pl. Grunwaldzki 2/4
50-384 Wrocław
tel. +48 71 375 74 01, +48 71 328 07 17
fax +48 71 375 74 29
e-mail: dyrekcja@math.uni.wroc.pl
www.math.uni.wroc.pl

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Geometria funkcyjnych przestrzeni Banacha

Michał Wojciechowski

IM PAN Warszawa, Polska

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BANACH SPACES OF SMOOTH FUNCTIONS

Michal Wojciechowski

Let $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x(1)^{\alpha(1)}\partial x(2)^{\alpha(2)}\ldots\partial x(n)^{\alpha(n)}}$ denote the partial derivative corresponding to the *multiindex* $\alpha = (\alpha(j))_{j=1}^n \in \mathbb{Z}_+^n$ where $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$; here $|\alpha| = \sum_{j=1}^n \alpha(j)$ is the *order* of the derivative ∂^α; we use the convention $x^\alpha := \prod_{j=1}^n \alpha(j)$.

Let Ω be an open set in $\mathbb{R}^n$. Let $C_{00}^\infty(\Omega)$ be the space of all infinitely many times differentiable scalar-valued functions on Ω with compact supports. A function $g:\Omega\to\mathbb{K}$ is said to be the α -th *distributional partial derivative* of an $f:\Omega\to\mathbb{K}$, in symbol $g = D^\alpha$, provided

$$\int_\Omega g\phi dx = (-1)^{|\alpha|}\int_\Omega f\partial^\alpha\phi dx \quad \text{for} \quad \phi\in C_{00}^\infty(\Omega)$$

The (distributional) derivative of order 0 of a function f coincides with f. If a partial derivative of f is continuous on Ω then the corresponding distributional derivative of f coincides with the partial derivative.

Let $1\le p\le\infty$. Let us put

$$L_{(k)}^p(\Omega) = \{f :\to \mathbb{K} : \quad D^\alpha f \quad \text{exists and} \quad D^\alpha f\in L^p(\Omega) \quad for \quad |\alpha|\le k\}.$$

We equip $L_{(k)}^p$ with the norm

$$||f||_{\Omega,(k),p} = \begin{cases} \big(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha f(x)|^p dx\big)^{1/p}, & \text{for } 1\le p<\infty \\ \max_{|\alpha|\le k} \operatorname{essup}_{x\in\Omega} |D^\alpha f(x)|, & \text{for } p=\infty. \end{cases}.$$

By $C_{0,(k)}$ we denote the closure of $C_{0,0}^\infty(\Omega)$ in the norm $||\cdot||_{\Omega,(k),\infty}$, and by $C_k(\Omega)$ a subspace of $L_{(k)}^\infty(\Omega)$ of functions having uniformly continuous distributional partial derivatives of order $\alpha\le k$. Clearly

$$C_{0,(k)}(\Omega)\subseteq C_{(k)}(\Omega)\subset L_{(k)}^\infty(\Omega).$$

The spaces $L_{(k)}^p(\Omega)$ $\quad(1\le p\le\infty)\quad C_{0,(k)}(\Omega),\quad C_{(k)}(\Omega)$ for $=1,2,\ldots$ are called *classical Sobolev spaces*. A routine argument gives:

Proposition 1.

(i) *The classical Sobolev spaces are Banach spaces;*
(ii) $C_{0,(k)}(\Omega)$ *and* $L_{(k)}^p$ *for* $1\le p<\infty$ *are separable;* $C_{(k)}(\Omega)$ *is separable iff* Ω *is bounded.*

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Given $f \in L^p_{(k)}(\Omega)$ the tuple $(D^\alpha f)_{|\alpha|\le k}$ icalled a *jet* of f. A jet can be regarded as a vector valued function from Ω into $\mathbb{K}^{K(n,k)}$ where $K(n,k)$ is the number of partial derivatives of order $\le k$ in n variables. Let $\bigoplus_{|\alpha|\le k} L^p(\Omega)$ be the product of $K(n,k)$ copies of the space $L^p(\Omega)$ equipped with the norm

$$||f_\alpha||_p = \begin{cases} \big(\sum_{|\alpha|\le k} ||f_\alpha||^p_{L^p(\Omega)}\big)^{1/p}, & \text{for } 1 \le p < \infty \\ max_{|\alpha|\le k}||f_\alpha||_{L^\infty(\Omega)}, & \text{for } p = \infty. \end{cases}$$

The spaces $\bigoplus_{|\alpha|\le k} C(\Omega)$ and $\bigoplus_{|\alpha|\le k} C_0(\Omega)$ are defined similarly. Clearly the space $\bigoplus_{|\alpha|\le k} L^p(\Omega)$ is naturally isomorphic to the vector-valued L^p-space $L^p(\Omega.l^2_{K(n,k)})$. Sometime is more convenient to work with the second model

The canonical embedding is the map

$$\mathcal{J} = \mathcal{J}_{\Omega,(k),p} : L^p_{(k)}(\Omega) \to \bigoplus_{|\alpha|\le k} L^p(\Omega)$$

defined by

$$\mathcal{J}(f) = (D^\alpha)_{|\alpha|\le k} \quad \text{for } f \in L^p_{(k)}(\Omega).$$

The space $\bigoplus_{|\alpha|} L^2(\Omega)$ is a Hilbert space. with the inner product defined by

$$\langle (f_\alpha), (g_\alpha)\rangle := \sum_{|\alpha|\le k} \int_\Omega f_\alpha \bar{g}_\alpha dx\,.$$

Thus $L^2_{(k)}(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{|\alpha|\le k} \int_\Omega D^\alpha f \overline{D^\alpha g} dx\,.$$

The orthogonal projection

$$\mathcal{P} = \mathcal{P}_{\Omega,(k)} : \bigoplus_{|\alpha|\le k} L^2(\Omega) \xrightarrow{onto} J\big(L^2_{(k)}(\Omega)\big)$$

is called the *Riesz-Sobolev projection* .

For regular domains Ω the Riesz-Sobolev projection has stronger analytic properties than boundedness in $\bigoplus_{|\alpha|\le k} L^2(\Omega)$. In particular we have

Theorem 2. . *Let $n = 1, 2, \ldots$; $k = 0, 1, \ldots$. Then*

(a) *The Riesz-Sobolev projection $\mathcal{P}_{\mathbb{R}^n,(k)}$ is of weak type $(1,1)$;*
(b) *$\mathcal{P}_{\mathbb{R}^n,(k)}$ is of strong type (p,p) for $1 < p < \infty$;*
(c) *if either $n = 1$ or $k = 0$ then $\mathcal{P}_{\mathbb{R}^n,(k)}$ is of strong type $(1,1)$ for $1 \le p \le \infty$.*

Positive results on isomorphic classification of Sobolev spaces heavily depend on linear extension theorems.

Recall that if X and Y are (Banach) function spaces on topological spaces S and T respectively with $S \subset T$ then a bounded linear operator $\mathcal{E} : X \to Y$ is called a *linear extension operator* provided $\mathcal{E}(f)(s) = f(s)$ for $s \in S$ and $f \in X$. Assuming that $\mathcal{E}$ exists and the restriction operator $R_{|S}$ is continuous and takes Y onto X we infer that X and $Y_0 = \{g \in Y : g(s) = 0\}$ are isomorphic to complemented subspaces of Y; the desired projection are $\mathcal{E} \circ R_{|S}$ and $Id_T - \mathcal{E}$. We formulate (not in full generality) the most useful results on linear extension operators for Sobolev spaces:

WET=Whitney Extension Theorem. *Let $\Omega \subset \mathbb{R}^n$ be open and non-empty ,let $k = 0, 1, \ldots$. Then there is a linear extension operator $\mathcal{E}^{\Omega}_{(k)} : \widetilde{C}_{(k)}(\Omega) \to C_{0,(k)}(\mathbb{R}^n)$.*

Here

$$\widetilde{C}_{(k)}(\Omega) = \begin{cases} C_{(k)}(\Omega), & \Omega \text{ bounded} \\ \{f \in C_{(k)}(\Omega) : \lim\limits_{|x|=\infty} \partial^{\alpha} f(x) = 0 \,\text{for}\, |\alpha| \le k\}, & \Omega \text{ unbounded.} \end{cases}$$

JET=Jones Extension Theorem. . *Let $\Omega \subset \mathbb{R}^n$ be the (ε, δ) domain. Then there is a linear extension operator*

$$\Lambda^{\Omega}_{(k),p} : L^p_{(k)} \to L^p_{(k)}(\mathbb{R}^n), \quad 1 \le p \le \infty$$

Recall that Ω is an (ε, δ) domain if there are $\varepsilon, \delta > 0$ such that if whenever $y, x \in \Omega$ with $|x-y| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ joining x to y such that $arclength(\gamma) \le \varepsilon^{-1}|x-y|$ and $\inf_{w \in \mathbb{R}^n \setminus \Omega} |z-w| \ge \frac{\varepsilon|x-z||z-y|}{|x-y|}$ for all $z \in \gamma$.

To the contrary with the spaces of one variable the non-reflexive Sobolev spaces of more than one variable are not isomorphic to corresponding L^1 and $C(K)$ spaces. The main analytic tool used in the proofs of next two theorems is a special case of the Sobolev Embedding Theorem . The assumption on integrability of derivatives of function yields "better" integrability of function itself. We begin with classical Sobolev embedding theorem discovered by S.L.Sobolev [S] for $p > 1$ and extended independently by Galiardo [G] and Nirenberg [N] to $p = 1$. Here "$\hookrightarrow$" stands for the set theoretical inclusion.

Theorem 3.

(i) *If $n \ge pk$, $q < \infty$ and $\frac{1}{p} \ge \frac{1}{q} \ge \frac{1}{p} - \frac{k}{n}$ then $L^p_{(k)}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$;*

(ii) *$L^1_n(\mathbb{R}^n) \hookrightarrow L^{\infty}(\mathbb{R}^n)$;*

(iii) *If $n < pk$ then $L^p_{(k)}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$, i.e. in that case every function from $L^p_{(k)}(\mathbb{R}^n)$ can be modified on the set of measure 0 to be continuous.*

The reader is referred to [Ad], [St], for various proofs of Theorem 4.

We use the following notation: The characters of the group $\mathbb{T}^n$ are identified with exponents $e_a : I^n \to \mathbb{C}$ defined for $a = (a_j) \in \mathbb{Z}^n$ by $e_a(x) = \exp 2\pi i \sum_{j=1}^{n} a_j x_j$ for $x = (x_j) \in I^n$. We put $\widehat{f}(a) = \int_{I^n} f(x) e_a(-x) dx \quad (a \in \mathbb{Z}^n; f \in L^1(I^n))$.

Theorem 4. *Let $k, n \quad (k = 1, 2, \ldots; n = 2, 3, \ldots)$ be given. Then $L^1_{(k)}(\Omega)$ is not a $\mathcal{L}_1$-space for ever non-empty open $\Omega \subset \mathbb{R}^n$*

Proof. Since every non-empty open Ω contains a cube $tI^n + x$ for some $t > 0$ and $x \in \Omega$, it follows from JET that $L^1_{(k)}(\Omega)$ contains a complemented subspace isomorphic to $L^1_{(k)}(I^n)$. Thus it is enough to restrict ourselves to $\Omega = I^n$.

We consider the case $k = 1; n = 2$ to which the general case reduces.

Define S_1 for C^∞ functions on I^2 by

$$S_1(f) = \sum_{a \in \mathbb{Z}^2} \widehat{f}(a) e_a,$$

By Theorem 3 the operator S_1 *extends to a bounded linear operator from* $L^1_{(1)}(I^2)$ *into* $L^2(I^2)$.

We shall denote the extension also by S_1. Assume the Claim. Define $T_1 : L^2(I^2) \to L^1_{(1)}(I^2)$ by

$$T_1(f) = \sum_{a \in \mathbb{Z}^2} \widehat{f}(a) e_a^{(1)} \quad \text{where } e_a^{(1)} = \frac{e_a}{\sqrt{1 + (2\pi a_1)^2 + (2\pi a_2)^2}}.$$

T_1 is bounded because $(e_a^{(1)})_{a \in \mathbb{Z}^2}$ is an orthonormal system in $Ł^2_{(1)}(I^2)$ hence

$$||f||^2_{L^2(I^2)} = || \sum_{a \in \mathbb{Z}^2} \widehat{f}(a) e_a^{(1)} ||^2_{L^2_{(1)}(I^2)} \geq ||T_1(f)||^2_{L^1_{(1)}(I^2)}.$$

Thus $S_1 T_1 : L^2(I^2) \to L^2(I^2)$ is not a Hilbert-Schmidt operator because

$$\sum_{a \in \mathbb{Z}^2} ||S_1 T_1(e_a)||^2_{L^2(I^2)} = \sum_{a \in \mathbb{Z}^2} (1 + (2\pi a_1)^2 + (2\pi a_2)^2)^{-1} = +\infty.$$

Thus $L^1_{(1)}(I^2)$ is not a $\mathcal{L}_1$-space because by a result of Grothendieck (cf. [Gr], [DJT], 4.12) every operator on a Hilbert space which factors through a $\mathcal{L}_1$-space is Hilbert-Schmidt.

Let μ be a complex-valued Borel measure on $\Omega \subset \mathbb{R}^n$ with finite variation. A measure ν is the distributional derivative corresponding to the multiindex α in symbol $\nu = D^\alpha \mu$ provided $\int_\Omega \phi d\nu = (-1)^{|}\alpha| \int_\Omega \partial^\alpha \phi d\mu$ for every $\phi \in C^\infty_{0,0}(\Omega)$. Denote by $BV_k(\Omega)$ the space of complex-valued Borel measures on Ω having all distributional derivatives of order $\leq k$ with the norm $||\mu||_{BV_k} = \sum_{|\alpha| \leq k} ||D^\alpha \mu||$ where the norm of a measure is its total variation $(k = 1, 2, \dots)$.

$BV_k(\Omega)$ is a Banach space which is a counterpart in L^1 norm of $L^\infty_{(k)}(\Omega)$. Clearly $BV_k(\Omega)$ contains naturally $L^1_{(k)}(\Omega)$ as a subspace. One can extend the Sobolev embedding operator on $BV_k(I^n)$; one can show in that way that there are bounded non-absolutely summing operators from $BV_k(\Omega)$ into a hilbert space.

The non-isomorphism of $C_{(1)}(S_n)$ with $C(K)$-spaces is due to Grothendieck who gave in [Gr 2] some indication for the proof. Henkin [He] published the complete proof that $C_{(k)}(S_n)$ is not isomorphic to any $C(K)$-space; he even showed that it is not uniformly homeomorphic. The Grothendieck-Henkin argument is simpler if one uses the spaces on $\mathbb{T}^n$ instead of on S_n. The idea based upon the following

Proposition 5 (folklore). *Let G be a compact abelian group, E a finite dimensional Hilbert space, X a translation invariant $\mathcal{L}_\infty$ subspace of $C(G, E)$. Then X is complemented in $C(G, E)$ via the orthogonal projection from $L^2(G, E)$ onto X, i.e this orthogonal projection is of strong type (∞, ∞).*

Outline of the proof. . The injectivity of l^∞_n and the definition of an ∞ spaces implies the existence of a net of finite dimensional operators $\{T_u : C(G, E) \to X : u \in \Sigma\}$ whose restrictions $T_u|X$ tends pointwise to the identity on X. Averaging the T_u's with respect to the Haar measure of G we get the net $\{\widetilde{T}_u : u \in \Sigma\}$ of invariant operators which tends to the desired projection. $\Box$.

Now specifying $G = \mathbb{T}^n$ and $X = C_{(k)}(\mathbb{T}^n)$ we conclude that $C_{(k)}(\mathbb{T}^n)$ is not an $\mathcal{L}_\infty$-space because the corresponding orthogonal projection which is $\mathcal{P}_{\mathbb{T}^n,(k)}$ is not of strong type (∞, ∞).

Next we discuss another property of Sobolev spaces in L^1-norms which differenties them from $L^1(\mu)$-spaces. The latter spaces are by the Lebesgue Decomposition Theorem complemented in their second duals. We identify a Banach space with its canonical image in its second dual.

Theorem 6. (cf. [PW]) *If $n = 2, 3, \dots ; k = 1, 2, \dots$ then for every non-empty open $\Omega \supset \mathbb{R}^n$ the space $L^1_{(k)}(\Omega)$ is uncomplemented in its secoand dual.*

Note that $L^1_{(k)}(\Omega)$ does not contain an isomorphic copy of c_0 because $L^1_{(k)}(\Omega)$ is isometric to a subspace of $\bigoplus_{|\alpha| \le k} L^1(\Omega)$ which is obviously isomorphic to an $L^1(\mu)$ space for some measure μ. Thus combining several facts on Banach lattices (cf. [LT], Proposition 1.c.6, Proposition 1.a.11, Theorem 1.b.16), with Theorem 6 we get

Corollary 7. *If Ω, k and n satisfy the assumption of Theorem 6 then $L^1_{(k)}(\Omega)$ is isomorphic to no complemented subspace of a Banach lattice.*

Next we introduce some notation. We represent $\mathbb{R}^n = R^{n-1} \times \mathbb{R}$ and we write $x = (y, x_n)$ with $y \in R^{n-1}$ and $x_n \in \mathbb{R}$. We identify R^{n-1} with the hyperplane $\{x = (y, x_n) \in \mathbb{R}^n : x_n = 0\}$. We put $\mathbb{R}^n_- = \{x = (y, x_n) \in \mathbb{R}^n : x_n < 0\}$ and $\mathbb{R}^n_+ = \{x = (y, x_n) \in \mathbb{R}^n : x_n > 0\}$. By $\mathcal{D}(\overline{\mathbb{R}^n_-})$ we denote the space of scalar-valued infinitely many times differentiable functions on $\mathbb{R}^n_-$ which together with all their partial derivatives are uniformly continuous on $\mathbb{R}^n_-$ and whose unique continuous extensions to $\overline{\mathbb{R}^n_-}$ have compact supports; we use the same symbol to denote the functions on $\mathbb{R}^n_-$ and their extensions to $\overline{\mathbb{R}^n_-}$. It is not hard to verify that $\mathcal{D}(\overline{\mathbb{R}^n_-})$ is dense in $L^1_{(1)}(\mathbb{R}^n_-)$ in the norm $||\cdot||_{L^1_{(1)}(\mathbb{R}^n_-)}$. The next result is due to Gagliardo (cf. [Ga2]); it belongs to so called "trace theorems".

Proposition 8. *There is the unique bounded linear surjection $Tr : L^1_{(1)}(\mathbb{R}^n_-) \to L^1(R^{n-1})$ such that $Tr(\phi) = \phi|_{R^{n-1}}$ for $\phi \in \mathcal{D}(\overline{\mathbb{R}^n_-})$.*

Recall that a right inverse of a bounded linear operator $T : X \to Y$ $(X, Y$ Banach spaces) is a bounded linear operator $S : Y \to X$ such that $TS = Id_Y$. The crutial analytic ingredient used in the proof of Theorem D is

Peetre Theorem (cf. [Pee2]). . *The trace $Tr : L^1_{(1)}(\mathbb{R}^n_-) \to L^1(R^{n-1})$ admits no right inverse.*

The simple proof of Peetre's could be find in [PW]. We also need the following result from the theory of Banach spaces

Lindenstrauss Lifting Principle=LLP. . *If a bounded linear surjection $Q : X \to Y$ $(X, Y$ Banach spaces) has the property that $\ker Q$ is complemented in $(\ker Q)^{**}$ then for every $\mathcal{L}_1$-space E every linear operator $T : E \to Y$ admits a lifting $\widetilde{T} : E \to Y$, i.e. $T = Q\widetilde{T}$. In particular if Y is isomorphic to $L^1(0,1)$, $E = Y$ and $T = Id_E$ then $\widetilde{T}$ is a right inverse for Q.*

Proof of Theorem 6. It is enough to show that some complemented subspace of $L^1_{(1)}(\mathbb{R}^n)$ is uncomplemented in its second dual. By Proposition 8 and Peetre Theorem and LLP, $\ker Tr$ is uncomplemented in its second dual. We show that $\ker Tr$ is isomorhic to a complemented subspace of $L^1_{(1)}(\mathbb{R}^n)$. Let ${}^oL^1_{(1)}(\mathbb{R}^n)$ denote the subspaces of $L^1_{(1)}(\mathbb{R}^n)$ consisting of the functions which are odd with respect to the variable x_n. This subspace is complemented in $L^1_{(1)}(\mathbb{R}^n)$ via the projection $f \to {}^of$ where ${}^of(y,x_n) = \big(f(y,x_n) - f(y,-x_n)\big)/2$ for $(y,x_n) \in R^{n-1} \times \mathbb{R}$ a.e. For $f \in \ker Tr$ we define $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ by

$$\widetilde{f}(y,x_n) = \begin{cases} f(y,x_n) & \text{for} x_n \le 0 \\ -f(y,-x_n), & \text{for} x_n > 0 \end{cases} \tag{8}$$

To prove that ${}^oL^1_{(1)}(\mathbb{R}^n_-)$ is isomorphic to $\ker Tr$ we show that

(i) the formula (8) defines a function in ${}^oL^1_{(1)}(\mathbb{R}^n)$;

(ii) the operator $f \mapsto \widetilde{f}$ is a surjection onto ${}^oL^1_{(1)}(\mathbb{R}^n)$.

Note that (i) holds for $f \in \mathcal{D}(\overline{\mathbb{R}^n_-}) \cap \ker Tr$. Thus it is enough to show that $\mathcal{D}(\overline{\mathbb{R}^n_-}) \cap \ker Tr$ is dense in $\ker Tr$. Fix $\varepsilon > 0$ and $f \in \ker Tr$. Since $\mathcal{D}(\overline{\mathbb{R}^n_-})$ is dense in $L^1_{(1)}(\mathbb{R}^n_-)$, there is $f_\varepsilon \in \mathcal{D}(\overline{\mathbb{R}^n_-})$ such that $\|f - f_\varepsilon\|_{L^1_{(1)}(\mathbb{R}^n_-)} < \varepsilon$. Since $Tr(f_\varepsilon) = Tr(f - f_\varepsilon)$, Proposition 8 yields $\|f_\varepsilon|_{\mathbb{R}^{n-1}}\|_{L^1(\mathbb{R}^{n-1})} \le \|f - f_\varepsilon\|_{L^1_{(1)}(\mathbb{R}^n_-)} < \varepsilon$. Therefore, by Proposition 8, there exists $g_\varepsilon \in \mathcal{D}(\overline{\mathbb{R}^n_-})$ such that $g_\varepsilon|_{\mathbb{R}^{n-1}} = f_\varepsilon|_{\mathbb{R}^{n-1}}$ and $\|g_\varepsilon\|_{L^1_{(1)}(\mathbb{R}^n_-)} < C\varepsilon$. Hence $(f_\varepsilon - g_\varepsilon)|_{\mathbb{R}^{n-1}} = 0$ and $\|f - (f_\varepsilon - g_\varepsilon)\|_{L^1_{(1)}(\mathbb{R}^n_-)} < \|f - f_\varepsilon\|_{L^1_{(1)}(\mathbb{R}^n_-)} + \|g_\varepsilon\|_{L^1_{(1)}(\mathbb{R}^n_-)} < (C+1)\varepsilon$.

For (ii) note that the map $f \to \widetilde{f}$ is an isomorphism because $\|f\|_{L^1_{(1)}(\mathbb{R}^n_-)} \le \|\widetilde{f}\|_{L^1_{(1)}(\mathbb{R}^n)} \le 2\|f\|_{L^1_{(1)}(\mathbb{R}^n_-)}$. For $\Phi \in {}^oL^1_{(1)}(\mathbb{R}^n) \cap \mathcal{D}(\mathbb{R}^n)$ one has $\widetilde{\Phi|_{\mathbb{R}^n_-}} = \Phi$ Thus it suffices to show that ${}^oL^1_{(1)}(\mathbb{R}^n) \cap \mathcal{D}(\mathbb{R}^n)$ is dense in ${}^oL^1_{(1)}(\mathbb{R}^n)$. Fix $\varepsilon > 0$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^1_{(1)}(\mathbb{R}^n)$, given $F \in {}^oL^1_{(1)}(\mathbb{R}^n)$ (hence satisfying ${}^oF = F$) there is a $\Phi \in \mathcal{D}(\mathbb{R}^n)$ such that $\|F - \Phi\|_{L^1_{(1)}(\mathbb{R}^n)} < \varepsilon$. Thus $\|F -^o \Phi\|_{L^1_{(1)}(\mathbb{R}^n)} = \|^o(F - \Phi)\|_{L^1_{(1)}(\mathbb{R}^n)} \le \|F - \Phi\|_{L^1_{(1)}(\mathbb{R}^n)} < \varepsilon$. $\square$

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