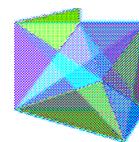




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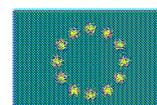
Metody teoriomnogościowe w przestrzeniach Banacha.

Piotr Koszmider



**KAPITAŁ LUDZKI**  
NARODOWA STRATEGIA SPÓJNOŚCI

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## **Note on lecturer**

Piotr Koszmider po ukończeniu studiów magisterskich w Uniwersytecie Warszawskim (1988) odbył studia doktoranckie w University of Toronto (Kanada), a następnie staż podoktorancki w York University (Kanada) i Hebrajskim Uniwersytecie w Jerozolimie (Izrael). Następnie pracował w USA (Auburn University i Ohio University). W tym okresie zajmował się zastosowaniami forcingu i kombinatoryki dwóch liczb kardynalnych w algebrach Boole'a i topologii teoriomnogościowej. W 1998 zaczął prace w Universidade de São Paulo (Brazylia) gdzie rozpoczął badania, które kontynuuje do tej pory nad zastosowaniami forcingu oraz w ogóle kombinatoryki nieskończonej i topologii mnogościowej w przestrzeniach Banacha, w szczególności w przestrzeniach Banacha funkcji ciągłych. Habilitował się w 1999 w Universidade de São Paulo i w 2006 w IMPAN w Warszawie. Od 2007 pracuje w Politechnice Łódzkiej.

## LECTURE NOTES

### SET-THEORETIC METHODS IN BANACH SPACES

PIOTR KOSZMIDER

Given at the Institute of Mathematics of the University of Wrocław  
as a part of the series *Set Theory Applied* within the programme of  
Środowiskowe Studia Doktoranckie z Nauk Matematycznych  
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The main idea of these lectures is to show how the Banach spaces of continuous functions can serve as the tool of applying combinatorial set-theory in the theory Banach spaces.

#### 0.1. The special place of Banach spaces of continuous functions among Banach spaces.

- (1) Every Banach space is a subspace of a  $C(K)$ , for example (Banach-Mazur) every separable Banach space is a subspace of  $C([0, 1])$
- (2) Some fundamental Banach spaces are  $C(K)$ s:  $l_\infty \equiv C(\beta N)$ ,  $c_0 \sim C([0, \omega])$
- (3) The supremum norm is "the simplest" norm
- (4) Many important examples and counter-examples are  $C(K)$ s
- (5)  $C(B_{X^*})$  carries lots of information about the Banach space  $X$ , especially if  $X$  is not separable
- (6) Many of these examples exploit topological or set-theoretic topological properties of  $K$

#### Some classical uses of $C(K)$ s as counter-examples

- (1) J. Schreier ([73]) Shows that  $C([0, \omega^\omega])$  is not isomorphic to  $C([0, \omega])$  giving the first example of nonisomorphic Banach spaces with isomorphic duals.
- (2) W. Johnson and J. Lindenstrauss ([33]) consider  $C(K)$  where  $K$  is induced by an almost disjoint family obtaining a WCG space with dual not a WCG space.
- (3) R. Pol ([64]) obtains a weakly Lindelöf  $C(K)$  which is not WCG
- (4) K. Kunen ([54]) obtains (under CH) a nonseparable  $C(K)$  with no uncountable biorthogonal system
- (5) M. Talagrand ([79]) obtains (under CH) among others a  $C(K)$  which has neither  $c_0$  nor  $l_\infty$  as a quotient.
- (6) R. Haydon ([28]) obtains among others a  $C(K)$  which has neither  $c_0$  nor  $l_\infty$  as complemented subspaces.

1. THE BANACH SPACES  $C(K)$ : 18.15-19.00H, 18.11.2010

ABSTRACT. Introductory definitions. Introductory examples. The Stone duality. Basic operations. Structural theorems.

## 1.1. Introductory definitions.

- (1)  $K$  - unless otherwise stated an infinite Hausdorff compact space
- (2)  $C(K)$  the set of all real-valued continuous functions from  $K$  into  $\mathbb{R}$  with the pointwise algebraic operations and the supremum norm
- (3) Morphisms among Banach spaces - isomorphisms ( $\sim$ ) but not isometries ( $\equiv$ )

If  $T: C(K) \rightarrow C(L)$  is an *isometry* (i.e., a linear isomorphism which preserves the norm),  $T$  induces a homeomorphism between  $K$  and  $L$  (Banach-Stone, see [69, 7.8.4]), but many non-homeomorphic  $K$ s have isomorphic  $C(K)$ s. The simplest examples are of two disjoint convergent sequences  $K$ , i.e.,  $(y_{2m}) \rightarrow \infty_1$  and  $(y_{2m-1}) \rightarrow \infty_2$  for  $m > 0$  with their respective distinct limit points  $\infty_1$  and  $\infty_2$  and one convergent sequence  $L$ , i.e.,  $(x_n) \rightarrow \infty$  with its limit point  $\infty$ . One can explicitly define an isomorphism  $T$ :

$$\begin{aligned} T(f)(x_0) &= f(\infty_1) - f(\infty_2), \\ T(f)(x_{2m}) &= f(y_{2m}) - \frac{f(\infty_1) - f(\infty_2)}{2}, \\ T(f)(x_{2m-1}) &= f(y_{2m-1}) + \frac{f(\infty_1) - f(\infty_2)}{2}, \\ T(f)(\infty) &= \frac{f(\infty_1) + f(\infty_2)}{2}. \end{aligned}$$

for  $m > 0$ . In particular, we noted that  $C([0, \omega + \omega]) \sim C([0, \omega]) \oplus C([0, \omega]) \sim C([0, \omega])$ . Working with  $C(K)$ s seems then, is working in poorer environment than with compact spaces, after all, we identify compact spaces with the same  $C(K)$  (in the isomorphic sense). It is just one side of the coin, on the other hand we get more “continuous” mappings from  $K$  to  $K$ , i.e., not all operators on  $C(K)$  come from a “usual” continuous mapping on  $K$ .

## 1.2. Introductory examples.

- (1) Examples of subspaces
- (2)  $l_\infty(\kappa)$ ,  $l_1(\kappa)$ ,  $c_0(\kappa)$  in  $C(K)$ s
- (3) Examples of isomorphisms, injections and surjections

**Proposition 1.1.** *Suppose  $K$  has a pairwise disjoint family cardinality  $\kappa$  if and only if  $C(K)$  has a subspace isomorphic to  $c_0(\kappa)$ .*

*Proof.* If  $(U_\xi)_{\xi < \kappa}$  is a pairwise disjoint family of open subsets of  $K$ , take  $f_\xi: K \rightarrow [0, 1]$  of norm one with supports included in  $U_\xi$ . It generates even an isometric copy of  $c_0(\kappa)$ .

Now, suppose  $T: c_0(\kappa) \rightarrow C(K)$  is an isomorphism. As it has a bounded inverse defined on its image, there is  $\varepsilon > 0$  such that for  $f_\xi = T(\chi_{\{\xi\}})$  we have  $\|f_\xi\| > \varepsilon$ . For  $\xi \in \kappa$  consider

$$V_\xi = f_\xi^{-1}[(\varepsilon/2, \|T\| + 1)].$$

Let  $k \in \mathbb{N}$  be such that  $k\varepsilon/2 > \|T\|$ . Note that no point  $x$  of  $K$  can be in  $k$  sets  $V_\xi$ , because then we would have  $T(\chi_{\{\xi_1, \dots, \xi_k\}})(x) > \|T\|$  which is impossible since  $\|\chi_{\{\xi_1, \dots, \xi_k\}}\| = 1$ . Let  $m \leq k$  be maximal such that

$$\left\{ \bigcap_{\xi \in F} V_\xi \neq \emptyset : F \in [\kappa]^m \right\}$$

has cardinality  $\kappa$ . It is easy to note that this family contains a subfamily of cardinality  $\kappa$  which is pairwise disjoint. □

**Proposition 1.2.** *Suppose that  $K$  has an independent family of cardinality  $\kappa$  of clopen sets. Then  $C(K)$  has a subspace isometric to  $l_1(\kappa)$ .*

*Proof.* Let  $\{a_\xi : \xi < \kappa\}$  be an independent family of  $Clop(K)$ , consider  $f_\xi = \chi_{a_\xi} - \chi_{K \setminus a_\xi}$ . □

Having an independent family of cardinality  $\kappa$  corresponds in the case of not necessarily totally disconnected space to the existence of a continuous surjection  $\phi : K \rightarrow [0, 1]^\kappa$ . Whether this gives an isomorphic or isometric copy of  $l_1(\kappa)$  depends on  $\kappa$  and extra set-theoretic assumptions (see e.g. [59])

**Proposition 1.3.** *Suppose that  $\mathcal{A}$  is a subalgebra of  $\wp(\mathbb{N})$ .  $C(K_{\mathcal{A}})$  contains  $l_\infty$  if and only if there is an infinite  $A \subseteq \mathbb{N}$  such that for each infinite  $B \subseteq A$  there is an element  $C \in \mathcal{A}$  such that  $C \cap A = B$ .*

### 1.3. The Stone duality.

**Definition 1.4.** *Suppose  $\mathcal{A}$  is a Boolean algebra. The Stone space of  $\mathcal{A}$  is denoted by  $K_{\mathcal{A}}$  and defined as*

$$K_{\mathcal{A}} = \{x \mid x : \mathcal{A} \rightarrow \{0, 1\}, x \text{ is a Boolean homomorphism}\}$$

*The basis of the topology consists of the sets of the form*

$$[a] = \{x \in K_{\mathcal{A}} : x(a) = 1\}$$

**Theorem 1.5.** *If  $\mathcal{A}$  is a Boolean algebra, then its Stone space  $K_{\mathcal{A}}$  is a compact Hausdorff space.*

We opted for such a definition instead of the usual where the points of the Stone space are the ultrafilters of  $\mathcal{A}$  because this way the Stone duality follows a general scheme of the dualities related to Banach spaces. Also “the way back” i.e., getting the Boolean algebra from a compact Hausdorff, totally disconnected space  $K$  follows a similar pattern: the Boolean algebra  $Clop(K)$  of clopen sets can be associated with the Boolean algebra  $mathcal{A}_K$  of continuous functions into  $\{0, 1\}$  with the operations of  $\max$ ,  $\min$ ,  $1-$ .

**Proposition 1.6.** *Let  $\mathcal{B}$  be a Boolean algebra and  $L$  be a compact Hausdorff totally disconnected space, then:*

- $\mathcal{A}_{K_{\mathcal{B}}} \equiv \mathcal{B}$ .
- $K_{\mathcal{A}_L} \equiv L$ .

**Definition 1.7.** *Suppose  $\mathcal{A}, \mathcal{B}$  are Boolean algebras and  $h : \mathcal{A} \rightarrow \mathcal{B}$  is a Boolean homomorphism. Then  $\phi_h : K_{\mathcal{B}} \rightarrow K_{\mathcal{A}}$  is defined by*

$$\phi_h(y) = y \circ h$$

**Proposition 1.8.** *Suppose  $h : \mathcal{A} \rightarrow \mathcal{B}$  is monomorphism, epimorphism, isomorphism. Then  $\phi_h : K_{\mathcal{B}} \rightarrow K_{\mathcal{A}}$  is surjective, injective, homeomorphism, respectively.*

Below, to get a symmetric definition, we identify clopen sets with the continuous functions into  $\{0, 1\}$  i.e. elements of  $\mathcal{A}_K$ .

**Definition 1.9.** *Suppose  $K, L$  are compact, Hausdorff, totally disconnected spaces and  $\phi : K \rightarrow L$  is a continuous mapping. Then  $h_\phi : \mathcal{A}_L \rightarrow \mathcal{A}_K$  is defined by*

$$h_\phi(a) = a \circ \phi$$

**Proposition 1.10.** *Suppose  $\phi : K \rightarrow L$  is surjective, injective, homeomorphism. Then  $h_\phi : Clop(L) \rightarrow Clop(K)$  is monomorphism, epimorphism, isomorphism, respectively.*

Thus passing from Boolean algebras to totally disconnected compact spaces inverts the direction of the arrows. The images of injections or monomorphisms are naturally identified with closed subsets or subalgebras. The induced objects are obtained by taking the intersection of the preimage of 1 like clopen set or an ultrafilter with the subobject. Thus a closed subset  $K \subseteq L$  induces an epimorphism from  $Clop(L)$  onto  $Clop(K)$  by  $a \rightarrow a \cap K$  and all homomorphic images of  $Clop(L)$  are of this form, i.e., can be associated with closed subsets of  $L$ , on the other hand a subalgebra  $\mathcal{A} \subseteq \mathcal{B}$  induces a surjective mapping from  $K_{\mathcal{B}}$  onto  $K_{\mathcal{A}}$  by  $u \rightarrow u \cap \mathcal{A}$  and all continuous images of  $K_{\mathcal{B}}$  are of this form, i.e., can be associated with subalgebras of  $\mathcal{B}$ .

**Theorem 1.11** (Weierstrass-Stone). *If  $\mathcal{C}$  is a subalgebra of  $C(K)$  which contains a constant function and separates the points of  $K$ , then  $\mathcal{C}$  is dense in  $C(K)$*

**Theorem 1.12.** *Suppose  $K$  is totally disconnected compact Hausdorff space. Then the simple functions (those assuming only finitely many values) are dense in  $C(K)$*

For more on the Stone duality see [35] or [21].

**1.4. Basic operations.** In a similar manner to the Stone duality we can induce morphism between Banach spaces  $C(K)$  given morphisms between  $K$ s. We separate the cases of monomorphisms and epimorphisms.

**Theorem 1.13.** *Suppose  $K$  and  $L$  are compact spaces and  $\phi : K \rightarrow L$  is a continuous surjection. Then  $T_\phi : C(L) \rightarrow C(K)$  given by*

$$T_\phi(f) = f \circ \phi$$

*for all  $f \in C(L)$  is an isometric isomorphism onto its range which is closed in  $C(K)$ . In particular  $C(L)$  is isometric to a subspace of  $C(K)$ .*

**Theorem 1.14.** *Suppose  $L \subseteq K$  Then  $T_L : C(K) \rightarrow C(L)$  given by*

$$T_L(f) = f|_L$$

*for all  $f \in C(K)$  is a surjection of norm one whose kernel is*

$$C_0(K||L) = \{f \in C(K) : \forall x \in L f(x) = 0\}.$$

*In particular  $C(L)$  is isometric to  $C(K)/C_0(K||L)$ .*

**Corollary 1.15.**  $l_\infty/c_0 \cong C(\mathbb{N}^*)$ .

**Theorem 1.16** (Banach-Stone). *Suppose that  $T : C(K) \rightarrow C(L)$  is an isometry. Then  $T$  is of the form  $\pm T_\phi$  where  $\phi : L \rightarrow K$  is a homeomorphism. In particular  $K$  and  $L$  are homeomorphic.*

Cambern and Amir independently proved that if  $T : C(K) \rightarrow C(L)$  is an isomorphism. such that  $\|T\|\|T^{-1}\| < 2$ , then  $K$  and  $L$  are homeomorphic. 2 is the best possible constant.

**Definition 1.17.** *If  $X$  is a Banach space, a bounded linear operator  $P : X \rightarrow X$  is called a projection if and only if  $P^2 = P$ . If  $\mathcal{A}$  is a Banach space, a Boolean homomorphism  $p : \mathcal{A} \rightarrow \mathcal{A}$  is called a (Boolean) projection if and only if  $p^2 = p$ . If  $K$  is a compact space, a continuous mapping  $r : K \rightarrow K$  is called a retraction if and only if  $r^2 = r$ .*

For a function  $f : A \rightarrow A$  the condition  $f^2 = f$  is equivalent to the condition  $f|_{Im(f)} = Id_{Im(f)}$ . The simplest example of a projection in a  $C(K)$  space is of the form  $T(f) = f\chi_A$  where  $A \subseteq K$  is a clopen set. Bounded projections in Banach spaces have closed ranges.

**Definition 1.18.** *We call a closed subspace  $Y$  of a Banach space  $X$  a complemented subspace if and only if there is another closed subspace  $Z \subseteq X$  such that  $X = Y \oplus Z$  (i.e.,  $Y \cap Z = \{0\}$  and  $Y + Z = X$ ).*

**Proposition 1.19.** *Let  $X$  be a Banach space and let  $Y \subseteq X$  be a closed subspace.  $Y$  is complemented in  $X$  if and only if there is a projection in  $X$  onto  $Y$ .*

The above proposition shows that projections in Banach spaces are much more important than retraction of compact spaces or projections in Boolean algebras. In the case of Banach space they correspond to decompositions of a Banach space into two simpler Banach spaces.

**Proposition 1.20.** *Let  $\mathcal{A}$  be a Boolean algebra.  $p : \mathcal{A} \rightarrow \mathcal{A}$  is a projection if and only if  $\phi_p : K_{\mathcal{A}} \rightarrow K_{\mathcal{A}}$  is a retraction. Let  $K$  be a compact totally disconnected space.  $r : K \rightarrow K$  is a retraction if and only if  $h_r : \mathcal{A}_K \rightarrow \mathcal{A}_K$  is a projection.*

**Proposition 1.21.** *Let  $K$  be a compact space. If  $r : K \rightarrow K$  is a retraction with its image  $L \subseteq K$ , then  $T_r : C(K) \rightarrow C(K)$  is a projection onto a subspace*

$$\{f \circ r : f \in C(K)\}$$

*which is isometric to  $C(L)$ . The kernel of the projection is  $C_0(K||L)$ .*

**Proposition 1.22.** *Finite codimensional closed subspaces of Banach spaces are complemented and if they have the same codimension, they are isomorphic to each other.*

### 1.5. Structural Theorems.

**Theorem 1.23** (Milutin, Bessaga, Pełczyński). *Classification of  $C(K)$ 's for  $K$  metrizable. The following are all representatives of all distinct isomorphism classes of Banach spaces  $C(K)$  for  $K$  compact metrizable:*

- (1)  $C(2^\omega) \sim C([0, 1]) \sim C(K)$ , for any  $K$  uncountable
- (2)  $C([0, \alpha])$  for  $\alpha < \omega_1$  such that if  $\beta < \alpha$  and  $n \in \omega$ , then  $\beta^n < \alpha$ .

**Theorem 1.24.** *Let  $f$  be any bounded function from the canonical basis of  $l_1(\kappa)$  into a Banach space  $X$ . Then there is a bounded linear operator  $T : l_1(\kappa) \rightarrow X$  which extends  $f$ .*

**Theorem 1.25.** *Let  $f$  be any function from the independent family of generators of the free Boolean algebra  $Fr(\kappa)$  into a Boolean algebra  $A$ . Then there is a homomorphism of Boolean algebras  $h : Fr(\kappa) \rightarrow A$  which extends  $f$ .*

**Theorem 1.26** (Sikorski Extension Theorem). *Suppose  $A \subseteq B$  are Boolean algebras  $C$  is a complete Boolean algebra and  $h : A \rightarrow C$  is a homomorphism. Then there is a homomorphism  $g : B \rightarrow C$  which extends  $h$ .*

**Theorem 1.27** (Goodman, Nachbin). *Suppose  $X \subseteq Y$  are Banach spaces  $K$  is the Stone space of a complete Boolean algebra and  $T : X \rightarrow C(K)$  is a bounded linear operator. Then there is a bounded linear operator  $S : Y \rightarrow C(K)$  which extends  $T$ .*

**Question 1.28.** *A Banach space which can be put in place of  $C(K)$  for  $K$  extremally disconnected above is called an injective Banach space. What are injective Banach spaces? What are injective Banach spaces of the form  $C(K)$ ? Are all injective Banach spaces of the form  $C(K)$  for  $K$  a Stone space of a complete Boolean algebra?*

**Corollary 1.29.**  *$l_1(2^\omega)$  is a subspace of a Banach space  $X$  if and only if  $l_\infty$  is a quotient of  $X$ .*

*Proof.* The cardinality of  $l_\infty$  is  $2^\omega$  and so there is a function from the basis of  $l_1(2^\omega)$  onto  $l_\infty$ . This can be extended to a surjective operator from  $l_1(2^\omega)$  onto  $l_\infty$  and this can be extended to an operator from  $X$  onto  $l_\infty$ .  $\square$

2. CONTINUOUS LINEAR FUNCTIONALS ON  $C(K)$ s: 19.15-20.00H, 18.11.2010

ABSTRACT. The Riesz representation theorem. Radon measures on totally disconnected spaces. Atomic measures. Radon measures on scattered spaces.

**2.1. The Riesz representation theorem.** The dual of a Banach space  $X$  is the set of all continuous functionals with the norm given by

$$\|\phi\| = \sup\{|\phi(x)|/\|x\| : x \in X\}$$

It is a standard elementary fact that it is indeed a Banach space. In the case of spaces  $C(K)$  we will interpret the elements of the dual as certain measures on  $K$ . This is due to the Riesz representation theorem which states that there is an isometry between the dual to a  $C(K)$  and a certain Banach space  $M(K)$  of measures on  $K$ . We will need some terminology concerning signed borel (i.e., defined on borel sets of a compact  $K$ ) measures. Such a measure is countably additive if  $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$  whenever  $A_n$ 's pairwise disjoint borel sets and the equality includes the statement that the series is absolutely convergent. We define a variation of measure  $\mu$  at a borel  $A \subseteq K$  as

$$|\mu|(A) = \sup\{|\mu(B) - \mu(C)| : B \cap C = \emptyset, B, C \subseteq A, B, C \text{ are borel}\}.$$

It turns out that if  $\mu$  is a bounded measure, then  $|\mu|$  is a bounded measure which is *nonnegative*, i.e., assumes only non-negative values. In case of signed measures the regularity condition can be stated as: for every borel  $A \subseteq K$  and every  $\varepsilon > 0$  there are compact  $F$  and open  $U$  satisfying  $F \subseteq A \subseteq U \subseteq K$  such that  $|\mu|(U \setminus A) < \varepsilon$ .

**Definition 2.1.** *Radon measure on a compact space  $K$  is a signed, finite, countably additive, borel, regular measure.*

**Theorem 2.2** (The Riesz representation theorem). *Every continuous linear functional  $\psi$  on a  $C(K)$  is of the form*

$$f \rightarrow \psi(f) = \int f d\mu_\psi \quad \forall f \in C(K)$$

for some Radon measure  $\mu_\psi$  on  $K$ . We have  $\|\psi\| = |\mu_\psi|$ . And so there is an isometry between the Banach space  $C(K)^*$  and  $M(K)$ .

**2.2. Radon measures and the Riesz theorem on totally disconnected spaces.** If  $\mathcal{A}$  is a Boolean algebra, then by a measure we will mean a finite signed measure i.e., a function  $\mu : \mathcal{A} \rightarrow \mathbb{R}$  such that  $\mu(0) = 0$  and  $\mu(a \vee b) = \mu(a) + \mu(b)$  whenever  $a \wedge b = 0$ . The latter property will be called the finite additivity of  $\mu$ . A measure is bounded if its range is a bounded subset of  $\mathbb{R}$ .

**Proposition 2.3.** *Suppose that  $K$  is totally disconnected. Let  $\mu$  be a Radon measure on  $K$  and  $\nu$  its restriction to the algebra of clopen sets of  $K$ . Then  $\nu$  is a bounded measure on  $\text{Clop}(K)$  of the same variation.*

**Proposition 2.4.** *Suppose that  $K$  is a totally disconnected compact Hausdorff space. Let  $\mu$  be a bounded measure on the Boolean algebra  $\text{Clop}(K)$ . Then  $\mu$  uniquely extends to a Radon measure on  $K$ .*

*Proof.* Now, let us see the uniqueness. Let  $\mu, \nu \in M(K)$  agree on clopen sets of  $K$ . Let  $A \subseteq K$  be Borel. For every  $\varepsilon > 0$  the regularity of the measures, the compactness and the total disconnectedness of  $K$  give a clopen  $U \subseteq K$  such that  $|\mu|(U \Delta A), |\nu|(U \Delta A) < \varepsilon$ . Since  $\mu(U) = \nu(U)$  we conclude that  $\mu(A)$  and  $\nu(A)$  are arbitrarily close to each other, and hence must be equal.  $\square$

### 2.3. Atomic measures.

**Proposition 2.5.** *Atomic measures form a closed subspace of the Banach space  $M(K)$  which is isometric to  $l_1(K)$ . Every atomic Radon measure  $\mu$  is of the form*

$$\mu = \sum_{i \in N} a_i \delta_{x_i}$$

where  $x_i \in K$  and  $\sum_{i \in N} |a_i| < \infty$ .

*Proof.* It is clear (by considering norm-one functions in  $C(K)$  which assume values  $\pm 1$  on given finitely many points) that to be a bounded measure (i.e., in particular Radon measure), the series  $\sum \{\pm \mu(\{x\}) : x \in X\}$  must be convergent to a sum  $\leq \|\mu\|$ . Thus the set of singletons of non-null  $\mu$ -measure is countable, say  $\{x_i : i \in N\}$  and  $\sum_{i \in N} |a_i| < \infty$  where  $a_i = \mu(\{x_i\})$ .

Note that no subset of  $K - \{x_i : i \in N\}$  can contain an atom, so  $\mu = \sum_{i \in N} a_i \delta_{x_i}$ .

Note that  $\mu = \sum_{i \in N} a_i \delta_{x_i}$  is always an atomic Radon measure since  $M(K)$  is a Banach space. So we get a function  $T : l_1(K) \rightarrow X$  where  $X$  is the space of atomic measures in  $M(K)$ . It is easy to see that it is linear, bounded and onto, so by the open mapping theorem it is an isomorphism and so  $X$  is a closed subspace of  $M(K)$ .

One can see explicitly that the subspace of atomic measure is closed. If a Radon measure  $\mu$  is not an atomic measure, let us see that its distance from the subspace of atomic measures is positive.

Namely, consider  $\nu = \mu - \sum_{i \in N} a_i \delta_{x_i}$ , where  $x_i$ 's and  $a_i$ 's are as above, that is  $\nu$  has no atoms, but is a nonzero measure. Let  $A \subset K - \{x_i : i \in N\}$  be of non-null  $\nu$ -measure.

Take any atomic measure  $\lambda = \sum_{i \in N} b_i \delta_{y_i}$ , so  $\nu(A - \{y_i : i \in N\}) = \nu(A)$ , i.e.,

$$(\mu - \lambda)(A - \{y_i : i \in N\}) = (\nu - \lambda)(A - \{y_i : i \in N\}) = \nu(A) \neq 0$$

which completes the proof of the proposition.  $\square$

**2.4. Radon measures on scattered spaces.** If  $K$  is scattered we can define its Cantor-Bendixson derivative  $X^{(\alpha)}$  for each ordinal  $\alpha$  by the inductive conditions:  $K^{(0)} = K$ ,  $K^{(\alpha+1)} = (K^{(\alpha)})'$  and  $K^{(\lambda)} = \bigcap_{\alpha < \lambda} K^{(\alpha)}$  where  $X'$  is the set of all nonisolated points of  $X$ . The minimal ordinal  $\alpha$  such that  $K^{(\alpha+1)} = \emptyset$  is called the height of  $K$  and denoted  $ht(K)$ . The supremum of the cardinalities of the sets  $K^{(\alpha)} \setminus K^{(\alpha+1)}$  is called the width of  $K$ .

Examples of scattered spaces:

- (1) Ordinal intervals  $[0, \alpha]$  with the order topology; For example  $ht([0, \omega_1]) = wd([0, \omega_1]) = \omega_1$ .
- (2) The Stone space of a Boolean algebra generated by  $Fincofin(\mathbb{N})$  and a maximal almost disjoint family  $\mathcal{A}$  of subsets of  $\mathbb{N}$ . This is so called  $\Psi$ -space or Isbell's space or Mrówka's space.  $ht(\Psi_{\mathcal{A}}) = 3$ ,  $wd(\Psi_{\mathcal{A}}) = |\mathcal{A}|$
- (3) the ladder system space
- (4) Specific constructions like Ostaszewski's space, Kunen's space, Ciesielski-Pol's space

**Theorem 2.6** (Mazurkiewicz-Sierpiński). *Every metrizable scattered compact space is homeomorphic to a space  $[0, \alpha]$  for some countable ordinal.*

**Theorem 2.7.** *Suppose that  $K$  is a compact Hausdorff space. The following are equivalent:*

- (1)  $K$  is scattered
- (2) There is no surjection  $f : K \rightarrow [0, 1]$
- (3)  $l_1$  is not a subspace of  $C(K)$

**Proposition 2.8.** *A compact scattered space is totally disconnected.*

**Theorem 2.9.**  *$K$  is scattered if and only if  $M(K)$  consists only of atomic measures if and only if  $M(K)$  is isometric to  $l_1(K)$ .*

*Proof.* We will only prove that every Radon measure on a scattered compact  $K$  has an atom. Then  $\nu = \mu - \sum_{i \in N} a_i \delta_{x_i}$  must be always the zero measure, where  $a_i = \mu(\{x_i\})$  and  $x_i$ 's are all atoms of  $\mu$ , so it follows that every Radon measure is atomic.

Let  $\mu \in M(K)$ . Let  $\alpha < ht(K)$  be minimal such that  $|\mu|(K^{(\alpha)} - K^{(\alpha+1)}) \neq 0$ . By the regularity, there is a compact set  $X \subseteq K^{(\alpha)} - K^{(\alpha+1)}$  which is  $\mu$ -non-null. However, as  $K^{(\alpha)} - K^{(\alpha+1)}$  is a discrete set of isolated points of  $K^{(\alpha)}$ , such a compact set may only be finite, i.e.,  $\mu$  has an atom.  $\square$

## 3. THE KUNEN SPACE AND BIORTHOGONAL SYSTEMS: 12.15-13H, 19.11.2010

ABSTRACT. The weak topology in  $C(K)$ . The Kunen space.

3.1. The weak topology in  $C(K)$ .

**Theorem 3.1.**  *$K$  is scattered if and only if the weak topology in  $C(K)$  coincides in bounded sets with the pointwise convergence topology.*

*Proof.* Suppose  $K$  is scattered. It is clear that given  $x_i \in K$  for  $i \leq k \in \mathbb{N}$  we have

$$\begin{aligned} & \{g \in C(K) : \forall i \leq k |f(x_i) - g(x_i)| < \varepsilon\} = \\ & = \{g \in C(K) : \forall i \leq k |\delta_{x_i}(f) - \delta_{x_i}(g)| < \varepsilon\}, \end{aligned}$$

i.e., open sets in the pointwise convergence topology are open in the weak topology. Now if  $\mu = \sum_{i \in \mathbb{N}} a_i \delta_{x_i}$  is a Radon measure on  $K$ ,  $f \in C(K)$ ,  $\|f\| < M$  and  $M \in \mathbb{R}$  we have

$$\begin{aligned} & \{g \in C(K) : \|g\| < M, |\int (f - g)d\mu| < \varepsilon\} \supseteq \\ & \supseteq \{g \in C(K) : \|g\| < M, \forall i \leq k |f(x_i) - g(x_i)| < \delta\} \end{aligned}$$

where  $k \in \mathbb{N}$  is such that  $\sum_{i > k} |a_i| < \varepsilon/4M$  and  $\delta < \varepsilon/2k$ . That is open sets in the weak topology are open in the topology of pointwise convergence if we work with bounded subsets of  $C(K)$ .

Conversely if  $K$  is not scattered, then there is an atomless Radon measure  $\mu$  over  $K$ . In this case one constructs a family  $\mathcal{F} \in C(K)$  which has 0 in its closure with respect to the topology of the pointwise convergence however  $\mu$  separates  $\mathcal{F}$  from 0.

For every finite set  $\{x_1, \dots, x_n\}$  of points in  $K$  consider an open  $U$  such that  $x_1, \dots, x_n \in U$  and  $|\mu|(U) < \varepsilon$ . The existence of such an  $U$  follows from the atomlessness of  $\mu$ . So choose a function  $f \in C(K)$  such that  $f|_{K-U} = 1$ ,  $0 \leq f \leq 1$  and  $f(x_i) = 0$  for all  $i \leq n$ . It follows that  $\int f d\mu > 1 - \varepsilon$ . It is clear that the functions as above have 0 in its pointwise closure.  $\square$

## 3.2. The Kunen space.

**Definition 3.2.** *Let  $X$  be a Banach space and  $X^*$  its dual.  $(x_i, x_i^*)_{i \in I}$  is called a biorthogonal system if and only if  $x_i^*(x_i) = 1$  for all  $i \in I$  and  $x_i^*(x_j) = 0$  for all distinct  $i, j \in I$ .*

**Lemma 3.3.** *If  $(x_i, x_i^*)_{i \in I}$  is biorthogonal, then  $\{x_i : i \in I\}$  is discrete in the weak topology.*

**Theorem 3.4.** *If a Banach space  $X$  has density bigger than  $2^\omega$ , then  $X$  has an uncountable biorthogonal system.*

**Theorem 3.5.** *Assume CH. There is a scattered compact nonmetrizable space  $K$  such that  $K^n$  is hereditarily separable for each  $n \in \mathbb{N}$ . In particular  $C(K)$  is Lindelöf in the weak topology and so it has no uncountable biorthogonal system.*

**Theorem 3.6** ([11]). *It is consistent that there is a scattered compact  $K$  of weight  $\omega_2$  such that  $K^n$  is hereditarily separable for each  $n \in \mathbb{N}$ . In particular it is consistent that there is a Banach space of density  $\omega_2$  without uncountable biorthogonal systems.*

It is not known if there could consistently exist Banach spaces of density  $\omega_3$  or bigger without uncountable biorthogonal systems.

4. UNCOUNTABLE BIORTHOGONAL SYSTEMS IN  $C(K)$ S UNDER MARTIN'S AXIOM:  
12.15-13.00H, 19.11.2010

**Theorem 4.1.** *Suppose  $K$  is compact. A biorthogonal system  $(x_i, x_i^*)_{i \in I}$  in  $C(K) \times M(K)$  is called nice if and only if  $x_i^* = \delta_{x_i} - \delta_{y_i}$  for some points  $x_i, y_i \in K$  and  $i \in I$ .*

**Theorem 4.2.** *If  $K$  is not hereditarily separable, then  $C(K)$  has a nice uncountable biorthogonal system*

**Definition 4.3.** *Let  $\mathcal{A}$  be a Boolean algebra.  $\mathcal{X} \subseteq \mathcal{A}$  is called  $n$  irredundant set if and only if no  $a \in \mathcal{X}$  belongs to the subalgebra generated by  $\mathcal{X} \setminus \{a\}$ .*

**Theorem 4.4.** *If a Boolean algebra  $\mathcal{A}$  has an uncountable irredundant set, then the Banach space  $C(K_{\mathcal{A}})$  has a nice uncountable biorthogonal system, and  $K^2$  has an uncountable discrete set.*

**Theorem 4.5** (Todorćević). *Assume  $MA+\neg CH$ . If  $C(K)$  is not separable then it has an uncountable biorthogonal system.*

*Proof.* We will only prove the theorem for the case of  $K$  totally disconnected, i.e.,  $K = K_{\mathcal{A}}$  for some Boolean algebra  $\mathcal{A}$ . It is enough to assume that  $\mathcal{A}$  has cardinality  $\omega_1$  and to prove that  $\mathcal{A}$  has an uncountable irredundant set. We may also assume that  $K_{\mathcal{A}}$  is hereditarily separable, in particular that tightness of  $K_{\mathcal{A}}^n$  is countable for each  $n \in \mathbb{N}$ .

Using a strictly increasing sequence of countable subalgebras of  $\mathcal{A}$  whose union is the entire  $\mathcal{A}$  and the fact that for each proper subalgebra  $\mathcal{B} \subseteq \mathcal{A}$  there are two distinct Boolean homomorphisms  $x, y : \mathcal{A} \rightarrow \{0, 1\}$  such that  $x|_{\mathcal{B}} = y|_{\mathcal{B}}$ , we can construct two sequences of distinct points  $x_\alpha, y_\alpha \in K_{\mathcal{A}}$  and elements of  $a_\alpha \in \mathcal{A}$  such that  $x_\alpha(a_\alpha) \neq y_\alpha(a_\alpha)$  (i.e.,  $x_\alpha \in [a_\alpha] \not\subseteq y_\alpha \in [a_\alpha]$ ) and  $x_\alpha(a_\beta) = y_\alpha(a_\beta)$  (i.e.,  $x_\alpha \in [a_\beta] \Leftrightarrow y_\alpha \in [a_\beta]$ ) for all  $\beta < \alpha < \omega_1$ .

We consider a partial order  $\mathbb{P}$  consisting of conditions of the form  $p$  being finite subsets of  $\omega_1$  where the order is defined by  $p \leq q$  if and only if  $p \supseteq q$ , and

$$\forall \beta \in q \forall \alpha \in p \setminus q \quad x_\alpha(a_\beta) = y_\alpha(a_\beta).$$

First we prove that it satisfies the c.c.c. Suppose that  $(p_\xi)_{\xi < \omega_1}$  is a sequence of elements of  $\mathbb{P}$ . Using the  $\Delta$ -system lemma we may assume that there is a finite  $\Delta \subseteq \omega_1$  such that

- (1)  $p_\xi \cap p_\eta = \Delta$  for each distinct  $\xi, \eta < \omega_1$ .
- (2)  $\Delta < p_\xi \setminus \Delta < p_\eta \setminus \Delta$  for  $\xi < \eta < \omega_1$
- (3)  $p_\xi \setminus \Delta$  has exactly  $k$  elements which in the increasing order are  $\alpha_1(\xi), \dots, \alpha_k(\xi)$ .

Consider

$$p_{\xi\eta} = p_\xi \cup p_\eta$$

and suppose that for no  $\xi < \eta < \omega_1$  we have  $p_{\xi\eta} \leq p_\xi, p_\eta$ . This means that for all  $\xi < \eta < \omega_1$  there is an  $\alpha \in p_\eta \setminus \Delta$  and there is  $\beta \in p_\xi \setminus \Delta$  such that

$$x_\alpha(a_\beta) \neq y_\alpha(a_\beta)$$

Introduce the following notation:

$$w_\xi = (x_{\alpha_1(\xi)}, y_{\alpha_1(\xi)}, \dots, x_{\alpha_k(\xi)}, y_{\alpha_k(\xi)})$$

$$W_\eta = \bigcup_{1 \leq i \leq k} K^2 \times \dots \times \left\{ [a_{\alpha_i(\eta)}] \times (K \setminus [a_{\alpha_i(\eta)}]) \cup (K \setminus [a_{\alpha_i(\eta)}]) \times [a_{\alpha_i(\eta)}] \right\} \times \dots \times K^2$$

Thus for each  $\eta < \omega_1$  we have that  $W_\eta$ s are clopen in  $K^{2k}$  and

- (1)  $\forall \xi > \eta \ w_\xi \notin W_\eta$ ,
- (2)  $\forall \xi \leq \eta \ w_\xi \in W_\eta$ ,

Hence  $(w_\xi)_{\xi < \omega_1}$  is a free sequence in  $K^{2k}$  in the sense that

$$\overline{\{w_\xi : \xi \leq \eta\}} \cap \overline{\{w_\xi : \xi > \eta\}} = \emptyset$$

which means that  $K^{2n}$  is not countably tight (see [31]). As countable tightness is preserved by finite products of compact spaces ([?]), we have that  $K$  is not countably tight, which contradicts the fact that  $K$  is hereditarily separable. This completes the proof that  $\mathbb{P}$  satisfies the c.c.c.

It is easy to note that  $D_\alpha = \{p \in \mathbb{P} : p \cap [\alpha, \omega_1) \neq \emptyset\}$  is a dense subset of  $\mathbb{P}$  (conditions trivially extend up). Now let  $G \subseteq \mathbb{P}$  be a filter which intersects all  $D_\alpha$ s. Let  $A = \bigcup G$ . Note that if  $\beta \in A$  and  $q_\beta \in G$  is such that  $\beta \in q$ , we note that  $x_\alpha(a_\beta) = y_\alpha(a_\beta)$  for all  $\alpha \in A \setminus (q_\beta \cap \beta)$ . Now use a version of the pressing down lemma to obtain an uncountable  $B \subseteq A$  such that  $q_\beta \cap \beta$  is constantly  $F \subseteq A$  for all  $\beta \in B$  and consider  $\{a_\beta : \beta \in B \setminus F\}$ . It is easy to see that it is the required irredundant set. □

**Definition 4.6.** [13] *Let  $K$  be a compact Hausdorff space and  $n \in \mathbb{N}$ . We say that the functionals of a sequence  $(f_\xi, \mu_\xi)_{\xi \in \omega_1} \subseteq C(K) \times M(K)$  are  $n$ -supported if each  $\mu_\xi$  is an atomic measure whose support consists of no more than  $n$  points of  $K$ .*

**Theorem 4.7.** [13] *For each natural  $n > 1$ , it is consistent that there is a compact Hausdorff space  $K_{2n}$  such that in  $C(K_{2n})$  there is no uncountable biorthogonal sequence whose functionals are  $2n - 1$ -supported, but there are biorthogonal systems whose functionals are  $2n$ -supported. Moreover,  $K_{2n}^n$  is hereditarily separable but  $K_{2n}^{n+1}$  has an uncountable discrete subspace. Neither the Banach algebra  $C(K_{2n})$  nor the Boolean algebra  $\text{Clop}(K_{2n})$  have an uncountable irredundant family. In particular,  $C(K_4)$  has an uncountable biorthogonal system but it has no uncountable nice biorthogonal system.*

In particular, we are unable to obtain  $K$ 's such that  $C(K)$  contains biorthogonal systems whose functionals are  $2n + 1$ -supported but does not contain one whose functionals are  $2n$ -supported.

On the other hand, if  $n = 1$  one has absolute results. If  $K$  is the split interval, then  $K$  is hereditarily separable and so cannot have an uncountable semibiorthogonal system whose functionals are 1-supported, but  $C(K)$  has an uncountable biorthogonal system (see [22]).

5. THE WEAK\* TOPOLOGY ON THE SET OF ALL CONTINUOUS LINEAR FUNCTIONALS ON  $C(K)$ s: 18.15-19H, 25.11.2010

**5.1. Basic properties of the weak\* topology in  $M(K)$ .** Suppose  $X$  is a Banach space and  $X^*$  is its dual, we introduce the weak\* topology on  $X^*$  by declaring the following sets as sub-basic open

$$[v, \varepsilon_1, \varepsilon_2] = \{v^* \in X^* : \varepsilon_1 < v^*(v) < \varepsilon_2\}$$

for each  $v \in X$  and  $\varepsilon_1 < \varepsilon_2$ .

$$B_{X^*} = \{v^* \in X^* : \|v^*\| \leq 1\}.$$

$\delta : K \rightarrow M(K)$  is defined by  $\delta(x) = \delta_x$  for all  $x \in K$ .

**Theorem 5.1** (Banach-Alaoglu). *Suppose  $X$  is a Banach space, then  $B_{X^*}$  is compact in the weak\* topology.*

**Proposition 5.2.**  $\delta[K] \subseteq B_{M(K)}$  is homeomorphic to  $K$  in the weak\* topology.

**Proposition 5.3.** *Let  $K$  be a compact space.  $\text{conv}(\delta[K])$  is weak\* dense in the dual ball  $B_{X^*}$ , hence  $\text{span}(\delta[K])$  is weak\* dense  $M(K)$ .*

**Proposition 5.4.** *Let  $X$  be a Banach space, then the dual balls  $nB_{X^*}$  are nowhere dense in the weak or weak\* topology.*

**Theorem 5.5** (Banach-Dieudonné). *Let  $X$  be a Banach space and  $X^*$  its dual. if  $A \subseteq X^*$  is convex and  $nB_{X^*} \cap A$  is weak\* closed for each  $n \in \mathbb{N}$ , then  $A$  is weak\* closed.*

**Theorem 5.6** (Josefson-Nissenzweig). *Let  $X$  be a Banach space. Then there is a sequence  $(x_n^*)$  of elements of the dual unit sphere  $S_{X^*}$  which converges to 0 in the weak\*-topology.*

**Proposition 5.7.** *Suppose that  $(\mu_n)$  is a bounded sequence in  $M(K)$ , such that  $\int f d\mu_n$  converges for every  $f \in C(K)$ . Then, the limit  $(\mu)$  is in  $M(K)$  and  $(\mu_n)$  converges to  $\mu$  in the weak\* topology.*

**Proposition 5.8.** *Suppose that  $\mathcal{A}$  is a Boolean algebra and  $(\mu_n)$  is a bounded sequence in  $M(K_{\mathcal{A}})$ , such that  $\mu_n(U)$  converges for every clopen  $U \subseteq K$ . Then, the limit  $(\mu)$  is in  $M(K)$  and  $(\mu_n)$  converges to  $\mu$  in the weak\* topology.*

Whether the assumption of the boundedness is necessary or not depends on the Boolean algebra  $\mathcal{A}$ . If it is not necessary, we say that the algebra has the Nikodym property. For example  $\sigma$ -complete Boolean algebras have this property, but also some quite incomplete algebras like the algebra of Jordan measurable subsets of  $[0, 1]$  share the Nikodym property.

6. THE WEAK TOPOLOGY ON THE SET OF ALL CONTINUOUS LINEAR FUNCTIONALS ON  $C(K)$ : 19.15-20.00H, 25.11.2010

ABSTRACT. Basic properties of the weak topology in  $M(K)$ . The compactness in the weak topology in  $M(K)$ . The Grothendieck property. Complemented copies of  $c_0$ .

**6.1. Basic properties of the weak topology in  $M(K)$ .** The weak topology on  $M(K)$  is defined as for a general Banach space that is it is the weakest topology on  $M(K)$  with respect to which all linear norm-continuous functionals are continuous. However it is not easy to imagine all the functionals on the measures from  $M(K)$ . The “easiest” elements of the bidual of  $C(K)$  are borel subsets of  $K$ . If  $A \subseteq K$  is borel, it is measurable with respect to any Radon measure  $\mu \in M(K)$  and so the mapping  $\mu \rightarrow \mu(A)$  is a well defined bounded linear functional on  $M(K)$  which we will denote by  $\theta_A$ .

**Proposition 6.1.**  $\delta[K]$  is discrete in the weak topology.

*Proof.*

$$\{\delta_x\} = \{\mu \in M(K) : \theta_{\{x\}}(\mu) > 0\} \cap \delta[K].$$

□

Looking at the definitions of the weak, the norm and the weak\* topology on  $M(K)$  (or any dual space) we obtain the following inclusions:

$$\text{the weak* topology} \subseteq \text{the weak topology} \subseteq \text{the norm topology}$$

**6.2. Compactness in the weak topology.**

**Theorem 6.2** (Eberlein-Smulian). *Suppose that  $X$  is a Banach space. Then the following are equivalent for an  $A \subseteq B$*

- (1)  $A$  is relatively weakly compact (i.e., the weak closure of  $A$  is compact),
- (2) Every sequence in  $A$  has a converging subsequence (not necessarily to an element of  $A$ ),
- (3) Every sequence in  $A$  has an accumulation point (not necessarily in  $A$ ).

**Theorem 6.3** (Grothendieck-Dieudonné). *A bounded sequence  $(\mu_k)$  of Radon measures in  $C(K)^*$  is relatively weakly compact if and only if for every pairwise disjoint sequence of open sets  $(U_n)$  in  $K$  we have*

$$\lim_{n \rightarrow \infty} \sup_{k \in N} |\mu_k(U_n)| = 0$$

**Proposition 6.4.** *A bounded  $(\mu_n)$  is weakly convergent to  $\mu \in M(K)$  if and only if  $\mu_n(U) \rightarrow \mu(U)$  for every open  $U \subseteq K$ .*

**6.3. The Grothendieck property.**

**Definition 6.5.** *We say that a Banach space  $X$  is Grothendieck (has the Grothendieck property) if and only if every sequence  $(x_n^*)$  of elements  $X^*$  which is weak\* convergent is also weak convergent.*

**Proposition 6.6.** *If a Banach space  $C(K)$  has the Grothendieck property, then  $K$  has no non-trivial convergent sequence.*

*Proof.* If  $x_n \rightarrow x$  in  $K$ , then  $\delta_{x_n} \rightarrow \delta_x$  because  $\delta : K \rightarrow M(K)$  is a homeomorphism onto its image  $\delta[K]$ . But  $\delta[K]$  is discrete in the weak topology.  $\square$

**Proposition 6.7.**  $c_0$  does not have the Grothendieck property.

*Proof.*  $c_0$  is isomorphic to  $C([0, \omega])$  and  $[0, \omega]$  has a convergent sequence.  $\square$

**Lemma 6.8** (Rosenthal). *Suppose  $\{m_n^k : n, k \in \mathbb{N}\}$  are non-negative real numbers such that there is an  $M$  satisfying for each  $k \in \mathbb{N}$*

$$\sum_{n \in \mathbb{N}} m_n^k < M.$$

*Then, for each  $\varepsilon > 0$  there is an infinite  $A \subseteq \mathbb{N}$  such that*

$$\sum_{n \in A \setminus \{k\}} m_n^k < \varepsilon,$$

*holds for each  $k \in \mathbb{N}$ .*

**Definition 6.9.** A Boolean algebra  $\mathcal{A}$  is said to have a subsequential completeness property if and only if whenever  $(a_n : n \in \mathbb{N}) \subseteq \mathcal{A}$  is such that  $a_n \wedge a_m = 0$  for all  $n \neq m$  then there is an infinite  $A \subseteq \mathbb{N}$  such that  $\bigvee_{n \in A} a_n$  exists in  $\mathcal{A}$ .

**Proposition 6.10.** *Suppose that a Boolean algebra  $\mathcal{A}$  has the subsequential completeness property. Then the Banach space  $C(K_{\mathcal{A}})$  has the Grothendieck property. In particular all spaces  $C(K_{\mathcal{A}})$  for  $\mathcal{A}$  complete or  $\sigma$ -complete have the Grothendieck property.*

*Proof.* Let  $(\mu_n)$  be a bounded sequence in  $M(K)$  which does not converge weakly but converges weakly\*. If  $\{\mu_n : n \in \mathbb{N}\}$  is weakly relatively compact then by the Eberlein-Smulian theorem it must have two convergent subsequences to two different limits. But then, they would be weakly\* convergent to the same limits contradicting the fact that the sequence is weakly\* convergent. So we may assume that  $\{\mu_n : n \in \mathbb{N}\}$  is not weakly relatively compact.

By the Dieudonné-Grothendieck theorem, there are pairwise disjoint open  $U_n$ s such that after renumerating the sequence of measures we have

$$|\mu_n(U_n)| > \varepsilon.$$

By the regularity of the measures we may assume that  $U_n = [a_n]$  for some  $a_n \in \mathcal{A}$ . Apply the Rosenthal lemma to  $m_n^k = \mu_k([a_n])$  for  $\varepsilon/3$  obtaining an infinite  $A \subseteq \mathbb{N}$  such that for every  $B \subseteq A$  we have

- (1)  $|\mu_k(\bigcup_{n \in B} [a_n])| < \varepsilon/3$  if  $k \notin B$  and
- (2)  $|\mu_k(\bigcup_{n \in B} [a_n])| > |\mu_k([a_k])| - \varepsilon/3 > 2\varepsilon/3$  if  $k \in B$

Now, if we could replace  $\bigcup_{n \in B} [a_n]$  by  $[\bigvee_{n \in B} a_n]$  we would obtain a contradiction with the fact that the sequence converges weakly\*. If we consider an uncountable almost disjoint family  $\{A_\xi : \xi < \omega_1\}$  of infinite subsets of  $\mathbb{N}$ , it is not difficult to prove that the sets

$$D_\xi = \left[ \bigvee_{n \in B} a_n \right] \setminus \bigcup_{n \in B} [a_n]$$

are pairwise disjoint, so there is one of them  $D_{\xi_0}$  where all the measures are zero, hence for a  $B \subseteq A_{\xi_0}$  we could perform the abovementioned replacement, hence we obtain a contradiction with the fact that the sequence converges weakly\*.

$\square$

**Proposition 6.11.** *The Grothendieck property is preserved by taking quotients of Banach spaces.*

**Proposition 6.12** (Schachermayer). *Suppose that  $M(K)$  contains a bounded sequence  $(\mu_n)_{n \in \mathbb{N}}$  which is weakly\* convergent but has no subsequence which is weakly convergent. Then  $C(K)$  has a complemented copy of  $c_0$ . In particular if a  $C(K)$  has the Grothendieck property if and only if it has a complemented copy of  $c_0$ .*

*Proof.* By Eberlein-Smulian theorem and the Grothendieck-Dieudonné theorem we may assume that there are pairwise disjoint open  $U_n \subseteq K$  and an  $\varepsilon > 0$  such that  $|\mu_n(U_n)| > \varepsilon$  for each  $n \in \mathbb{N}$ . Let  $\mu$  be the weak\* limit of  $(\mu_n)$ s and  $\nu_n = \mu_n - \mu$ . By going to a smaller subsequence we may assume that  $|\nu_n(U_n)| > \varepsilon/2$ . Consider bounded sequence of functions  $f_n \in C(K)$  such that supports of  $f_n$ s are included in  $U_n$ s and  $\int f_n d\nu_n = 1$ . Of course the space generated by  $(f_n)$  is isomorphic to  $c_0$ . Define

$$P(f) = \sum_{n \in \mathbb{N}} \left[ \int_{U_n} f d(\nu_n) \right] f_n.$$

$P$  is the required projection.

To prove the last part of the proposition note that no Grothendieck space has a complemented copy of  $c_0$  because of 6.11 and 6.7. If  $C(K)$  is not Grothendieck, then it has a weakly\* convergent sequence (and so bounded) which is not weakly convergent. If it is not relatively weakly compact, then it has subsequence with no convergent subsequence by the Grothendieck-Dieudonné theorem. If it is relatively weakly compact and not convergent, then it has at least subsequences which are weakly convergent to different limits, but then they are weakly\* convergent to these different limits contradicting the fact that the entire sequence is weakly\* convergent.  $\square$

**Corollary 6.13.**  *$c_0$  is complemented in a  $C(K)$  space if and only if it is a quotient of it.*

*Proof.* Of course a complemented subspace is a quotient. Now suppose that  $c_0$  is a quotient of a  $C(K)$ , hence the  $C(K)$  does not have the Grothendieck property because it is preserved by taking quotients and  $c_0$  does not have it. So by the Schachermayer characterization  $C(K)$  has a complemented copy of  $c_0$ .  $\square$

**Theorem 6.14** (C. Brech). *It is consistent that there are Boolean algebra  $\mathcal{A}$  of cardinality smaller than  $2^\omega$  such that the Banach space  $C(K_{\mathcal{A}})$  has the Grothendieck property.*

**Proposition 6.15** (A. Aviles, F. Cabello, J. Castillo, M. Gonzalez, Y. Moreno). *If a Boolean algebra  $\mathcal{A}$  of cardinality smaller than  $\mathfrak{s}$ , then the Banach space  $C(K_{\mathcal{A}})$  does not have the Grothendieck property.*

**Theorem 6.16** (Sobczyk).  *$c_0$  is complemented in any separable superspace. In particular if  $K$  is metrizable, then  $C(K)$  has complemented copies of  $c_0$ .*

7. DICHOTOMIES RELATED TO  $c_0$  AND  $l_\infty$ : 12.15-13.00H, 26.11.2010

The following basic topological question is still not completely settled:

**Question 7.1** (Efimov). *Suppose that  $K$  is an infinite compact Hausdorff space. Does it contain  $\beta\mathbb{N}$  or a nontrivial convergent sequence?*

There are many examples of compact spaces without nontrivial convergent sequences nor copies of  $\beta\mathbb{N}$  but all were obtained under some special set-theoretic assumptions. The first example was due to Fedorchuk and obtained assuming  $\diamond$ . Now we have examples under CH,  $s = \omega_1$  and more but no example under  $MA + \neg CH$ . For recent progress on the Efimov problem see the survey paper of K. P. Hart.

This question is natural because  $\beta\mathbb{N}$  and  $[0, \omega]$  are in a sense the opposite examples of compact Hausdorff spaces. Similar questions appeared in the Banach space theory as well. Pelczyński asked if every  $C(K)$  has a complemented copy of  $c_0$  or a complemented copy of  $l_\infty$ . Haydon has obtained the following result:

**Theorem 7.2** (Haydon). [28] *There is a Banach space  $C(K_{\mathcal{A}})$  which does not have  $c_0$  nor  $l_\infty$  as complemented subspaces.*

This gives some information about compact spaces: there is a totally disconnected compact  $K_{\mathcal{A}}$  which does not have nontrivial convergent sequences nor it has  $\beta\mathbb{N}$  as a continuous image. This is because, a convergent sequence gives a complemented copy of  $c_0$  and a continuous image  $\beta\mathbb{N}$  gives  $C(\beta\mathbb{N}) \sim l_\infty$  as a subspace of  $C(K)$ , it is necessarily complemented since  $l_\infty$  is injective. Actually the question about quotients would be more relevant to the Efimov's question. We know that for  $c_0$  being a quotient or a complemented subspace of  $c_0$  are equivalent.

**Theorem 7.3** (Talagrand). [79] *Assume CH. There is an infinite totally disconnected compact Hausdorff  $K_{\mathcal{A}}$  such that neither  $c_0$  nor  $l_\infty$  are quotients of  $C(K)$*

**Theorem 7.4** (Haydon, Levy, Odell). [29] *Assume  $MA + \neg CH$ . Suppose that  $K$  is an infinite compact Hausdorff space. Then  $C(K)$  has  $l_\infty \sim C(\beta\mathbb{N})$  as a quotient or it has  $c_0$  as a quotient.*

**Proposition 7.5.** *Let  $X$  be a Banach space.  $l_\infty$  is a quotient of  $X$  if and only if  $\beta\mathbb{N}$  is a subspace of  $B_{X^*}$ .*

So the problem whose independence was proved through the above results of Talagrand and Haydon-Levy-Odell could be worded as follows: If  $X$  is an infinite dimensional Banach space, is it true that  $B_{X^*}$  either contains a copy of  $\beta\mathbb{N}$  or a nontrivial convergent sequence. This is true if we adopt a new definition of a nontrivial sequence in  $B_{X^*}$  as a usual nontrivial weakly\* convergent sequence which does not converge weakly.

Another version of Pelczyński's question was if every Banach space above every separable subspace has a complemented subspace isomorphic to a subspace of  $l_\infty$ . This was answered by Johnson and Lindenstrauss in the negative without the use of any additional set-theoretic assumptions.

## 8. GROTHENDIECK SPACES UNDER MARTIN'S AXIOM: 13.15-14.00H, 26.11.2010

## 8.1. Preliminaries.

**Theorem 8.1** (MA+¬CH). *Suppose that  $K$  is an infinite compact Hausdorff space. Then  $C(K)$  has  $l_\infty \sim C(\beta\mathbb{N})$  as a quotient or it has  $C(\text{convergent sequence}) \sim c_0$  as a quotient.*

During the course of the proof we will need the following definitions and facts which will not be proved:

**Definition 8.2.** *Suppose  $(v_n)$  is a linearly independent sequence of vectors in a vector space. We say that  $(w_n)$  is a convex block subsequence of  $(v_n)$  if there are finite subsets  $B_n \subseteq \mathbb{N}$  and non-negative real numbers  $a_i$  such that the following hold for each  $n \in \mathbb{N}$ :*

- (1)  $\max(B_n) < \min(B_{n+1})$ ,
- (2)  $w_n = \sum_{i \in B_n} a_i v_i$ ,
- (3)  $\sum_{i \in B_n} a_i = 1$ .

*If in addition all  $a_i$ s are rational we say that  $(w_n)$  is a rational convex block subsequence. We write  $(w_n) \prec (v_n)$  if  $(w_n)$  is eventually a convex block subsequence of  $(v_n)$ .*

- (1) A lemma of Rosenthal:

Suppose  $A$  is a set and  $F_n : A \rightarrow [-M, M]$  are functions such that no convex block subsequence of  $(F_n)$  converges pointwise on  $A$ . Then there are  $c \in \mathbb{R}$ ,  $\delta > 0$  and a convex block subsequence  $(G_n)$  of  $(F_n)$  such that

- (a)  $\sup_{a \in A} \text{osc}(G_n(a)) = 2\delta$ ,
- (b) For each convex block subsequence  $(H_n)$  of  $(G_n)$  for each  $\delta' < \delta$  there is  $a \in A$  such that

$$\liminf_{n \in \mathbb{N}} H_n(a) < c - \delta' < c + \delta' < \limsup_{n \in \mathbb{N}} H_n(a).$$

- (2) Argyros-Bourgain-Zachariades result: Let  $\kappa$  be a cardinal greater than  $\omega_1$  and  $(f_\alpha)_{\alpha < \kappa}$  be a bounded collection of functions in a  $C(K)$  such that there is a Radon measure  $\mu$  on  $K$  with  $\int |f_\alpha - f_\beta| d\mu > \varepsilon > 0$  for all  $\alpha < \beta < \kappa$ , then there is  $A \subseteq \kappa$  of cardinality  $\kappa$  such that  $(f_\alpha)_{\alpha \in A}$  is equivalent to the canonical basis of  $l_1(A)$ , in particular  $C(K)$  contains  $l_1(\kappa)$ .
- (3) Properties of the pseudointersections

## 8.2. Application of pseudointersections to convex block subsequences.

**Lemma 8.3.** *Suppose that  $(u_n) \prec (w_n) \prec (v_n)$ , then  $(u_n) \prec (v_n)$ .*

**Lemma 8.4.** *Let  $\kappa < \mathfrak{p}$  be a cardinal and let  $v_n$  be a linearly independent sequence in a vector space. Suppose that  $(w_n^\alpha)$  is a rational convex block subsequence of  $(v_n)$  for each  $\alpha < \kappa$ . Suppose that for each  $k \in \mathbb{N}$  and each  $\alpha_1 < \dots < \alpha_k < \kappa$  there is*

$$(u_n) \prec (w_n^{\alpha_1}), \dots, (w_n^{\alpha_k}).$$

*Then there is  $(u'_n)$  such that  $(u'_n) \prec (w_n^\alpha)$  for each  $\alpha < \kappa$ .*

### 8.3. The main argument.

**Theorem 8.5.** *Let  $K$  be a compact space and  $\mathcal{F} \subseteq B_{C(K)}$ . Let  $(\lambda_n)$  be a sequence in  $B_{M(K)}$  such that no convex block subsequence of  $(\lambda_n)$  weakly\* converges.*

*Then there is  $\eta > 0$ , a positive Radon measure  $\mu$  and a sequence  $\{f_\alpha : \alpha < p\} \subseteq C(K)$  such that for all distinct  $\alpha, \beta < \kappa$*

$$\int |f_\alpha - f_\beta| d\mu > \eta.$$

*Proof.* We may assume that all the  $\lambda_n$ s are positive measures. Apply the lemma of Rosenthal for  $A = C(K)$ ,  $F_n = \lambda_n$ , obtaining  $c \in R$  and  $\delta > 0$  and  $(\mu_n) \prec (\lambda_n)$ . In particular for  $\delta' = \delta/2$  and  $(\nu_n) \prec (\mu_n)$ , there is always an  $f \in C(K)$  such that

$$\liminf_{n \in \mathbb{N}} \int f d\nu_n < c - \delta' \quad \text{and} \quad c + \delta' < \limsup_{n \in \mathbb{N}} \int f d\nu_n.$$

Now construct by transfinite induction on  $\alpha < p$  a transfinite sequence of sequences  $(\mu_n^\alpha) \prec (\mu_n)$  and a transfinite sequence of  $f_\alpha \in C(K)$  which satisfy the following:

- (1)  $(\mu_n^\beta) \prec (\mu_n^\alpha)$  for all  $\alpha < \beta < p$ ,
- (2)  $\inf f_\beta d\mu_n^\alpha \rightarrow c$  for all  $\alpha < \beta < p$
- (3)  $\lim_{n \in \mathbb{N}} \int f_\alpha d\mu_{2n}^\alpha < c - \delta'$  and  $c + \delta' < \lim_{n \in \mathbb{N}} \int f_\alpha d\mu_{2n+1}^\alpha$ .

For each  $\alpha < p$  define

$$K_\alpha = \bigcap_{m \in \mathbb{N}} \overline{\text{conv}(\{\mu_n^\alpha : n \geq m\})}$$

Where the closure is taken with respect to the weak\* topology and *conv* means the convex hull, i.e., the set of all convex combinations. Note that  $K_\beta \subseteq K_\alpha$ , so by the compactness there is a  $\mu \in \bigcap_{\alpha < p} K_\alpha$ . It is not difficult to prove that as  $\int |f_\alpha - f_\beta| d\mu_n^\alpha > \delta'$  for sufficiently large  $n \in \mathbb{N}$  and all  $\alpha > \beta$ , the same must hold (otherwise we would get a weak\* separation) for  $\mu$ , i.e., we have  $\int |f_\alpha - f_\beta| d\mu > \delta'$  as required. □

**Proposition 8.6.** *Suppose that  $K$  is compact Hausdorff space and  $(x_n)$  is a discrete sequence of its points, then no convex block subsequence of  $(\delta_{x_n})_{n \in \mathbb{N}}$  converges weakly.*

To prove the theorem of Haydon, Levy and Odell take an infinite dimensional  $C(K)$ . If it is not Grothendieck, then it has  $c_0$  as a quotient. If it is Grothendieck, by the above proposition  $M(K)$  has a sequence of measures with no convex block subsequence which converges weakly\*, so by the main argument and Argyros, Bourgain and Zachariades result  $C(K)$  contains  $l_1(2^\omega)$  so by 1.29  $C(K)$  has  $l_\infty$  as a quotient.

Applying the fact that every Banach space embeds into a Banach space of the form  $C(K)$  one can use the above result to prove the following:

**Theorem 8.7.** [29] *Suppose  $MA + \neg CH$ . Every nonreflexive Grothendieck space has  $l_\infty$  as a quotient.*

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