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Constructing the List of the 50 Fake Projective Planes

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(Wrocław, 7 – 10.04.2014)

50 Fake Planes

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completing a project started by

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17–20 March 2009

A **fake projective plane** is a smooth, closed, compact, complex manifold of complex dimension 2 (so of real dimension 4) which has the same Betti numbers as the complex projective plane, $\mathbf{P}^2(\mathbf{C})$, but which is not holomorphically equivalent to $\mathbf{P}^2(\mathbf{C})$.

In fact, a fake projective plane is necessarily an algebraic variety, to wit an **algebraic surface**. Algebraic surfaces were one of the main objects of study of the Italian school of algebraic geometry active in the late 1800's and the first decades of the 1900's. Enriques and Kodaira are perhaps the two most famous of many famous names associated with the study of algebraic surfaces.

The desired Betti numbers are:

$$b_0 = \dim H^0(\mathbf{P}^2(\mathbf{C}), \mathbf{Q}) = 1$$

$$b_1 = \dim H^1(\mathbf{P}^2(\mathbf{C}), \mathbf{Q}) = 0$$

$$b_2 = \dim H^2(\mathbf{P}^2(\mathbf{C}), \mathbf{Q}) = 1$$

$$b_3 = \dim H^3(\mathbf{P}^2(\mathbf{C}), \mathbf{Q}) = 0$$

$$b_4 = \dim H^4(\mathbf{P}^2(\mathbf{C}), \mathbf{Q}) = 1$$

Any copy of the complex projective line, $\mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^2(\mathbf{C})$ is a generator of $H_2(\mathbf{P}^2(\mathbf{C})) \cong \mathbf{Z}$.

Any smooth, closed, compact, manifold X of real dimension 4 has $b_0(X) = b_4(X) = 1$ and $b_1(X) = b_3(X)$. To have $b_1(X) = b_3(X) = 0$, it is necessary and sufficient that $H_1(X)$ be a finite group.

For those who know something about surface theory, note that the **Hodge diamond** of a fake plane is just like that of $\mathbf{P}^2(\mathbf{C})$:

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

The true projective plane has **Kodaira dimension** $-\infty$, but fake planes have Kodaira dimension 2, i.e. they are surfaces of **general type**.

Theorem: Up to holomorphic and anti-holomorphic equivalence, there are exactly 50 fake projective planes; up to holomorphic equivalence, exactly 100.

The calculations and estimates in

[Prasad, Yeung, 2007] *Fake projective planes*, Invent. Math. **168**, 321–370

show that any fake plane has to lie in one of 28 "classes".

Our further calculations work out exactly how many fake planes lie in each class. For each fake plane X, we know the fundamental group, $\pi_1(X)$: we have group generators expressed as explicit matrices in $U(2,1)$, and a group presentation in terms of those generators.

[Cartwright, Steger, 2010] *Enumeration of the 50 fake projective planes*, C. R. Acad. Sci. Paris, Ser. I **348**, 11–13.

The first fake plane, known as **Mumford's fake plane**, was constructed in:

[Mumford, 1979] *An algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$*, Am. J. Math. **101**, 233–244.

Besides producing a particularly interesting algebraic surface, Mumford's construction is a tour de force of technique.

First he constructs a certain rigid analytic space over $\mathbf{Q}_2$, the 2-adics, by gluing together affine algebraic varieties, one for each vertex of the building associated to $\mathrm{PGL}(3, \mathbf{Q}_2)$. Next, he constructs a certain discrete cocompact subgroup of $PGL(3, \mathbf{Q}_2)$, and looks at the corresponding quotient of the rigid analytic space. He proves that this quotient is a smooth algebraic variety over $\mathbf{Q}_2$.

In fact he proves it is a smooth algebraic variety over $\mathbf{Q}$. By the magic of algebraic geometry, algebraical-geometrical Euler characteristics and Betti numbers which he can calculate for the 2-adic variety pass over and become ordinary Euler characteristics and Betti numbers for the corresponding variety over $\mathbf{C}$, thought of as a manifold of complex dimension 2.

By the way, before working out this example using the building of $\mathrm{PGL}(3, \mathbf{Q}_2)$, Mumford worked out some similar examples using the trees which are the buildings of $\mathrm{PGL}(2, \mathbf{Q}_p)$:

[Mumford, 1972] *An analytic construction of degenerating curves over complete local rings*, Compositio Math. **24**, 129–174.

After Mumford's work of 1979, two new examples were produced in:
[Ishida, Kato, 1998] *The strong rigidity theorem for non-archimedean uniformization*, Tohoku Math. J. **50**, 537–555.

using two new subgroups of $\mathrm{PGL}(3, \mathbf{Q}_2)$ exhibited in:
[Cartwright, Mantero, Steger, Zappa, 1993] *Groups acting simply transitively on the vertices of a building of type* $\widetilde{A}_2$ *I and II.*, Geometriae Dedicata **47** (1993), 143–166 and 167–223.

It is trivial to see that these groups have the same properties as Mumford's group. The work of Ishida and Kato is mostly dedicated to the proof that the fake planes obtained as quotients by the two new groups and by the one old group are all inequivalent.

Finally:

[Keum, 2006] *A fake projective plane with an order 7 automorphism*, Topology **45**, 919–927.

constructs one more fake plane. Keum's fake plane is a finite quotient of a finite index covering surface of Mumford's original fake plane.

The project initiated by Prasad and Yeung has full credit for 46 out of 50 fake planes!

Based on the proof of the Calabi conjecture:

[Yau, 1978] *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*, Comm. Pure Appl. Math. **31**, 339–411.

one knows that the universal cover of any fake plane, X, is holomorphically equivalent to the unit ball $B(\mathbf{C}^2)$. Thus, $\pi_1(X)$ must act freely, discretely, and holomorphically on $B(\mathbf{C}^2)$.

What is the group of holomorphic diffeomorphisms of $B(\mathbf{C}^2)$?

Let $U(2,1)$ act on $\mathbf{C}^3$, preserving the unitary form $|z|^2 = -|z_1|^2 - |z_2^2| + |z_3|^2$. Then $U(2,1)$ preserves the subset of vectors of positive norm:

$$\{z \in \mathbf{C}^3\,;\ |z_1|^2 + |z_2|^2 < |z_3|^2\}$$

Projectivizing this set (looking at equivalence classes modulo scalar multiplication) we obtain $B(\mathbf{C}^2)$. This exhibits the well-known action of $U(2,1)$ on $B(\mathbf{C}^2)$. The kernel of that action consists of the scalar matrices in $U(2,1)$. Dividing out by that we get $\mathrm{PU}(2,1)$.

It is well-known, and not hard to prove, that $\mathrm{PU}(2,1)$ is the full group of holomorphic diffeomorphisms of $B(\mathbf{C}^2)$. Alternatively, we can view $\mathrm{PU}(2,1)$ as

$$\mathrm{PSU}(2,1) = \mathrm{SU}(2,1)/\{1, \omega, \omega^2\} \text{ where } \omega = e^{2\pi i/3}.$$

Just because it is a fundamental group $\pi_1(X) \subset \mathrm{PU}(2,1)$ must be discrete. Because X is compact, $\pi_1(X)$ must be **cocompact** in $\mathrm{PU}(2,1)$: $\pi_1(X)\backslash \mathrm{PU}(2,1)$ must be compact. In other words, $\pi_1(X)$ must be a **uniform lattice** in $\mathrm{PU}(2,1)$.

Any torsion element of $PU(2,1)$ fixes some point in $B(\mathbf{C}^2)$. This is easy to check directly, but it also follows from the fact that $B(\mathbf{C}^2)$ has an invariant metric with negative sectional curvatures. Since $\pi_1(X) \subseteq \mathrm{PU}(2,1)$ acts freely on $B(\mathbf{C}^2)$, it must be torsion free.

By Hurewicz's Theorem

$$H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

the abelianization of $\pi_1(X)$. This must be finite.

Recap: we are looking for groups $\pi_1(X) \subset \mathrm{PU}(2,1)$ such that

- $\pi_1(X)$ is a uniform lattice in $\mathrm{PU}(2,1)$
- $\pi_1(X)$ is torsion free and
- $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ is finite.

Vice versa, if we find such a lattice in $\mathrm{PU}(2,1)$, then the associated quotient of $B(\mathbf{C}^2)$ is a smooth, compact, closed, complex manifold of complex dimension 2, with $b_0(X) = b_4(X) = 1$ and $b_1(X) = b_3(X) = 0$.

What about $b_2(X)$?

By a result of Chern, the Euler characteristic of any quotient of $B(\mathbf{C}^2)$ is proportional to its volume:

$$b_0(X) - b_1(X) + b_2(X) - b_3(X) + b_4(X) = \chi(X) = 3\,\mathrm{vol}(X)$$

where the volume element on X is that obtained by pushing down the invariant volume element on $B(\mathbf{C}^2)$. When X is a fake plane, $\chi(X) = 3$, so $\mathrm{vol}(X) = 1$. The result of Chern, which is a generalization of Gauss's result for surfaces, is one of the precursors of the Atiyah–Singer Index Theorem.

By definition, $\mathrm{covol}(\pi_1(X))$ is the the volume of any fundamental domain for the action of $\pi_1(X)$ on $B(\mathbf{C}^2)$. That is $\mathrm{covol}(\pi_1(X)) = \mathrm{vol}(X)$.

Thus, if $\pi_1(X) \subset \mathrm{PU}(2,1)$ satisfies:

- $\pi_1(X)$ is a uniform lattice in $\mathrm{PU}(2,1)$,
- $\pi_1(X)$ is torsion free,
- $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ is finite, and
- $\mathrm{covol}(\pi_1(X)) = 1$

then X, the associated quotient of $B(\mathbf{C}^2)$ will be a smooth, closed, compact, complex manifold of complex dimension 2 with $b_0(X) = b_4(X) = 1$, $b_1(X) = b_3(X) = 1$, and $\chi(X) = 3$, which implies that $b_2(X) = 1$. In short, X will be a fake plane.

Here is the precise normalization of the invariant volume element on $B(\mathbf{C}^2)$ needed to make Chern's result true. It is determined by the Hirzebruch Proportionality Principal.

$$
\begin{aligned}
r &= \operatorname{arctanh}|z| = \frac{1}{2}\log\frac{1+|z|}{1-|z|} \\
\Theta &= z/|z| \in S^3 \\
d\Theta &= \text{usual measure on } S^3\text{, total volume}{=}2\pi^2 \\
d\,\mathrm{vol}(z) &= \frac{2}{\pi^2}\sinh^3 r\cosh r\,dr\,d\Theta
\end{aligned}
$$

Here $r = d_{\mathsf{inv}}(0, z)$, the invariant distance from 0 to z.

The unitary group $U(3)$ acts on $\mathbf{C}^3$, and this action descends to an action on the true projective plane, $\mathbf{P}^2(\mathbf{C})$. One finds the following, analogous, formulas for the volume element invariant under $U(3)$.

$$
\begin{aligned}
r' &= \arctan|z| \\
\Theta &= z/|z| \in S^3 \\
d\Theta &= \text{usual measure on } S^3\text{, total volume}{=}2\pi^2 \\
d\,\mathrm{vol}(z) &= \frac{2}{\pi^2}\sin^3 r' \cos r'\, dr'\, d\Theta
\end{aligned}
$$

where now $z \in \mathbf{C}^2$, the affine plane, a set of full measure in $\mathbf{P}^2(\mathbf{C})$. With this normalization, one checks that $\mathrm{vol}(\mathbf{P}^2(\mathbf{C})) = 1$, so $3 = \chi(\mathbf{P}^2(\mathbf{C})) = 3\,\mathrm{vol}(\mathbf{P}^2(\mathbf{C}))$.

Non-arithmetic uniform lattices in $PU(2,1)$ are exhibited in

[Mostow, 1980] *Mostow, G. D. On a remarkable class of polyhedra in complex hyperbolic space*, Pacific J. Math. **86**, 171–276.

and further explicated in subsequent work of Mostow and Deligne. These examples are fascinating, but *not* relevant to the study of fake projective planes.

[Klingler, 2003] *Sur la rigidité de certains groupes fondamentaux, l'arithméticité des réseaux hyperboliques complexes, et les 'faux plans projectifs'*, Invent. Math. **153**, 105–143.

and

[Yeung, 2004] *Integrality and arithmeticity of co-compact lattices corresponding to certain complex two-ball quotients of Picard number one*, Asian J. Math. **8**, 107–130.

prove independently that $\pi_1(X) \subset \mathrm{PU}(2,1)$ is an **arithmetic** lattice whenever X is a fake plane.

“Arithmetic” means that $\pi_1(X)$ can be constructed in a certain rather complicated way, where the starting point is some linear algebraic group defined over $\mathbf{Q}$, such that the associated group over $\mathbf{R}$ is isomorphic to $SU(2,1) \times K$, with K compact.

The basic example for all arithmetic lattices is $SL(n,\mathbf{Z}) \subset SL(n,\mathbf{Q}) \subset SL(n,\mathbf{R})$. In a certain sense, all arithmetic lattices “come from” this basic example. To say what “come from” means is to give the actual definition of “arithmetic lattice”.

At least in principle, the arithmetic lattices of $PU(2,1)$ can be listed. One would have to start with a list of pairs (k,ℓ) such that $\mathbf{Q} \subset k \subset \ell$ is a tower of finite algebraic extensions, $[\ell : k] = 2$, k is totally real, and ℓ is totally complex.

For arithmetic lattices, we have Prasad's Covolume Formula from:

[Prasad, 1989] *Volumes of S-arithmetic quotients of semi-simple groups*, Publ. Math., Inst. Hautes Etud. Sci. **69**, 91–117

Using this as their basic tool, Prasad and Yeung constructed a list of 25 *maximal* arithmetic subgroups $\Gamma \subset \mathrm{PU}(2,1)$ so that any $\pi_1(X)$ has to be a subgroup of one of the Γ.

The point is that most of the countably many maximal arithmetic subgroups of $\mathrm{PU}(2,1)$ are too sparse — most of them have covolume which is too big.

For each of the 25 maximal arithmetic subgroups Γ. identified by Prasad and Yeung we

- used a computer to search for elements of Γ,
- used the group property to construct a list of all elements shifting $0 \in B(\mathbf{C}^2)$ by not too much,
- used numerical integration to calculate the covolume of the group generated by our elements,
- and so verified that we had generators for Γ.

For these steps we used our own programs, written in `C`.

Next we

- calculated numerically the radius of a Dirichlet fundamental domain for Γ,
- and used this to construct a list of conjugacy classes of finite order elements,
- as well as a big enough set of relations to give a presentation of Γ,
- which we then simplified somewhat, with a mixture of automatic and hand methods.

This was done with further `C` programs, plus `MAGMA` code.

Finally we

- used MAGMA, GAP, lowx, and/or C programs to find all subgroups of Γ of the right index to have covolume 1,
- determined which of them were torsion free,
- and which had finite abelianizations.

These last two calculations are already built in to MAGMA and GAP, and run very fast.

```
From: Tim Steger <steger@uniss.it>
To: donaldic@math.rutgers.edu
Subject: Numerical Results, etc.
Date: Sat, 22 Apr 2006 19:50:30 -0400

Dear Donald,


...


According to my calculations, that $2 r_H$ corresponds to a
Hilbert-Schmidt norm of $40.99118$.  Since the numerical
calculations are not necessarily valid to more than
2~places, that should probably be rounded up to~$42$.


...
```

```
From: donaldic@math.rutgers.edu
To: steger@uniss.it
Subject: progress report
Date: Tue, 25 Apr 2006 14:39:34 -0400 (EDT)

Dear Tim,

        I wrote a program to find all the relations of the
form xyz=1, where x, y and z are in the 4 nontrivial
double cosets whose squared Hilbert Schmidt norm is less
than 42. Incidentally, isn’t it nice that 42 is the answer!

...
```

From here on, we discuss one particular maximal arithmetic subgroup from [Prasad, Yeung]'s list. This subgroup is denoted by $(a = 7, p = 2, \{7\})$. The notation $a = 7$ means that

$$k = \mathbf{Q} \qquad \qquad \ell = \mathbf{Q}[\sqrt{-7}]$$

[Prasad, Yeung]'s construction of the arithmetic subgroup starts with a **central simple algebra** $\mathcal{D}$ of degree 3 over $\ell = \mathbf{Q}[\sqrt{-7}]$. Such a $\mathcal{D}$ is a certain sort of associative algebra of dimension 9 over ℓ, with center equal to ℓ.

[Weil, 1974] Basic Number Theory, Third Edition, Die Grundlehren der mathematischen Wissenschaften, Band 144, Springer-Verlag, Berlin.

has an excellent exposition of central simple algebras over number fields.

Identify $\mathcal{D}$ with ℓ^9 by fixing a basis $(e_j)_{1 \leq j \leq 9}$. The algebra structure is given by

$$e_j e_k = \sum_m c_{jk}^m e_m$$

for some structure constants $c_{jk}^m \in \ell$. The tensor product $\mathcal{D} \otimes_\ell \mathbf{C}$ can be obtained by considering $\mathbf{C}^9$ as an algebra, using these same structure constants. One property characterizing central simple algebras is this:

$$\mathcal{D} \otimes_\ell \mathbf{C} \cong \mathrm{Mat}_{3\times 3}(\mathbf{C})$$

There are many possibilities for $\mathcal{D}$. These are classified, up to isomorphism, by their **Hasse invariants**. In the notation $(a = 7, p = 2, \{7\})$, $p = 2$ specifies the Hasse invariant of $\mathcal{D}$. Without giving the general definition of the Hasse invariant, here is the meaning of $p = 2$.

In the field $\mathbf{Q}_2$ of 2-adic numbers, there exist two square roots of -7. These give two inclusions $\ell = \mathbf{Q}[\sqrt{-7}] \hookrightarrow \mathbf{Q}_2$. For each of these, it is required that

$$\mathcal{D} \otimes_\ell \mathbf{Q}_2 \ncong \mathrm{Mat}_{3\times 3}(\mathbf{Q}_2)$$

General theory says that this will be true if and only if $\mathcal{D} \otimes_\ell \mathbf{Q}_2$ is a division algebra. A fortiori, $\mathcal{D}$ must also be a division algebra.

Moreover, for any prime $q \neq 2$, if there is a square root of -7 in $\mathbf{Q}_q$, then it is required that

$$\mathcal{D} \otimes_\ell \mathbf{Q}_q \cong \mathrm{Mat}_{3\times 3}(\mathbf{Q}_q)$$

and if there is no square root of -7 in $\mathbf{Q}_q$, it is required that

$$\mathcal{D} \otimes_\ell \mathbf{Q}_q[\sqrt{-7}] \cong \mathrm{Mat}_{3\times 3}(\mathbf{Q}_q[\sqrt{-7}])$$

The general classification of central simple algebras over number fields guarantees that exactly one such algebra exists, up to isomorphism and anti-isomorphism.

Concrete work requires a concrete presentation. Here is the simplest presentation we could come up with. The general form is that of a **cyclic central simple algebra**.

Let ζ be a 7th root of unity, $\zeta = e^{2\pi i/7}$. Let

$$m = \mathbf{Q}[\zeta]$$

The irreducible equation for ζ is

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$$

so $[m : \mathbf{Q}] = 6$.

Since

$$\sqrt{-7} = \zeta + \zeta^2 + \zeta^4 - \zeta^3 - \zeta^6 - \zeta^5$$

one knows that $\ell = \mathbf{Q}[\sqrt{-7}] \subset m$ with $[m : \ell] = 3$. Moreover m is Galois over ℓ, with cyclic 3 element Galois group generated by $\phi : m \to m$, where

$$\phi(\zeta) = \zeta^2$$

The desired presentation of $\mathcal{D}$ is:

$$\mathcal{D} = \langle m, \sigma \rangle$$

$$\sigma^3 = \frac{3 + \sqrt{-7}}{4}$$

$$\sigma x \sigma^{-1} = \phi(x) \qquad \text{for } x \in m$$

or equivalently

$$\mathcal{D} = \langle \mathbf{Q}, \zeta, \sigma \rangle$$

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$$

$$\sigma^3 = \frac{3 + \sqrt{-7}}{4}$$

$$\sigma y = y \sigma \qquad \text{for } y \in \mathbf{Q} \text{ (or for } y \in \ell\text{)}$$

$$\sigma \zeta = \zeta^2 \sigma$$

One can realize this algebra inside $Mat_{3\times 3}(m)$ by

$$\zeta \mapsto \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$$

$$\sigma \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3+\sqrt{-7}}{4} & 0 & 0 \end{pmatrix}$$

In particular, this identifies $\mathcal{D}^\times$ as a subgroup $GL(3, \mathbf{C})$. What we want is a subgroup lying in $U(2, 1)$, and for this we need an operation on $\mathcal{D}$ which looks like the adjoint map on $GL(3, \mathbf{C})$.

An **involution of second kind** on $\mathcal{D}$ is an involutory antiautomorphism $\iota : \mathcal{D} \to \mathcal{D}$, antilinear over $\ell = \mathbf{Q}[\sqrt{-7}]$.

General theory for central simple algebras guarantees the existence of an involution of the second type on $\mathcal{D}$. Actually, the above matrix realization of $\mathcal{D}$ has the nice property that the usual adjoint on $\mathrm{Mat}_{3\times 3}(\mathbf{C})$ stabilizes $\mathcal{D}$. This gives the following involution of the second kind, $\iota_0 : \mathcal{D} \to \mathcal{D}$:

$$\iota_0(\zeta) = \zeta^{-1} \qquad\qquad \iota_0(\sigma) = \sigma^{-1}$$

or equivalently

$$\iota_0(x) = \bar{x} \qquad \text{for } x \in m$$
$$\iota_0(\sigma) = \sigma^{-1}$$

Antilinearity holds because $\iota_0(\sqrt{-7}) = -\sqrt{-7}$.

Now that we have ι_0, define

$$U(\mathcal{D},\iota_0) = \{x \in \mathcal{D}\,;\ \iota_0(x)x = 1\}$$
$$PU(\mathcal{D},\iota_0) = \{x \in \mathcal{D}\,;\ \iota_0(x)x \in \mathbf{Q}^\times\}/\ell^\times$$

Using the Skolem–Noether Theorem, which states that all automorphisms of $\mathcal{D}$ over ℓ are inner, one can elegantly redefine $PU(\mathcal{D},\iota_0)$ as

$\{\bar{x} : \mathcal{D} \to \mathcal{D}\,;\ \bar{x}$ is an algebra automorphism over ℓ and $\bar{x}\iota_0 = \iota_0\bar{x}\}$

i.e. $\mathrm{Aut}(\mathcal{D},\iota_0)$.

Using the above matrix realization, we get $PU(\mathcal{D},\iota_0) \subset PU(3)$. This $\mathrm{PU}(\mathcal{D},\iota_0)$ is one of many possible versions of $PU(3)$ defined over the rationals. Think of it as (one possibility for) the subgroup of $PU(3)$ with “rational” entries.

We want a subgroup of $PU(2,1)$, not of $PU(3)$. It will be enough to use a different involution of the second kind. By Skolem–Noether, this necessarily has the form:

$$\iota(x) = F^{-1}\iota_0(x)F$$

for some $F \in \mathcal{D}$ satisfying $\iota_0(F) = F$.

Many such F, hence many such ι are possible. Which should we choose? There is a remarkable **Hasse principle**, which says that it doesn't matter. Any two choices of ι which (at the real place) give the group $PU(2,1)$ are conjugate under some automorphism of $\mathcal{D}$. (The same holds for any two choices giving $PU(3)$.)

The simplest possibility we could think of was $F = \zeta + \zeta^{-1}$. In the matrix realization

$$F \mapsto \begin{pmatrix} \zeta + \zeta^{-}1 & 0 & 0 \\ 0 & \zeta^2 + \zeta^{-2} & 0 \\ 0 & 0 & \zeta^4 + \zeta^{-4} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\pi/7) & 0 & 0 \\ 0 & \cos(4\pi/7) & 0 \\ 0 & 0 & \cos(8\pi/7) \end{pmatrix}$$

$$= \begin{pmatrix} 0.6234\ldots & 0 & 0 \\ 0 & -0.2225\ldots & 0 \\ 0 & 0 & -0.9009\ldots \end{pmatrix}$$

showing that the associated form has signature $(2, 1)$.

Define

$$PU(\mathcal{D},\iota) = \{x \in \mathcal{D}\,;\ \iota(x)x \in \mathbf{Q}^\times\}/\ell^\times$$

Equivalently, this is $\mathrm{Aut}(\mathcal{D},\iota)$.

Our objective is, as from the start, to arrive at a concrete understanding of the maximal arithmetic subgroup indicated by $(a = 7, p = 2, \{7\})$, which we will denote by Γ. If $PU(\mathcal{D},\iota)$ is a rational version of $PU(2,1)$, then $\Gamma \subset PU(\mathcal{D},\iota)$ is an "integer" version of $PU(2,1)$.

Any basis for $\mathcal{D}$ over $\mathbf{Q}$ determines a bijection $\mathcal{D} \cong \mathbf{Q}^{18}$. For a very particular basis, we have

$$\Gamma = \{\bar{x} \in PU(\mathcal{D},\iota)\,;\ \bar{x}(\mathbf{Z}^{18}) = \mathbf{Z}^{18}\}$$

There are many possible bases. If two of them are conjugate under the action of $PU(\mathcal{D}, \iota)$, they give rise to conjugate arithmetic subgroups. But not all of them give *maximal* arithmetic subgroups. This looks terribly complicated.

Fortunately, a prime by prime analysis, based on the **Strong Approximation Theorem** and the Bruhat–Tits theory of buildings, brings order out of chaos.

One fundamental point is that given any two bases of $\mathcal{D}$, the matrix in $GL(18, \mathbf{Q})$ converting one to the other is integral in $\mathbf{Q}_q$ for all but finitely many primes q. The bad primes are the ones appearing in the denominators of the entries.

The Strong Approximation Theorem is a super-duper version of the Chinese Remainder Theorem. It implies that for each prime q we can choose which q-adic integrality condition to use, and these choices can be made independently, so long as we make the “standard” choice for all but finitely many primes. Also, the overall condition will be determined, up to conjugacy, by the conjugacy classes of the various q-adic conditions.

The maximal arithmetic subgroup under discussion is specified by the triple $(a = 7, p = 2, \{7\})$. The third item of that triple, $\{7\}$ means that for any prime $q \neq 7$ we should use the “standard” condition, but for $q = 7$ we should use a certain “alternative” condition.

From a theoretical point of view, it is pleasant to consider $PU(\mathcal{D},\iota)$ as a subgroup of $GL(\mathcal{D}) = GL(9,\mathbf{Q}[\sqrt{-7}]) \subseteq GL(18,\mathbf{Q})$.

However, for concrete calculations, it is simpler to deal with $SU(\mathcal{D},\iota)$, whose elements can be seen as 3×3 matrices over, for example, the complexes.

In particular, our calculations are easier and shorter in practice (although messier theoretically) if we think about the integrality conditions satisfied by elements of $U(\mathcal{D},\iota)$, which are in the preimage of $\Gamma \subseteq PU(\mathcal{D},\iota)$.

Consider $q = 7$. Just as one can embed $\mathcal{D}^\times \subset GL(3, \mathbf{C})$ using the inclusion $\ell \hookrightarrow \mathbf{C}$, one can use the inclusion $\ell \hookrightarrow \mathbf{Q}_7[\sqrt{-7}]$ and embed $\mathcal{D}^\times \hookrightarrow GL(3, \mathbf{Q}_7[\sqrt{-7}])$. From ι, one gets a symmetric sesquilinear form on $\mathbf{Q}_7[\sqrt{-7}]^3$, which we denote by $\langle \cdot, \cdot \rangle$.

Let $\mathcal{O}_7 = \mathcal{O}(\mathbf{Q}_7[\sqrt{-7}])$ be the subring of integers. If $(e_j)_{1 \leq j \leq 3}$ is any basis of $\mathbf{Q}_7[\sqrt{-7}]^3$, then $L_7 = \mathcal{O}_7 e_1 + \mathcal{O}_7 e_2 + \mathcal{O}_7 e_3$ is called a **lattice** in $\mathbf{Q}_7[\sqrt{-7}]^3$. The **dual lattice** is

$$L_7' = \{v \in \mathbf{Q}_7[\sqrt{-7}]^3 \,;\, \langle v, L_7 \rangle \subseteq \mathcal{O}_7\}$$

The 7-adic integrality condition we impose on $x \in \mathcal{D}$ is simply $xL_7 = L_7$. Thus, in some basis, we ask that the 7-adic matrix corresponding to x be integral. The choice of the lattice L_7 determines the choice of basis.

What sort of lattice do we use?

Because $7 \in \{7\}$, we want an integrality condition over $\mathbf{Q}_7$ which is "alternative". The choice of lattice L_7 determines the integrality condition, and by definition the "alternative" integrality condition arises when we choose L_7 so that:

$$\sqrt{-7}L_7 \subset L_7' \subset L_7$$

with strict inequality.

To proceed further, it was necessary to identify one particular such lattice, and to translate the above condition into concrete form.

Now consider $q = 5$. Everything goes just as for $q = 7$. We must choose a lattice $L_5 \subseteq \mathbf{Q}_5[\sqrt{-7}]^3$ and our 5-adic integrality condition on $x \in \mathcal{D}$ will be $xL_5 = L_5$.

However, because $5 \notin \{7\}$, the lattice we use must correspond to the "standard" integrality condition at $\mathbf{Q}_5$, which means by definition that it must satisfy the simpler condition:

$$L_5' = L_5$$

With a certain simple choice of L_5 satisfying the above condition, and for

$$x = \sum_{0 \leq j \leq 5} \sum_{0 \leq k \leq 2} c_{jk} \zeta^j \sigma^k$$

the condition on x is simply that all the rational coefficients c_{jk} are 5-adically integral: that they have no powers of 5 in their denominators.

Now consider $q = 11$, which is different because -7 has a square root mod 11: in $\mathbf{Z}/11$, $\pm\sqrt{-7} = \pm 2$. From **Hensel's Lemma**, we know that each of these extends to a square root of -7 in $\mathbf{Q}_{11}$. Each of these gives an embedding $\ell \hookrightarrow \mathbf{Q}_{11}$, and putting them together we have an embedding $\ell \hookrightarrow \mathbf{Q}_{11} \times \mathbf{Q}_{11}$. From this we get an embedding of $\mathcal{D}^\times \hookrightarrow GL(3, \mathbf{Q}_{11}) \times GL(3, \mathbf{Q}_{11})$. The involution ι corresponds to exchanging the two copies $GL(3, \mathbf{Q}_{11})$. Consequently, the condition $\iota(x)x = 1$ requires that the two coordinates of x be inverses.

From this analysis it follows that the 11-adic version of $U(\mathcal{D}, \iota)$ is isomorphic to $GL(3, \mathbf{Q}_{11})$, rather than to some unitary group over $\mathbf{Q}_{11}[\sqrt{-7}]$ (?!). We have a mapping $U(\mathcal{D}, \iota) \hookrightarrow GL(\mathbf{Q}_{11})$ and the 11-adic integrality condition on x is a condition on the image of x in $GL(\mathbf{Q}_{11})$.

The “standard” integrality condition in these circumstances is once again that x should preserve some fixed lattice $L_{11} \subseteq \mathbf{Q}_{11}^3$. Since $GL(\mathbf{Q}_{11})$ acts transitively on the set of such lattices, any two lattices give rise to integrality conditions which are conjugate.

We must pick one such lattice $L_{11} = \mathcal{O}_{11}e_1 + \mathcal{O}_{11}e_2 + \mathcal{O}_{11}e_3$ and translate the condition $xL_{11} = L_{11}$ into a concrete condition on x.

Finally, consider $q = 2$. As indicated by the second item in the triple $(a = 7, p = 2, \{7\})$, we constructed $\mathcal{D}$ so that its 2-adic version is a division algebra, not a matrix algebra. It is a consequence that any $x \in \mathcal{D}$ satisfying $x\iota(x) = 1$ automatically satisfies a 2-adic integrality condition.

No actual calculations are necessary for $q = 2$. When searching for elements of $\tilde{\Gamma}$, no check needs to be done for 2-adic integrality.

An arbitrary element of $\mathcal{D}$ is given by:

$$x = \sum_{0 \le j \le 5} \sum_{0 \le k \le 2} c_{jk} \zeta^j \sigma^k = \sum_{0 \le j \le 5} \sum_{0 \le k \le 2} (c'_{jk} + c''_{jk}\sqrt{-7}) \zeta^j \sigma^k$$

for 18 rational coefficients (c'_{jk}) and (c''_{jk}).

A naive integrality condition would be to require $c'_{jk}, c''_{jk} \in \mathbf{Z}$. The q-adic version would be that the coefficients ought not to have factors of q in their denominators. It is a fact that for all but finitely many primes q, the naive q-adic condition is equivalent to a "standard" integrality condition for that prime q. The same is true even if we use some other basis for $\mathcal{D}$ over Q. This is why the "standard" conditions are standard.

Fortunately, it is easy enough to work out the finite set of primes where the naive condition won't work. For $(a = 2, p = 7)$, the answer is $\{2, 3, 5, 7\}$. Consequently, the first condition on our 18 coefficients is that they have no primes except 2, 3, 5, and 7 in their denominators.

As explained above, there is no need to impose any condition on factors of 2 in the denominators.

Now consider 3. A long calculation expresses the desired 3-adic condition as the condition that 18 particular linear combinations of the 18 original coefficients must not have any 3's in their denominators. This calculation is so long that it is difficult to imagine doing it without computer assistance, but there are no computationally intensive steps.

The situation is analogous for 5 and for 7.

Concretely, the "alternative" 7-adic condition for $(a = 7, p = 2)$:

The following matrix defines 18 linear combinations of the 18 coefficients. The condition is that those combinations have no 7's in their denominators.

```
[ 9, 1, 3, 9, 1, 28, 34, 1, 3, 43, 0, 6, 9, 46, 3, 24, 47, 30 ],
[ 43, 0, 6, 43, 0, 40, 21, 0, 6, 31, 2, 28, 43, 1, 6, 6, 48, 43 ],
[ 0, 0, 0, 30, 48, 32, 4, 47, 22, 20, 39, 26, 15, 3, 44, 45, 9, 41 ]
[ 0, 0, 0, 24, 2, 13, 13, 4, 12, 9, 48, 18, 12, 43, 24, 8, 45, 2 ],
...
[ 35, 0, 21, 38, 44, 6, 1, 3, 16, 15, 3, 30, 38, 2, 20, 35, 42, 7 ],
[ 42, 0, 42, 29, 3, 44, 5, 43, 17, 12, 1, 10, 1, 3, 9, 21, 0, 28 ],
[ 22, 1, 6, 36, 46, 9, 22, 2, 28, 12, 1, 9, 31, 0, 31, 27, 47, 33 ],
[ 18, 0, 12, 25, 1, 34, 18, 47, 24, 3, 0, 34, 28, 2, 46, 15, 48, 0 ]
```

Summary:

- The triple $(a = 7, p = 2, \{7\})$ specifies a particular maximal arithmetic subgroup $\Gamma \subseteq PU(2,1)$.
- [Prasad, Yeung] worked out which Γ need to be looked at to find all fake planes. There are 25 of them, and each is specified by the data from a triple like that of our example.
- $$\Gamma = \{x \in PU(\mathcal{D}, \iota)\,;\; x \text{ satisfies certain integrality conditions}\}$$

To proceed further, we have to find some nontrivial elements of Γ. How?