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Topological methods in Banach spaces

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TOPOLOGIES ON A BANACH SPACE

$(X, \|\cdot\|)$ a Banach space

There are several topologies on X:

- norm topology – the topology generated by the metric induced by the norm.

- Weak topology: the weakest topology making all the norm continuous linear functionals continuous .. denoted by w or $\sigma(X, X^*)$ (X^* = the dual Banach space to X)

- weaker topologies: given $A \subset X^*$ denote by $\sigma(X, A)$ the weakest topology on X making all the functionals from A continuous

  $\sigma(X, A)$ is a locally convex topology generated by seminorms $x \mapsto |a(x)|$, $a \in A$

  basis of neighborhoods of 0 is formed by sets $\{x \in X : |a_i(x)| < \varepsilon_i, i = 1, \dots, n\}$

  $$a_1, \dots, a_n \in A, \ \varepsilon_1, \dots, \varepsilon_n > 0, \ n \in \mathbb{N}$$

  $\sigma(X, A)$ need not be Hausdorff. It is Hausdorff iff A separates points of X,

  i.e. $\forall x \in X, x \neq 0 \ \exists a \in A : a(x) \neq 0$

Examples: • $X = Y^*$ for a Banach space Y
Then $X^* = Y^{**}$. Take $A = Y \subset Y^{**}$
... $\sigma(Y^*, Y)$, the weak* topology

• $X = c_0(\Gamma)$ for a set Γ
$A = \{e_\gamma : \gamma \in \Gamma\} \subset \ell_1(\Gamma) = X^*$
canonical unit vectors
$\sigma(X, A)$ is the topology of pointwise convergence on Γ. ($\tau_p(\Gamma)$)

• $X = C(K)$, K compact Hausdorff space
$A = „K" = \{\delta_k : k \in K\} \subset X^*$

δ_k ... Dirac measure supported by k,
i.e. the evaluation functional $\delta_k(f) = f(k)$.
$\sigma(X, A)$ ... the pointwise convergence topology
$\tau_p(K)$ or τ_p

• $X = C(K)$, $A \subset K$ dense
$A \approx \{\delta_k : k \in A\} \subset X^*$
$\sigma(X, A)$ ... $\tau_p(A)$ ... topology of pointwise convergence on A

DUALITY OF BANACH SPACES AND COMPACT SPACES

- X a Banach space $\Rightarrow$ (B_{X^*}, w^*) is compact

the closed unit ball of X^*

[Banach – Alaoglu]

- K compact $\Rightarrow$ $(C(K), \|\cdot\|_\infty)$ is a Banach space

continuous functions on K (real or complex)

<u>The duality:</u> $X \hookrightarrow \mathcal{C}(B_{X^*}, w^*)$

$J: x \mapsto f_x$

$f_x(x^*) = x^*(x)$, $x^* \in B_{X^*}$, $x \in X$

Then J is a linear isometry of X into $\mathcal{C}(B_{X^*}, w^*)$

Moreover, J is a w–to–τ_p homeomorphism

$J(X)$ is τ_p-closed in $\mathcal{C}(B_{X^*}, w^*)$

[$f \in J(X) \Leftrightarrow$ f is affine & $f(0) = 0$ in the real case

& $f(\alpha x^*) = \alpha f(x^*)$ for $|\alpha| = 1$ in the complex case]

(Banach–Dieudonné theorem is used)

<u>Conversely:</u> $K \hookrightarrow (C(K)^*, w^*)$ $\quad z \mapsto \delta_z$

Then δ is a homeomorphism of K onto $(\delta(K), w^*)$

$\delta(K) \subset$ the unit sphere of $C(K)^*$.

SOME PROPERTIES OF WEAK TOPOLOGIES

<u>Theorem 1</u> X a linear space, $M \subset X^{\#}$ (the algebraic dual) separates points of X $\Rightarrow$ $(X, \sigma(X,M))$ satisfies ccc (countable chain condition)

– i.e. there is no uncountable system of pairwise disjoint nonempty open sets

<u>Scheme of the proof</u>:

1. $\mathbb{R}^{\mathbb{R}}$ is separable, thus it has ccc
2. $\mathbb{R}^{\Gamma}$ has ccc for any set Γ
3. Let B be an algebraic basis of span M

   Let $\Phi : X \to \mathbb{R}^{B}$ be defined by
   $$\Phi(x)(b) = b(x)$$
   $\Rightarrow \Phi$ is a homeomorphism of $(X, \sigma(X,M))$ onto $\Phi(X) \subset \mathbb{R}^{B}$ and $\Phi(X)$ is dense in $\mathbb{R}^{B}$

<u>Theorem 2</u> K compact $\Rightarrow (C(K), \tau_p)$ has countable tightness, i.e. whenever $A \subset C(K)$, $f \in \overline{A}^{\tau_p}$ $\Rightarrow \exists\, C \subset A$ countable : $f \in \overline{C}^{\tau_p}$.

<u>Corollary</u>: X Banach space $\Rightarrow (X, w)$ has countable tightness.

<u>Example</u>: (X^*, w^*) need not have countable tightness

Γ uncountable, $X = \ell_1(\Gamma)$, $X^* = \ell_\infty(\Gamma)$

$A = \{ f \in \ell_\infty(\Gamma) : \|f\| \leq 1, \text{ supp } f \text{ countable} \}$

Then $\overline{A}^{w^*} = B_{\ell_\infty(\Gamma)} \supsetneq A$

& $\forall C \subset A$ countable $\overline{C}^{w^*} \subset A$

Proof of Theorem 2 Let $A \subset C(K)$ and $g \in \overline{A}^{\tau_p}$

For $f \in C(K)$ and $k, n \in \mathbb{N}$ set

$$G_f^{k,n}(f) = \{ (x_1, \dots, x_n) \in K^n : |f(x_i) - g(x_i)| < \frac{1}{k} \text{ for } i = 1, \dots, n \}$$

Then $G^{k,n}(f)$ is open in K^n

Moreover, for fixed k, n

$\{ G^{k,n}(f) : f \in A \}$ is a cover of K^n (as $g \in \overline{A}^{\tau_p}$)

K^n compact $\Rightarrow$ there exists $F_{k,n} \subset A$ finite s.t.

$$\bigcup_{f \in F_{k,n}} G^{k,n}(f) = K^n$$

Set $C := \bigcup_{k,n \in \mathbb{N}} F_{k,n} \Rightarrow C$ is countable and $g \in \overline{C}^{\tau_p}$

Remark: It would be enough to have $F_{k,n}$ countable, i.e. to suppose that K^n is Lindelöf for each $n \in \mathbb{N}$.

TOPOLOGICAL CHARACTERIZATIONS OF SOME CLASSES OF BANACH SPACES

X is a Banach space

① X is separable $\Leftrightarrow$ (X, w) is separable $\Leftrightarrow$ (X, w) has countable network

② $\dim X < \infty$ $\Leftrightarrow$ (X, w) is metrizable $\Leftrightarrow$ (X, w) is first countable

③ X^* is separable $\Leftrightarrow$ (B_X, w) is metrizable
$\Leftrightarrow$ $X = \bigcup_{n=1}^{\infty} F_n$: F_n weakly closed, (F_n, w) metrizable

④ X is Asplund (i.e. $\forall Y \subset\subset X$ separable : Y^* is separable)
$\Leftrightarrow$ $\forall A \subset X$ weakly separable : $A = \bigcup_{n=1}^{\infty} F_n$,
F_n weakly closed in A, (F_n, w) metrizable

⑤ X is reflexive $\Leftrightarrow$ (B_X, w) compact $\Leftrightarrow$ (X, w) σ-compact

⑥ X does not contain any subspace isomorphic to ℓ_1
$\Leftrightarrow$ (B_X, w) is Fréchet-Urysohn
$\Leftrightarrow$ $X = \bigcup_{n=1}^{\infty} F_n$: F_n weakly closed, (F_n, w) Fréchet-Urysohn

(Topological space T is Fréchet-Urysohn
$\overset{\text{def}}{\Leftrightarrow}$
$\forall A \subset T$ $\forall x \in T$: $x \in \overline{A} \Rightarrow \exists (x_n) \subset A : x_n \to x$)

DUALITY OF SOME CLASSES:

(1) X separable $\Leftrightarrow$ (B_{X^*}, w^*) metrizable
K metrizable $\Leftrightarrow$ $C(K)$ separable

(2) Reflexive spaces, Asplund spaces cannot be characterized using just (B_{X^*}, w^*):

ℓ_2 is reflexive, c_0 is Asplund and not reflexive, ℓ_1 is not Asplund

But $(B_{\ell_2^*}, w^*)$, $(B_{c_0^*}, w^*)$ and $(B_{\ell_1^*}, w^*)$

are homeomorphic by Keller's theorem

WEAK COMPACTNESS AND ANGELICITY

<u>Def:</u> T topological space, $A \subset T$

A is
- relatively compact in T if $\overline{A}$ is compact
- relatively countably compact in T if $\forall (x_n) \subset A$ $\exists x \in T$ a clusterpoint of (x_n)
- relatively sequentially compact in T if $\forall (x_n) \subset A$ $\exists (x_{n_k})$: $x_{n_k} \to x \in T$

<u>Def:</u> T topological space is called <u>angelic</u> if $\forall A \subset T$ relatively countably compact:
- $\overline{A}$ is compact
- $\forall x \in \overline{A}$ $\exists (x_n) \subset A$: $x_n \to x$

<u>Theorem 3</u> L compact Hausdorff space $\Rightarrow (C(L), \tau_p)$ is angelic.

<u>PROOF:</u> <u>STEP 1</u> $A \subset C(L)$ relatively τ_p-countably compact

$\Rightarrow$
- A relatively τ_p-sequentially compact
- $\forall f \in \overline{A}^{\tau_p} \exists (f_n) \subset A : f_n \to f$

WLOG A separable (in fact countable)

[relative sequential compactness is defined using sequences
Thm 2 $\Rightarrow \forall f \in \overline{A}^{\tau_p} \exists C \subset A$ countable $f \in \overline{C}^{\tau_p}$]

WLOG L metrizable:

$S \subset A$ dense countable

Define $\varphi : L \to \mathbb{R}^S$

$$\varphi(x)(f) = f(x) \quad , f \in S, x \in L$$

$\Rightarrow \varphi$ is continuous $\Rightarrow M = \varphi(L)$ is a metrizable compact space

Let $\varphi^*: C(M) \to C(L)$ be defined $\varphi^*(g) = g\circ\varphi$, $g \in C(M)$

Then φ^* is a linear isometry and $\tau_p - \tau_p$ homeomorphism
Moreover,

$\varphi^*(C(M)) = \{h \in C(L) : \forall \ell, \ell' \in L : \varphi(\ell) = \varphi(\ell') \Rightarrow h(\ell) = h(\ell')\}$

[$\subset$ obvious

$\supset$: $h \in$ $\Rightarrow \exists\, g: M \to \mathbb{R}$ $g\circ\varphi = h$

h continuous, φ closed $\Rightarrow$ g continuous]

Thus $\varphi^*(C(M))$ is τ_p-closed in $C(L)$,

it contains S, so it contains $A \subset \overline{S}^{\tau_p}$

Thus WLOG $M = L$, i.e. L is metrizable

- L metrizable $\Rightarrow$ L is separable. Let $T \subset L$ be countable dense set. Then $(C(L), \tau_p(T))$ is metrizable as $\mathbb{R}^T$ is metrizable

A is τ_p-rel. countably compact $\Rightarrow$ A is $\tau_p(T)$-rel. countably compact $\Rightarrow$ A is $\tau_p(T)$-rel. sequentially compact (by metrizability of $\tau_p(T)$)

<u>Claim:</u> $(f_n) \subset A$: $(f_n \xrightarrow{\tau_p(T)} f \Leftrightarrow f_n \xrightarrow{\tau_p} f$

[$\Leftarrow$ obvious

$\Rightarrow$ Let $f_n \xrightarrow{\tau_p(T)} f$.

(f_{n_k}) any subsequence $\Rightarrow \exists\, g$ τ_p-cluster point

then g is also $\tau_p(T)$-cluster point

$f_{n_k} \xrightarrow{\tau_p(T)} f \Rightarrow f = g$

Thus f is a τ_p-cluster point of any subsequence of (f_n)

Thus $f_n \xrightarrow{\tau_p} f$]

It follows that A is τ_p- ~~seq~~ rel. sequentially compact

Further $f \in \overline{A}^{\tau_p} \Rightarrow f \in \overline{A}^{\tau_p(T)}$

$\Rightarrow \exists\ (f_n) \subset A : f_n \xrightarrow{\tau_p(T)} f \Rightarrow f_n \xrightarrow{\tau_p} f$

The end of STEP 1

STEP 2: If A is moreover bounded, then A is relatively weakly sequentially compact

$(f_n) \subset \mathcal{C}(K)$ bounded, $f_n \xrightarrow{\tau_p} f \in \mathcal{C}(K)$

$\Rightarrow f_n \xrightarrow{w} f$

[Riesz representation theorem + Lebesgue dominated convergence theorem]

STEP 3 Lemma (Eberlein): X Banach space

$A \subset X$ bounded

A rel. sequentially weakly compact $\Rightarrow$ A rel. weakly compact

(or rel. countably weakly compact)

<u>Proof</u> Let $A \subset X$ be bounded

Suppose A is not rel. weakly compact.

Then

$$\overline{A}^{w^*} \not\subset X \qquad (w^* = \sigma(X^{**}, X^*)$$
$$X \text{ embedded in } X^{**})$$

Fix $x^{**} \in \overline{A}^{w^*} \setminus X$

$\delta := \operatorname{dist}(x^{**}, X) > 0$

H-B $\Rightarrow \exists x^{***} \in X^{***}$, $\|x^{***}\| = 1$,

$x^{***}|_X = 0$, $x^{***}(x^{**}) = \delta$

Fix $\varepsilon \in (0, \frac{1}{2}) \Rightarrow$ There are $(x_k) \subset A$, $(x_k^*) \subset B_{X^*}$

s.t.

(i) $x^{**}(x_n^*) > \delta(1-\varepsilon)$

(ii) $|x_n^*(x_k)| < \delta \cdot \varepsilon$, $k < n$

(iii) $x_n^*(x_k) > \delta(1-\varepsilon)$, $k \geq n$

[x_1^* exists due to Goldstine theorem

given x_n^*, find $x_n \in A$ to satisfy (iii)

using $x^{**} \in \overline{A}^{w^*}$

then find x_{n+1}^* to satisfy (i) and (ii) by Goldstine theorem]

Then (x_k) has no weakly cluster point

Let x be such a cluster point

Let x^* be a w^*-cluster point of (x_n^*) in B_{X^*}

$$\left.\begin{array}{l}(ii) \Rightarrow \forall k: |x^*(x_k)| \leq \delta\varepsilon \Rightarrow |x^*(x)| \leq \delta\varepsilon \\ (iii) \Rightarrow \forall n: |x_n^*(x)| \geq \delta(1-\varepsilon) \Rightarrow |x^*(x)| \geq \delta(1-\varepsilon)\end{array}\right\} \text{a contradiction}$$

COMPLETION OF THE PROOF:

$A \subset C(L)$ rel. countably compact

by STEP 1 $\forall f \in \overline{A}^{\tau_p}$ $\exists (f_n) \subset A : f_n \xrightarrow{\tau_p} f$

It remains to prove that $\overline{A}^{\tau_p}$ is compact (in τ_p)

A bounded $\Rightarrow$ by STEPS 1 and 2 A is rel. weakly sequentially compact $\Rightarrow$ by STEP 3 A is rel. weakly compact, i.e. $\overline{A}^{w}$ is weakly compact

$\tau_p \subset w \Rightarrow$ on $\overline{A}^{w}$ topologies τ_p and w coincide

thus $\overline{A}^{w} = \overline{A}^{\tau_p}$ is τ_p-compact

A unbounded $\quad B = \{\operatorname{arctg} \circ f : f \in A\}$

$f \mapsto \operatorname{arctg} \circ f$ is a τ_p-to-τ_p homeomorphism $C(K, \mathbb{R})$ onto $C(K, (-\frac{\pi}{2}, \frac{\pi}{2}))$

B bdd and rel. τ_p-countably compact

$\Rightarrow \overline{B}^{\tau_p}$ is τ_p-compact

If we know that $\overline{B}^{\tau_p} \subset C(K, (-\frac{\pi}{2}, \frac{\pi}{2}))$,

then $\overline{A}^{\tau_p} = \{ \operatorname{tg} \circ f : f \in \overline{B}^{\tau_p} \}$ is τ_p-compact

$f \in \overline{B}^{\tau_p} \Rightarrow \exists (f_n) \subset B : f_n \xrightarrow{\tau_p} f$

(by STEP 1)

$(\operatorname{tg} \circ f_n) \subset A \Rightarrow$ ex. $(\operatorname{tg} \circ f_{n_k}) \xrightarrow{\tau_p} g$

Then $f = \operatorname{arctg} \circ g$

Corollary 1 X Banach space $\Rightarrow (X, w)$ angelic

$[(X,w) \subset (C((B_{X^*}, w^*)), \tau_p)$ as a closed subset$]$

Corollary 2 X a Banach space, $A \subset X$

TFAE: (i) A weakly compact

(ii) A weakly countably compact

(iii) A weakly sequentially compact

Corollary 3 K a compact space, $A \subset C(K)$

TFAE (i) A τ_p-compact

(ii) A τ_p-countably compact

(iii) A τ_p-sequentially compact

If A is bounded (in the norm), then

(iv) A is weakly compact

Remark: • Weakly compact sets are norm-bounded

• There are unbounded τ_p-compact sets in $C(K)$

Corollary 4 A a τ_p-compact subset of $C(K)$

$\Rightarrow$ A is homeomorphic to a weakly compact subset of a Banach space

Def: A compact space L is called *Eberlein* if L is homeomorphic to a weakly compact subset of a Banach space

WCG SPACES

<u>Def:</u> A Banach space X is called weakly compactly generated (WCG) if $\exists\, K \subset X$ weakly compact s.t. $\overline{\operatorname{span} K} = X$

<u>Examples:</u> ① X separable .... $(x_n) \subset B_X$ norm-dense
$K = \{\frac{1}{n} x_n\} \cup \{0\} \Rightarrow K$ is norm compact
$\overline{\operatorname{span} K} = X$

<u>Remark:</u> X is separable $\Leftrightarrow \exists K \subset X$ norm compact
$\overline{\operatorname{span} K} = X$

② X reflexive ... $K = B_X$

<u>Remark:</u> X is reflexive $\Leftrightarrow \exists\, K \subset X$ weakly compact s.t. $\operatorname{span} K = X$

③ $X = c_0(\Gamma)$, $K = \{e_\gamma : \gamma \in \Gamma\} \cup \{0\}$

④ $X = L^1(\mu)$, μ finite measure

[$T: L^2(\mu) \to L^1(\mu)$ the identity map
$\Rightarrow T$ is continuous, hence w-to-w continuous
$K = T(B_{L^2(\mu)})$]

Theorem Let X Banach space. TFAE

(1) X is WCG

(2) $\exists A \subset X$ weakly σ-compact $\|\cdot\|$-dense

(3) $\exists A \subset X$ weakly σ-compact weakly dense

Proof (1) $\Rightarrow$ (2) K weakly compact, $\overline{\operatorname{span} K} = X$

Then $\operatorname{span} K$ is weakly σ-compact

(2) $\Rightarrow$ (3) obvious

(3) $\Rightarrow$ (2) A weakly σ-compact weakly dense

$\Rightarrow \operatorname{span} A$ is weakly σ-compact $\|\cdot\|$-dense

(2) $\Rightarrow$ (1) $A = \bigcup_n K_n$, K_n weakly compact

K_n bounded $\Rightarrow K_n \subset c_n \cdot B_X$

$$L := \{0\} \cup \bigcup_{n=1}^{\infty} \frac{1}{n c_n} B_X$$

$\Rightarrow L$ is weakly compact and $\overline{\operatorname{span} L} = X$

THEOREM 5

(1) X is WCG $\Longrightarrow$ (B_{X^*}, w^*) is Eberlein

(2) $C(K)$ is WCG $\Longleftrightarrow$ K is Eberlein

(3) (B_{X^*}, w^*) is Eberlein $\Longrightarrow$ X is isomorphic to a subspace of a WCG space

<u>Proof</u> (1) X is WCG $\Rightarrow \exists L \subset X$ weakly compact $\overline{\operatorname{span} L} = X$

Take $\Phi : X^* \longrightarrow C(L, w)$

$\Phi(x^*)(x) = x^*(x)$, i.e. $\Phi(x^*) = x^*|_L$

Then Φ is w^*-to-τ_p continuous and one-to-one (as $\overline{\operatorname{span} L} = X$)

$\Rightarrow \Phi|_{(B_{X^*}, w^*)}$ is a homeomorphism onto $(\Phi(B_{X^*}), \tau_p)$. So, it is an Eberlein compact

(2) $\Rightarrow$ follows from (1) as $K \subset (B_{C(K)^*}, w^*)$

$\Leftarrow$ K Eberlein $\Rightarrow$ WLOG $K \subset (C(L), \tau_p)$ for a compact L

Take $\Phi : L \longrightarrow C(K)$

$\Phi(l)(k) = k(l)$

Φ is continuous L-to-$(C(K), \tau_p)$

$\Rightarrow \Phi(L)$ is a τ_p-compact subset of $C(K)$ separating points of K

$\Rightarrow$ alg$(\{1\} \cup L)$ is norm-dense in $C(K)$, it is τ_p-σ-compact, thus weakly σ-compact.

(3) follows from (2) as $X \subset C(B_{X^{**}}, w^*)$

Remarks: • A continuous image of an Eberlein compact is again Eberlein

• A subspace of a WCG space need not be WCG

Corollary: In (3) we have $\Longleftrightarrow$

In (1) $\Leftarrow$ need not hold.

Theorem 6: X WCG $\Rightarrow$ X is $F_{\sigma\delta}$ in (X^{**}, w^*)

K Eberlein $\Rightarrow$ $C(K)$ is $F_{\sigma\delta}$ in $\mathbb{R}^K$

Proof: ~~$\overline{\operatorname{span}\, L} = X$, L weakly compact~~

K_n weakly compact in X, $X = \overline{\bigcup_n K_n}^{\|\cdot\|}$

$$\text{Then } X = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left(K_n + \tfrac{1}{m} B_{X^{**}}\right)$$

$X = C(K)$:

$$X = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left(K_n + \left[-\tfrac{1}{m}, \tfrac{1}{m}\right]^K\right)$$

Corollary: X WCG $\Rightarrow$ (X, w) is Lindelöf.