Geometria funkcjonych przestrzeni Banacha

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BANACH SPACES OF SMOOTH FUNCTIONS

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Let \( \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha(1)} \cdots \partial x_n^{\alpha(n)}} \) denote the partial derivative corresponding to the multiindex \( \alpha = (\alpha(j))_{j=1}^n \in \mathbb{Z}_+^n \) where \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \); here \( |\alpha| = \sum_{j=1}^n \alpha(j) \) is the order of the derivative \( \partial^\alpha \); we use the convention \( x^\alpha := \prod_{j=1}^n x_j^{\alpha(j)} \).

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). Let \( C_{00}^\infty(\Omega) \) be the space of all infinitely many times differentiable scalar-valued functions on \( \Omega \) with compact supports. A function \( g : \Omega \rightarrow \mathbb{K} \) is said to be the \( \alpha \)-th distributional partial derivative of an \( f : \Omega \rightarrow \mathbb{K} \), in symbol \( g = D^\alpha \), provided

\[
\int_\Omega g \phi \, dx = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \phi \, dx \quad \text{for} \quad \phi \in C_{00}^\infty(\Omega)
\]

The (distributional) derivative of order 0 of a function \( f \) coincides with \( f \). If a partial derivative of \( f \) is continuous on \( \Omega \) then the corresponding distributional derivative of \( f \) coincides with the partial derivative.

Let \( 1 \leq p \leq \infty \). We put

\[
L_{p k}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{K} : D^\alpha f \text{ exists and } D^\alpha f \in L^p(\Omega) \text{ for } |\alpha| \leq k \right\}.
\]

We equip \( L_{pk}(\Omega) \) with the norm

\[
| | f | |_{\Omega,(k),p} = \begin{cases} 
\left( \sum_{|\alpha| \leq k} \left| \int_\Omega |D^\alpha f(x)|^p \, dx \right|^{1/p}, & \text{for } 1 \leq p < \infty \\
\max_{|\alpha| \leq k} \left| \int_\Omega |D^\alpha f(x)| \, dx \right|, & \text{for } p = \infty.
\end{cases}
\]

By \( C_{0,k}(\Omega) \) we denote the closure of \( C_{0,0}^\infty(\Omega) \) in the norm \( | | . | |_{\Omega,(k),\infty} \), and by \( C_k(\Omega) \) a subspace of \( L_{k}^\infty(\Omega) \) of functions having uniformly continuous distributional partial derivatives of order \( \alpha \leq k \). Clearly

\[
C_{0,(k)}(\Omega) \subseteq C_{(k)}(\Omega) \subseteq L_{(k)}^\infty(\Omega).
\]

The spaces \( L_{(k)}^p(\Omega) \) \( (1 \leq p \leq \infty) \), \( C_{0,(k)}(\Omega) \), \( C_{(k)}(\Omega) \) for \( = 1, 2, \ldots \) are called classical Sobolev spaces. A routine argument gives:

**Proposition 1.**

(i) The classical Sobolev spaces are Banach spaces;
(ii) \( C_{0,(k)}(\Omega) \) and \( L_{(k)}^p(\Omega) \) for \( 1 \leq p < \infty \) are separable; \( C_{(k)}(\Omega) \) is separable iff \( \Omega \) is bounded.
Given $f \in L^p_{(k)}(\Omega)$ the tuple $(D^\alpha f)_{|\alpha| \leq k}$ is called a jet of $f$. A jet can be regarded as a vector valued function from $\Omega$ into $\mathbb{R}^{K(n,k)}$ where $K(n,k)$ is the number of partial derivatives of order $\leq k$ in $n$ variables. Let $\bigoplus_{|\alpha| \leq k} L^p(\Omega)$ be the product of $K(n,k)$ copies of the space $L^p(\Omega)$ equipped with the norm

$$
\|f\|_p = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}, \quad \text{for } 1 \leq p < \infty
$$

$$
\max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, \quad \text{for } p = \infty.
$$

The spaces $\bigoplus_{|\alpha| \leq k} C(\Omega)$ and $\bigoplus_{|\alpha| \leq k} C_0(\Omega)$ are defined similarly. Clearly the space $\bigoplus_{|\alpha| \leq k} L^p(\Omega)$ is naturally isomorphic to the vector-valued $L^p$-space $L^p(\Omega, K(n,k))$. Sometimes it is more convenient to work with the second model.

The canonical embedding is the map

$$
\mathcal{J} = \mathcal{J}_{\Omega,(k),p} : L^p_{(k)}(\Omega) \to \bigoplus_{|\alpha| \leq k} L^p(\Omega)
$$

defined by

$$
\mathcal{J}(f) = (D^\alpha f)_{|\alpha| \leq k} \quad \text{for } f \in L^p_{(k)}(\Omega).
$$

The space $\bigoplus_{|\alpha|} L^2(\Omega)$ is a Hilbert space with the inner product defined by

$$
\langle (f_a), (g_a) \rangle := \sum_{|\alpha| \leq k} \int_\Omega f_a \overline{g}_\alpha \, dx.
$$

Thus $L^2_{(k)}(\Omega)$ is a Hilbert space with the inner product

$$
\langle f, g \rangle = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha f D^\alpha g \, dx.
$$

The orthogonal projection

$$
\mathcal{P} = \mathcal{P}_{\Omega,(k)} : \bigoplus_{|\alpha| \leq k} L^2(\Omega) \xrightarrow{\text{onto}} J(L^2_{(k)}(\Omega))
$$

is called the Riesz-Sobolev projection.

For regular domains $\Omega$ the Riesz-Sobolev projection has stronger analytic properties than boundedness in $\bigoplus_{|\alpha| \leq k} L^2(\Omega)$. In particular we have

**Theorem 2.** Let $n = 1, 2, \ldots, k = 0, 1, \ldots$. Then

(a) The Riesz-Sobolev projection $\mathcal{P}_{\mathbb{R}^n,(k)}$ is of weak type $(1,1)$;

(b) $\mathcal{P}_{\mathbb{R}^n,(k)}$ is of strong type $(p,p)$ for $1 < p < \infty$;

(c) if either $n = 1$ or $k = 0$ then $\mathcal{P}_{\mathbb{R}^n,(k)}$ is of strong type $(1,1)$ for $1 \leq p \leq \infty$.
Positive results on isomorphic classification of Sobolev spaces heavily depend on linear extension theorems.

Recall that if $X$ and $Y$ are (Banach) function spaces on topological spaces $S$ and $T$ respectively with $S \subset T$ then a bounded linear operator $E : X \to Y$ is called a linear extension operator provided $E(f)(s) = f(s)$ for $s \in S$ and $f \in X$. Assuming that $E$ exists and the restriction operator $R|_S$ is continuous and takes $Y$ onto $X$ we infer that $X$ and $Y_0 = \{g \in Y : g(s) = 0\}$ are isomorphic to complemented subspaces of $Y$; the desired projection are $E \circ R|_S$ and $1 \circ T - E$. We formulate (not in full generality) the most useful results on linear extension operators for Sobolev spaces:

**WET=Whitney Extension Theorem.** Let $\Omega \subset \mathbb{R}^n$ be open and non-empty, let $k = 0, 1, \ldots$. Then there is a linear extension operator $E^W_{(k)} : C(\Omega) \to C_{0,(k)}(\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be the $(\varepsilon, \delta)$ domain. Then there is a linear extension operator $A^W_{(k),p} : L^p(\Omega) \to L^p(\mathbb{R}^n), \quad 1 \leq p \leq \infty$

Recall that $\Omega$ is an $(\varepsilon, \delta)$ domain if there are $\varepsilon, \delta > 0$ such that if whenever $y, x \in \Omega$ with $|x - y| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ joining $x$ to $y$ such that $\text{arclength}(\gamma) \leq \varepsilon^{-1}|x - y|$ and $\inf_{z \in \mathbb{R}^n \setminus \Omega} |z - w| \geq \varepsilon^{-1}|x - y|$ for all $z \in \gamma$.

To the contrary with the spaces of one variable the non-reflexive Sobolev spaces of more than one variable are not isomorphic to corresponding $L^1$ and $C(K)$ spaces. The main analytic tool used in the proofs of next two theorems is a special case of the Sobolev Embedding Theorem. The assumption on integrability of derivatives of function yields “better” integrability of function itself. We begin with classical Sobolev embedding theorem discovered by S.L. Sobolev [S] for $p > 1$ and extended independently by Galiardo [G] and Nirenberg [N] to $p = 1$. Here ”$\hookrightarrow$” stands for the set theoretical inclusion.

**Theorem 3.**

(i) If $n \geq pk$, $q < \infty$ and $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{k}{n}$ then $L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$;

(ii) $L^1(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$;

(iii) If $n < pk$ then $L^p(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$, i.e. in that case every function from $L^p(\mathbb{R}^n)$ can be modified on the set of measure 0 to be continuous.

The reader is referred to [Ad], [St], for various proofs of Theorem 4.

We use the following notation: The characters of the group $\mathbb{Z}^n$ are identified with exponents $e_a : \mathbb{I}^n \to \mathbb{C}$ defined for $a = (a_j) \in \mathbb{Z}^n$ by $e_{a}(x) = \exp 2\pi i \sum_{j=1}^n a_j x_j$ for $x = (x_j) \in \mathbb{I}^n$. We put $\tilde{f}(a) = \int_{\mathbb{I}^n} f(x) e_{a}(-x)dx \quad (a \in \mathbb{Z}^n; f \in L^1(\mathbb{I}^n))$.

**Theorem 4.** Let $k, n \quad (k = 1, 2, \ldots; n = 2, 3, \ldots)$ be given. Then $L^1_{(k)}(\Omega)$ is not a $\mathcal{L}_1$-space for every non-empty open $\Omega \subset \mathbb{R}^n$.

**Proof.** Since every non-empty open $\Omega$ contains a cube $tI^n + x$ for some $t > 0$ and $x \in \Omega$, it follows from JET that $L^1_{(k)}(\Omega)$ contains a complemented subspace isomorphic to $L^1_{(k)}(I^n)$. Thus it is enough to restrict ourselves to $\Omega = I^n$.
We consider the case $k = 1; n = 2$ to which the general case reduces. Define $S_1$ for $C^\infty$ functions on $I^2$ by

$$S_1(f) = \sum_{a \in \mathbb{Z}^2} \hat{f}(a)e_a,$$

By Theorem 3 the operator $S_1$ extends to a bounded linear operator from $L^1(I^2)$ into $L^2(I^2)$.

We shall denote the extension also by $S_1$. Assume the Claim. Define $T_1 : L^2(I^2) \to L^1(I^2)$ by

$$T_1(f) = \sum_{a \in \mathbb{Z}^2} \hat{f}(a)e_a^{(1)} \quad \text{where} \quad e_a^{(1)} = \frac{e_a}{\sqrt{1 + (2\pi a_1)^2 + (2\pi a_2)^2}}.$$

$T_1$ is bounded because $(e_a^{(1)})_{a \in \mathbb{Z}^2}$ is an orthonormal system in $L^2(I^2)$ hence

$$\|f\|_{L^2(I^2)}^2 = \sum_{a \in \mathbb{Z}^2} \hat{f}(a)e_a^{(1)}||e_a^{(1)}||_{L^2(I^2)}^2 \geq \|T_1(f)\|_{L^1(I^2)}^2.$$

Thus $S_1T_1 : L^2(I^2) \to L^2(I^2)$ is not a Hilbert-Schmidt operator because

$$\sum_{a \in \mathbb{Z}^2} \|S_1T_1(e_a)\|_{L^2(I^2)}^2 = \sum_{a \in \mathbb{Z}^2} (1 + (2\pi a_1)^2 + (2\pi a_2)^2)^{-1} = +\infty.$$

Thus $L^1(I^2)$ is not a $L_1$-space because by a result of Grothendieck (cf. [Gr], [DJT], 4.12) every operator on a Hilbert space which factors through a $L_1$-space is Hilbert-Schmidt.

Let $\mu$ be a complex-valued Borel measure on $\Omega \subset \mathbb{R}^n$ with finite variation. A measure $\nu$ is the distributional derivative corresponding to the multiindex $a$ in symbol $\nu = D^a \mu$ provided $\int_\Omega \phi d\nu = (-1)^{|a|}\int_\Omega \partial^a \phi d\mu$ for every $\phi \in C^\infty_0(\Omega)$. Denote by $BV_k(\Omega)$ the space of complex-valued Borel measures on $\Omega$ having all distributional derivatives of order $\leq k$ with the norm $\|\nu\|_{BV_k} = \sum_{|a| \leq k} \|D^a \nu\|$ where the norm of a measure is its total variation ($k = 1, 2, \ldots$).

$BV_k(\Omega)$ is a Banach space which is a counterpart in $L^1$ norm of $L^1_k(\Omega)$. Clearly $BV_k(\Omega)$ contains naturally $L^1_k(\Omega)$ as a subspace. One can extend the Sobolev embedding operator on $BV_k(I^n)$; one can show in that way that there are bounded non-absolutely summing operators from $BV_k(\Omega)$ into a hilbert space.

The non-isomorphism of $C(\Omega)$ with $C(K)$-spaces is due to Grothendieck who gave in [Gr 2] some indication for the proof. Henkin [He] published the complete proof that $C(\Omega)$ is not isomorphic to any $C(K)$-space; he even showed that it is not uniformly homeomorphic. The Grothendieck-Henkin argument is simpler if one uses the spaces on $T^n$ instead of on $\Omega$.

**Proposition 5 (folklore).** Let $G$ be a compact abelian group, $E$ a finite dimensional Hilbert space, $X$ a translation invariant $L_\infty$ subspace of $C(G,E)$. Then $X$ is complemented in $C(G,E)$ via the orthogonal projection from $L^2(G,E)$ onto $X$, i.e this orthogonal projection is of strong type ($\infty, \infty$).

**Outline of the proof.** The injectivity of $L^\infty$ and the definition of an $\infty$ spaces implies the existence of a net of finite dimensional operators $\{T_u : C(G,E) \to X : u \in \Sigma\}$ whose...
restrictions $T_u|X$ tends pointwise to the identity on $X$. Averaging the $T_u$’s with respect to the Haar measure of $G$ we get the net $\{\tilde{T}_u : u \in \Sigma\}$ of invariant operators which tends to the desired projection. □

Now specifying $G = \mathbb{T}^n$ and $X = C_c(\mathbb{T}^n)$ we conclude that $C_c(\mathbb{T}^n)$ is not an $L_\infty$-space because the corresponding orthogonal projection which is $P_{\mathbb{T}^n}(x)$ is not of strong type $(\infty, \infty)$.

Next we discuss another property of Sobolev spaces in $L^1$-norm which differentiates them from $L^1(\mu)$-spaces. The latter spaces are by the Lebesgue Decomposition Theorem complemented in their second dual. We identify a Banach space with its canonical image in its second dual.

**Theorem 6.** (cf. [PW]) If $n = 2, 3, \ldots ; k = 1, 2, \ldots$ then for every non-empty open $\Omega \supset \mathbb{R}^n$ the space $L^1_{(k)}(\Omega)$ is uncomplemented in its second dual.

Note that $L^1_{(k)}(\Omega)$ does not contain an isomorphic copy of $c_0$ because $L^1_{(k)}(\Omega)$ is isometric to a subspace of $\bigoplus_{|\alpha| \leq k} L^1(\Omega)$ which is obviously isomorphic to an $L^1(\mu)$ space for some measure $\mu$. Thus combining several facts on Banach lattices (cf. [LT], Proposition 1.c.6, Proposition 1.a.11, Theorem 1.b.16), with Theorem 6 we get

**Corollary 7.** If $\Omega, k$ and $n$ satisfy the assumption of Theorem 6 then $L^1_{(k)}(\Omega)$ is isomorphic to no complemented subspace of a Banach lattice.

Next we introduce some notation. We represent $\mathbb{R}^n = R^{n-1} \times \mathbb{R}$ and we write $x = (y, x_n)$ with $y \in R^{n-1}$ and $x_n \in \mathbb{R}$. We identify $R^{n-1}$ with the hyperplane $\{x = (y, x_n) \in \mathbb{R}^n : x_n = 0\}$. We put $\mathbb{R}^n_+ = \{x = (y, x_n) \in \mathbb{R}^n : x_n < 0\}$ and $\mathbb{R}^n_- = \{x = (y, x_n) \in \mathbb{R}^n : x_n > 0\}$. By $\mathcal{D}(\mathbb{R}^n_-)$ we denote the space of scalar-valued infinitely many times differentiable functions on $\mathbb{R}^n_-$ which together with all their partial derivatives are uniformly continuous on $\mathbb{R}^n$ and whose unique continuous extensions to $\mathbb{R}^n_- \times \mathbb{R}^n_+$ have compact supports; we use the same symbol to denote the functions on $\mathbb{R}^n_-$ and their extensions to $\mathbb{R}^n_- \times \mathbb{R}^n_+$. It is not hard to verify that $\mathcal{D}(\mathbb{R}^n_-)$ is dense in $L^1_{(1)}(\mathbb{R}^n_-)$ in the norm $\|\|_{L^1_{(1)}(\mathbb{R}^n_-)}$. The next result is due to Gagliardo (cf. [Ga2]); it belongs to so-called ”trace theorems”.

**Proposition 8.** There is the unique bounded linear surjection $Tr : L^1_{(1)}(\mathbb{R}^n_-) \to L^1(R^{n-1})$ such that $Tr(\phi) = \phi_{|R^{n-1}}$ for $\phi \in \mathcal{D}(\mathbb{R}^n_-)$.

Recall that a right inverse of a bounded linear operator $T : X \to Y$ (X,Y Banach spaces) is a bounded linear operator $S : Y \to X$ such that $TS = Id_Y$. The crucial analytic ingredient used in the proof of Theorem D is

**Peetre Theorem** (cf. [Pee2]). The trace $Tr : L^1_{(1)}(\mathbb{R}^n_-) \to L^1(R^{n-1})$ admits no right inverse.

The simple proof of Peetre’s could be found in [PW]. We also need the following result from the theory of Banach spaces

**Lindenstrauss Lifting Principle**=LLP. If a bounded linear surjection $Q : X \to Y$ (X,Y Banach spaces) has the property that $\ker Q$ is complemented in $(\ker Q)^{**}$ then for every $L_1$-space $E$ every linear operator $T : E \to Y$ admits a lifting $\tilde{T} : E \to Y$, i.e. $T = QT$. In particular if $Y$ is isomorphic to $L^1(0,1)$, $E = Y$ and $T = Id_E$ then $\tilde{T}$ is a right inverse for $Q$. 5
Proof of Theorem 6. It is enough to show that some complemented subspace of $L^{1}_{(1)}(\mathbb{R}^{n})$ is uncomplemented in its second dual. By Proposition 8 and Peetre Theorem and LLP, $\ker Tr$ is uncomplemented in its second dual. We show that $\ker Tr$ is isomorphic to a complemented subspace of $L^{1}_{(1)}(\mathbb{R}^{n})$. Let $oL^{1}_{(1)}(\mathbb{R}^{n})$ denote the subspaces of $L^{1}_{(1)}(\mathbb{R}^{n})$ consisting of the functions which are odd with respect to the variable $x_{n}$. This subspace is complemented in $L^{1}_{(1)}(\mathbb{R}^{n})$ via the projection $f \to o f$ where $o f(y, x_{n}) = (f(y, x_{n}) - f(y, -x_{n}))/2$ for $(y, x_{n}) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a.e. For $f \in \ker Tr$ we define $\tilde{f} : \mathbb{R}^{n} \to \mathbb{R}$ by

$$
\tilde{f}(y, x_{n}) = \begin{cases} f(y, x_{n}) & \text{for } x_{n} \leq 0 \\ -f(y, -x_{n}) & \text{for } x_{n} > 0 \end{cases}
$$

To prove that $oL^{1}_{(1)}(\mathbb{R}^{n})$ is isomorphic to $\ker Tr$ we show that

(i) the formula (8) defines a function in $oL^{1}_{(1)}(\mathbb{R}^{n})$;

(ii) the operator $f \to \tilde{f}$ is a surjection onto $oL^{1}_{(1)}(\mathbb{R}^{n})$.

Note that (i) holds for $f \in D(\mathbb{R}^{n}) \cap \ker Tr$. Thus it is enough to show that $D(\mathbb{R}^{n}) \cap \ker Tr$ is dense in $\ker Tr$. Fix $\varepsilon > 0$ and $f \in \ker Tr$. Since $D(\mathbb{R}^{n})$ is dense in $L^{1}_{(1)}(\mathbb{R}^{n})$, there is $f_{\varepsilon} \in D(\mathbb{R}^{n})$ such that $\|f - f_{\varepsilon}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} < \varepsilon$. Since $Tr(f_{\varepsilon}) = Tr(f - f_{\varepsilon})$, Proposition 8 yields $\|f_{\varepsilon}\|_{L^{0}(\mathbb{R}^{n-1})} \leq \|f - f_{\varepsilon}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} < \varepsilon$. Therefore, by Proposition 8, there exists $g_{\varepsilon} \in D(\mathbb{R}^{n})$ such that $g_{\varepsilon} \in L^{0}(\mathbb{R}^{n-1})$ and $\|g_{\varepsilon}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} < C\varepsilon$. Hence $\|f_{\varepsilon} - g_{\varepsilon}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} = 0$ and $\|f - (f_{\varepsilon} - g_{\varepsilon})\|_{L^{1}_{(1)}(\mathbb{R}^{n})} < \|f - f_{\varepsilon}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} + \|g_{\varepsilon}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} < (C + 1)\varepsilon$.

For (ii) note that the map $f \to \tilde{f}$ is an isomorphism because $\|\tilde{f}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} \leq \|\tilde{f}\|_{L^{1}_{(1)}(\mathbb{R}^{n})} \leq 2\|f\|_{L^{1}_{(1)}(\mathbb{R}^{n})}$. For $\Phi \in oL^{1}_{(1)}(\mathbb{R}^{n}) \cap D(\mathbb{R}^{n})$ one has $\tilde{\Phi}\big|_{\mathbb{R}^{n}} = \Phi$. Thus it suffices to show that $oL^{1}_{(1)}(\mathbb{R}^{n}) \cap D(\mathbb{R}^{n})$ is dense in $oL^{1}_{(1)}(\mathbb{R}^{n})$. Fix $\varepsilon > 0$. Since $D(\mathbb{R}^{n})$ is dense in $L^{1}_{(1)}(\mathbb{R}^{n})$, given $F \in oL^{1}_{(1)}(\mathbb{R}^{n})$ (hence satisfying $o\Phi = F$) there is a $\Phi \in D(\mathbb{R}^{n})$ such that $\|F - \Phi\|_{L^{1}_{(1)}(\mathbb{R}^{n})} < \varepsilon$. Thus $\|F - o\Phi\|_{L^{1}_{(1)}(\mathbb{R}^{n})} = \|o(F - \Phi)\|_{L^{1}_{(1)}(\mathbb{R}^{n})} \leq \|F - \Phi\|_{L^{1}_{(1)}(\mathbb{R}^{n})} < \varepsilon$. □


