

Let $D \in \mathbb{D}(G/K)^G$ be the Laplacian introduced in the proof of Theorem 3.1. If $A$ is a relatively compact open set in $G/K$, and $A'$ is another open set with $C \subset A' \subset A$, we can use the Sobolev immersion theorem and the regularity theorem for elliptic operators to conclude that, if $d = \dim G/K$ and $2m > n + \frac{d}{2}$, we have the estimate:

$$\|f\|_{C^n(C)} \leq C_n \|f\|_{H^{2m}(A')} \leq C_m \|D^{m}_{\mathcal{F}} f\|_{L^2(A)} .$$

(4.1)
Hence
\[ \| \varphi - \varphi_0 \|_{C^m(C)} \leq C_m \| D^m \varphi - D^m \varphi_0 \|_{L^2(A)} \]
\[ = C_m \| \lambda_D(\varphi)^m \varphi - \lambda_D(\varphi_0)^m \varphi_0 \|_{L^2(A)} \]
\[ \leq C_m \left( \| \lambda_D(\varphi_0)^m \|_{L^2(A)} + \| \lambda_D(\varphi)^m - \lambda_D(\varphi_0)^m \|_{L^2(A)} \right) \]
\[ \leq C_m |A|^{\frac{1}{2}} \left( \| \lambda_D(\varphi_0)^m \|_{C(A)} + \| \lambda_D(\varphi)^m - \lambda_D(\varphi_0)^m \|_{C(A)} \right). \]

Using the first part of the proof, we conclude that \( \{ \varphi : \| \varphi - \varphi_0 \|_{C(A)} < \delta \} \subset U_{\varphi_0}(C, n, \varepsilon) \) if \( \delta \) is sufficiently small.

Let \( D = (D_1, \ldots, D_k) \) be a finite set of generators of \( \mathbb{D}(G/K)^G \) (which exists by Corollary 1.4).

The map \( \rho_D \) which assigns to a spherical function \( \varphi \) the \( k \)-tuple
\[ (4.2) \quad \rho_D(\varphi) = (\lambda_{D_1}(\varphi), \ldots, \lambda_{D_k}(\varphi)) \in \mathbb{C}^k \]
is injective, by Theorem 3.1.

It is assumed that \( \Xi \) is endowed with the compact-open (or \( \mathcal{E} \)-) topology.

**Theorem 4.2.** The map \( \rho_D \) is a homeomorphism from \( \Xi \) to its image in \( \mathbb{C}^k \), and \( \rho_D(\Xi) \) is closed.

**Proof.** Lemma 4.1 implies that \( \rho_D \) is continuous.

We preliminarily observe that the topology on \( \Xi \) is first-countable. Since \( G/K \) is assumed to be connected, the connected component of the identity \( G_0 \) of \( G \) acts transitively on \( G/K \). So, if \( V \) is a symmetric, connected, open neighborhood of the identity in \( G \),
\[ G/K = G_0 \cdot x_0 = \bigcup_{n \geq 1} V^n \cdot x_0, \]
and, for every \( \varphi_0 \in \Xi \), the sets \( U_n = \{ \varphi : \| \varphi - \varphi_0 \|_{C(V^n \cdot x_0)} < \frac{1}{n} \} \) form a fundamental system of neighborhoods in the compact-open topology.

Let now \( \{ \zeta_n \} \) be a sequence of points of \( \Xi \) converging to \( \zeta \in \mathbb{C}^k \). We prove that the functions \( \varphi_n = \rho_D^{-1}(\zeta_n) \) converge to a spherical function. This will prove both statements simultaneously.

Let \( D \) be the Laplacian used above. There is a polynomial \( Q \) such that \( D = Q(D_1, \ldots, D_r) \). For every \( n, m \), \( D^m(\varphi_n) = Q(\zeta_n)^m \varphi_n \). Since the \( \zeta_n \) are bounded, it follows from (4.1) that the \( C^1 \)-norms of the \( \varphi_n \) are uniformly bounded on every compact set.

By the Ascoli-Arzelà theorem, given a subsequence \( \{ \zeta_{n_k} \} \), there is a sub-subsequence \( \{ \varphi_{n_{k_j}} \} \) such that \( \{ \varphi_{n_{k_j}} \} \) converges uniformly on compact sets. The limit function \( \varphi \) satisfies (3.1), is bounded, and \( \varphi(x_0) = 1 \), i.e., \( \varphi \) is spherical. Moreover \( \rho_D(\varphi) = \zeta \), and this implies that \( \varphi \) does not depend on the choice of the subsequence.
Corollary 4.3 (*). The spectrum $\Delta$ of $(G, K)$ is homeomorphic to its image $\rho_D(\Sigma) \subset \rho_D(\Xi)$, and the sets $\rho_D(\Delta)$, $\rho_D(\Delta^+)$, are closed.

If the $D_j$ are formally self-adjoint, then $\rho_D(\Delta^+) \subset \mathbb{R}^k$.

5. Examples

5.1. The hyperbolic plane.

Consider the pair $(SL_2(\mathbb{R}), SO_2)$ of Examples 1.e and 3.c in Ch. 1.

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ splits under the adjoint action of $SO_2$ as $\mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{so}_2 = \mathbb{R} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathfrak{p}$ consists of the symmetric matrices of trace 0:

$$v = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

The adjoint action of $SO_2$ on $\mathfrak{p}$ is given by rotations in the $(a, b)$-plane. So the algebra of $\text{Ad}(SO_2)$-invariant polynomials on $\mathfrak{p}$ is generated by the unique element $P(a, b) = a^2 + b^2$. The corresponding operator on $G/K$ is a $G$-invariant Laplacian, which we call $L$. Under the identification of $G/K$ with the upper half plane $M$ of Example 1.e, $L$ is the hyperbolic Laplacian, equal (up to a constant factor) to

$$L = y^2(\partial_x^2 + \partial_y^2).$$

For $\gamma \in \mathbb{C}$, the function

$$u_{\gamma}(x + iy) = y^{\frac{1}{2} + i\gamma}$$

is an eigenfunction of $L$, with eigenvalue $-\gamma^2 - \frac{1}{4}$. Therefore,

$$\varphi_{\gamma}(z) = \int_{SO_2} u_{\gamma}(k \cdot z) \, dk = \frac{1}{2\pi} \int_0^{2\pi} u_{\gamma}\left(\frac{\cos t \, z + \sin t}{-\sin t \, z + \cos t}\right) \, dt$$

is an $SO_2$-invariant eigenfunction. Since $SO_2$ is the stabilizer of $z_0 = i$ and $u_{\gamma}(i) = 1$, we have $\varphi_{\gamma}(i) = 1$, and $\varphi_{\gamma}$ is a spherical function.

By Theorem 3.1, we have

$$\varphi_{\gamma} = \varphi_{\gamma'}, \iff \gamma = \pm \gamma',$$

and we have found all spherical functions.

We state without proof the following theorem$^4$.

Theorem 5.1. The spherical function $\varphi_{\gamma}$ is bounded if and only if $-\frac{1}{2} \leq \text{Im} \, \gamma \leq \frac{1}{2}$, and positive definite if and only if $\gamma^2 + \frac{1}{4} \geq 0$, i.e., $\gamma \in \mathbb{R} \cup \left(i\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$.

$^4$cf. Helgason and Faraut.
Notice that, taking $\mathcal{D} = \{L\}$, $\rho_D(\Delta)$ is the image of the strip $-\frac{1}{2} \leq \text{Im} \gamma \leq \frac{1}{2}$ under the map $\lambda(\gamma) = -\gamma^2 - \frac{1}{4}$, i.e. a parabolic region in $\mathbb{C}$, and $\rho_D(\Delta^+) = [0, +\infty)$.

5.2. Pairs $(K \ltimes \mathbb{R}^n, K)$.

Consider a Gelfand pair of the type presented in Section 10 of Ch. 1. The homogeneous space is identified with $H$ and the algebra $\mathbb{D}(H)^K$ consists of the left- and $K$-invariant differential operators on $H$. They are obtained by symmetrization from $K$-invariant polynomials on $\mathfrak{h} (= \mathfrak{p})$.

The simplest situation occurs when $H = \mathbb{R}^n$ and $K \subset O_n$. The algebra $\mathbb{D}(\mathbb{R}^n)^K$ consists of the constant-coefficient operators $D_P = P(i^{-1} \partial_x)$, where $P$ is a $K$-invariant polynomial.

We proceed as in the previous section in order to identify the spherical functions. Given $P$ $K$-invariant and $\zeta \in \mathbb{C}^n$, the function

$$u_\zeta(x) = e^{ix \cdot \zeta} = e^{i \sum_{j} x_j \zeta_j}$$

is an eigenfunction of $D_P$ with eigenvalue $P(\zeta)$. Then the generalized Bessel function

$$\varphi_\zeta(x) = \int_K u_\zeta(kx) \, dk$$

is a spherical function with the same eigenvalue. Obviously,

$$\varphi_\zeta = \varphi_{\zeta'} \iff \zeta \in K\zeta'$$

and we have found all spherical functions.

Proposition 5.2 (*). The spherical function $\varphi_\zeta$ is bounded if and only if $\zeta \in \mathbb{R}^n$. Let $\mathcal{D} = \{D_{P_1}, \ldots, D_{P_k}\}$ be a set of real generators of $\mathbb{D}(\mathbb{R}^n)^K$. Then

$$\rho_D(\Delta) = \{(P_1(\xi), \ldots, P_k(\xi)) : \xi \in \mathbb{R}^n\} \subset \mathbb{R}^k$$

6. Joint spectral analysis of $\mathcal{D}$

Let $(G, K)$ be a Gelfand pair.

Consider the following resolution of the identity of $L^2(G/K)$ based on $\Delta^+$: given a Borel subset $\omega$ of $\Delta^+$ and $f \in L^2(G/K)$, set

$$E(\omega)f(g) = \int_\omega \hat{f}(\varphi)\varphi(g) \, d\sigma(\varphi)$$

where $\sigma$ is the Plancherel measure.

Theorem 6.1. Let $D \in \mathbb{D}(G/K)^G$, defined on $C^\infty_c(G/K)$. Then the operator

$$\tilde{D} = \int_{\Delta^+} \lambda_D(\varphi) \, dE(\varphi)$$

(6.1)
is a closed extension of $D$. If $D$ is formally self-adjoint, \( \tilde{D} \) is its unique self-adjoint extension.

The operators \( \tilde{D} \) strongly commute among themselves.

**Proof.** Combining together the Plancherel-Godement Theorem 8.2 with the identity

\[
\tilde{D} \hat{f} = \tilde{\lambda}_D \hat{f} ,
\]

for \( f \in C_c^\infty(G/K)^K \), we obtain that \( \tilde{D} \) is unitarily equivalent to multiplication by \( \tilde{\lambda}_D \). The conclusion follows from spectral properties of multiplication operators and the last part of Corollary 4.3. \( \Box \)

Let now \( P_1, \ldots, P_k \) be real valued polynomials generating \( \mathcal{P}(\mathfrak{p})^{\text{Ad}(K)} \). Define \( \mathcal{D} = \{D_{P_1}, \ldots, D_{P_k}\} \) according to (1.3) and \( \rho_D \) according to (4.2).

Denote by \( \mathcal{D}_D \) the push-forward of the projection-valued measure \( E \) form \( \Delta^+ \) to \( \rho_D(\Delta^+) \). Then (6.1) applied to \( D = D_j \in \mathcal{D} \) takes the form

\[
\tilde{D}_j = \int_{\rho_D(\Delta^+)} \xi_j d\mathcal{D}_D(\xi) ,
\]

where \( \xi_j \) is the \( j \)-th component of \( \xi \in \mathbb{R}^k \).

**Corollary 6.2 (\textit{*}).** The support of \( \mathcal{D}_D \) inside \( \rho_D(\Delta^+) \) is the joint \( L^2 \)-spectrum of the operators \( \tilde{D}_j \in \mathcal{D} \).

Recall that the joint spectrum of \( \tilde{D}_1, \ldots, \tilde{D}_k \) is defined as the set of \( k \)-tuples \( (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k \) for which there exist bounded operators \( A_1, \ldots, A_k \) such that

\[
(\tilde{D}_1 - \xi_1 I)A_1 + \cdots + (\tilde{D}_k - \xi_k I)A_k = I .
\]

**Examples.**

(6.a) For the hyperbolic plane of Section 5.1, the spectrum of \( \tilde{L} \) is the half line \([\frac{1}{4}, +\infty)\), properly contained in \( \rho_D(\Delta^+) \).

(6.b) (\textit{*}) For the pairs \( (K \rtimes \mathbb{R}^n, K) \) of Section 5.2, the joint spectrum coincides with \( \rho_D(\Delta) \).
CHAPTER 3

STRUCTURE OF LIE GROUPS FORMING A GELFAND PAIR

1. Rôle of the nilradical of \( G \)

In this section we state several results which will require a rather deep and extended analysis to be proved. We refer to Wolf’s book for details. We only present some statements which are preliminary to the main theorems and help to understand their nature.

Consider a homogeneous space \( G/K \), with \( K \) compact and \( G \) connected. We add two further assumptions:

- \( G/K \) is simply connected (which implies that \( K \) is also connected);
- the action of \( G \) on \( G/K \) is effective (equivalently, \( K \) is its own normalizer in \( G \)).

Under these hypotheses, the Lie algebra \( \mathfrak{g} \) of \( G \) admits a form of Chevalley decomposition, i.e., a modified version of the Levi decomposition \( \mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r} \), which takes into account the presence of \( K \) and, at the same time, emphasizes the nilradical \( \mathfrak{n} \) rather than the solvable radical \( \mathfrak{r} \).

**Proposition 1.1.** There is a reductive subalgebra \( \mathfrak{l} = \mathfrak{h} \oplus \mathfrak{s} \) of \( \mathfrak{g} \), with \( \mathfrak{h} \) abelian, containing \( \mathfrak{t} \), and such that \( \mathfrak{g} = \mathfrak{n} + \mathfrak{l} \) and

\[
\mathfrak{g} \cong (\mathfrak{l} \oplus \mathfrak{n})/\{ (X,-X) : X \in \mathfrak{l} \cap \mathfrak{n} \}.
\]

In the same way, setting \( U = \{ (u,u^{-1}) : u \in L \cap N \} \),

\[
G = LN \cong (L \ltimes N)/U.
\] (1.1)

In general, \( \mathfrak{l} \) and \( \mathfrak{n} \) can have a nonempty intersection. In any case, however, the adjoint action of an element of \( \mathfrak{l} \cap \mathfrak{n} \) on \( \mathfrak{n}^C \) must be nilpotent and diagonalizable at the same time, so it must be trivial. This implies the following.

**Lemma 1.2.** \( \mathfrak{l} \cap \mathfrak{n} \) is contained in the center of \( \mathfrak{n} \) and in the abelian component of \( \mathfrak{l} \). In particular \( L \cap N \), as a subgroup of \( L \), acts trivially on \( N \).

**Proposition 1.3 (⋆).** Assume that \( (G,K) \) is a Gelfand pair. Then also \( (L,K) \) and \( (K \ltimes N,K) \) are Gelfand pairs.
The proof requires the characterization of Gelfand pairs in terms of commutativity of orbital measures (Proposition 3.2 and Corollary 10.2 in Ch. 1), Lemma 1.2, together with the remark that $U \cap (L \times \{e\})$ and $U \cap (\{e\} \times N)$ are trivial.

**Proposition 1.4** \(^1\) If $(G, K)$ is a Gelfand pair and $N$ is simply connected, then $N$ is at most step two.

**Proof.** By Proposition 1.3, we may assume that $G = K \ltimes N$. Let $m$ be the step of $\mathfrak{n}$. The terms $\mathfrak{n}^{(j)}$ of the descending central series are $K$-invariant. For each $j \leq m - 1$, we take a $K$-invariant complement $\mathfrak{v}_j$ to $\mathfrak{n}^{(j+1)}$ in $\mathfrak{n}^{(j)}$. Then each $\mathfrak{v}_j$ is nontrivial and $[\mathfrak{v}_1, \mathfrak{n}^{(m-1)}] = \mathfrak{n}^{(m)} \neq \{0\}$.

Choose $X \in \mathfrak{v}_1$, $Y \in \mathfrak{n}^{(m-1)}$ such that $[X, Y] \neq 0$, and let $x = \exp_N(X)$, $y = \exp_N(Y)$. We prove that, if $m \geq 3$, $yx \not\in (Kx)(Ky)$. This implies that the two orbital measures $\mu_x$ and $\mu_y$ do not commute (take a nonnegative continuous function equal to 1 at $xy$ and with support disjoint from $(Kx)(Ky)$, and make it $K$-invariant by averaging over $K$).

Applying the BCH formula, we obtain the identity

$$Y + X + \frac{1}{2} [Y, X] = k_1 \cdot X + k_2 \cdot Y + \frac{1}{2} [k_1 \cdot X, k_2 \cdot Y].$$

If $m \geq 3$, the subspaces $\mathfrak{v}_1, \mathfrak{n}^{(m-1)}, \mathfrak{n}^{(m)}$ are complementary, hence it must be $k_1 X = X$, $k_2 Y = Y$, and hence $[k_1 \cdot X, k_2 \cdot Y] = [X, Y]$. This contrasts with the assumption that $[X, Y] = 0$. \hfill \Box

2. The Vinberg and Yakimova theorems

We state now Vinberg’s theorem\(^2\).

**Theorem 2.1.** Let $G$ be connected, $K \subset G$ compact, with $G/K$ simply connected and $G$ acting effectively on it. Assume that $(G, K)$ is a Gelfand pair. Then

1. the group $U$ in (1.1) is trivial, hence $G = L \ltimes N$;
2. $N$ is simply connected and of step at most 2;
3. the actions of $L$ and $K$ on $N$ are orbit-equivalent;
4. $(L, K)$ and $(K \ltimes N, K)$ are Gelfand pairs.

Condition (3) means that $L$ and $K$ produce the same orbits on $N$. This is equivalent to either of the identities

$$\mathcal{P}(\mathfrak{n})^K = \mathcal{P}(\mathfrak{n})^L, \quad \mathcal{D}(N)^K = \mathcal{D}(N)^L.$$

\(^1\)Taken from C. Benson, J. Jenkins, G. Ratcliff, *On Gelfand pairs associated with solvable Lie groups*, Trans.AMS, 321 (1990), 85-116.

This is a very strong restriction. For instance, notice that $L$ needs not be compact in general. However, if $|x|$ is a $K$-invariant norm on $n$, it is also $L$-invariant. Hence $L$ acts on $N$ as a compact group.

Theorem 2.1 tells us that there are two basic types of Gelfand pairs:

- the **reductive Gelfand pairs**, in which $N$ is trivial; these include the **symmetric pairs**, for which $G/K$ is a symmetric space (not containing a Euclidean factor);
- the **nilpotent Gelfand pairs**, where $K = L$.

Besides these, there are many other candidates to being Gelfand pairs, on the basis of the fact that they satisfy condition (1)-(4). First of all, product pairs $(G_1 \times G_2, K_1 \times K_2)$, with $(G_i, K_i)$ of different types. Less obvious examples are

$$
(U_n \ltimes \mathbb{C}^n, SU_n), \quad (SO_{2n} \ltimes \mathbb{R}^{2n}, U_n), \quad (SU_{2n} \ltimes \mathbb{C}^{2n}, Sp_n),
$$

$$
(U_n \ltimes H_n, SU_n), \quad (SU_{2n} \ltimes H_{2n}, Sp_n)
$$

where $H_n$ is the $(2n + 1)$-dimensional Heisenberg group.

Vinberg’s theorem only gives necessary conditions for a more general pair $(G, K)$ to be a Gelfand pair. In order to state a necessary and sufficient condition, we must introduce some further notation.

The groups $L$ and $K$ act on the Lie algebras $l, \mathfrak{l}, \mathfrak{n}$ through the restrictions of the adjoint representation of $G$. Then they also act on the quotient vector space $l/\mathfrak{l}$, and on the dual spaces $l^*, \mathfrak{l}^*, \mathfrak{n}^*, (l/\mathfrak{l})^*$.

Given an element $\gamma$ in one of these spaces, denote by $L_{\gamma}, K_{\gamma}$ the stabilizer of $\gamma$ in $L, K$, respectively.

The following theorem is due to O. Yakimova.\(^3\)

**Theorem 2.2.** Let $G = L \ltimes N$, where $L$ is reductive and $N$ is nilpotent, connected and simply connected. Let also $K$ be a compact subgroup of $L$. Then $(G, K)$ is a Gelfand pair if and only if the following conditions are satisfied:

1. the actions of $L$ and $K$ on $N$ are orbit-equivalent;
2. for every $\gamma \in n^*$, $(L_{\gamma}, K_{\gamma})$ is a Gelfand pair;
3. for every $\beta \in (l/\mathfrak{l})^*$, $(K_{\beta} \ltimes N, K_{\beta})$ is a Gelfand pair.

So, assuming we know how to identify reductive Gelfand pairs on one side and on nilpotent Gelfand pairs on the other, Yakimova’s theorem gives the recipe to identify the more general ones, under mild topological hypotheses.

**Examples (\(*\)).**

If $N$ is abelian, condition (3) is obviously satisfied. Then the pairs in the first line of (2.1) are Gelfand pairs (for the last two, the verification of (2) makes use of the fact that $(L, K)$ is a symmetric pair).

---

One also shows that the second line of (2.1) also consists of Gelfand pairs.

3. Nilpotent Gelfand pairs

Consider now a nilpotent pair \((K \ltimes N, K)\). If \(N\) is abelian this obviously a Gelfand pair. Suppose therefore that \(N\) is two-step. Then \(\mathfrak{n}\) properly contains its center \(\mathfrak{z}\).

Given a linear functional \(\lambda \in \mathfrak{z}^*\), let \(\mathfrak{r}_\lambda\) be the radical of the skew-symmetric form

\[
B_\lambda(X, X') = \lambda([X, X'])
\]

Then \(\mathfrak{r}_\lambda\) contains \(\mathfrak{z}\) and is invariant under the stabilizer \(K_\lambda\) of \(\lambda\) in \(K\).

Let \(\mathfrak{v}_\lambda\) be a \(K_\lambda\)-invariant complement of \(\mathfrak{r}_\lambda\) in \(\mathfrak{n}\). Then the restriction of \(B_\lambda\) to \(\mathfrak{v}_\lambda\) is a \(K_\lambda\)-invariant symplectic form. Then \(\mathfrak{v}_\lambda\) admits a complex structure \(J_\lambda\) and a hermitian inner product \(\langle , \rangle_\lambda\) under which \(K_\lambda\) acts unitarily and such that

\[
B_\lambda(V, V') = \text{Im} \langle V, V' \rangle_\lambda
\]

**Theorem 3.1 (**) The following conditions are equivalent:

1. \((K \ltimes N, K)\) is a Gelfand pair;
2. for every \(\lambda \in \mathfrak{z}^*\), \((K_\lambda \ltimes (N/\ker \lambda), K_\lambda)\) is a Gelfand pair;
3. for every \(\lambda \in \mathfrak{z}^*\), the space \(\mathcal{P}(\mathfrak{v}_\lambda)\) of polynomials on the complex space \((\mathfrak{v}_\lambda, J_\lambda)\) decomposes without multiplicities under the action of \(K_\lambda\).

**Example (**) Take

\[ N = \mathbb{C}^n \oplus \mathfrak{u}_n, \]

where \(\mathfrak{u}_n\) is the Lie algebra of skew-hermitean \(n \times n\) complex matrices. The product on \(N\) is

\[
(v, Z)(v', Z') = \left(v + v', Z + Z' + \frac{1}{2}(vv'^* - v'v^*)\right)
\]

\((v \text{ and } v' \text{ are column vectors})\). Take \(K = U_n\) and, for \(k \in K\), define the automorphism of \(N\)

\[
k \cdot (v, Z) = (kv, kZk^*)
\]

On the basis of Theorem 3.1, \((K \ltimes N, K)\) is a Gelfand pair.