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Potential theory of subordinated Brownian motions

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(Wrocław, 3.10.2013 – 28.11.2013)

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
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FUNDUSZ SPOŁECZNY

Potential theory of subordinated Brownian motions

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December 2, 2013

Abstract

The present notes contain the material covered in the series of lectures for PhD students attending the Joint Doctoral Studies Project. The purpose was to make a concise exposition of the potential theory of Lévy processes and to make the presentation as self-contained as possible. Needless to say that we did not pretend that either the presentation or the bibliography is exhaustive. The first section is introductory and is devoted to the classical potential theory. In the next one we introduce the basic potential-theoretic concepts in the probabilistic setting with the aid of Brownian motion. In Section 3 we introduce Feynman-Kac semigroups and outline their connections with the theory of Schrödinger operator. Section 4 deals with the fundamental example of isotropic stable Lévy process. It is the place where the whole theory began. In the Section 5 we present another important example of Lévy process, namely the relativistic stable Lévy process. The last section deals with the subordinated Brownian motion processes - the most regular and best known class among general Lévy processes. We restrict our attention to the processes with Laplace exponent being a complete Bernstein function. We compute basic potential-theoretic object for these processes and indicate what are essential steps in proving Harnack inequality in this setting.

1 Classical potential theory

1.1 Beginning of potential theory

- **Newton** (1687): Law of universal gravitation, study of $F(x)$ - **force acting on a unit mass** at $x \in \mathbb{R}^d$, $d \geqslant 3$.
- **Lagrange** (1773): The above vector field (of forces) is **a gradient** of a certain function $U := U_2(x) = \mathcal{A}_{d,2}|x|^{2-d}$.
- **Green** (1828) named U **potential function**.
- **Gauss** (1840) named U **potential**.
- **Gauss**: **potential method** is suitable to resolve many complicated problems **of mathematical physics**, not only problems of gravitation or electrostatics.

For a field generated by a charge located according to a measure μ we define the **potential** of μ:

$$U_2\mu(x) = \mathcal{A}_{d,2} \int\limits_{\mathbf{R}^d} |x-y|^{2-d}\, d\mu(y)\,, \quad \mathcal{A}_{d,2} = \frac{\Gamma(d/2-1)}{2\pi^{d/2}}\,.$$

Harmonicity of potentials

Physically, $U_2(x)$ corresponds to the potential at the point x generated by the unit charge placed at the point $0 \in \mathbb{R}^d$.

By a direct differentiation we check that the function $U_2(x-y)$ is **harmonic** for $x \in \mathbb{R}^d \setminus \{y\}$, i.e. it satisfies the Laplace equation:

$$\Delta_x U_2(x-y) = 0, \quad x \neq y,$$

where $\Delta = \sum_{i=1}^d \partial_i^2$.

More generally, $U_2\mu(x)$, potential of a measure μ, is harmonic outside the support of μ.

The same (Laplace) equation is satisfied by a stabilized temperature $T(x)$ of the body D with no inner heat sources, when heated only by the surface. To determine the temperature of the body requires to solve **the Dirichlet problem**.

1.2 Radial harmonic functions

Radial harmonic functions on $\mathbb{R}^d \setminus \{0\}$, $d \geqslant 1$, (depending only on $|x|$) are of the form

$$\begin{aligned} C_1|x| + C_2 &\quad \text{in } \mathbb{R}^1, \\ C_1 \ln|x| + C_2 &\quad \text{in } \mathbb{R}^2, \\ C_1|x|^{2-d} + C_2 &\quad \text{in } \mathbb{R}^d, d \geqslant 3. \end{aligned}$$

To justify this statement, we write the Laplace equation for the function $h(r)$, where $r = |x|$. By a direct differentiation, we obtain

$$\frac{d^2h}{dr^2} + \frac{d-1}{r}\frac{dh}{dr} = 0$$

Solving this differential equation, we obtain the conclusion.

1.3 Equivalent definitions of harmonicity

Definition 1. Let D be a domain (i.e. connected open subset) in $\mathbb{R}^d$, $d \geqslant 1$. A Borel function f, defined on $\mathbb{R}^d$ is called **harmonic** on D if $f \in C^2(D)$ and $\Delta f \equiv 0$ on D.

Equivalent definition:

Definition 2. A Borel function f on $\mathbb{R}^d$, $|f| < \infty$, is harmonic on a domain D iff it satisfies **mean value property** on D; that is for every ball $B(x,r) \subset\subset D$ we have

$$f(x) = \int_{S(x,r)} f(y)\,\sigma_r(dy) = \int_{S(0,1)} f(x+ry)\,\sigma_1(dy).$$

Here σ_r is the (normalized) uniform surface measure on the sphere $S(x,r) = \partial B(x,r)$; analogously σ_1 - on the unit sphere $S(0,1)$.

Remark. Spherical integration over $S(x,r)$ can be replaced by integration over $B(x,r)$ with respect to the Lebesgue measure.

1.4 Basic properties of harmonic functions

- **Maximum Principle.** Let f be harmonic in a domain $D \subset \mathbb{R}^d$ and continuous in $\overline{D}$. Then either $f(x) < sup_{u\in D} f(u)$, for $x \in D$, or $f(x) \equiv const.$ over the whole set D;
- **Harnack Inequality.** Let f be a positive harmonic in a domain $D \subset \mathbb{R}^d$. Then for every compact subset $K \subset D$ there is a constant $C > 0$ such that for every $x_1, x_2 \in K$ we have
$$C^{-1} f(x_1) \leqslant f(x_2) \leqslant C f(x_1) .$$
- **Harnack Theorem.** Let f_n be an increasing sequence of harmonic functions in a domain $D \subset \mathbb{R}^d$. Then either f_n is convergent to a harmonic function on D, uniformly on compact subsets of D, or f_n is everywhere divergent to $+\infty$ on D.

1.5 Dirichlet Problem (1850)

D – domain in $\mathbb{R}^d$,

φ - continuous function on ∂D (boundary of D).

Problem: Find a function $f : D \to \mathbb{R}^d$ which

- is harmonic in the domain D, that is, for $x \in D$ it satisfies
$$\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f(x)}{\partial x_i^2} = 0,$$
- f is continuous on $\overline{D}$ and such that $f|_{\partial D} = \varphi$.
- Solution (if it exists) is unique - Maximum principle!
- Remark: not for all domains such a function exists.

1.6 Solution of the Dirichlet Problem

If the boundary of the set D is "smooth", then there exists a function of two variables $G_D(x,y)$ such that **the solution** $f(x)$ can be expressed in the form

- $$f(x) = \int_{\partial D} \varphi(y) \frac{\partial G_D(x,y)}{\partial \vec{n}_y} \, d\sigma(y),$$
- $G_D(x,y)$ is **the Green function** of the set D,
- $\vec{n}_y$ is **normal vector** at the point y of the boundary,
- σ is **the normalized surface measure** on ∂D.
- function
$$P_D(x,y) = \frac{\partial G_D(x,y)}{\partial \vec{n}_y}$$
is **the Poisson kernel** of the set D.

1.7 Properties of Green function

Definition 3. A function $G_D(x,y)$ defined on $\overline{D} \times D$, for a domain $D \subset \mathbb{R}^d$ is called **the Green function of** D if it satisfies

- $G_D(\cdot,y)$ is harmonic on $D \setminus \{y\}$,
- $G_D(\cdot,y)$ is continuous on $\overline{D} \setminus \{y\}$ and vanishes on ∂D,
- $G_D(\cdot,y) - U_2(\cdot,y)$ remains harmonic at the point $\{y\}$.

Remark. If the Green function for a domain D exists, it is unique. Indeed, for a fixed $y \in D$ the function

$$w_y(x) = G_D(x,y) - U_2(x,y)$$

is a solution of the Dirichlet problem

$$\Delta w_y = 0 \quad \text{in} \quad D, \quad w_y(x) = U_2(x,y) \quad \text{in} \quad \partial D.$$

Physically, $G_D(x,y)$ is the potential at the point x generated by the unit charge placed at $y \in D$ and the charge on the grounded (potential 0) conducting surface ∂D.

1.8 Potentials, case $D = \mathbb{R}^d$

When $d \geqslant 3$, then **the Green function** of the whole space (traditionally called **potential** and denoted by U) is given by the formula

$$U(x,y) = \frac{\Gamma(\frac{d-2}{2})/2\pi^{d/2}}{|x-y|^{d-2}}.$$

When $d = 2$ **potential** $U(x,y) = -\frac{1}{\pi}\log|x-y|$.

When $d = 1$ **potential** $U(x,y) = -|x-y|$.

1.9 Green function - halfspace

Green function and **Poisson kernel** are expressed by explicit formulas also for D being a **halfspace** or a **ball** in $\mathbb{R}^d$. Let $D = H$, $H = \{x \in R^d : x_d > 0\}$. For $y = (y_1, .., y_{d-1}, y_d) \in H$ put

$$y^* = (y_1, .., y_{d-1}, -y_d) \qquad \text{(symmetry with respect } \{y_d = 0\}).$$

Theorem 1. Green function of halfspace: for $x, y \in H$ and $d \geqslant 3$

- $$G_H(x,y) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|x-y^*|^{d-2}} \right).$$

Proof. We subtract another unit charge placed at such a point that the resulting potential at ∂H is 0. The same apply to the case of a ball. We check that $G_H(x,y)$ is harmonic for $x \in H \setminus \{y\}$, vanishes at ∂H and $G_H(x,y) - \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \frac{1}{|x-y|^{d-2}}$ is harmonic for all $x \in H$. □

1.10 Green function for $D = B(0,1)$

If $y \in B(0,1)$, $y \neq 0$, put $y^* = y/|y|^2$ - inversion with respect to sphere $\{|x| = 1\}$. We have $|y|/|y^*| = 1$ and $y/|y| = y^*/|y^*|$.

Theorem 2. *Green function of $B(0,1)$: for $x,\ y \in B(0,1)$, $y \neq 0$, and $d \geqslant 3$*

- $$G_{B(0,1)}(x,y) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}}\left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|y|^{d-2}|x-y^*|^{d-2}}\right).$$

  We put $G_{B(0,1)}(x,0) = \Gamma(\frac{d-2}{2})/2\pi^{d/2}(|x|^{2-d} - 1)$.

Proof. We have $|y|^2|x-y^*|^2 = |y|^2|x|^2 - 2(x, y^*/|y^*|)|y^*||y|^2 + |y|^2|y^*|^2 = |y|^2|x|^2 - 2(x, y/|y|)|y|^2|y^*| + |y|^2|y^*|^2 = |y|^2|x|^2 - 2(x,y) + 1$. Hence, $\lim_{0\neq y\to 0} G_{B(0,1)}(x,y) = G_{B(0,1)}(x,0)$ for $x \in B(0,1)$ at $y = 0$ and $G_{B(0,1)}(x,y) = 0$, for $|x| = 1$. It also satisfies all the remaining conditions. □

1.11 Properties of Poisson kernel

Definition 4. A positive and continuous function function $K(x,y)$ defined on $D \times \partial D$, for a domain $D \subset \mathbb{R}^d$, is called **the Poisson kernel for** D if it satisfies

- $K(\cdot, z)$ is harmonic in D, for every $z \in \partial D$,
- $\int_{\partial D} K(x,z)\,\sigma(dz) = 1$, for every $x \in D$,
- $\lim_{D\ni x\to w} \int_{\partial D\cap B(w,\delta)^c} K(x,z)\,\sigma(dz) = 0$, for every $w \in \partial D$ and $\delta > 0$.

  Here σ denotes the normalized surface measure on ∂D.

Remark. If the Poisson kernel for a domain D exists, it is unique (again, it is the unique solution of of the Dirichlet problem with the given boundary condition). If the Poisson kernel for a bounded domain D exists, then the solution of the Dirichlet problem with a boundary value $f \in C(\partial D)$ can be expressed by

$$f(x) = \int_{\partial D} K(x,z)\,f(z)\,\sigma(dz)\,.$$

1.12 Poisson kernel

For $x = (x_1, ..., x_d)$ put $\tilde{x} = (x_1, ..., x_{d-1})$.

Theorem 3. For $x \in H,\ y \in \partial H$ we have the formula for Poisson kernel for H:

- $$P_H(x,y) = \frac{\Gamma(d/2)}{\pi^{d/2}}\frac{x_d}{(x_d^2 + |\tilde{y} - \tilde{x}|^2)^{\frac{d}{2}}}.$$
- *When $d = 2$, then $P_H(x,y)$ is the density of the Cauchy distribution on the line $\{(x,y) : y = 0\}$.*

For **a ball** $B := B(0, r)$ in $\mathbb{R}^d$, $d \geqslant 1$, **Poisson kernel** is determined by the formula:

Theorem 4. For $x \in B(0, r)$ and $z \in \partial B(0, r)$, i.e. $|z| = r$ we obtain

$$P_B(x, z) = \frac{\Gamma(d/2)}{\pi^{d/2} r} \frac{r^2 - |x|^2}{|x - z|^d}.$$

1.13 Solution of the Dirichlet problem in a ball

The explicit formula for the Poisson kernel in a ball gives us the possibility of write down the form of the solution for the Dirichlet problem.

Theorem 5. Solution of the Dirichlet problem in $B(x_0, r)$ with the boundary value f is given by the formula:

$$u(y) = \int_{\partial B(x_0, r)} f(x) \frac{r^2 - |y - x_0|^2}{r|y - x|^d} \, d\sigma(x) \,,$$

where σ is the normed uniform surface measure on $\partial B(x_0, r)$, and f is defined and continuous on $\partial B(x_0, r)$.

A direct consequence of the above formula is the Harnack Inequality and Harnack Theorem for a ball and, consequently, for compact subsets.

2 Potential theory and Brownian motion

2.1 Brownian motion and the Dirichlet problem

In 40-ties of XX century **S.Kakutani**, and in 50-ties **J.L.Doob** explained how to solve **the Dirichlet problem** in terms of **Brownian motion** . Foundations of the contemporary potential theory of Markov processes are due to **G.Hunt** (1957, 1958).

Let $W(t)$ be the Brownian motion starting from $\mathbb{R}^d$ and let D be a (regular) domain in $\mathbb{R}^d$. Assume that Brownian motion starts from the point $x \in D$ and put

- $\tau_D = \inf\{t > 0 : \ W_t \notin D\}$ — the first exit time from the set D.
- Function
$$f(x) = \mathbb{E}^x\left(\varphi(W_{\tau_D})\right)$$
- is the solution of the Dirichlet problem for D and φ.

2.2 Stopping time

Let (Ω, Σ, P) be a probability space, $\Omega_\tau \subseteq \Omega$, T - an interval $\overline{\mathbb{Z}}$ or $\overline{\mathbb{R}}$; $\{\mathcal{F}_t; t \in T\}$ - increasing family of sub σ-algebras Σ.

Definition 5. A positive random variable $\tau : \Omega_\tau \longrightarrow T$ is called a stopping time, (Markov time) if $\{\tau \leqslant t\} \in \mathcal{F}_t$, $t \in T$

We also define $\mathcal{F}_\tau := \{A \subseteq \Omega_\tau; A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$, for every $t \in T\}$.
Remark. 1. When τ is countably valued then the above definition is equivalent to the following: τ is stopping time with respect to $\{\mathcal{F}_n\}$ if for every n the following holds: $\{\tau = n\} \in \mathcal{F}_n$.
2. $\mathcal{F}_\tau$ is a σ-algebra $\subseteq \Omega_\tau \cap \Sigma$.
3. $\tau : (\Omega_\tau, \mathcal{F}_\tau) \longrightarrow (T, \mathcal{B}_T)$ is measurable.

2.3 Markov property of the process $X = \{X_t; t \in T\}$

Let $\theta_s : (\Omega, \Sigma) \longrightarrow (\Omega, \Sigma)$ acts as a "shift" on the basic probability space according to the rule: $X_t \circ \theta_s = X_{t+s}$.

The easiest way to perceive these operators is to work on the standard probability space $(\mathbb{R}^{[0,\infty)}, \otimes_{t\geqslant 0}\mathcal{B}_\mathbb{R}, \mu)$, where μ is the distribution of the process X. Then $X_t(\omega) = \omega(t)$ and $X_t(\omega) \circ \theta_s = \omega(t+s)$.

Further, we consider the process with the initial distribution $X(0) = Y$ - an arbitrary random variable. Conditional expectation (probability) with respect to a process with the initial distribution Y we denote by $\mathbb{E}^Y[\cdot]$, $(P^Y(\cdot))$. When $Y = x \in \mathbb{R}^d$ we write $\mathbb{E}^x[\cdot]$, $(P^x(\cdot))$.

Theorem 6. Markov property of $\{X_t; t \geqslant 0\}$: for $Z \geqslant 0$, $\mathcal{F}_\infty$-measurable

$$\mathbb{E}^x[Z \circ \theta_t | \mathcal{F}_t] = \mathbb{E}^{X_t}[Z],$$

where $\mathcal{F}_t = \sigma\{X_s; s \leqslant t\}$, $\mathcal{F}_\infty = \sigma\{X_s; s \geqslant 0\}$.

2.4 Strong Markov property of the process X

Theorem 7. For τ - $\mathcal{F}_t$-stopping time and $Z \geqslant 0$, $\mathcal{F}_\infty$-measurable random variables, we have

$$\mathbb{E}^x[Z \circ \theta_\tau | \mathcal{F}_\tau] = \mathbb{E}^{X_\tau}[Z],$$

where $\mathcal{F}_t = \sigma\{X_s; s \leqslant t\}$, $\mathcal{F}_\infty = \sigma\{X_s; s \geqslant 0\}$.

Remark. When $\{X_t; \mathcal{F}_t; t \geqslant 0\}$ has a Markov property and $\mathcal{F}_t$, $\mathcal{F}_t$ is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_t$ and complete, and X_t is a normal Markov process, then $\{X_t; \mathcal{F}_t; t \geqslant 0\}$ **has the strong Markov property**.
Normal Markov process - phase space S is compact, metric and separable, process has a Feller property and P_t defined by $P_t f(x) = \int f(y) P_t(x, dy) = \mathbb{E}^x f(X_t)$ acts on $C(S)$ as a strongly continuous contraction semigroup (process is stochastically continuous).

2.5 Regular points of the process

Let $\{X_t\}_{t \geqslant 0}$ be a stochastic process with values in $\mathbb{R}^d$ and $D \subseteq \mathbb{R}^d$ - a Borel subset.

Definition 6. The first exit time from D is defined as follows:

$$\tau_D = \inf\{t > 0; X_t \notin D\}.$$

Definition 7. The point $x \in \mathbb{R}^d$ is called regular for D when $P^x(\tau_D = 0) = 1$.

We further assume that $X = W$ is a Wiener process in $\mathbb{R}^d$ and $\mathcal{F}_t = \sigma\{W_s; s \leqslant t\}$. We note that $\{\tau_D = 0\} \in \mathcal{F}_{0+}$ so by the $0-1$ Blumenthal law we have either $P^x(\tau_D = 0) = 0$ or 1. The set of all regular points of the set D is denoted by D^r. When $x \in Int(D^c)$ then $x \in D^r$. When $x \in Int(D)$ then the Wiener process remains certain time in $Int(D)$ so $Int(D) \subseteq (D^r)^c$. The only problem is to determine what is the behaviour of the process at $x \in \partial D$. Typically, the process oscilates wildly in the vicinity of the point $x \in \partial D$, hence it leaves immediately from D.

2.6 Exterior cone property

Definition 8. Let $V_a = \{(x_1, \ldots, x_d); x_1 > 0; |(x_2, \ldots, x_d)| < a x_1\}$. A cone V in $\mathbb{R}^d$ is a translation and a rotation of V_a.

Lemma 8. Let $z \in \partial D$. If there exists a cone V with the vertex z such that $V \cap B(z, r) \subseteq D^c$ for a $r > 0$, then z is regular.

Proof. Put $C = \frac{\sigma_r(V \cap \mathbb{S}_r(z))}{\sigma_r(\mathbb{S}_r(z))}$ and $B_n = B(z, r/n)$, $V_n = V \cap \mathbb{S}_{r/n}(z)$. By the rotational invariance (with respect to the starting point) of the distribution of the Wiener process, $W_{\tau_{B(x,r)}}$is also rotationally invariant hence it is the normed spherical measure on $\mathbb{S}_r(x)$. Hence $P^x(W_{\tau_{B(x,r)}} \in V) = C$. At the same time,

$$P^z(\tau_D = 0) \geqslant P^z(\limsup\{W_{\tau_{B_n}} \in V_n\}) \geqslant \limsup P^z(\{W_{\tau_{B_n}} \in V_n\})$$

so $P^z(\tau_D = 0) \geqslant C > 0$ thus $0-1$ Blumenthal law implies $P^z(\tau_D = 0) = 1$. □

2.7 Probabilistic solution of the Dirichlet problem

Let D be a bounded in $\mathbb{R}^d$ and $f \in L^\infty(\partial D)$. The function $H_D f(x)$ defined by

$$H_D f(x) = \mathbb{E}^x[\tau_D < \infty; f(W_{\tau_D})]$$

is harmonic in D. If z is regular and f - continuous at $z \in \partial D$ then $\lim_{D \ni x \to z} H_D f(x) = f(x)$

Proof of harmonicity. Let $x \in D$, $B \subset\subset B(x,r)$. Since $\tau_B < \tau_D$ - P^x a.s. so $\tau_D = \tau_B + \tau_D \circ \theta_{\tau_B}$. Moreover, $W_{\tau_D} \circ \theta_{\tau_B} = W_{\tau_B + \tau_D \circ \theta_{\tau_B}} = W_{\tau_D}$. Let $\Psi = \mathbf{1}_{\{\tau_D < \infty\}} f(W_{\tau_D})$. It holds $\Psi \circ \theta_{\tau_B} = \Psi$ hence $\mathbb{E}^x[\Psi] = \mathbb{E}^x[\mathbb{E}^x[\Psi \circ \theta_{\tau_B} | \mathcal{F}_{\tau_B}]] = \mathbb{E}^x[\mathbb{E}^{W_{\tau_B}}[\Psi]] = \mathbb{E}^x[H_D f(W_{\tau_B})]$. Since the distribution of W_{τ_B} is the uniform normalized spherical measure, therefore $H_D f(x) = \frac{1}{r^{d-1}\omega_d} \int_{\mathbb{S}_r(x)} H_D f(y) \sigma_r(dy)$, and $H_D f$ has the mean value property in D, so it is harmonic in D.

2.8 Convergence at regular points - auxiliary lemmas

Lemma 1. If $f \in L^\infty(\mathbb{R}^d)$ or $f \in L^1(\mathbb{R}^d)$ then $P_t f(\cdot) \in C(\mathbb{R}^d)$ for every $t > 0$.
Proof. For $f \in L^\infty(\mathbb{R}^d)$ and $x_n \to x$ we obtain $|P_t f(x_n) - P_t f(x)| \leqslant ||f||_\infty \int_{\mathbb{R}^d} |p_t(x_n, y) - p_t(x,y)| dy$ and the integral on the left-hand side converges since $\int_{B(0,r)^c} |\dots| < \varepsilon$, for large r. We also have $\int_{B(0,r)} |\dots| \to 0$, by bounded convergence theorem. For $f \in L^1(\mathbb{R}^d)$ we again apply bounded convergence theorem. Here $p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} e^{-||x-y||^2/2t}$ - the transition density of the Wiener process in $\mathbb{R}^d$.

Corollary. Proces X_t has both Feller and strong Feller property, i.e. $P_t : C_0(\mathbb{R}^d) \longrightarrow C_0(\mathbb{R}^d)$ and $P_t : L^\infty(\mathbb{R}^d) \longrightarrow C(\mathbb{R}^d)$

Feller property of the process.
We show that $P_t : C_0(\mathbb{R}^d) \longrightarrow C_0(\mathbb{R}^d)$. To do this, observe that if $f \in C_0(\mathbb{R}^d)$ then for every $\varepsilon > 0$ there exists $r > 0$ such that $|f(y)| < \varepsilon$ if only $|y| > r$. Then we have

$$|P_t f(x)| \leqslant \varepsilon + ||f||_\infty \int_{B(0,r)} p(t; x, y)\, dy\,.$$

The integral on the right-hand side tends to 0 when $x \to \infty$.
The property that $\lim_{t \to 0} ||P_t f - f||_\infty = 0$, for $f \in C_0(\mathbb{R}^d)$ follows from the uniform continuity of functions in $C_0(\mathbb{R}^d)$. This finishes the proof of the Feller property of the process.

Lemma 2. The function $x \longrightarrow \mathbb{E}^x[\kappa \circ \theta_t]$ is continuous on $\mathbb{R}^d$ for $t > 0$ and κ bounded and $\mathcal{F}_\infty$-measurable.

Proof. Let $f(x) = \mathbb{E}^x[\kappa]$. It holds $f \in L^\infty(\mathbb{R}^d)$. Applying the Markov property: $\mathbb{E}^x[\kappa \circ \theta_t] = \mathbb{E}^x[\mathbb{E}^x[\kappa \circ \theta_t | \mathcal{F}_t]] = \mathbb{E}^x[\mathbb{E}^{W_t}[\kappa]] = \mathbb{E}^x[f(W_t)] = P_t f(x) \in C(\mathbb{R}^d)$.

Remark. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is called upper semicontinuous if it is a decreasing limit of continuous functions. We have $\limsup_{x \to x_0} \phi(x) \leqslant \phi(x_0)$.

Lemma 3. The function $\phi : x \longrightarrow P^x(\tau_D > t)$ is upper semicontinuous on $\mathbb{R}^d$ for $t > 0$ and any D open in $\mathbb{R}^d$.
Proof. We show that $P^x(\tau_D > t) = \lim_{s \downarrow 0} \downarrow P^x(\tau_D \circ \theta_s > t - s) = \lim_{s \downarrow 0} \mathbb{E}^x[\mathbf{1}_{(t-s,\infty)}(\tau_D) \circ \theta_s]$. We note that $\inf\{t > s; W_t \notin D\} = s + \tau_D \circ \theta_s$. Let $x \in D^r$, i.e. $P^x(\tau_D = 0) = 1$. There

exists a sequence $s_n \downarrow 0$ such that $W_{s_n} \in D^c$ thus for $s < s_n$ it holds $s + \tau_D \circ \theta_s < s_n$. Now, $\{s + \tau_D \circ \theta_s > t\}_{0<s<t}$ increases in s, so

$\lim_{s \downarrow 0} \downarrow P^x(\tau_D \circ \theta_s > t - s) = P^x(0 > t) = 0 = P^x(\tau_D > t)$.

Let now $x \notin D^r$, i.e. $\tau_D > 0$ P^x a.e. For $s < t$ it holds $\{\tau_D > s\} \supset \{\tau_D > t\}$. If $\tau_D > s$ then $\tau_D = s + \tau_D \circ \theta_s$ hence $\tau_D \circ \theta_s > t - s$. Moreover, $\tau_D \circ \theta_s > t - s$, for $s < t$, which with $\tau_D > s_0$, for some $s_0 < t$ ($\tau_D > 0$) yields $\tau_D > t$. This justifies our formula, hence ϕ is a decreasing limit of continuous functions (Lemma 2) - consequently, it is upper semicontinuous.

2.9 Proof of convergence at regular points.

Let z be a regular point from ∂D and let f be continuous at z. For $\varepsilon > 0$ there exists $\delta > 0$ such that for $w \in \partial D \cap B(z, \delta)$ we have $|f(w) - f(z)| < \varepsilon/2$. Put $M = ||f||_\infty$. Since $P^x(\tau_{B(x,\delta/2)} > 0) = P^0(\tau_{B(0,\delta/2)} > 0) = 1$ we see that there exists $s > 0$ such that for every x it holds $P^x(\tau_{B(x,\delta/2)} \leqslant s) < \varepsilon/8M$. By Lemma 3 $\limsup_{x \to z} P^x(\tau_D > s) \leqslant P^z(\tau_D > s) = 0$. Thus, there exists $\delta' > 0$ such that if $|x - z| < \delta'$ then $P^x(\tau_D > s) < \varepsilon/8M$.

Moreover, $P^x(\tau_{B(x,\delta/2)} \leqslant \tau_D) \leqslant P^x(\tau_{B(x,\delta/2)} \leqslant s) + P^x(\tau_D > s)$. Therefore, when $|x - z| < \delta/2$ to $\tau_{B(x,\delta/2)} \leqslant \tau_{B(z,\delta)}$ then $P^x(\tau_{B(z,\delta)} \leqslant \tau_D) \leqslant P^x(\tau_{B(x,\delta/2)} \leqslant \tau_D) \leqslant \varepsilon/8M + \varepsilon/8M = \varepsilon/4M$.

If $x \in \overline{D}$ and $|x - z| < \delta \wedge (\delta/2)$ we obtain $\mathbb{E}^x[\tau_D < \infty; |f(X_{\tau_D}) - f(z)|] \leqslant P^x(\tau_D < \tau_{B(z,\delta)})\varepsilon/2 + P^x(\tau_{B(z,\delta)} \leqslant \tau_D)2M \leqslant \varepsilon/2 + (\varepsilon/4M)2M$ and this concludes the proof.

As an application, for the solution $u_{1,0}$ of the Dirichlet problem with boundary values $f \equiv 1$ on $\mathbb{S}_\delta(0)$ and $f \equiv 0$ on $\mathbb{S}_R(0)$ we obtain

- **(i)** $u_{1,0}(x) = \frac{\ln R - \ln ||x||}{\ln R - \ln \delta}$, $x \in \overline{D}$, **for** $d = 2$,
- **(ii)** $u_{1,0}(x) = \frac{||x||^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}}$, $x \in \overline{D}$, **for** $d \geqslant 3$.

2.10 Recurrence and transitivity of the Wiener process

From the uniqueness of the solution of Dirichlet problem, we obtain $u_{1,0}(x) = P^x(||W_{\tau_D}|| = \delta) = P^x(W_t \text{ hits } \mathbb{S}_\delta \text{ before hitting } \mathbb{S}_R)$. Fix $\delta > 0$ and let $R \to \infty$. For $\mathbb{R}^2$ (case (i)): $\lim_{R\to\infty} u_{1,0}(x) = 1$. When $d \geqslant 3$ (case (ii)): $\lim_{R\to\infty} u_{1,0}(x) = (\delta/||x||)^{d-2}$. Hence

- $d = 2$: $P^x(W_t \text{ hits } \mathbb{S}_\delta \text{, for some } t > 0) = 1$.
- $d \geqslant 3$: $P^x(W_t \text{ hits } \mathbb{S}_\delta \text{, for some } t > 0) = (\delta/||x||)^{d-2}$.

Let now $d = 2$, $R > 0$ and $\delta \to 0$. We obtain $\lim_{\delta\to 0} u_{1,0}(x) = 0$. so $P^x(W_t \text{ hits } 0) = 0$. We repeat the same arguments for every translation of D so we obtain
Two-dimensional Wiener process, starting from x, does not hit any fixed point $y \neq x$, almost surely.

2.11 Killed process

Let X_t be a Markov process with the transition density function $p(t;x,y)$. Define the first exit time from the set D:

$$\tau_D = \inf\{t > 0 : \ X(t) \notin D\}$$

and the process killed at the time of first exit from D:

$$X_D(t) = \begin{cases} X(t), & \text{gdy } 0 \leqslant t < \tau_D, \\ \partial, & \text{gdy } t \geqslant \tau_D. \end{cases}$$

where ∂ is a "cemetery" – a certain, isolated state of the space of the values of the process X. Its transition function is of the form

$$P_t^D(x,A) = P^D(t;x,A) = P^x(t < \tau_D; X_t \in A),\ t > 0,\ x \in D,\ ;$$

and its transition density (if X has one) is given by
Hunt's Formula

$$p^D(t;x,y) = p(t;x,y) - \mathbb{E}^x[\tau_D < t; p(t-\tau_D; X_{\tau_D}, y)].$$

2.12 Basic properties of killed process

Justification of the Hunt's Formula
For a bounded Borel function f we obtain

$$\begin{aligned} & \int_D \mathbb{E}^x[\tau_D < t; p(t-\tau_D; X_{\tau_D}, y)]\, f(y)\, dy \\ = \ & \mathbb{E}^x[\tau_D < t; \int_D p(t-\tau_D; X_{\tau_D}, y)\, f(y)\, dy] \\ = \ & \mathbb{E}^x[\tau_D < t; \mathbb{E}^{X_{\tau_D}}[f(X_s)]|_{s=t-\tau_D}] \\ = \ & \mathbb{E}^x[\mathbb{E}^x[\tau_D < t; f(X_{s+\tau_D})|\mathcal{F}_{\tau_D}]|_{s=t-\tau_D}] \\ = \ & \mathbb{E}^x[\mathbb{E}^x[\tau_D < t; f(X_t)|\mathcal{F}_{\tau_D}]] = \mathbb{E}^x[\tau_D < t; f(X_t)]\,. \end{aligned}$$

Subtracting from the first part of the formula, with $f(y)$ integrated over D, we obtain

$$\mathbb{E}^x[f(X_t)] - \mathbb{E}^x[\tau_D < t; f(X_t)] = \mathbb{E}^x[t < \tau_D; f(X_t)]\,.$$

2.13 Feller properties of killed process

Theorem. For regular D the killed process has Feller and strong Feller property
Semigroup, Feller and strong Feller properties.
For $f \in L^\infty(D)$ and $0 < s < t$

$$\begin{aligned} & P_t^D f(x) = \mathbb{E}^x[s < \tau_D; \mathbb{E}^{X_s}[t-s < \tau_D; f(X_{t-s})]] \\ = \ & P_s^D P_{t-s}^D f(x) = \mathbb{E}^x[s < \tau_D; \phi_{t-s}(X_s)] = P_s^D\, \phi_{t-s}(x), \end{aligned}$$

where $\phi_s(x) = \mathbb{E}^x[s < \tau_D; f(X_s)]]$. This proves the semigroup property of P_t^D. Furthermore, $P_s\phi_{t-s} \in C_b(\mathbb{R}^d)$ and

$$|P_s\, \phi_{t-s}(x) - P_t^D f(x)| = |P_s\, \phi_{t-s}(x) - P_s^D \phi_{t-s}(x)| \leqslant P^x(\tau_D \leqslant s)\, ||f||_\infty\,.$$

We show that $P^x(\tau_D \leqslant s)$ converges uniformly to zero, as $s \to 0$, on any compact subset of D. This will show that $P_t^D f$ is continuous in D, so $P_t^D f \in C_b(D)$.

$P^x(\tau_D \leqslant s)$ **converges uniformly to zero, as** $s \to 0$**, on any compact subset of** D. Indeed, for $x \in D$ and small $r > 0$ we have $\tau_{B(x,r)} \leqslant \tau_D$ hence $\{\tau_D \leqslant s\} \subseteq \{\tau_{B(x,r)} \leqslant s\}$ and we obtain, as $s \to 0$,

$$\begin{aligned} & P^x(\tau_D \leqslant s) \leqslant P^x(\tau_{B(x,r)} \leqslant s) \\ = \quad & P^0(\tau_{B(0,r)} \leqslant s) \to 0 \,. \end{aligned}$$

By compactness arguments, we obtain the conclusion.
Now, by lower semicontinuity of $x \to P^x(\tau_D > t)$ we obtain for any $z \in \partial D$

$$\begin{aligned} \limsup_{x\to z} P_t^D f(x)| &\leqslant ||f||_\infty \limsup_{x\to z} P_t^x(\tau_D > t) \\ &\leqslant ||f||_\infty P_t^z(\tau_D > t) \end{aligned}$$

and the last expression is 0 if z is regular. This, along with the strong continuity of the semigroup, proves the Feller property.

2.14 Characteristics of the killed process

Stopping or killing the process

- $\tau_D = \inf\{t > 0 : \; X(t) \notin D\}$ - first exit time (from D)
- $X_{\tau_D \wedge t}$ - stopped process (when exiting from D)
- $X_t, t < \tau_D$ - killed process (when exiting from D)

The simplest (conceptually) object - first exit time τ_D.

The most widely used object - X_{τ_D} - the stopped process (at the first hitting time). The density of distribution of X_{τ_D} - called Poisson kernel of the set D - provides the solution of the Dirichlet problem.
Killed process - very difficult to investigate.
The Hunt's formula indicates that if we know the distribution of (τ_D, X_{τ_D}) then we are able to determine the transition density of the killed process.
The basic example - Brownian motion and D -
a halfspace . The starting point – reflection principle for Brownian motion.

2.15 Reflection Principle for Brownian motion

Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion in $\mathbb{R}^1$ (starting from 0) and τ - a stopping time with respect to W. Put

$$\rho_\tau W_t = \begin{cases} W_t, & t \leqslant \tau, \\ 2W\tau - W_t, & t > \tau. \end{cases}$$

Reflection Principle: $\rho_\tau W_t$ **is a Brownian motion**
Corollary.

$$P(\max_{s\leqslant t} W_s > a) = 2P(W_t > a) = P(|W_t| > a).$$

Remark. We apply Reflection Principle to compute the distribution of the first exit time from the halfspace (a, ∞), where $a > 0$ and the process starts from 0. We denote $\tau_a := \tau_{(a,\infty)}$.

2.16 Density function of distribution of τ_a

We have

$$\{\max_{s\leqslant t} W_t > a\} = \{\tau_a \leqslant t\}$$

We compute the density function of the random variable τ_a:

$$\begin{aligned}
\frac{d}{dt}P\{\tau_a \leqslant t\} &= \frac{d}{dt}P\{\max_{s\leqslant t} W_t > a\} = \\
\frac{d}{dt}[\sqrt{\frac{2}{\pi t}}\int_a^\infty e^{-x^2/2t}dx] &= -\frac{1}{2}\sqrt{\frac{2}{\pi t^3}}\int_a^\infty e^{-x^2/2t}dx + \\
\sqrt{\frac{2}{\pi t}}\int_a^\infty \frac{x^2}{2t^2}e^{-x^2/2t}dx &\overset{int.byparts}{=} \frac{a}{\sqrt{2\pi t^3}}e^{-a^2/2t}
\end{aligned}$$

2.17 Laplace transform of τ_a

$$\begin{aligned}
\mathbb{E}[e^{-\lambda^2\tau_a}] &= \frac{a}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda^2 u}e^{-a^2/2u}u^{-3/2}du \\
&= \frac{a}{\sqrt{2\pi}}2\left(\frac{a^2}{2\lambda^2}\right)^{-1/4}\mathbf{K}_{-1/2}(\sqrt{2}\lambda a) \\
&= \frac{a}{\sqrt{2\pi}}2\sqrt{\frac{\sqrt{2}\lambda}{a}}\sqrt{\frac{\pi}{2\sqrt{2}a\lambda}}\,e^{-\sqrt{2}\lambda a} = e^{-\sqrt{2}\lambda a}\,.
\end{aligned}$$

where $\mathbf{K}_\vartheta$ - the modified Bessel function of second kind:

$$\int_0^\infty e^{-au}e^{-b/u}u^{\nu-1}du = 2(b/a)^{\nu/2}\mathbf{K}_\nu(2\sqrt{ab})$$

Moreover, $\mathbf{K}_{-1/2}(x) = \mathbf{K}_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$.

2.18 Transition density of the killed process

The most fundamental object in potential theory - transition probability P^D of the process $X_D(t)$ killed at the first exit time from the set D: for $x, y \in D$ we put

$$P^D(x; A) = P^x[t < \tau_D; X_t \in A]\,.$$

When X has the transition density $p(t; x, y)$ then the transition density of the killed process can be expressed by the formula:

$$p^D(t; x, y) = p(t; x, y) - \mathbb{E}^x[\tau_D < t; p(t - \tau_D; X_{\tau_D}, y)].$$

Kelvin's symmetry principle gives us $p^D(t; x, y)$ for halfspace $D = H = \{x \in \mathbb{R}^d \,:\, x_d > 0\}$. For $y = (y_1, .., y_{d-1}, y_d) \in H$ put

$$y^* = (y_1, .., y_{d-1}, -y_d)\,.$$

We then obtain

$$p^D(t; x, y) = p(t; x, y) - p(t; x, y^*)\,.$$

Exercise: direct computation of transition density of the killed process
As an exercise we compute $p^D(t;x,y)$ directly from Hunt's formula. The time τ_D is determined by the last coordinate of the process; consequently, it does not depend on the first $(d-1)$ coordinates of the process. Now, $y \to \tilde{y}$ denotes the projection onto first $(d-1)$ coordinates. We thus obtain

$$\begin{aligned}
& \mathbb{E}^x[\tau_D < t; p(t-\tau_D; X_{\tau_D}, y)] \\
= \; & \mathbb{E}^x[\int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}}\, p(t-s, (\tilde{X}_s, 0), y)\, ds] \\
= \; & \int_{\mathbb{R}^{d-1}} \int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}} \frac{e^{-|z-\tilde{x}|^2/2s}}{(2\pi s)^{(d-1)/2}} \frac{e^{-(|z-\tilde{y}|^2+y_d^2)/2(t-s)}}{(2\pi(t-s))^{d/2}}\, ds\, dz \\
= \; & \int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}} \frac{e^{-y_d^2/2(t-s)}}{\sqrt{2\pi(t-s)}} \\
& \left\{ \int_{\mathbb{R}^{d-1}} \frac{e^{-|z-\tilde{x}|^2/2s}}{(2\pi s)^{(d-1)/2}} \frac{e^{-|z-\tilde{y}|^2/2(t-s)}}{(2\pi(t-s))^{(d-1)/2}}\, dz \right\} ds\,.
\end{aligned}$$

Now, the expression in parentheses is the convolution of two $(d-1)$-dimensional Gaussian densities hence is equal to

$$\frac{e^{-|\tilde{x}-\tilde{y}|^2/2t}}{(2\pi t)^{(d-1)/2}}\,.$$

To compute the remaining expression

$$\int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}} \frac{e^{-y_d^2/2(t-s)}}{\sqrt{2\pi(t-s)}}\, ds$$

we take the Laplace transform and, after changing order of integration and variables we obtain

$$\int_0^\infty \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}}\, e^{-\lambda s}\, ds \int_0^\infty e^{-\lambda u} \frac{e^{-y_d^2/2u}}{\sqrt{2\pi u}}\, du$$

The last expression can be expressed in terms of modified Bessel function $\mathbf{K}_{1/2}$ of the second order as follows

$$\frac{2}{\sqrt{2\pi}} \left(\frac{y_d^2}{2\lambda}\right)^{1/4} \mathbf{K}_{1/2}(\sqrt{2\lambda}\, y_d) = \frac{e^{-\sqrt{2\lambda}\, y_d}}{\sqrt{2\lambda}}$$

while the first one is of the form

$$\frac{2\, x_d}{\sqrt{2\pi}} \left(\frac{x_d^2}{2\lambda}\right)^{-1/4} \mathbf{K}_{-1/2}(\sqrt{2\lambda}\, x_d) = e^{-\sqrt{2\lambda}\, y_d}\,.$$

Hence, after multiplication we obtain

$$\frac{e^{-\sqrt{2\lambda}\,(x_d+y_d)}}{\sqrt{2\lambda}} = \mathcal{L}\left(\frac{e^{-(x_d+y_d)^2/2t}}{\sqrt{2\pi t}}\right)\,.$$

Thus, the whole expression is of the form $(2\pi t)^{-d/2}\, e^{-|x-y^*|^2/2t}$ where $y^* = (y_1, y_2, ..., y_{d-1}, -y_d)$.

2.19 Green function and Poisson kernel

Poisson kernel and **Green function** of the set D have simple explanations in terms of the process killed or stopped when exiting D:

- $$P_D(x,y) = P^x\left(X_{\tau_D} \in dy\right)$$
  the density of the distribution of hitting the boundary of the set D;
- $$G_D(x,y) = \int_0^\infty p^D(t;x,y)\,dt$$
  ,,density" of occupying time of the process at y.

Green operator
For bounded Borel functions $f:\mathbb{R}^d \to \mathbb{R}$ and domain D put **Green operator**:

- $$G_D f(x) = \mathbb{E}^x\left(\int_0^{\tau_D} f(X_t)\,dt\right).$$
- Green function is the kernel of this operator:
  $$G_D f(x) = \int_D G_D(x,y)\,f(y)\,dy.$$
- In particular, when $f = 1_A$, we obtain
  $$G_D 1_A(x) = \mathbb{E}^x\left[\int_0^{\tau_D} 1_A(X_t)\,dt\right] = \int_D G_D(x,y)\,1_A(y)\,dy$$
  is the mean occupying time of the process, starting from x, within the set A .

Properties of Green function
For $d \geqslant 3$ we have
$$G_D(x,y) = U_2(x,y) - \mathbb{E}^x[\tau_D < \infty; U_2(X_{\tau_D}, y)]\,.$$
Indeed, denote
$$r^D(t;x,y) = \mathbb{E}^x[\tau_D < t; p(t-\tau_D; X_{\tau_D}, y);]\,.$$
We obtain, for $x \neq y$,
$$\int_0^\infty r^D(t;x,y)\,dt = \mathbb{E}^x[\tau_D < \infty; \int_0^\infty p(u; X_{\tau_D}, y)\,du]\,.$$
which justifies the formula, if we show that this expression is finite. Putting $\delta = \rho(y, \partial D)$ we obtain
$$\mathbb{E}^x[\tau_D < \infty; U(X_{\tau_D}, y)] \leqslant U(\delta) < \infty\,.$$
For $d = 1, 2$ the corresponding formulas are also valid but in terms of compensated potentials instead.

2.20 Potential operator

When $D = \mathbb{R}^d$, $d \geqslant 3$, computing as before, we obtain **the potential operator** and **the potential function**:

-
$$U_2 f(x) = \int_0^\infty \mathbb{E}^x[f(X_t)]\,dt.$$

- Potential function is the kernel of this operator:
$$U_2 f(x) = \int_{\mathbb{R}^d} U_2(x,y)\,f(y)\,dy.$$

- We obtain
$$U_2(x,y) = \int_0^\infty \frac{1}{(2\,\pi\,t)^{d/2}}\,e^{-|x-y|^2/2t}\,dt = \frac{1}{2\pi^{d/2}}\frac{\Gamma(d/2-1)}{|x-y|d-2}.$$

In this way, we obtained the same object as at the beginning, thus exemplifying the connection between analytical and probabilistic theories. For $d = 1, 2$ analogous formulas are valid, but with different proofs.

2.21 Brownian motion: $\mathfrak{U}f = \frac{1}{2}f''$ on $\mathfrak{D}_{\mathfrak{U}} = C^{(2)}$

Generator of Brownian motion.
$S = \mathbf{R}^*$, $P_t(x,E) = \frac{1}{\sqrt{2\pi t}}\int_{E-x} e^{-y^2/2t}dy$, $P_t(\infty,\infty) = 1$.
We have
$P_t f(x) = \frac{1}{\sqrt{2\pi t}}\int f(y)e^{-(y-x)^2/2t}dy$, f -bounded Borel. If $f \in C^{(2)}$ then

$$\frac{1}{t}[T_t f(x) - f(x)] = \frac{1}{\sqrt{2\pi t}}\int e^{-y^2/2t}\,\frac{f(x+y)-f(x)-f'(x)\,y}{t}\,dy$$
$$= \frac{1}{\sqrt{2\pi t}}\int e^{-y^2/2t}\,\frac{f''(x+\theta y)}{2t}y^2 dy$$
$$= \frac{1}{2\sqrt{2\pi t}}\int e^{-u^2/2}\,f''(x+\theta u\sqrt{t})u^2 du \to \frac{1}{2}f''(x),$$

for every x, when $t \to 0$. We thus have

$$f \in C^{(2)} \subseteq \mathfrak{D}_{\mathfrak{U}}$$

and $\mathfrak{U}f = \frac{1}{2}f''$ on $C^{(2)}$. Furthermore, $\mathfrak{D}_{\mathfrak{U}} = C^{(2)}$.
For $g \in C$, we solve in f: $\lambda f - \frac{1}{2}f'' = g$.
We obtain

$$f(x) = \mathfrak{R}_\lambda g(x) = \frac{1}{\sqrt{2\pi}}\int[\int_0^\infty e^{-\lambda t}e^{-(y-x)^2/2t}t^{-1/2}dt]g(y)dy$$
$$= \frac{1}{\sqrt{2\lambda}}\int g(y)e^{-\sqrt{2\lambda}|x-y|}dy,$$

since $\int_0^\infty e^{-au}e^{-b/u}u^{\nu-1}du = 2(b/a)^{\nu/2}\mathbf{K}_\nu(2\sqrt{ab})$ (modified Bessel function of 2-nd order), and we have $\mathbf{K}_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$. We obtain that $f \in C^{(2)}$ solves $\lambda f - f''/2 = g$.

2.22 Fundamental solution of $\frac{1}{2}\Delta = \delta_0$

For $f \in C_c(\mathbb{R}^d)$ the following holds

$$\frac{1}{2}\Delta\, U_2 f(x) = -f(x)\,.$$

Proof. We obtain

$$\begin{aligned} P_t\, U_2\, f(x) &= \mathbb{E}^x[\int_0^\infty \mathbb{E}^{X_t}[f(X_s)]\, ds] \\ &= \int_0^\infty \mathbb{E}^x[\mathbb{E}^{X_t}[f(X_s)]]\, ds = \int_0^\infty \mathbb{E}^x[f(X_{t+s})]\, ds \\ &= \int_t^\infty \mathbb{E}^x[f(X_u)]\, du\,. \end{aligned}$$

We thus have obtained

$$P_t\, U_2\, f(x) - U_2\, f(x) = -\int_0^t \mathbb{E}^x[f(X_u)]\, du\,.$$

After dividing by t, we obtain the conclusion when $t \to 0$.

2.23 $\Delta(G_D\phi) = -2\phi$ for $\phi \in C_c(D)$

We use the following representation of the Green function

$$G_D(x,y) = U_2(x,y) - \mathbb{E}^x[\tau_D < \infty; U_2(X_{\tau_D}, y)]\,.$$

From the previous result we obtain

$$\Delta\, U_2\phi(x) = -2\phi(x)\,.$$

However, the second term in the representation of G_D, acting on ϕ, gives the harmonic function so the result follows.
If $\phi \in C_c^\infty(D)$ then $G_D\phi \in C_c^\infty(D)$ and we also have

$$G_D(\Delta\phi) = -2\phi\,.$$

For $d = 1$ and $d = 2$ additional assumptions are required, e.g. boundedness of the domain D.

2.24 Bibliographical notes

The material contained in this section is standard. For analytical informations about harmonicity: see [4]; for the probabilistic part a good source is [3]. The monograph [2] contains a rigorous and exhaustive treatment of the foundations of Markov processes. The book [1] contains an outline of interplay between Analysis and Probability. Papers [5] and [6] contain a pioneering exposition of potential theory of Markov processes.

2.24.1 Bibliography

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3 Schrödinger operator and Feynman-Kac semigroups

3.1 Schrödinger Equation

Let D be a domain in $\mathbb{R}^d$, $d \geqslant 1$. We consider the following:
Schrödinger Equation

$$\frac{1}{2}\Delta u(x) + q(x)u(x) = 0, \quad x \in D\,,$$

where Δ is the Laplace operator and q is a Borel measurable function on D.
The function q is called a "potential". We restrict potentials q to a suitable class of functions to obtain a reasonable potential theory of this operator. The above equation is usually understood in weak sense, i.e. in terms of distributions. The Schrödinger operator is one of the most important objects in mathematical physics.

3.2 Kato class $\mathcal{J}$

We say that a Borel function q belongs to the **Kato class** $\mathcal{J}$ if q satisfies either of the two equivalent conditions:

$$(i) \qquad \lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leqslant r} |q(y)|\, U_2(x-y)\, dy = 0\,,$$

$$(ii) \qquad \lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_0^t P_s|q|(x)\, ds = 0\,.$$

We write $q \in \mathcal{J}_{loc}$ if for every bounded Borel subset $B \subseteq \mathbb{R}^d$ holds $\mathbf{1}_B\, q \in \mathcal{J}$.
If $q \in \mathcal{J}$ then $\sup_{x\in\mathbb{R}^d} \int_{|x-y|<1} |q(y)|\, dy < \infty$; in particular, $\mathcal{J}_{loc} \subseteq L^1_{loc}$. If $f \in L^\infty(\mathbb{R}^d)$ then $f, f\,q \in \mathcal{J}$.

Let D be a domain in $\mathbb{R}^d$ and $q \in \mathcal{J}$. For $t > 0$ we denote

$$A(t) = \int_0^t q(X_s)\, ds \quad \text{and} \quad e_q(t) \;=\; \exp\left(\int_0^t q(X_s)\, ds\right)\,.$$

We call A an **additive** and e_q **multiplicative functionals**, respectively: $e_q(t+s) = e_q(t)\, e_q(s)\circ\theta_t$. We will always assume that D is either bounded or of finite Lebesgue measure.

3.3 Gauge function

The function

$$u(x) \;=\; \mathbb{E}^x e_q(\tau_D)\,,$$

is called the **gauge** (function) for (D,q), and when it is bounded in D, we say that (D,q) is **gaugeable**. The gauge u is the principal object of the potential theory of the Schrödinger operator.

Khasminski Lemma. Let τ be a stopping time of the process X satisfying for every $t > 0$ the condition

$$\tau \leqslant t + \tau\circ\theta_t \quad \text{on} \quad \{t < \tau\}\,.$$

Suppose that $q \geqslant 0$ and $\mathbb{E}^x[A(\tau)] < \infty$ for all x. If

$$\sup_x \mathbb{E}^x[A(\tau)] = \alpha < 1$$

then

$$\sup_x \mathbb{E}^x[e^{A(\tau)}] \leqslant \frac{1}{1-\alpha}\,.$$

Proof. We obtain

$$\frac{1}{n+1}\,A(\tau)^{n+1} = \int_0^\tau [A(\tau) - A(t)]^n\,dA(t).$$

For $t < \tau$ we obtain

$$\begin{aligned} A(\tau) - A(t) &\leqslant A(t + \tau\circ\theta_t) - A(t) = \int_t^{t+\tau\circ\theta_t} q(X_s)\,ds \\ &= [\int_0^\tau q(X_s)\,ds]\circ\theta_t = A(\tau)\circ\theta_t\,. \end{aligned}$$

By Fubini's theorem

$$\begin{aligned} \frac{1}{n+1}\,\mathbb{E}^x[A(\tau)^{n+1}] &\leqslant \mathbb{E}^x\int_0^\tau [A(\tau)\circ\theta_t]^n\,dA(t) \\ &= \int_0^\infty \mathbb{E}^x\{t<\tau; [A(\tau)\circ\theta_t]^n\,q(X_t)\}\,dt\,. \end{aligned}$$

We apply the Markov property to the last integral and obtain

$$\begin{aligned} &\int_0^\infty \mathbb{E}^x\{t<\tau; [A(\tau)\circ\theta_t]^n\,q(X_t)\}\,dt \\ =\ &\int_0^\infty \mathbb{E}^x\{t<\tau; \mathbb{E}^{X_t}[A(\tau)^n]\,q(X_t)\}\,dt \\ \leqslant\ &\int_0^\infty \mathbb{E}^x\{t<\tau;\ q(X_t)\}\,dt\ \sup_x \mathbb{E}^x[A(\tau)^n] \\ =\ &\mathbb{E}^x[A(\tau)]\ \sup_x \mathbb{E}^x[A(\tau)^n]\,. \end{aligned}$$

It follows that

$$sup_x\mathbb{E}^x[A(\tau)^{n+1}] \leqslant (n+1)\,sup_x\mathbb{E}^x[A(\tau)]\,sup_x\mathbb{E}^x[A(\tau)^n]\,.$$

The proof is completed by induction on n.

3.4 Killed Feynman-Kac semigroups

Let X_t be a Feller process with the induced contraction semigroup $P_tf(x) = \mathbb{E}^x f(X_t)$ and let $e_q(t) = exp(\int_0^t q(X_s)\,ds)$ be the corresponding multiplicative functional. We always assume that $q \in \mathcal{J}$. The semigroup of the process X is denoted by P_t (P_t^D when X is killed when exiting D).

Now, by $(T_t) = (T_t^D)$ we denote the Feynman-Kac semigroup killed on exiting D. Thus, for nonnegative Borel f we have

$$T_t f(x) = \mathbb{E}^x[t < \tau_D;\, e_q(t) f(X_t)]\,.$$

(T_t) is a strongly continuous semigroup of bounded operators on each of the appropriate spaces for the semigroup (P_t^D) and for each $1 \leqslant p \leqslant \infty$ we have $||T_t||_p \leqslant ||T_t||_\infty \leqslant e^{C_0 + C_1 t}$. T_t is also a bounded operator from L^p into L^∞, for each $t > 0$. There exists a symmetric bounded and continuous kernel u_t of T_t. Moreover, $u_t \in C_0(D \times D)$, whenever D is regular.

For each $f \in L^p$, $(1 \leqslant p \leqslant \infty)$ we have

$$T_t f(x) = \int_D u_t(x,y)\, f(y)\, dy\,.$$

T_t maps L^1 into $C_0(D)$, for $t > 0$.

For the semigroup (T_t) we introduce the potential operator V as follows:

$$Vf(x) = \int_0^\infty T_t f(x)\, dt = \mathbb{E}^x[\int_0^{\tau_D} e_q(t)\, f(X_t)\, dt]\,,$$

for nonnegative and Borel measurable f on D. We call V the **q-Green operator**.

If $\int_0^\infty ||T_t||_\infty\, dt < \infty$ then V is bounded on L^p, $1 \leqslant p \leqslant \infty$. In particular, $V\mathbf{1} \in L^\infty(D)$. Then V has a symmetric density $V(x,y)$ called the **q-Green function** given by the formula

$$V(x,y) = \int_0^\infty u_t(x,y)\, dt\,.$$

3.5 Gauge function

Gauge function of D.

$$u(x) = \mathbb{E}^x e_q(\tau_D)\,.$$

The following fundamental theorem about gauge function is due to K.L. Chung and K.M. Rao (1968).

Gauge Theorem

For bounded D and $q \in \mathcal{J}$ either $u(x) = \infty$ for all $x \in D$ or $u \in L^\infty(D)$. Moreover, the following conditions are equivalent:

- (i) $u \in L^\infty(D)$
- (ii) $\int_0^\infty ||T_t||_\infty\, dt < \infty$;
- (iii) $V\mathbf{1} \in L^\infty(\mathbb{R}^d)$;
- (iv) $V|q| \in L^\infty(\mathbb{R}^d)$.

3.6 Green function and q-Green function

Assume that $q \in \mathcal{J}$ and $V\mathbf{1} \in L^\infty(D)$. Then we have the following relation
Assume $V(|q|G_D|f|)(x) < \infty$, for $x \in D$. Then

$$V f(x) = G_D f(x) + V(qG_D f)(x)\,.$$

Proof. Application of Fubini's theorem and Markov property yields

$$\begin{aligned} V\,q\,G_D\,f\,(x) &= \mathbb{E}^x \int_0^{\tau_D} e_q(t)\,q(X_t)\,\mathbb{E}^{X_t} \int_0^{\tau_D} f(X_s)\,ds\,dt \\ &= \mathbb{E}^x \int_0^{\tau_D} e_q(t)\,q(X_t) \int_t^{\tau_D} f(X_s)\,ds\,dt \\ &= \mathbb{E}^x \int_0^{\tau_D} f(X_s) \int_0^{s} e_q(t)\,q(X_t)\,dt\,ds \\ &= \mathbb{E}^x \int_0^{\tau_D} f(X_s)\,[e_q(s)-1]\,ds = V\,f(x) - G_D\,f(x)\,. \end{aligned}$$

The relation below is another version of the statement above and has the same proof.
If $G_D(|q|V|f|)(x) < \infty$, for $x \in D$, then

$$V f(x) = G_D f(x) + G_D(qV f)(x)\,.$$

Properties of q-Green function V
Assume that (D,q) is gaugeable. Then
(i) $V(\cdot,\cdot)$ is finite, symmetric and continuous in $(x,y) \in D \times D$, $x \neq y$;
(ii) For any $x \in D$, $\lim_{y\to\partial D} V(x,y) = 0$;
(iii) There exists $C > 0$ such that $V(x,y) \leqslant C\,U_2(x,y)$, $(x,y) \in D \times D$.

3.7 Uniqueness of solutions of Schrödinger equation

Definition. Let $q \in \mathcal{J}_{loc}$. For $u \in C^2(D)$ such that $u\,q\,\mathbf{1}_D \in L^1$ we define, in distributional sense, the operator Su by the formula

$$Su = -(\frac{1}{2}\Delta + q)\,u$$

and call S the (weak) Schrödinger operator on D.

Theorem. Let D be a bounded and regular domain in $\mathbb{R}^d$ and let $q \in \mathcal{J}_{loc}$. Assume that $V\mathbf{1} \in L^\infty$ and $\phi \in C(\overline{D})$, $\phi = 0$ on ∂D and $S\,\phi$ vanishes in D. Then $\phi = 0$ on D.

Proof. Put $f = \phi - G_D(q\phi)$. Since $q \in \mathcal{J}_{loc}$, we have $q\,\phi\,\mathbf{1}_D \in \mathcal{J}$. Hence $G_D(q\,\phi) \in C_0(D)$. Moreover, we have

$$\frac{1}{2}\Delta\,f = \frac{1}{2}\Delta\phi - \frac{1}{2}\Delta\,G_D(q\phi) = (\frac{1}{2}\Delta + q)\,\phi = -S\phi = 0$$

in D. By maximum principle we obtain that $f = 0$ in D, so $\phi = G_D(q\,\phi)$. By the property of V we obtain:

$$V(q\,\phi) = G_D(q\,\phi) + V(q\,G_D(q\phi)) = \phi + V(q\,\phi)\,.$$

Since $\phi = 0$ on ∂D, we have obtained $\phi = 0$ on D.

3.8 q-harmonicity

Let u be a Borel function on a domain D. It is called **q-harmonic** if for every open subset $U \subset\subset D$ we have

$$u(x) = \mathbb{E}^x[\tau_U < \infty; e_q(\tau_U)\, u(X_{\tau_U})]\,, \quad x \in U\,.$$

The following property shows that q-harmonic functions are solutions of the Schrödinger equation in D.

Theorem. For a domain $D \subset \mathbb{R}^d$ and $q \in \mathcal{J}_{loc}(D)$ we have: If u is q-harmonic in D then for every open bounded U with the exterior cone property such that $\overline{U} \subset D$ we obtain

$$u(x) = \mathbb{E}^x u(X_{\tau_U}) + G_U(qu)(x)\, a.e.$$

with the right-hand side continuous. This shows that $\frac{1}{2}\Delta u(x) + q(x)u(x) = 0$ for $x \in D$.

Proof. We denote

$$\Phi(t) = \mathbf{1}_{\{t<\tau_U\}}\, q(X_t) \exp[\int_t^{\tau_U} q(X_s)\, ds] u(X_{\tau_U})$$

By Fubini's Theorem we obtain

$$\begin{aligned}
& \int_0^\infty E^x[\Phi(t)]\, dt \\
= & \int_0^\infty E^x[t < \tau_U;\, q(X_t)\, \{e_q(\tau_U)\, u(X_{\tau_U})\} \circ \theta_t\,]\, dt \\
= & E^x[\int_0^{\tau_U} q(X_t)\, E^{X_t}[e_q(\tau_U)\, u(X_{\tau_U})]\, dt] \\
= & E^x[\int_0^{\tau_U} q(X_t)\, u(X_t)\, dt] \\
= & G_U(q\, u)(x)\,.
\end{aligned}$$

At the same time

$$\frac{d}{dt}\left(\exp[\int_t^{\tau_U} q(X_s)\, ds]\right) = -q(X_t)\, \exp[\int_t^{\tau_U} q(X_s)\, ds]$$

so we obtain

$$\begin{aligned}
& \int_0^\infty E^x[\Phi(t)]\, dt \;=\; E^x[\{e_q(\tau_U) - 1\}\, u(X_{\tau_U})] \\
= & E^x[e_q(\tau_U)\, u(X_{\tau_U})] \;-\; E^x[u(X_{\tau_U})] \\
= & u(x) \;-\; E^x[u(X_{\tau_U})]\,.
\end{aligned}$$

This shows the formula . Since u is bounded on U, $qu\mathbf{1}_U \in \mathcal{J}$. $G_U(q\,u)$ is continuous on U. This proves the first part of the theorem. Now, if f is a nonnegative bounded Borel measurable function on ∂D and $u(x) = E^x[e_q(\tau_D)\, f(X_{\tau_D})]$ then u is regular q-harmonic and in the above arguments we may replace U by D. If f is additionally continuous at ∂D then $u \in C(\overline{D})$.

3.9 Conditional Brownian motion

One of the most important tools when working with potential theory of Schrödinger operator is the Doob h-transform which we introduce here as a kind of "conditioning".

Let h be harmonic and positive on a bounded domain D; by $p_t^D(x,y)$ we denote the transition density function of (Y_t) killed on exiting D. **Definition.** For $x, y \in D$ and $t > 0$ we define

$$p_h^D(t;x,y) \;=\; h(x)^{-1} p^D(t;x,y)\, h(y)\,, \quad t>0\,, x,y \in D\,.$$

We first check that p_h^D is a density for a transition sub-probability. The semigroup property is fairly obvious:

$$\begin{aligned} & \int_D p_h^D(t;x,y)\, p_h^D(s;y,z)\, dy \\ = \;& h(x)^{-1} \int_D p^D(t;x,y)\, p^D(s;y,z)\, dy\, h(z) \\ = \;& h(x)^{-1} p^D(t+s;x,z)\, h(z) \;=\; p_h^D(t+s;x,z)\,. \end{aligned}$$

It remains to show that

$$\int_D p_h^D(t;x,y)\, dy \;\leqslant\; 1\,,$$

for all $x \in D$ and all $t > 0$ and this is stated in the lemma below.

Lemma. For any $t > 0$ and $x \in D$ we have

$$\mathbb{E}^x[t < \tau_D;\, h(X_t)] \;\leqslant\; h(x)\,.$$

Proof. Let D_n be a sequence of bounded open domains with the properties: $\overline{D}_n \subseteq D_{n+1} \subseteq D$ and $\bigcup D_n = D$. Since h is harmonic in D, we have $h(y) = \mathbb{E}^y[h(X_{\tau_{D_n}})]$, for $y \in D_n$. Thus, if $x \in D_n$

$$\begin{aligned} & \mathbb{E}^x[t < \tau_{D_n};\, h(X_t)] \;= \mathbb{E}^x[t < \tau_{D_n};\, \mathbb{E}^{X_t}[h(X_{\tau_{D_n}})]\,] \\ = \;& \mathbb{E}^x[t < \tau_{D_n};\, h(X_{\tau_{D_n}})] \leqslant \mathbb{E}^x[h(X_{\tau_{D_n}})] \;= h(x)\,. \end{aligned}$$

If now $n \to \infty$, we obtain the conclusion by monotone convergence.

According to the general theory of Markov processes p_h^D defines a Markov process on the state space $D_\partial = D \cup \{\partial\}$, where ∂ is the extra point ("cemetery") needed to define the transition probabilities. This proces is called h-conditioned Brownian motion. Its lifetime is defined as $T_\partial = \tau_D$. The process remains at ∂ in $[\tau_D, \infty)$. The h-conditioned process is still denoted by X_t, but we use notation P_h^x and $\mathbb{E}_h^x$ to indicate the corresponding probabilities and expectations. Thus, for any bounded Borel measurable function f on D we have

$$\begin{aligned} \mathbb{E}_h^x[f(X_t)] \;&=\; h(x)^{-1} \int_D p^D(t;x,y)\, h(y)\, f(y)\, dy \\ &=\; h(x)^{-1} \mathbb{E}^x[t < \tau_D;\, f(X_t)\, h(X_t)]\,. \end{aligned}$$

3.10 h-Brownian motion

We begin with some elementary properties of h-Brownian motion.

Proposition. Let $\Phi \geqslant 0$ be an $\mathcal{F}_t$-measurable function.
For every $t > 0$ and $x \in D$ we have

$$\mathbb{E}_h^x[t < \tau_D;\, \Phi] = h(x)^{-1}\mathbb{E}^x[t < \tau_D;\, \Phi\, h(X_t)]\,.$$

Proof. By a routine argument based on the monotone class theorem, it is enough to prove the above property when Φ is the indicator of the set $\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}$, where B_i are Borel subsets of D, $0 < t_1 < t_2 < \cdots < t_n \leqslant t$, $i = 1, \cdots, n$.

For $n = 1$, our property reduces to the previous one (definition) . Suppose that our property is true for $n-1$, $n \geqslant 2$. By the Markov property of our h-conditioned process and that of original Brownian motion process as well as induction hypothesis we obtain

$$\begin{aligned}
& P_h^x\{t < \tau_D, X_{t_1} \in B_1, X_{t_2-t_1} \circ \theta_{t_1} \in B_2, \cdots, X_{t_n-t_1} \circ \theta_{t_1} \in B_n\} \\
= \;& \mathbb{E}_h^x[t_1 < \tau_D,\, X_{t_1} \in B_1; \\
& P_h^{X_{t_1}}\{t - t_1 < \tau_D, X_{t_2-t_1} \in B_2, \cdots, X_{t_n-t_1} \in B_n\}] \\
= \;& h(x)^{-1}\mathbb{E}^x[t_1 < \tau_D, X_{t_1} \in B_1; \\
& \mathbb{E}^{X_{t_1}}[t - t_1 < \tau_D, X_{t_2-t_1} \in B_2, \cdots, X_{t_n-t_1} \in B_n; h(X_{t-t_1})]\,] \\
= \;& h(x)^{-1}\mathbb{E}^x[t < \tau_D, X_{t_1} \in B_1; X_{t_2} \in B_2, \cdots, X_{t_n} \in B_n; h(X_t)].
\end{aligned}$$

Now, let us recall that for a given Markov time τ by $\mathcal{F}_\tau$ we denote the family of all sets $A \in \mathcal{F}$ with the property $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$, for every $t > 0$. In our further considerations we will also use σ-field $\mathcal{F}_{\tau-}$ which is generated by $\mathcal{F}_{0+}$ and sets $A_t \cap \{\tau > t\}$, where $A_t \in \mathcal{F}_t$. It is easy to see that $\mathcal{F}_{\tau-} \subseteq \mathcal{F}_\tau$ and that functions $f_t \mathbf{1}_{(t,s]}(\tau)$ are measurable with respect to $\mathcal{F}_{\tau-}$, where f_t is $\mathcal{F}_t$-measurable and $0 < t < s$.

From this remark we obtain that X_τ is measurable with respect $\mathcal{F}_{\tau-}$, whenever (X_t) has left-continuous sample paths. Thus, $X_{\tau-}$ is measurable with respect to $\mathcal{F}_{\tau-}$, whenever (X_t) is "cadlag". Generalizing the previous statement we also obtain

Theorem. For any Markov time T and any $\mathcal{F}_T$-measurable function $\Phi \geqslant 0$

$$\mathbb{E}_h^x[T < \tau_D;\, \Phi] = h(x)^{-1}\mathbb{E}^x[T < \tau_D;\, \Phi\, h(X_T)]\,.$$

To choose a suitable conditioning we further assume that D is a **Lipschitz domain**, i.e. it can be represented locally as a part below the graph of a Lipschitz function. For such domains D and a fixed point $x_0 \in D$, there exists the unique function $K(x,z)$, $x \in D$, $z \in \partial D$, called **the Martin's kernel** which, for any fixed $z \in \partial D$ is a strictly positive harmonic function of $x \in D$, for any fixed $x \in D$, $K(x,\cdot)$ is a continuous function in ∂D, $K(x_0, z) = 1$, for all $z \in \partial D$ and $\lim_{D \ni x \to w} K(x,z) = 0$, for any $w \neq z \in \partial D$.

3.11 z-Brownian motion

For each $z \in \partial D$ the $K(\cdot, z)$-conditioned Brownian motion is called the z-Brownian motion and associated probability and expectation is denoted by P_z^x and $\mathbb{E}_z^x$. The z-Brownian motion plays a dominant rôle, since the process hits the boundary, on exiting from D . Sometimes we need another version of conditioned process, namely the y-Brownian motion, for $y \in D$. This is h-conditioned process, as before, but with $h(\cdot) = G(\cdot, y)$, where G is the Green function of D, instead of kernel K. The y-process is defined on $D \setminus \{y\}$.

In the sequel we write the z-version of corresponding theorems but the y-version is also available, with $\tau_{D\setminus\{y\}}$ instead of τ_D. The next theorem underlines the importance of this object in the study of Feynman-Kac semigroups:

Theorem.
For bounded Lipschitz domain $D \subset \mathbb{R}^d$ and $q \in \mathcal{J}$: If $\Phi \geqslant 0$, measurable with respect to $\mathcal{F}_{\tau_D -}$, $f \in \mathcal{B}_+(\partial D)$ and $x \in D$, then we have

$$\mathbb{E}^x[f(X_{\tau_D})\,\Phi] \;=\; \mathbb{E}^x[f(X_{\tau_D})\,\mathbb{E}^x_{X_{\tau_D}}[\Phi]\,]\,.$$

3.12 Conditional gauge

The previous theorem enables us to perform a kind of "separation of variables" which is exemplified as the next corollary:

Corollary.
Let D be a bounded Lipschitz domain and $q \in \mathcal{J}$. For any $f \in \mathcal{B}_+(\partial D)$ we have

$$\mathbb{E}^x[e_q(\tau_D)\,f(X_{\tau_D})] \;=\; \mathbb{E}^x[\mathbb{E}^x_{X_{\tau_D}}[e_q(\tau_D)]\,f(X_{\tau_D})\,]\,.$$

Now, the function

$$u(x,z) \;=\; \mathbb{E}^x_z[e_q(\tau_D)]\,, \quad (x,z) \in D \times \partial D\,,$$

will be called the **conditional gauge**.

If we know that the function u is bounded over domain D, then the potential theory of the Schrödinger operator can be deduced from the existing potential theory of the Laplace operator. This, however, requires a more advanced theory. An indispensable tool here is the so-called Boundary Harnack Principle.

3.13 Boundary Harnack Principle (BHP)

Boundary Harnack Principle
Let D be a Lipschitz domain in $\mathbb{R}^d$, $d \geqslant 2$, and V be an open set. For every compact set $K \subset V$, there exists a positive constant $C = C(\alpha, D, V, K)$, such that for all nonnegative functions u and v in $\mathbf{R}^d$, which are continuous in V, α-harmonic in $D \cap V$, vanish on $D^c \cap V$, and satisfy $u(x_0) = v(x_0)$ for some $x_0 \in D \cap K$, we have

$$C^{-1}u(x) \leqslant v(x) \leqslant Cu(x), \quad x \in D \cap K.$$

BHP is an important tool in excursion theory and potential theory. It was first proved for Laplacian and Lipschitz domains in 1977 by Dahlberg. Later, this result was generalized to other elliptic operators and a methodology of such proofs, based on estimates for the Green function, was given.

3.14 $3G$ Theorem

$3G$ **Theorem**
Let D be a bounded Lipschitz domain, $D \subseteq \mathbb{R}^d$, $d \geqslant 3$, and G_D its Green function. There exists C such that for every $x, y, z \in D$ the following holds

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leqslant C\left(|x-y|^{2-d} + |y-z|^{2-d}\right)$$

and

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leqslant C\left(\frac{|x-z|}{|x-y||y-z|}\right)^{d-2}.$$

A version for the z-Brownian motion (conditioning by Martin's kernel):
$3G$ **Theorem**
Under the same assumptions we obtain

$$\frac{G_D(x,y)K(y,z)}{K(x,z)} \leqslant C\left(|x-y|^{2-d} + |y-z|^{2-d}\right)$$

Applications
Green function of z-Brownian motion

$$(G_D)_z(x,y) = \frac{G_D(x,y)K(y,z)}{K(x,z)}$$

3.15 Green function of z-Brownian motion

Proof. We compute the Green operator for the process conditioned by $X_{\tau_D} = z$, $z \in \partial D$. For a bounded Borel f we obtain

$$\begin{aligned}
(G_D)_z f(x) = \int_0^\infty \mathbb{E}_z^x f(X_t^D)dt &= \int_0^\infty \mathbb{E}_z^x[t < \tau_D; f(X_t)]dt \\
= \quad & K(x,z)^{-1}\int_0^\infty \mathbb{E}^x[t<\tau_D; f(X_t)K(X_t,z)]dt \\
= \quad & K(x,z)^{-1}G_D f(x)K(\cdot,z) \\
= \quad & K(x,z)^{-1}\int_D G_D(x,y)f(y)K(y,z)dy\,.
\end{aligned}$$

Analogously we obtain the y-Green function:
Green function of y-Brownian motion

$$(G_D)_y(x,u) = \frac{G_D(x,u)G_D(u,y)}{G_D(x,y)}\,.$$

3.16 Conditional Gauge Teorem (CGT)

Theorem (Conditional Gauge Theorem)
Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geqslant 2$, and $q \in \mathcal{J}_{loc}$. If (D,q) is gaugeable then $u(x,y)$ is symmetric in $D \times D$ and bounded in $D \times \overline{D}$ for $x \neq y$.

Applying **(CGT)** and the usual Harnack Inequality we obtain
Theorem (Harnack Principle)
There are constants C_1 and C_2 such that for every ball $B(x, 2r)$ with $r < C_2$ and every function $u \geqslant 0$, q-harmonic in $B(x, 2r)$ we have

$$u(y) \leqslant C_1 u(x)\,, \quad y \in B(x, r)\,.$$

In a similar way we derive the boundary Harnack principle for nonnegative q-harmonic functions and Lipschitz domains.

3.17 Potential theory of the operator $\mathcal{S}$

Theorem (Boundary Harnack Principle)
There exist constants C_3 and R such that for all $\xi \in \partial D$, $r \in (0, R/2)$ and nontrivial functions $u, v \geqslant 0$, q-harmonic in $D \cap B(\xi, 2r)$ which vanish continuously in $D^c \cap B(\xi, 2r)$ we have

$$u(x)/v(x) \leqslant C_3 u(y)/v(y)\,, \quad x, y \in D \cap B(\xi, r)$$

and $\lim u(x)/v(x)$ exists as $D \ni x \to \xi$.
Proof. For $r > 0$ small enough every Lipschitz domain D_r such that $D \cap B(\xi, r/R) \subseteq D_r \subseteq D \cap B(\xi, r)$ with $R = R(D)$ is gaugeable with the corresponding constants C_1/C_2 and C_2/C_1. By **BHP** and Harnack chain inequality for harmonic functions , the present result follows.

3.18 Bibliographical notes

Much of the material is taken from [3]. The paper [4] is recommended for people interested in derivation of 3G Theorem. Readers interested in Boundary Harnack Principle are referred to the paper [6]. Paper [5] contains historically the first proof of this theorem.

3.18.1 Bibliography

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4 Potential theory of isotropic Lévy stable processes

4.1 $(\tau_a)_{a>0}$ is Lévy stochastic process

- $(X(t))_{t\geqslant 0}$ is called **Lévy process (motion)**, when it has the following properties:
- **independent increments**: for disjoint intervals $(t_1,t_2),\ldots,(t_{n-1},t_n)$, **increments** $X_{t_2}-X_{t_1},\ldots,X_{t_n}-X_{t_{n-1}}$ **are independent random vectors**;
- **homogeneity**: **distribution of the increment** $X_{t+s}-X_t$ does not depend on s and is **the same** as the distribution of X_s.

We recall:

$$\tau_a := \tau_{(-\infty,a)} = \inf\{t>0; W_t>a\}\,.$$

Theorem. $(\tau_a)_{a>0}$ is Lévy stochastic process

Proof. Let $0<a<b$. Then $\tau_a<\tau_b$ so $\tau_b=\tau_a+\tau_b\circ\theta_{\tau_a}$, P^0 a.e. hence we obtain for Borel sets A, B:

$$\begin{aligned}
&P^0(\tau_a\in A;\tau_b-\tau_a\in B)=P^0(\tau_a\in A;P^0(\tau_b-\tau_a\in B|\mathcal{F}_{\tau_a}))\\
=\ &P^0(\tau_a\in A;P^0(\tau_b\circ\theta_{\tau_a}\in B|\mathcal{F}_{\tau_a}))=P^0(\tau_a\in A;P^{W_{\tau_a}}(\tau_b\in B))\\
=\ &P^0(\tau_a\in A)\,P^a(\tau_b\in B)=P^0(\tau_a\in A)\,P^0(\tau_{b-a}\in B)\,.
\end{aligned}$$

Thus, $(\tau_a)_{a>0}$ is a Lévy process (homogenous, with independent increments), $\tau_0=0$. It is an asymmetric 1/2-stable stochastic process with increasing sample paths; i.e. **stable subordinator**. Analogous calculations are valid for multi-dimensional increments.

4.2 Isotropic Cauchy process

We define the following subordination of Brownian motion $(B_t)_{t>0}$ by the independent process $(\tau_a)_{a>0}$:

$$Y_a=B_{\tau_a}\,.$$

Y_a **- homogeneous isotropic Cauchy process**
One-dimensional distributions:

$$\begin{aligned}
&P^0(B_{\tau_a}\in A)=\int_0^\infty P^0(B_s\in A)\,\frac{a\,e^{-a^2/2s}}{\sqrt{2\pi s^3}}\,ds\\
=\ &\int_0^\infty\int_A\frac{e^{-y^2/2s}}{\sqrt{2\pi s}}\,dy\,\frac{a\,e^{-a^2/2s}}{\sqrt{2\pi s^3}}\,ds=\int_0^\infty\int_A\frac{e^{-(a^2+y^2)/2s}}{2\pi s^2}\,dy\,ds\\
=\ &\int_A\frac{1}{\pi}\frac{a}{a^2+y^2}\int_0^\infty e^{-u}\,du\,dy=\frac{1}{\pi}\int_A\frac{a}{a^2+y^2}\,dy\,.
\end{aligned}$$

4.3 β-stable subordinators

β-stable subordinator: 1-dimensional β-stable Lévy motion $(\eta_t^{(\beta)})$ given by its Laplace transform:

$$\mathbb{E}^0e^{-u\eta_t^{(\beta)}}=e^{-tu^\beta}\,,\quad 0<\beta<1\,,\quad u>0\,,$$

$\mathbb{E}^x$ - expectation of the process starting from $x \in \mathbb{R}$.

Denote by $\theta_\beta(t,x)$ the transition density of $\eta_t^{(\beta)}$. We have
Potential of the subordinator:

$$\int_0^\infty \theta_\beta(t,x)\,dt = \frac{x^{\beta-1}}{\Gamma(\beta)}\,, \quad x > 0\,.$$

Justification : Apply Laplace transform for both sides.

Potential operator of $\eta_t^{(\beta)}$:

$$\tilde{U}_\beta f(x) = \mathbb{E}^x \int_0^\infty f(\eta_t^{(\beta)})\,dt = \frac{1}{\Gamma(\beta)} \int_x^\infty (y-x)^{\beta-1} f(y)\,dy\,.$$

4.4 Isotropic stable Lévy motion

Stable subordination

Let $B = (B_t)$ be a d-dimensional Brownian motion, $(B_t) = (B_t^1, \ldots, B_t^d)$, $\mathbb{E}^0 B_t^i = 0, \mathbb{E}^0 (B_t^i)^2 = 2t$. We put $g_t(x)$, $x \in \mathbb{R}^d$, $t > 0$, be the density of B_t:

$$g_t(x) = \frac{1}{(4\pi t)^{d/2}}\, e^{-|x|^2/4t} \quad \text{hence} \quad \mathbb{E}^0 e^{i(\xi, B_t)} = e^{-t|\xi|^2}\,.$$

Isotropic α-stable Lévy motion

For $0 < \alpha < 2$ we define isotropic ("symmetric") α-stable Lévy motion Y_t in $\mathbb{R}^d$ by the formula

$$(Y_t) = (B_{\eta_t^{(\alpha/2)}})\,,$$

with B and $\eta^{(\alpha/2)}$ independent.

4.5 Characteristic function of stable Lévy motion

Characteristic function Y_t

$$\mathbb{E}^0 e^{i(Y_t, \xi)} = e^{-t|\xi|^\alpha}\,.$$

Fractional Laplace operator: generator of Y

$$\begin{aligned} \Delta^{\alpha/2} u(x) &= \lim_{t \downarrow 0} \frac{\mathbb{E}^x u(Y_t) - u(x)}{t} \\ &= \mathcal{A}(d,-\alpha) \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}} dy\,, \quad u \in C_b^2(\mathbb{R}^d)\,, \end{aligned}$$

$\mathcal{A}(d,\gamma) = \Gamma((d-\gamma)/2)/(2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|)$.

4.6 Riesz potentials

Riesz potential U_α: for $\alpha < d$ we define on $\mathbb{R}^d$:

$$U_\alpha(z) = \mathcal{A}(d,\alpha)\,\frac{1}{|z|^{d-\alpha}}\,,$$

where $\mathcal{A}(d,\alpha) = \Gamma((d-\alpha)/2)/(2^\alpha \pi^{d/2}\Gamma(\alpha/2))$.

Computing the potential operator for the process Y we obtain:

$$\begin{aligned}\int_0^\infty \mathbb{E}^x f(Y_t)dt &= \int_{\mathbb{R}^d}\int_0^\infty\int_0^\infty f(x+y)g_u(y)\,\theta_{\alpha/2}(t,u)dudydt\\ &= \frac{1}{2^d\pi^{d/2}\Gamma(\alpha/2)}\int_{\mathbb{R}^d} f(z)\int_0^\infty \frac{e^{-|x-z|^2/4u}}{u^{\frac{d-\alpha}{2}+1}}\,du\,dz\\ &= \mathcal{A}(d,\alpha)\int_{\mathbb{R}^d}\frac{f(z)}{|x-z|^{d-\alpha}}\,dz = U_\alpha f(x)\,.\end{aligned}$$

Thus, the Riesz potential is the **kernel** of the **potential operator** for Y.

4.7 α-harmonicity

α-harmonicity

A Borel function f on $\mathbb{R}^d$ is called α-harmonic in (open) set D iff it is continuous in D and

$$\Delta^{\alpha/2} f = 0 \quad \text{in} \quad D\,.$$

Fractional laplasjan $\Delta^{\alpha/2}$ is understood in the distributional sense. Potential theory $\Delta^{\alpha/2}$ **is not local!** $f, g, \ldots$ are **always** defined on the whole of $\mathbb{R}^d$ and are always assumed to be Borel measurable.

Equivalently - averaging property

A Borel function f on $\mathbb{R}^d$ is called α-harmonic in (open) set D if for every open $U \subseteq \mathbb{R}^d$ such that $\overline{U} \subseteq D$ the following holds

$$f(x) = \mathbb{E}^x f(Y_{\tau_U})\,.$$

4.8 Poisson kernels

An exquisite feature of the isotropic stable process is that there are explict formulas both for Poisson kernels and Green functions of balls and halfspaces. Their derivation is, however, much more difficult and requires a lot of clever calculations with the use of a special version of Kelvin's transform.

Poisson kernel of $B(0,r)$

For $|x| < r$, $|y| > r$

$$P_r(x,y) = \frac{\Gamma(d/2)\sin \pi\alpha/2}{\pi^{1+d/2}}\left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^{\alpha/2}\frac{1}{|x-y|^d}\,.$$

Poisson kernel of $\mathbb{H}$

For $\quad y_d < 0 < x_d$

$$P_H(x,y) = \frac{\Gamma(d/2)\sin\pi\alpha/2}{\pi^{1+d/2}}\left(\frac{x_d}{-y_d}\right)^{\alpha/2}\frac{1}{|x-y|^d}\,.$$

4.9 Green functions

Green function of $B(0,1)$

For $|x| < 1$ and $|y| < 1$

$$G(x,y) = \frac{\Gamma(d/2)}{2^\alpha\pi^{d/2}[\Gamma(\alpha/2)]^2}\frac{1}{|x-y|^{d-\alpha}}\int_0^{\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}}\frac{r^{(\alpha/2)-1}}{(r+1)^{d/2}}dr\,.$$

Green function of $\mathbb{H}$

For $x_d > 0$, $y_d > 0$ the following holds

$$G_{\mathbb{H}} = \frac{\Gamma(d/2)}{2^\alpha\pi^{d/2}[\Gamma(\alpha/2)]^2}\frac{1}{|x-y|^{d-\alpha}}\int_0^{\frac{4x_dy_d}{|x-y|^2}}\frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{\frac{d}{2}}}dt.$$

4.10 Derivation of Poisson kernel of $B(0,1)$ for $\alpha < d$

The following theorem is essential in obtaining the explicit form of the Poisson kernel for balls. Its proof consists of calculations in spherical coordinates and is enclosed in the end of this section (Appendix).

Theorem. For $|y| < r$ we have

$$I(y) = C_\alpha^d\int_{|z|\leqslant r}\frac{dz}{(r^2-|z|^2)^{\alpha/2}|z-y|^{d-\alpha}} = 1\,,\quad C_\alpha^d = \frac{\Gamma(d/2)\sin(\pi\alpha/2)}{\pi^{1+d/2}}\,. \tag{1}$$

For fixed $x \in \mathbb{R}^d$, $|x| < r$, and $r > 0$ we define the inversion $u \to u^*$ with respect to the sphere with center x and radius $\sqrt{r^2-|x|^2}$ (Kelvin's transform) as follows:

Kelvin's transform

$$u^* = x + \frac{(r^2-|x|^2)(u-x)}{|u-x|^2}\,. \tag{2}$$

Directly from the definition we obtain

$$|u^*-x||u-x| = r^2-|x|^2 \tag{3}$$

Lemma. For every $u, y \in \mathbb{R}^d$, $|u| < r$ and $|x| < r$:

$$\frac{|y^* - u^*|}{|y-u|} = \frac{|u* - x|}{|y-x|}\,, \tag{4}$$

$$|u^*|^2 = r^2 + \frac{(r^2-|x|^2)(r^2-|u|^2)}{|u-x|^2}\,, \tag{5}$$

$$\frac{du^*}{|u^*-y|^d} = \frac{du}{|u-y|^d}\,. \tag{6}$$

In the sequel we put $r = 1$ and define

$$f(x,y) = C_\alpha^d \left|\frac{|1-|x|^2|}{|1-|y|^2|}\right|^{\alpha/2} \frac{1}{|x-y|^d} \tag{7}$$

Proof of the Lemma. Writing the defining formula (2) for y^* and for u^* and taking $|y^* - u^*|^2$ we obtain

$$\begin{aligned}
\frac{|y^*-u^*|^2}{(r^2-|x|^2)^2} &= \left|\frac{y-x}{|y-x|^2} - \frac{u-x}{|u-x|^2}\right|^2 \\
&= \frac{1}{|y-x|^2} - 2\frac{(y-x,u-x)}{|y-x|^2|u-x|^2} + \frac{1}{|u-x|^2} \\
&= \frac{|u-x|^2 + 2(y-x,u-x) + |y-x|^2}{|y-x|^2|u-x|^2} \\
&= \frac{|(u-x)-(y-x)|^2}{|y-x|^2|u-x|^2} = \frac{|u-y|^2}{|y-x|^2|u-x|^2}\,.
\end{aligned}$$

Taking into account the property (3), we obtain the first formula stated in the lemma.

Now, using again the defining formula, we take squares of the norms at both sides of the formula (2) and after writing the right-hand side with the common denominator, we obtain, after multiplication by this denominator

$$\begin{aligned}
&|u^*|^2|u-x|^2 \\
=\ & |x|^2|u-x|^2 + (2|x|^2 - 2(u,x))(r^2-|x|^2) + (r^2-|x|^2)^2 \\
=\ & |x|^2(|u|^2 - 2(u,x) + |x|^2) + (|x|^2 + r^2 - 2(u,x))(r^2-|x|^2) \\
=\ & |x|^2(|u|^2 - r^2) + (|x|^2 + r^2 - 2(u,x))r^2 \\
=\ & |x|^2|u|^2 + r^4 - |x|^2r^2 - |u|^2r^2 + |u-x|^2r^2 \\
=\ & (r^2-|x|^2)(r^2-|u|^2) + |u-x|^2r^2\,,
\end{aligned}$$

which completes the proof of the formula (5).

To prove the formula (6), it is enough to show that Kelvin's transform maps balls onto balls with appropriate radius. We indicate how to prove the corresponding statement for spheres. Let $S = \{u : |u-u_1| = t\}$. Using the formula (4) we obtain for $u \in S$:

$$\frac{|u^*-u_1^*|}{|u-u_1|} = \frac{|u^*-y|}{|u_1-y|} = \lambda|u^*-y|\,,$$

with $\lambda = t/(|u_1 - y|)$. After some calculations we obtain the equation of the sphere S^* with the center u_λ and radius R_λ where

$$u_\lambda = \frac{u_1^\lambda - \lambda^2 y}{1-\lambda^2}\,, \quad R_\lambda = \frac{\lambda}{1-\lambda^2}\,|u_1 - y|^2\,.$$

With an extra effort we can show that for small λ the ball inside S is transformed onto ball inside S^*. Taking the quotient of volumes of the corresponding balls and letting $t \to 0$ we obtain the jacobian of the transformation $u \to u^*$ as follows:

$$\lim_{t\to 0} \frac{R_\lambda^d}{t^d} = \lim_{t\to 0} \frac{t^d}{t^d\,(1-\frac{t^2}{|u_1-y|^2})^d}\,\frac{|u^*-y|^d}{|u_1-y|^d} = \frac{|u^*-y|^d}{|u_1-y|^d}\,,$$

which proves the formula (6).

We now assume that $r = 1$ and define λ to be

$$\lambda(u) \quad = \quad \begin{cases} (1-|u|^2)^{-\alpha/2}, & |u| < 1; \\ 0, & |u| \geqslant 1. \end{cases}$$

We define Kelvin's transform of the measure λ by the formula

$$d\lambda^*(u^*) = \left(\frac{|u-x|}{\rho}\right)^{\alpha-d} d\lambda(u)\,.$$

Let $|x| < 1$. We consider inversion $u \to u^*$ with respect to sphere with center x and radius $\rho = \sqrt{1-|x|^2}$. Kelvin's transform of the measure λ is the measure λ^*:

$$d\lambda^*(u^*) = \left(\frac{|u-x|}{\rho}\right)^{\alpha-d} d\lambda(u) \overset{(7)}{=} \rho^{d-\alpha}\frac{f(x,u^*)}{C_\alpha^d}\,du^*\,, \quad |u^*| > 1.$$

$$\begin{aligned} d\lambda^*(u^*) \quad &= \quad \left(\frac{|u-x|}{\rho}\right)^{\alpha-d} d\lambda(u) = \frac{|u-x|^\alpha \rho^{d-\alpha}}{(1-|u|^2)^{\alpha/2}}\frac{du}{|u-x|^d} \\ &\overset{(6)}{=} \quad \frac{|u-x|^\alpha \rho^{d-\alpha}}{(1-|u|^2)^{\alpha/2}}\frac{du^*}{|u^*-x|^d} \\ &= \quad \left(\frac{|u-x|^2}{(1-|u|^2)(1-|x|^2)}\right)^{\alpha/2} \frac{\rho^d}{|u^*-x|^d}du^* \\ &\overset{(5)}{=} \quad \frac{\rho^d}{(|u^*|^2-1)^{\alpha/2}|u^*-x|^d}du^* \\ &= \quad \rho^{d-\alpha}\left(\frac{1-|x|^2}{|u^*|^2-1}\right)^{\alpha/2} \frac{du^*}{|u^*-x|^d} = \rho^{d-\alpha}\frac{f(x,u^*)}{C_\alpha^d}du^*. \end{aligned}$$

Hence

$$U_\alpha^{\lambda^*}(y^*) = \mathcal{A}(d,\alpha)\int_{\mathbb{R}^d} \frac{d\lambda^*(u^*)}{|u^*-y^*|^{d-\alpha}}$$

$$\begin{aligned} &= \quad \mathcal{A}(d,\alpha)\int_{|u^*|\geqslant 1} \frac{\rho^{d-\alpha}f(x,u^*)}{C_\alpha^d|u^*-y^*|^{d-\alpha}}du^* \\ &= \quad \frac{\rho^{d-\alpha}\mathcal{A}(d,\alpha)}{C_\alpha^d}\int_{|u^*|\geqslant 1} \frac{f(x,u^*)}{|u^*-y^*|^{d-\alpha}}du^*. \end{aligned} \tag{8}$$

On the other hand,

$$U_\alpha^{\lambda^*}(y^*) = \left(\frac{|y-x|}{\rho}\right)^{d-\alpha} U_\alpha^\lambda(y)\,.$$

Indeed, we obtain

$$\begin{aligned} U_\alpha^{\lambda^*}(y^*) &= \mathcal{A}(d,\alpha)\int_{\mathbb{R}^d}\frac{d\lambda^*(u^*)}{|u^*-y^*|^{d-\alpha}} \\ &= \mathcal{A}(d,\alpha)\int_{|u|\leqslant 1}\left(\frac{|u-x|}{\rho}\right)^{\alpha-d}\frac{1}{(1-|u|^2)^{\alpha/2}}\frac{du}{|u^*-y^*|^{d-\alpha}} \end{aligned}$$

$$\begin{aligned} &\overset{(4)}{=} \mathcal{A}(d,\alpha)\int_{|u|\leqslant 1}\left(\frac{\rho}{|u-x|}\right)^{d-\alpha}\left(\frac{|y-x|}{|y-u||u^*-x|}\right)^{d-\alpha} d\lambda(u) \\ &\overset{(3)}{=} \mathcal{A}(d,\alpha)\int_{|u|\leqslant 1}\rho^{d-\alpha}\left(\frac{|y-x|}{|y-u|\,\rho^2}\right)^{d-\alpha} d\lambda(u) \\ &= \left(\frac{|y-x|}{\rho}\right)^{d-\alpha} U_\alpha^\lambda(y)\,. \end{aligned}$$

Since $|y-x||y^*-x| = \rho^2$ we thus obtain

$$U_\alpha^{\lambda^*}(y^*) = \left(\frac{\rho}{|y^*-x|}\right)^{d-\alpha} U_\alpha^\lambda(y)\,.$$

Note that for $|y|\leqslant 1$ the following holds

$$\begin{aligned} U_\alpha^\lambda(y) &= \mathcal{A}(d,\alpha)\int_{\mathbb{R}^d}\frac{d\lambda(u)}{|u-y|^{d-\alpha}} \\ &= \mathcal{A}(d,\alpha)\int_{|u|\leqslant 1}\frac{du}{(1-|u|^2)^{\alpha/2}|u-y|^{d-\alpha}} \overset{(1)}{=} \frac{\mathcal{A}(d,\alpha)}{C_\alpha^d}. \end{aligned}$$

Transforming the above formula we obtain

$$U_\alpha^{\lambda^*}(y^*) \overset{(1)}{=} \frac{\rho^{d-\alpha}}{|y^*-x|^{d-\alpha}}\frac{\mathcal{A}(d,\alpha)}{C_\alpha^d}. \tag{9}$$

Now, we say that a finite measure ν, concentrated on a closed subset A of $\mathbb{R}^d$ has the finite energy if

$$\int_A\int_A |u-y|^{\alpha-d}|\nu|(du)\,|\nu|(dy) < \infty\,.$$

If now $|y|\leqslant 1$, then using once again Lemma, formula (5), we see that $|y^*|\geqslant 1$, so from the formulas (8) and (9) we obtain that for $|x|\leqslant 1\leqslant |y^*|$ we get

$$\int_{|u^*|\geqslant 1}\frac{f(x,u^*)du^*}{|u^*-y^*|^{d-\alpha}} = \frac{1}{|y^*-x|^{d-\alpha}}\,.$$

This, however, proves our theorem, in view of the following

Sweeping out Principle: Uniqueness

Suppose that for a $x \in B(0,1)$ and a measure ν with finite energy, concentrated on $B(0,1)^c$, we obtain for all $|z| > 1$

$$\int_{\mathbb{R}^d} \frac{d\nu(y)}{|z-y|^{d-\alpha}} = \frac{1}{|z-x|^{d-\alpha}}\,.$$

Then $\nu = \nu_x$ is the value of the hitting distribution $X_{\tau_{B(0,1)}}$ with the starting point x.

4.11 Harnack Inequality

Harnack Inequality for a ball

Let f be a nonnegative, α-harmonic function in $B(0,r)$.Then

$$f(x) \leqslant \left(\frac{r^2-|x|^2}{r^2-|y|^2}\right)^{\alpha/2} \left(\frac{r-|x|}{r+|y|}\right)^{-d} f(y)\,, \quad x,y \in B(0,r)\,.$$

Harnack Inequality for compact subsets of $K \subseteq D$

Let f,g be nonnegative α-harmonic functions in D and such that $f(x_0) = g(x_0)$ for some $x_0 \in K \subseteq D$.Then

$$cf(y) \leqslant g(y) \leqslant c^{-1} f(y)\,, \quad y \in K\,.$$

4.12 Boundary Harnack Principle (BHP)

BHP
Assume that $\alpha \in (0,2)$, $d \geqslant 2$. Let D be a Lipschitz domain in $\mathbb{R}^d$ and V. For every compact $K \subset V$ there exist $C = C(\alpha, D, V, K)$ such that for every nonnegative functions u, v that are continuous on V, α-harmonic in $D \cap K$,vanish on $D^c \cap V$ and satisfy $u(x_0) = v(x_0)$ for some $x_0 \in D \cap K$ the following holds:

$$C^{-1} u(x) \leqslant v(x) \leqslant C\,u(x), \quad x \in D \cap K\,.$$

($\alpha = 2$: B. Dahlberg 1977; $\alpha \in (0,2)$: K. Bogdan 1997, T.B.& K. Bogdan 1999, K. Bogdan, T. Kulczycki, M. Kwaśnicki 2008)

4.13 Martin's Representation

Martin's Representation (K. Bogdan 1999)
Let D be a Lipschitz domain in $\mathbb{R}^d$. For every nonnegative finite Borel measure μ on ∂D the function

$$u(x) = \int_{\partial D} K(x,Q)\,\mu(dQ)\,, \quad x \in \mathbb{R}^d$$

is singular α-harmonic on D.Moreover, for every $u \geqslant 0$, singular α-harmonic on D there exists unique nonnegative finite Borel measure μ na D such that the above representation holds.

The existence of Martin's kernel follows from the Boundary Harnack Principle. Indeed, the kernel K is defined as

$$K(x,Q) = \lim_{y \to Q \ni \partial D} \frac{G_D(x,y)}{G_D(x_0,y)}$$

and the existence of the limit above is a direct consequence of the Boundary Harnack Principle.

4.14 Kato class $\mathcal{J}^\alpha$

We say that a Borel function q belongs to the **Kato class** $\mathcal{J}^\alpha$ if q satisfies either of the two equivalent conditions:

$$(i) \qquad \lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leqslant r} |q(y)|\, U_\alpha(x-y)\, dy = 0\,,$$

$$(ii) \qquad \lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_0^t P_s|q|(x)\, ds = 0\,.$$

We write $q \in \mathcal{J}^\alpha{}_{loc}$ if for every bounded Borel subset $B \subseteq \mathbb{R}^d$ holds $\mathbf{1}_B\, q \in \mathcal{J}^\alpha$.

If $q \in \mathcal{J}^\alpha$ then $\sup_{x\in\mathbb{R}^d} \int_{|x-y|<1} |q(y)|\, dy < \infty$; in particular, $\mathcal{J}^\alpha_{loc} \subseteq L^1_{loc}$. If $f \in L^\infty(\mathbb{R}^d)$ then $f, f\, q \in \mathcal{J}^\alpha$.

Let D be a domain in $\mathbb{R}^d$ and $q \in \mathcal{J}^\alpha$. For $t > 0$ we denote

$$A(t) = \int_0^t q(X_s)\, ds \quad \text{and} \quad e_q(t) \;=\; \exp\left(\int_0^t q(X_s)\, ds\right)\,.$$

We call A an **additive** and e_q **multiplicative functionals**, respectively: $e_q(t+s) = e_q(t)\,(e_q(s)\circ \theta_t)$. We will always assume that D is either bounded or of finite Lebesgue measure.

4.15 Gauge function

The function

$$u(x) \;=\; \mathbb{E}^x e_q(\tau_D)\,,$$

is called the **gauge** (function) for (D, q), and when it is bounded in D, we say that (D, q) is **gaugeable**. The gauge u is the principal object of the potential theory of the Schrödinger operator $\Delta^{\alpha/2} + q$ on D.

Khasminski Lemma.
Let τ be a stopping time of the process X satisfying for every $t > 0$ the condition

$$\tau \leqslant t + \tau\circ\theta_t \quad \text{on} \quad \{t < \tau\}\,.$$

Suppose that $q \geqslant 0$ and $\mathbb{E}^x[A(\tau)] < \infty$ for all x. If

$$\sup_x \mathbb{E}^x[A(\tau)] = \delta < 1$$

then

$$\sup_x \mathbb{E}^x[e^{A(\tau)}] \leqslant \frac{1}{1-\delta}\,.$$

Gauge Theorem (K.L.Chung, Z. Zhao)

Let D be a bounded domain in $\mathbb{R}^d$ and $q \in \mathcal{J}^\alpha$. If $u(x) < \infty$ for some $x \in D$ then $u \in L^\infty(D)$.

Let u be a Borel function on a domain D. It is called **q-harmonic** if for every open subset $U \subset\subset D$ we have

$$u(x) = \mathbb{E}^x[\tau_U < \infty; e_q(\tau_U)\, u(X_{\tau_U})]\,, \quad x \in U\,.$$

The following property shows that q-harmonic functions are solutions of the Schrödinger equation $\Delta^{\alpha/2} + q = 0$ in D.

Theorem. For a domain $D \subset \mathbb{R}^d$ and $q \in \mathcal{J}^\alpha_{loc}(D)$ we have:

If u is q-harmonic in D then for every open bounded U with the exterior cone property such that $\overline{U} \subset D$ we obtain

$$u(x) = \mathbb{E}^x u(X_{\tau_U}) + G_U(qu)(x)\, a.e.$$

with the right-hand side continuous.

4.16 z-stable motion

For each $z \in \partial D$ the $K(\cdot, z)$-conditioned Lévy stable motion is called the z-stable motion and associated probability and expectation is denoted by P^x_z and $\mathbb{E}^x_z$.

Unlike the Brownian motion, isotropic stable process does not hit the boundary, when exiting D, at least for regular D. Therefore, the dominant rôle here is played by the y-stable Lévy motion, with $y \in D$. This is h-conditioned process, as before, but with $h(\cdot) = G(\cdot, y)$, where G is the Green function of D, instead of kernel K. The y-process is defined on $D \setminus \{y\}$.

In the sequel we write y-versions of corresponding theorems but the z-version is also available, with τ_D instead of $\tau_{D\setminus\{y\}}$. Although we write modifications of statements valid for Brownian motion, proofs are usually different, due to non-locality of corresponding operators.

Theorem. For bounded Lipschitz domain $D \subset \mathbb{R}^d$ and $q \in \mathcal{J}^\alpha$:

If $\Phi \geqslant 0$, measurable with respect to $\mathcal{F}_{\tau_D-}$, $f \in \mathcal{B}_+(D^c)$ and $x \in D$, then we have

$$\mathbb{E}^x[f(X_{\tau_D})\,\Phi] \;=\; \mathbb{E}^x[f(X_{\tau_D})\,\mathbb{E}^x_{X_{\tau_{D-}}}[\Phi]\,]\,.$$

4.17 Conditional gauge

The previous theorem enables us to perform a kind of "separation of variables" which we state as the next corollary:

Corollary.

Let D be a bounded Lipschitz domain and $q \in \mathcal{J}^\alpha$. For any $f \in \mathcal{B}_+(D^c)$ we have

$$\mathbb{E}^x[e_q(\tau_D)\, f(X_{\tau_D})] \;=\; \mathbb{E}^x[\mathbb{E}^x_{X_{\tau_{D-}}}[e_q(\zeta)]\, f(X_{\tau_D})\,]\,.$$

Here ζ is equal τ_D for z-conditioning and $\tau_{D\setminus\{y\}}$ for y-conditioning.

Now, the function

$$u(x, z) \;=\; \mathbb{E}^x_z[e_q(\zeta)]\,, \quad (x, z) \;\in D \times \overline{D}\,,$$

will be called the **conditional gauge**.

If we know that the function u is bounded over $D \times D$, then the potential theory of the Schrödinger operator $\Delta^{\alpha/2}+q$ can be deduced from the existing potential theory of the fractional Laplace operator. This, however, requires a more advanced theory. An indispensable tool here is the so-called Conditional Gauge Theorem for the fractional Laplacian.

4.18 Conditional Gauge Teorem (CGT)

Theorem (Conditional Gauge Theorem)
Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geqslant 2$, and $q \in \mathcal{J}^{\alpha}_{loc}$. If (D, q) is gaugeable then $u(x, y)$ is symmetric and bounded in $D \times D$ for $x \neq y$.

Applying **(CGT)** and the usual Harnack Inequality we obtain
Theorem (Harnack Principle)
There are constants C_1 and C_2 such that for every ball $B(x, 2r)$ with $r < C_2$ and every function $u \geqslant 0$, q-harmonic in $B(x, 2r)$ we have

$$u(y) \leqslant C_1 \, u(x) \,, \quad y \in B(x, r) \,.$$

In a similar way we derive the boundary Harnack principle for nonnegative q-harmonic functions and Lipschitz domains.

4.19 Potential theory of the operator $\mathcal{S}^{\alpha}$

Theorem (Boundary Harnack Principle)
There exist constants C_3 and R such that for all $\xi \in \partial D$, $r \in (0, R/2)$ and nontrivial functions $u, v \geqslant 0$, q-harmonic in $D \cap B(\xi, 2r)$ which vanish continuously in $D^c \cap B(\xi, 2r)$ we have

$$u(x)/v(x) \leqslant C_3 u(y)/v(y) \,, \quad x, y \in D \cap B(\xi, r)$$

and $\lim u(x)/v(x)$ exists as $D \ni x \to \xi$.
Proof. For $r > 0$ small enough every Lipschitz domain D_r such that $D \cap B(\xi, r/R) \subseteq D_r \subseteq D \cap B(\xi, r)$ with $R = R(D)$ is gaugeable with the corresponding constants C_1/C_2 and C_2/C_1. By **BHP** and Harnack chain inequality for harmonic functions , the present result follows.

4.20 $3G$ Theorem

$3G$ **Theorem**
Let D be a Lipschitz domain, $D \subseteq \mathbb{R}^d$ and G_D the Green function of D.There exists C such that for $x, y, z \in D$ the following holds

$$\frac{G_D(x,y) G_D(y,z)}{G_D(x,z)} \leqslant C \left(U_\alpha(x-y) + U_\alpha(y-z) \right)$$

and

$$\frac{G_D(x,y) G_D(y,z)}{G_D(x,z)} \leqslant C \left(\frac{|x-z|}{|x-y||y-z|} \right)^{d-\alpha} \,.$$

4.21 Appendix (calculation of $I(y)$)

For $|y| \leqslant r$ the following holds

$$I(y) = C^d_\alpha \int_{|x| \leqslant r} \frac{dx}{(r^2 - |x|^2)^{\alpha/2} |x-y|^{d-\alpha}} = 1 \,, \quad C^d_\alpha = \frac{\Gamma(d/2) \sin(\pi\alpha/2)}{\pi^{1+d/2}} \,.$$

Proof. We rely on technique developed in derivation of Poisson kernel for balls. We use inversion $x \to x^*$ with respect to the sphere with center at y and radius $R = \sqrt{r^2 - |y|^2}$. From the formula (5) it follows

$$\frac{|x-y|^2}{(r^2 - |x|^2)(r^2 - |y|^2)} = \frac{1}{|x^*|^2 - r^2}. \tag{10}$$

Using the formula (6) we obtain

$$I(y) = C^d_\alpha (r^2 - |y|^2)^{\alpha/2} \int_{|x^*|\geqslant r} \frac{dx^*}{(|x^*|^2 - r^2)^{\alpha/2}|x^* - y|^d}$$

We introduce now spherical coordinates in $\mathbb{R}^d$ with the north pole at y and we obtain

$$I(y) = \frac{C^d_\alpha (r^2 - |y|^2)^{\alpha/2}\omega_d}{\int_0^\pi \sin^{d-2}\theta\, d\theta} \int_r^\infty \frac{\rho^{d-1} d\rho}{(\rho^2 - r^2)^{\alpha/2}} \int_0^\pi \frac{\sin^{d-2}\theta\, d\theta}{(\rho^2 - 2|y|\rho\cos\theta + |y|^2)^{d/2}}$$

After substituting $\rho = \rho_1|y|$, $r = r_1|y|$ the above integral takes the form

$$I(y) = \frac{C^d_\alpha (r_1^2 - 1)^{\alpha/2}\omega_d}{\int_0^\pi \sin^{d-2}\theta\, d\theta} \int_{r_1}^\infty \frac{\rho_1^{d-1} d\rho_1}{(\rho_1^2 - r_1^2)^{\alpha/2}} \underbrace{\int_0^\pi \frac{\sin^{d-2}\theta\, d\theta}{(\rho_1^2 - 2\rho_1\cos\theta + 1)^{d/2}}}_{I_1(\rho_1)} \tag{11}$$

We prove that the following equality holds

$$I_1(\rho_1) = \int_0^\pi \frac{\sin^{d-2}\theta\, d\theta}{(\rho_1^2 - 2\rho_1\cos\theta + 1)^{d/2}} = \frac{1}{\rho_1^{d-2}} \frac{1}{\rho_1^2 - 1} \int_0^\pi \sin^{d-2} t dt, \tag{12}$$

and substituting (12) to (11) we obtain

$$I(y) = C^d_\alpha (r_1^2 - 1)^{\alpha/2}\omega_d \int_{r_1}^\infty \frac{\rho_1\, d\rho_1}{(\rho_1^2 - 1)(\rho_1^2 - r_1^2)^{\alpha/2}}. \tag{13}$$

To justify (12), we make substitution

$$\frac{\sin\theta}{\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1}} = \frac{\sin t}{\rho_1}. \tag{14}$$

After differentiation of both sides we obtain

$$\left(\frac{\rho_1\cos\theta}{\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1}} - \frac{\rho_1^2 \sin^2\theta}{\left(\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1}\right)^3} \right) d\theta = \cos t\, dt,$$

Next, we apply the substition (14):

$$\left(\frac{\rho_1\cos\theta - \sin^2 t}{\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1}} \right) d\theta = \cos t\, dt.$$

Taking squares of both sides of (14) we obtain

$$\begin{aligned}
\rho_1^2 \sin^2\theta &= \sin^2 t(\rho_1^2 - 2\rho_1\cos\theta + 1) \\
\rho_1^2(1 - \cos^2\theta) &= (1 - \cos^2 t)(\rho_1^2 - 2\rho_1\cos\theta + 1) \\
\rho_1^2 - \rho_1^2\cos^2\theta &= \rho_1^2 - 2\rho_1\cos\theta + 1 - \rho_1^2\cos^2 t + 2\rho_1\cos\theta\cos^2 t - \cos^2 t,
\end{aligned}$$

hence

$$\begin{aligned}
& (\rho_1\cos\theta - \sin^2 t)^2 \\
= \;& -2\rho_1\cos\theta\sin^2 t + \sin^4 t + 2\rho_1\cos\theta + \rho_1^2\cos^2 t - 2\rho_1\cos\theta\cos^2 t + 1 - \sin^2 t \\
= \;& 2\rho_1\cos\theta(1 - \sin^2 t - \cos^2 t) + \sin^4 t + \rho_1^2\cos^2 t - \sin^2 t \\
= \;& \sin^2 t(\sin^2 t - 1) + \rho_1^2\cos^2 t = \cos^2 t(\rho_1^2 - \sin^2 t)\,,
\end{aligned}$$

so we obtain

$$\rho_1 \cos\theta - \sin^2 t = \cos t\sqrt{\rho_1^2 - \sin^2 t}. \tag{15}$$

Carrying further our calculations we multiply both sides of (15) by (-2) and add to both sides expression $\rho_1^2 - 2\sin^2 t + 1$. We obtain

$$\begin{aligned}
\rho_1^2 - 2\rho_1\cos\theta + 1 &= \rho_1^2 - 2\sin^2 t + 1 - 2\cos t\sqrt{\rho_1^2 - \sin^2 t} \\
&= (\rho_1^2 - \sin^2 t) + \cos^2 t - 2\cos t\sqrt{\rho_1^2 - \sin^2 t} \\
&= \left(\sqrt{\rho_1^2 - \sin^2 t} - \cos t\right)^2.
\end{aligned}$$

Therefore

$$\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1} = \sqrt{\rho_1^2 - \sin^2 t} - \cos t. \tag{16}$$

Substituting (15) and (16) to (14) we obtain

$$\frac{\cos t\sqrt{\rho_1^2 - \sin^2 t}}{\sqrt{\rho_1^2 - \sin^2 t} - \cos t}\, dt = \cos t\, dt$$

Thus

$$\frac{d\theta}{\left(\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1}\right)^2} = \frac{dt}{\sqrt{\rho_1^2 - \sin^2 t}\left(\sqrt{\rho_1^2 - \sin^2 t} - \cos t\right)}$$

$$= \frac{dt\left(\sqrt{\rho_1^2 - \sin^2 t} + \cos t\right)}{\sqrt{\rho_1^2 - \sin^2 t}(\rho_1^2 - 1)} = \frac{1}{\rho_1^2 - 1}\left(1 + \frac{\cos t}{\sqrt{\rho_1^2 - \sin^2 t}}\right) dt$$

Thus, we obtain

$$\begin{aligned}
I_1(\rho_1) &= \int_0^\pi \frac{\sin^{d-2}\theta\, d\theta}{\left(\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1}\right)^d} \\
&= \int_0^\pi \left(\frac{\sin\theta}{\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1}}\right)^{d-2} \frac{d\theta}{(\sqrt{\rho_1^2 - 2\rho_1\cos\theta + 1})^2} \\
&= \int_0^\pi \left(\frac{\sin t}{\rho_1}\right)^{d-2} \frac{1}{\rho_1^2 - 1}\left(1 + \frac{\cos t}{\sqrt{\rho_1 - \sin^2 t}}\right) dt \\
&= \frac{1}{\rho_1^{d-2}} \frac{1}{\rho_1^2 - 1}\left(\int_0^\pi \sin^{d-2} t\, dt + \int_0^\pi \frac{\sin^{d-2} t \cos t\, dt}{\sqrt{\rho_1 - \sin^2 t}}\right).
\end{aligned}$$

If we now show that

$$\int_0^\pi \frac{\sin^{d-2} t\cos t\, dt}{\sqrt{\rho_1 - \sin^2 t}} = 0,$$

then we finally obtain the equality (12).
We note that

$$\int_0^\pi \frac{\sin^{d-2} t\cos t\, dt}{\sqrt{\rho_1 - \sin^2 t}} = \int_0^{\pi/2} \frac{\sin^{d-2} t\cos t\, dt}{\sqrt{\rho_1 - \sin^2 t}} + \int_{\pi/2}^\pi \frac{\sin^{d-2} t\cos t\, dt}{\sqrt{\rho_1 - \sin^2 t}}.$$

However,

$$\begin{aligned}
\int_{\pi/2}^{\pi} \frac{\sin^{d-2} t \cos t \, dt}{\sqrt{\rho_1 - \sin^2 t}} &= \int_0^{\pi/2} \frac{\sin^{d-2}(\pi-\phi)\cos(\pi-\phi)}{\sqrt{\rho_1 - \sin^2(\pi-\phi)}} d\phi \\
&= \int_0^{\pi/2} \frac{\sin^{d-2}\phi(-\cos\phi)}{\sqrt{\rho_1 - \sin^2\phi}} d\phi
\end{aligned}$$

what gives (12).

We are coming back to the formula (13). We put $s = \rho_1^2 - r_1^2$. Then

$$\begin{aligned}
& \frac{1}{C_\alpha^d} I(y) = (r_1^2 - 1)^{\alpha/2} \frac{\omega_d}{2} \int_0^\infty \frac{ds}{(s + r_1^2 - 1) s^{\alpha/2}} \\
= & \frac{\omega_d}{2} \int_0^\infty \frac{ds_1 (r_1^2 - 1)}{(s_1 + 1)(r_1^2 - 1) s_1^{\alpha/2}} = \frac{\omega_d}{2} \int_0^\infty \frac{ds_1}{(s_1 + 1) s_1^{\alpha/2}} \\
= & \frac{\omega_d}{2} \int_0^1 \frac{dw}{w(\frac{1}{w} - 1)^{\alpha/2}} = \frac{\omega_d}{2} \int_0^1 (1-w)^{-\alpha/2} w^{(\alpha/2)-1} dw \\
= & \frac{\omega_d}{2} B(1 - \alpha/2, \alpha/2) = \frac{\omega_d}{2} \frac{\Gamma(\alpha/2)\Gamma(1-\alpha)}{\Gamma(1)} = \frac{\omega_d}{2} \frac{\pi}{\sin(\pi\alpha//2)} \\
= & \frac{2\pi^{d/2}}{2\Gamma(d/2)} \frac{\pi}{\sin(\pi\alpha/2)} = \frac{\pi^{1+d/2}}{\Gamma(d/2)\sin(\pi\alpha/2)},
\end{aligned}$$

therefore $I(y) = 1$, which finishes the proof.

4.22 Bibliographical notes

Analytical aspects of elementary potential theory based on Riesz kernels are presented in great depths in [7]. Some probabilistic explanations as well as the treatment of the recurrent cases $\alpha \geqslant d = 1$ is contained in [5]. For readers interested in a more detailed proofs we recommend [6] (in [6] there are also some corrections and additional explanations, especially for the logarithmic case, missing in [5]). The advanced theory originated from the proof of the Boundary Harnack Principle [1]. For purely probabilistic proof ("box method") we refer to [2]. Schrödinger operator, Feynman-Kac semigroups, and Conditional Gauge Theorem in the setting of isotropic α-stable process were obtained in the papers [3] and [4] and our presentation is taken from these two papers. We refer to the paper [8] as containing the pioneering proof of the Poisson kernel for multidimensional isotropic α-stable process (in an analytical setting).

4.22.1 Bibliography

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5 Relativistic α-stable Lévy processes

Let $\theta_t^\alpha(u)$ be a density of $\alpha/2$-stable subordinator S_t^α with Laplace transform $\mathbb{E}^0 e^{-\lambda S_t^\alpha} = e^{-t\lambda^{\alpha/2}}$.

For $m > 0$ we define a density of a "new" subordinator $T_t^{\alpha,m}$ by

$$\theta_t^{\alpha,m}(u) = e^{mt}\,\theta_t^\alpha(u)\,e^{-m^{2/\alpha}u}, \quad u > 0.$$

We note that $\theta_t^{\alpha,m}$ is a probabilistic density function:

$$\int_0^\infty \theta_t^{\alpha,m}\, tu = e^{mt}\int_0^\infty e^{-(m^{2/\alpha})u}\,\theta_t^\alpha(u)\,du$$
$$= \quad e^{mt}\mathbb{E}^0 e^{-m^{2/\alpha}\,S_t^\alpha} = e^{mt}\,e^{-t\,(m^{2/\alpha})^{\alpha/2}} = e^{mt}\,e^{-mt} = 1\,.$$

Let B_t be d-dimensional Brownian Motion independent of $T_t^{\alpha,m}$. Process

$$X_t^m = B_{T_t^{\alpha,m}}$$

is called *relativistic α-stable process* (with parameter m).

5.1 Generator and Lévy measure

The infinitesimal generator of X_t^m is given by

$$mI - (m^{2/\alpha}I - \Delta)^{\alpha/2}.$$

The density function $\nu^m(x)$ of the Lévy measure of the relativistic α-stable process is of the form:

$$\nu^m(x) = \frac{\alpha 2^{\frac{\alpha-d}{2}}}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}\frac{m^{\frac{d+\alpha}{2\alpha}}\,K_{\frac{d+\alpha}{2}}(m^{1/\alpha}|x|)}{|x|^{\frac{d+\alpha}{2}}}\,.$$

where K_ϑ is the modified Bessel function of the third kind

$$K_\vartheta(z) = 2^{-\vartheta-1}z^\vartheta\int_0^\infty e^{-t}e^{-\frac{z^2}{4t}}\,t^{-\vartheta-1}\,dt, \quad |\arg z| < \frac{\pi}{4}.$$

5.2 m-potentials

For $\lambda \geqslant 0$ we define λ-potential by

$$U_\lambda^m(x) = \int_0^\infty e^{-\lambda t}p_t^m(x)dt,$$

where $p_t^m(x) = P^0(X_t^m \in dx)$.

For $\lambda = m$ we get Bessel potentials

$$U_m^m(x) = \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}}\frac{m^{\frac{d-\alpha}{2\alpha}}\,K_{(d-\alpha)/2}(m^{1/\alpha}|x|)}{|x|^{(d-\alpha)/2}}.$$

5.3 Harmonic measures and Green functions

Let $\tau_D = \{t \geqslant 0 : X_t^m \notin D\}$.

Definition (m-harmonic measure)

$$P_D^m(x, A) = \mathbb{E}^x[\tau_D < \infty; e^{-m\tau_D}\mathbf{1}_A(X_{\tau_D}^m)].$$

The density of the above measure is called **m-Poisson kernel** of the set D and will be denoted by $P_D^m(x, y)$.

m-Green function

$$G_D^m(x, y) = \int_0^\infty e^{-mt} p_t^{m,D}(x, y)\, dt,$$

$p_t^{m,D}$ - the transition density function of the killed process X_t^m when exiting the set D.

5.4 m-Poisson kernel and Green function for $(0, \infty)$

We consider a positive half-line $D = (0, \infty)$ in $\mathbb{R}$.

Theorem 1. For $x > 0$ and $y < 0$ we have

$$P_{(0,\infty)}^m(x, y) = \frac{\sin(\pi\alpha/2)}{\pi} \left(\frac{x}{-y}\right)^{\alpha/2} \frac{e^{-m^{1/\alpha}(x-y)}}{x - y}.$$

Theorem 2
For $x, y \in (0, \infty)$ we have

$$G_{(0,\infty)}^m(x, y) = \frac{|x - y|^{\alpha-1}}{2^\alpha \Gamma(\frac{\alpha}{2})^2} \int_0^{\frac{4xy}{(x-y)^2}} e^{-m^{\frac{1}{\alpha}}|x-y|(t+1)^{\frac{1}{2}}} t^{\frac{1}{\alpha}-1}(t + 1)^{-\frac{1}{2}} dt.$$

5.5 Results for a half-space

Consider a half-space $\mathbb{H} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$.

Theorem 3. For $x \in \mathbb{H}$ and $y \in \mathbb{H}^c$ we have

$$P_{\mathbb{H}}^m(x, y) = C_1 \cdot m^{d/2\alpha} \left(\frac{x_d}{-y_d}\right)^{\alpha/2} \frac{K_{d/2}(m^{1/\alpha}|x - y|)}{|x - y|^{d/2}}.$$

Theorem 4. For $x, y \in \mathbb{H}$ we have

$$G_{\mathbb{H}}^m(x, y) = C_2 \cdot \frac{|x - y|^{\alpha-\frac{d}{2}}}{m^{\frac{2\alpha}{d}}} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1} K_{\frac{d}{2}}(m^{\frac{1}{\alpha}}|x - y|(t + 1)^{\frac{1}{2}})}{(t + 1)^{\frac{d}{4}}} dt.$$

5.6 Sobolev spaces

Definition (classical Sobolev spaces)

Sobolev spaces are defined as

$$L^2_k(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \partial^\alpha f \in L^2(\mathbb{R}^d), |\alpha| \leqslant k\}$$

for $k = 0, 1, 2, \ldots,$ or equivalently

$$L^2_k(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : (1 + |x|^2)^{k/2} f^{\hat{}} \in L^2(\mathbb{R}^d)\}.$$

Definition (fractional Sobolev spaces)

For $s \geqslant 0$ and $1 \leqslant p \leqslant \infty$ we define

$$L^p_s(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : (1 + |x|^2)^{s/2} f^{\hat{}} \in L^p(\mathbb{R}^d)\}.$$

5.7 Bessel potentials

For $s > 0$ consider the operator $(I - \Delta)^{s/2}$ and let $J_s = (I - \Delta)^{-s/2}$ be the (formal) inverse operator.

Sobolev space $L^p_s(\mathbb{R}^d)$ consists of these functions from $L^p(\mathbb{R}^d)$, which can be written in the form $f = J_s(g)$ for $g \in L^p(\mathbb{R}^d)$.

For every $f \in L^p(\mathbb{R}^d)$ we have $J_s(f) = f * G_s$, where G_s has the following kernel:
Bessel potential

$$G_s(x) = \frac{2^{1-(d+s)/2}}{\Gamma(s/2)\pi^{d/2}} \frac{K_{(d-s)/2}(|x|)}{|x|^{(d-s)/2}}, \quad |x| > 0.$$

Here K_ϑ denotes the modified Bessel function of the third kind.

5.8 Connections with stochastic processes

- Fractional laplacian $(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$.
- Riesz potentials
$$\frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)} \frac{1}{|x|^{d-\alpha}}, \quad \alpha < d.$$
- Potential theory of symmetric α-stable processes (with the infinitesimal generator $-(-\Delta)^{\alpha/2}$).
- Operator $(I - \Delta)^{\alpha/2}$, $\alpha \in (0, 2)$.
- Bessel potentials.
- 1 - potential theory for relativistic α-stable process (with the infinitesimal generator $I - (I - \Delta)^{\alpha/2}$).

5.9 Relations between $(-\Delta)^{\alpha/2}$ and $(I-\Delta)^{\alpha/2}$

Lemma [E. M. Stein, Singular Integrals and ...., Ch. V, 3.2]
There exist finite measures μ_α, ν_α such that

$$(I-\Delta)^{\alpha/2} = \nu_\alpha + (-\Delta)^{\alpha/2}\,\mu_\alpha\,.$$

Lemma [M. Ryznar, Pot. Analysis 17, 2002]
There exists a probability measure σ_α such that

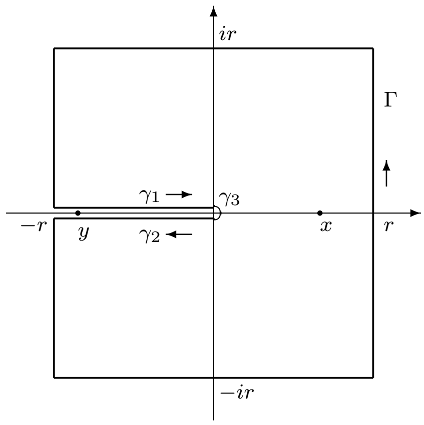
$$(I-\Delta)^{\alpha/2} = \sigma_\alpha + I + (-\Delta)^{\alpha/2}\,.$$

5.10 Appendix - derivation of Poisson kernel

Theorem 1
For $x>0$ and $y<0$ we have

$$P^m_{(0,\infty)}(x,y) = \frac{\sin(\pi\alpha/2)}{\pi}\left(\frac{x}{-y}\right)^{\alpha/2}\frac{e^{-m^{1/\alpha}(x-y)}}{x-y}.$$

- It is enough to consider only the case $m=1$ since the general case follows from the scaling property.
$$P^m_{(0,\infty)}(x,y) = m^{1/\alpha}P^1_{(0,\infty)}(m^{1/\alpha}x, m^{1/\alpha}y), \quad x>0, y<0.$$
- 1-Poisson kernel is uniquely determined by the following equality
$$\int_{-\infty}^0 P^1_{(0,\infty)}(x,u)U^1_1(u-y)du = U^1_1(x-y), \quad x>0, y<0.$$
- It is enough to show the following equality
$$\frac{\sin(\pi\alpha/2)}{\pi}\int_{-\infty}^0\left(\frac{x}{-u}\right)^{\alpha/2}\frac{e^{u-x}}{x-u}\frac{K_{\frac{1-\alpha}{2}}(|y-u|)}{|y-u|^{\frac{1-\alpha}{2}}}du = \frac{K_{\frac{1-\alpha}{2}}(x-y)}{(x-y)^{\frac{1-\alpha}{2}}}$$
for every $x>0$ i $y<0$.
- We consider the following function of complex variable z:
$$f(z) = \frac{x^{\alpha/2}}{z^{\alpha/2}}\frac{e^{z-x}}{z-x}\frac{K_{\frac{1-\alpha}{2}}(z-y)}{(z-y)^{\frac{1-\alpha}{2}}}, \quad z\in\mathbb{C}\backslash((-\infty,0]\cup\{x\})$$
- We integrate the function
$$f(z) = \frac{x^{\alpha/2}}{z^{\alpha/2}}\frac{e^{z-x}}{z-x}\frac{K_{\frac{1-\alpha}{2}}(z-y)}{(z-y)^{\frac{1-\alpha}{2}}}$$
over the contour described below.



5.10.1 Results for a half-space

Consider a half-space $\mathbb{H} = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}$.
Theorem 3
For $x \in \mathbb{H}$ and $y \in \mathbb{H}^c$ we have

$$P^m_{\mathbb{H}}(x, y) = C_1 \cdot m^{d/2\alpha} \left(\frac{x_d}{-y_d}\right)^{\alpha/2} \frac{K_{d/2}(m^{1/\alpha}|x-y|)}{|x-y|^{d/2}}.$$

Theorem 4
For $x, y \in \mathbb{H}$ we have

$$G^m_{\mathbb{H}}(x, y) = C_2 \cdot \frac{|x-y|^{\alpha - \frac{d}{2}}}{m^{\frac{2\alpha}{d}}} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1} K_{\frac{d}{2}}(m^{\frac{1}{\alpha}}|x-y|(t+1)^{\frac{1}{2}})}{(t+1)^{\frac{d}{4}}} dt.$$

Computing the mass of the m-Poisson kernel we get the following
Corollary

$$E^x e^{-m\tau_{\mathbb{H}}} = \frac{1}{\Gamma(\alpha/2)} \int_{m^{1/\alpha} x_d}^{\infty} t^{\alpha/2-1} e^{-t}\, dt, \quad x \in \mathbb{H}\,.$$

- The function given above is called incomplete gamma function.
- This function is harmonic on $\mathbb{H}$ for the operator $(m^{2/\alpha} I - \Delta)^{\alpha/2}$.

5.10.2 Sketch of the proof of theorem 3

We have to prove that

$$\int_{\overline{\mathbb{H}}^c} P^1_{\mathbb{H}}(x, u) U_1(u-y)\, du = U_1(x-y), \quad x \in \mathbb{H}, y \in \mathbb{H}^c\,.$$

Taking the $(d-1)$-dimensional Fourier transform of the both sides with respect to $(y_1, \dots, y_{d-1})$ we get

$$\frac{\sin(\pi\alpha/2)}{\pi} \int_{-\infty}^{0} \left(\frac{x_d}{-u_d}\right)^{\alpha/2} \frac{e^{\kappa(u_d - x_d)}}{x_d - u_d} \frac{K_{\frac{1-\alpha}{2}}(\kappa|y_d - u_d|)}{|y_d - u_d|^{\frac{1-\alpha}{2}}} du_d = \frac{K_{\frac{1-\alpha}{2}}(\kappa(x_d - y_d))}{(x_d - y_d)^{\frac{1-\alpha}{2}}}$$

where $\kappa = (1 + |z|^2)^{1/2}$.

5.10.3 Results for α-stable process

Taking $m \to 0$ in the formula for m-Poisson kernel for a half-space

$$P^m_{\mathbb{H}}(x,y) = C_1 \cdot m^{d/2\alpha} \left(\frac{x_d}{-y_d}\right)^{\alpha/2} \frac{K_{d/2}(m^{1/\alpha}|x-y|)}{|x-y|^{d/2}}$$

we get the following formula for Poisson kernel for a half-space for symmetric α-stable process

$$P^\alpha_{\mathbb{H}}(x,y) = \tilde{C}_1 \cdot \left(\frac{x_d}{-y_d}\right)^{\alpha/2} \frac{1}{|x-y|^d}.$$

Analogously, taking $m \to 0$ in the formula for m-Green function

$$G^m_{\mathbb{H}}(x,y) = C_2 \cdot \frac{|x-y|^{\alpha-\frac{d}{2}}}{m^{\frac{2\alpha}{d}}} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1} K_{\frac{d}{2}}(m^{\frac{1}{\alpha}}|x-y|(t+1)^{\frac{1}{2}})}{(t+1)^{\frac{d}{4}}} dt$$

we get the formula for Green function for a half-space for α-stable process

$$G^\alpha_{\mathbb{H}}(x,y) = \tilde{C}_2 \cdot |x-y|^{\alpha-d} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{\frac{d}{2}}} dt.$$

5.10.4 0-Green function

Resolvent Equation

$$G^0_{\mathbb{H}}(x,y) - G_{\mathbb{H}}(x,y) = \int_{\mathbb{H}} G^0_{\mathbb{H}}(x,u)\, G_{\mathbb{H}}(u,y)\, du,\ x,y \in \mathbb{H}$$

Iterating, we obtain

Series Representation

$$G^0_{\mathbb{H}}(x,y) = \sum_{n=1}^{\infty} (G_{\mathbb{H}})^{(n)}(x,y),\ x,y \in \mathbb{H}$$

Not very useful for estimating $G_{\mathbb{H}}(x,y)$. We use the formula $G^0_{\mathbb{H}}(x,y) = \int_0^\infty p^{\mathbb{H}}_t(x,y)\, dt$ and estimate $p^{\mathbb{H}}_t(x,y)$.

5.10.5 One-dimensional case: $\mathbb{H} = (0,\infty)$

Theorem

For $x, y > 0$,

$$G^0_{(0,\infty)}(x,y) \approx G_{(0,\infty)}(x,y) + (x \wedge y) \vee (x \wedge y)^{\alpha/2}.$$

Equivalently

$$G^0_{(0,\infty)}(x,y) \;\approx\; \begin{cases} G_{(0,\infty)}(x,y), & x \wedge y \leqslant 1,\ |x-y| < 1; \\ G_{(0,\infty)}(x,y) + x \wedge y, & x \wedge y > 1,\ |x-y| < 1\ ; \\ (x \wedge y) \vee (x \wedge y)^{\alpha/2}, & |x-y| \geqslant 1. \end{cases}$$

5.10.6 Multidimensional case of $\mathbb{H}$

Theorem

For $d \geqslant 3$ oraz $x, y \in \mathbb{H}$:

- If $|x-y| > 1$, then

$$G^0_{\mathbb{H}}(x,y) \approx \min\left\{ \frac{(x_d \vee x_d^{\alpha/2})(y_d \vee y_d^{\alpha/2})}{|x-y|^d}, \frac{1}{|x-y|^{d-2}} \right\}$$

- If $|x-y| \leqslant 1$, then

$$G^0_{\mathbb{H}}(x,y) \approx \left[\left(\frac{x_d \wedge y_d}{|x-y|} \right)^{\alpha/2} \wedge 1 \right] \frac{1}{|x-y|^{d-\alpha}}$$

For $d = 2$ oraz $x, y \in \mathbb{H}$:

- If $|x-y| > 1$, then

$$G^0_{\mathbb{H}}(x,y) \approx \ln\left(1 + 4\frac{(x_2 \vee x_2^{\alpha/2})(y_2 \vee y_2^{\alpha/2})}{|x-y|^2} \right)$$

- If $|x-y| \leqslant 1$, then

$$G^0_{\mathbb{H}}(x,y) \approx \left[\left(\frac{x_2 \wedge y_2}{|x-y|} \right)^{\alpha/2} \wedge 1 \right] \frac{1}{|x-y|^{2-\alpha}} + \ln(1 \vee (x_2 \wedge y_2)).$$

5.11 Bibliographical notes

The relativistic Lévy process is the second important example of discontinuous Lévy processes and received much of attention in recent years. It is of some interest to physicists as a probabilistic model of relativistic phenomena in connection with the stability of matter ([7], [11], [12]). It may be also of some interest to analysts as it is related to the potential theory based on Bessel potentials ([1], [2]). The paper was one of the first papers on the potential theory of these processes. Much of the material presented in this section is taken from the paper [3]. It may be interesting to notice that the paper [4] contains purely probabilistic proofs of results in [3] as well as provides connections between our isotropic stable and relativistic processes.

5.11.1 Bibliography

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6 Subordinated Lévy processes

6.1 Subordinators

Let $S = (S_t : t \geqslant 0)$ be a subordinator, that is, an increasing Lévy process taking values in $[0, \infty]$ with $S_0 = 0$. We remark that our subordinators are what some authors call killed subordinators. The Laplace transform of the law of S_t is given by the formula

$$\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)), \quad \lambda > 0. \tag{17}$$

The function $\phi : (0, \infty) \to \mathbb{R}$ is called the Laplace exponent of S, and it can be written in the form

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt). \tag{18}$$

Here $a, b \geqslant 0$, and μ is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1)\, \mu(dt) < \infty. \tag{19}$$

The constant a is called the killing rate, b the drift, and μ the Lévy measure of the subordinator S.

6.2 Complete Bernstein functions (CBF)

Recall that a C^∞ function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \leqslant 0$ for every $n \in \mathbb{N}$. It is well known that a function $\phi : (0, \infty) \to \mathbb{R}$ is a Bernstein function if and only if it has the representation given by (18).

A function $\phi : (0, \infty) \to \mathbb{R}$ is called a complete Bernstein function (CBF - in brief) if there exists a Bernstein function η such that

$$\phi(\lambda) = \lambda^2 \mathcal{L}\eta(\lambda), \quad \lambda > 0,$$

where $\mathcal{L}$ stands for the Laplace transform of the function η: $\mathcal{L}\eta(\lambda) = \int_0^\infty e^{-\lambda t}\eta(t)\, dt$. It is known that every complete Bernstein function is a Bernstein function and that the following three conditions are equivalent:

- (i) ϕ is a complete Bernstein function iff its Lévy measure
- (ii) $\psi(\lambda) := \lambda/\phi(\lambda)$ is a complete Bernstein function;
- (iii) ϕ is a Bernstein function whose Lévy measure μ is given by $\mu(dt) = \int_0^\infty e^{-st}\gamma(ds)dt$ with a measure γ on $(0, \infty)$ satisfying $\int_0^\infty \min\{\frac{1}{s}, \frac{1}{s^2}\}\, \gamma(ds) < \infty$.

6.3 Examples of subordinators

Example 1. (**Stable subordinators**)

We have here well-known stable subordinators. For $0 < \alpha < 2$, let $\phi(\lambda) = \lambda^{\alpha/2}$. We obtain

$$\lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1 - \alpha/2)} \int_0^\infty (1 - e^{-\lambda t})\, t^{-1-\alpha/2}\, dt,$$

i.e, the Lévy measure $\mu(dt)$ of ϕ has a density given by $(\alpha/2)/\Gamma(1-\alpha/2)\, t^{-1-\alpha/2}$. Since $t^{-1-\alpha/2} = \int_0^\infty e^{-ts} s^{\alpha/2}/\Gamma(1+\alpha/2)\, ds$, it follows that ϕ is a complete Bernstein function. The subordinator S corresponding to ϕ is called an $\alpha/2$-*stable subordinator*. It is known that the distribution $\eta_1(ds)$ of the $\alpha/2$-stable subordinator has a density $\eta_1(s)$ with respect to the Lebesgue measure. Moreover,

$$\eta_1(s) \sim 2\pi\Gamma\left(1+\frac{\alpha}{2}\right)\sin\left(\frac{\alpha\pi}{4}\right) s^{-1-\alpha/2}, \quad s \to \infty, \tag{20}$$

and for some positive constant $c > 0$

$$\eta_1(s) \leqslant c(1 \wedge s^{-1-\alpha/2}), \quad s > 0, \tag{21}$$

Example 2. (**Relativistic stable subordinators**)

For $0 < \alpha < 2$ and $m > 0$, let $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$. By integration

$$(\lambda + m^{2/\alpha})^{\alpha/2} - m = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{-\lambda t})\, e^{-m^{2/\alpha} t}\, t^{-1-\alpha/2}\, dt,$$

i.e., the Lévy measure $\mu(dt)$ of ϕ has a density given by $(\alpha/2)/\Gamma(1-\alpha/2)\, e^{-m^{2/\alpha}t}\, t^{-1-\alpha/2}$. This Bernstein function appeared in the study of the stability of relativistic matter. The subordinator $\tilde{S}$ corresponding to the complete Bernstein function $m + \phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2}$ is obtained by killing S at an independent exponential time with parameter m. By checking tables of Laplace transforms we obtain

$$\frac{1}{m+\phi(\lambda)} = \int_0^\infty e^{-\lambda t}\, \frac{1}{\Gamma(\alpha/2)}\, e^{-m^{2/\alpha}t}\, t^{-1+\alpha/2}\, dt,$$

hence the potential measure $\tilde{U}$ of the subordinator $\tilde{S}$ has the density $\tilde{u}$:

$$\tilde{u}(t) = \frac{1}{\Gamma(\alpha/2)}\, e^{-m^{2/\alpha}t}\, t^{-1+\alpha/2}.$$

Example 3.(**Gamma subordinator**)

Let $\phi(\lambda) = \log(1+\lambda)$. By use of Frullani's integral it follows that

$$\log(1+\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{e^{-t}}{t}\, dt,$$

i.e., the Lévy measure of ϕ has a density given by e^{-t}/t. Note that $e^{-t}/t = \int_0^\infty e^{-st} 1_{(1,\infty)}(s)\, ds$, hence the density of the Lévy measure μ is completely monotone. Therefore, ϕ is a complete Bernstein function. The corresponding subordinator S is called a *gamma subordinator*. The distribution $\eta_t(ds)$, $t > 0$, is given by

$$\eta_t(ds) = \frac{1}{\Gamma(t)}\, s^{t-1} e^{-s}\, ds, \quad s > 0. \tag{22}$$

Composition of subordinators

Before proceeding to the next two examples, let us briefly discuss composition of subordinators. Suppose that $S^1 = (S_t^1)$ and $S^2 = (S_t^2)$ are two independent subordinators with Laplace exponents ϕ^1, respectively ϕ^2, and convolution semigroups (η_t^1), respectively (η_t^2). Define the new process $S=(S_t)$ by $S_t = S^1_{S^2_t}$, subordination of S^1 by S^2. S is another subordinator. The distribution η_t of S_t is given by

$$\eta_t(ds) = \int_0^\infty \eta_t^2(du)\eta_u^1(ds)\,. \tag{23}$$

An elementary calculation shows that the Laplace exponent ϕ of S is given by $\phi(\lambda)=\phi^2(\phi^1(\lambda))$.

Example 4.(Iterated geometric stable subordinators)

Let $0<\alpha\leqslant 2$. Define,

$$\phi^{(1)}(\lambda)=\phi(\lambda)=\log(1+\lambda^{\alpha/2})\,,\quad \phi^{(n)}(\lambda)=\phi(\phi^{(n-1)}(\lambda))\,,\quad n\geqslant 2\,.$$

Since $\phi^{(n)}$ is a complete Bernstein function, we have that $\phi^{(n)}(\lambda)=\int_0^\infty(1-e^{-\lambda t})\mu^{(n)}(t)\,dt$ for a completely monotone Lévy density $\mu^{(n)}(t)$. The exact form of this density is not known.

Let $S^{(n)}=(S_t^{(n)}: t\geqslant 0)$ be the corresponding (iterated) subordinator, and let $U^{(n)}$ denote the potential measure of $S^{(n)}$. Since $\phi^{(n)}$ is a complete Bernstein function, $U^{(n)}$ admits a completely monotone density $u^{(n)}$. The explicit form of the potential density $u^{(n)}$ is not known.

Example 5.(Bessel subordinators)

The Bessel subordinator $S_I=(S_I(t))$
is a subordinator with no drift, no killing and Lévy density

$$\mu_I(t)=\frac{1}{t}\,I_0(t)\,e^{-t},$$

where for any real number ν, I_ν is the modified Bessel function. Since μ_I is the Laplace transform of the function $\gamma(t)=\int_0^{t\wedge 2}\pi^{-1}(2s-s^2)^{-1/2}\,ds$; the Laplace exponent of S_I is a complete Bernstein function; it is given by

$$\phi_I(\lambda)=\log((1+\lambda)+\sqrt{(1+\lambda)^2-1}).$$

For any $t>0$, the density of $S_I(t)$ is given by

$$f_t(x)=\frac{t}{x}I_t(x)e^{-x}.$$

The Bessel subordinator $S_K=(S_K(t))$
is a subordinator with no drift, no killing and Lévy density

$$\mu_K(t)=\frac{1}{t}K_0(t)e^{-t},$$

where for any real number ν, K_ν is the modified Bessel function. Since μ_K is the Laplace transform of the function

$$\gamma(t)=\begin{cases}0, & t\in(0,2],\\ \log(t-1+\sqrt{(t-1)^2+1}), & t>2\,,\end{cases}$$

the Laplace exponent of S_K is a complete Bernstein function; this is given by

$$\phi_K(\lambda)=\frac{1}{2}\left(\log((1+\lambda)+\sqrt{(1+\lambda)^2-1})\right)^2\,.$$

For any $t > 0$, the density of $S_K(t)$ is given by $f_t(x) = \sqrt{\frac{2\pi}{t}}\,\vartheta_x(\frac{1}{t})\frac{e^{-x}}{x}$, where

$$\vartheta_v(t) = \frac{v}{\sqrt{2\pi^3 t}} \int_0^\infty \exp(\frac{\pi^2 - \xi^2}{2t}) \exp(-v\cosh(\xi)) \sinh(\xi) \sin(\frac{\pi\xi}{t}) d\xi.$$

Example 6.

For any $\alpha \in (0,2)$ and $\beta \in (0, 2-\alpha)$, it follows from the properties of complete Bernstein functions that

$$\phi(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{\beta/2}$$

is a complete Bernstein function.

Example 7.

For any $\alpha \in (0,2)$ and $\beta \in (0,\alpha)$, it follows from the properties of complete Bernstein functions that

$$\phi(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{-\beta/2}$$

is a complete Bernstein function.

6.4 Slowly varying functions

A Borel measurable function $l : (0,\infty) \to (0,\infty)$ is called slowly varying at infinity if for every $\lambda > 0$ the following holds:

$$\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1\,.$$

Later on we will work with the following assumption:

Assumption (H): There exists $\alpha \in (0,2)$ and a function $l : (0,\infty) \to (0,\infty)$, slowly varying at infinity, locally bounded from above and below by positive constants and such that

$$\phi(\lambda) \approx \lambda^{\alpha/2}\, l(\lambda), \quad \lambda \to \infty\,.$$

The following lemma is crucial in the sequel:

Lemma 1. Let $w(t)$ be a completely monotone function given by

$$w(t) = \int_0^\infty e^{-st} f(s)\, ds$$

with $f \geqslant 0$, decreasing. Then

$$f(s) \leqslant (1-e^{-1})^{-1} s^{-1}\, w(s^{-1})\,, \quad s > 0.$$

If, furthermore

$$w(\lambda t) \leqslant a\,\lambda^{-\delta} w(t), \quad \text{for all } \lambda \geqslant 1,$$

and for some $\delta \in (0,1)$ and $a > 0$, $s_0 > 0$, then

$$f(s) \geqslant c_2 s^{-1} w(s^{-1})\,,$$

for $c_2 = c_2(w, f, a, s_0, \delta)$.

Proof of Lemma 1. Taking into account the representation of $w(t)$ and the fact that $f \geqslant 0$ is decreasing we obtain

$$\begin{aligned} w(t) &= \frac{1}{t}\int_0^\infty e^{-s} f(s/t)\,ds \geqslant \frac{1}{t}\int_0^r e^{-s} f(s/t)\,ds \\ &\geqslant \frac{1}{t} f(r/t)\,(1-e^{-r})\,. \end{aligned}$$

Thus, we obtained $f(r/t) \leqslant (1-e^{-r})^{-1}\, t\, w(t)$ and this proves the first inequality in the lemma. In the sequel we need the following form of the inequality:

$$f(s/t) \leqslant (1-e^{-1})^{-1}\,(t/s)\,w(t/s)\,, \quad s>0\,, t>0\,. \tag{24}$$

On the other hand, for $r \in (0,1)$ we have

$$\begin{aligned} tw(t) = \int_0^r e^{-s} f(s/t)\,ds + \int_r^\infty e^{-s} f(s/t)\,ds \leqslant \int_0^r e^{-s} f(s/t)\,ds \\ + \quad f(r/t)\,e^{-r} \leqslant (1-e^{-1})^{-1} t \int_0^r e^{-s}\, w(t/s)\,ds/s + f(r/t)\,e^{-r}\,, \end{aligned}$$

where in the last line we applied (24). If we now apply the upper estimate for $w(\lambda t)$ we obtain $w(t/s) \leqslant a\, s^\delta\, w(t)$, for $t \geqslant 1/s_0$ and $s<r$. Thus, for $r \in (0,1]$ we have

$$tw(t) \leqslant a\,(1-e^{-1})^{-1} tw(t) \int_0^r e^{-s}\, s^{\delta-1}\,ds + f(r/t)\,e^{-r}\,.$$

Choosing $r \in (0,1]$ so small that $a\,(1-e^{-1})^{-1} tw(t) \int_0^r e^{-s}\, s^{\delta-1}\,ds \leqslant 1/2$ we obtain $f(r/t) \geqslant c_1 tw(t)$, for some $c_1>0$. Since $w(t)$ is decreasing, we have

$$f(s) \geqslant c_1\, \frac{r}{s}\, w(\frac{r}{s}) \geqslant c_2 s^{-1} w(s^{-1}), \quad s \leqslant r s_0,$$

where $c_2 = c_1 r$. From that we obtain that for another $c_3 > 0$

$$f(s) \geqslant c_3\, s^{-1}\, w(s^{-1}), \quad s \leqslant s_0\,.$$

6.5 Potential of subordinator

Let $u(s)$ - potential of subordinator with Laplace exponent $\phi(\lambda)$. Then

$$\int_0^\infty e^{-s\lambda}\, u(s)\,ds = \frac{1}{\phi(\lambda)} \tag{25}$$

Indeed, if $p(t,s)$ denotes the transition density of the subordinator S_t then $u(s) = \int_0^\infty p(t,s)\,dt$ hence

$$\begin{aligned} \int_0^\infty e^{-s\lambda}\, u(s)\,ds = \int_0^\infty [\int_0^\infty e^{-s\lambda}\, p(t,s)\,ds]\,dt \\ = \int_0^\infty \mathbb{E}^0 e^{-\lambda S_t}\,dt = \int_0^\infty e^{-t\,\phi(\lambda)} dt = \frac{1}{\phi(\lambda)}\,. \end{aligned}$$

The above relation and the Lemma 1 yield the following:
If ϕ satisfies (H) then

$$u(t) \approx t^{-1}\,\phi(t^{-1})^{-1} \approx \frac{t^{\alpha/2-1}}{l(t^{-1})}\,, \quad t \to 0^+\,.$$

6.6 Lévy measure of subordinator

Lévy measure density $\mu(t)$ of subordinator S_t satisfies

$$\mu(t) \approx t^{-1}\,\phi(t^{-1}) \approx t^{-\alpha/2-1}\,l(t^{-1})\,, \quad t\to 0^+\,,$$

when ϕ is a CBF with zero killing satisfying the assumption (H).
Proof. $\psi(\lambda)=\lambda/\phi(\lambda)$ is also a CBF. Let $v(t)$ be its potential. We have

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\,v(t)\,dt &= \frac{1}{\psi(\lambda)} = \frac{\phi(\lambda)}{\lambda} = \frac{1}{\lambda}\int_0^\infty (1-e^{-\lambda t})\,\mu(t)\,dt\\ &\overset{int.\,by\,parts}{=} [(-\int_t^\infty \mu(s)\,ds)\,\frac{1-e^{-\lambda t}}{\lambda}]_0^\infty + \int_0^\infty e^{-\lambda t}\,\mu(t,\infty)\,dt\\ = &\int_0^\infty e^{-\lambda t}\,\mu(t,\infty)\,dt\,.\end{aligned}$$

Thus, $v(t)=\mu(t,\infty)=\int_t^\infty \mu(s)\,ds \approx t^{-1}\,\psi(t^{-1}) \approx \phi(t^{-1})$.

To recover the asymptotic of the density $\mu(t)$ we proceed as follows:
Using inequality $1-e^{-cy}\leqslant c(1-e^{-y})$, valid for all $c\geqslant 1$ and all $y\geqslant 0$ we obtain $\phi(c\lambda)\leqslant c\phi(\lambda)$. Thus, $\phi(s^{-1})=\phi(2(2s)^{-1})\leqslant 2\phi((2s)^{-1})$, for all $s>0$. Therefore, for all $s\in(0,1/2)$ and some constants c_1, c_2

$$v(s)\leqslant c_1\phi(s^{-1})\leqslant 2c_1\phi((2s)^{-1})\leqslant c_2 v(2s)\,.$$

Since $v(t/2)\geqslant v(t/2)-v(t)=\int_{t/2}^t \mu(s)\,ds\geqslant (t/2)\mu(t)$, we obtain for all $t\in(0,1)$:

$$\mu(t)\leqslant 2t^{-1}v(t/2)\leqslant c_2 t^{-1}v(t)\leqslant c_3\phi(t^{-1})\,.$$

Since $\phi(\lambda t)\geqslant a\lambda^\delta\phi(t)$, for $\lambda\geqslant 1$ and $t\geqslant 1/s_0$, we obtain

$$\phi(s^{-1})=\phi(\lambda(\lambda s)^{-1})\geqslant a\lambda^\delta\phi((\lambda s)^{-1})\,,$$

for $s\leqslant s_0/\lambda$. But we know that $c_4^{-1}\phi(s^{-1})\leqslant v(s)\leqslant c_4\phi(s^{-1})$, for $s<1$, for some constant c_4. Fix $\lambda=2^{1/\delta}((c_4a^{-1})\vee 1)^{1/\delta}$. Then for $s\leqslant(s_0\wedge 1)/\lambda$

$$\begin{aligned}v(\lambda s) &\leqslant c_4\phi((\lambda s)^{-1})\leqslant c_4a^{-1}\lambda^{-\delta}\phi(s^{-1})\\ &\leqslant c_4^2a^{-1}\lambda^{-\delta}v(s)=\frac{1}{2}\frac{c_4^2a^{-1}}{c_4^2a^{-1}\wedge 1}\,v(s)\leqslant\frac{1}{2}\,v(s)\,.\end{aligned}$$

Moreover,

$$(\lambda-1)s\mu(s)\geqslant\int_s^{\lambda s}\mu(t)\,dt=v(s)-v(\lambda s)\geqslant v(s)-\frac{1}{2}v(s)=\frac{1}{2}v(s)\,.$$

But this implies that for all small $t>0$

$$\mu(t)\geqslant\frac{t^{-1}v(t)}{2(\lambda-1)}=c_5t^{-1}v(t)\geqslant c_6t^{-1}\phi(t^{-1})\,.$$

6.7 Subordination

Let $B=(B_t,P^x)$ be a d-dimensional Brownian motion with transition density

$$g_t(x,y)=\frac{1}{(4\pi t)^{d/2}}\,e^{-|x-y|^2/4t}\,.$$

and let (S_t) be a subordinator with the transition density $p_t(s)$.
Subordinated Brownian motion
We define subordinated Brownian motion X_t in $\mathbb{R}^d$ by the formula

$$(X_t) = (B_{S_t}),$$

with B and S independent.
Characteristic function of X_t

$$\mathbb{E}^0 e^{i(X_t,\xi)} = e^{-t\,\phi(|\xi|^2)}.$$

Potential operator of X_t

$$\begin{aligned} Gf(x) = \mathbb{E}^x \int_0^\infty f(X_t)\,dt &= \mathbb{E}_1^x \mathbb{E}_2^0 \int_0^\infty f(B_{S_t})\,dt \\ &= \mathbb{E}_1^x \int_0^\infty \int_0^\infty f(B_s)\,p_t(s)\,ds\,dt \\ &= \int_{\mathbb{R}^d} \int_0^\infty g_s(x,y)\,[\int_0^\infty p_t(s)\,dt]\,f(y)\,dy\,ds \\ &= \int_{\mathbb{R}^d} [\int_0^\infty g_s(x,y)\,u(s)\,ds]\,f(y)\,dy\,. \end{aligned}$$

Thus, potential (kernel of potential operator) is given by the formula:

$$G(x,y) = \int_0^\infty g_s(x,y)\,u(s)\,ds\,,$$

where $u(s)$ is the potential of the subordynator S.

Lévy measure of X

$$\nu(A) = \int_A \int_0^\infty g_t(x)\,\mu(dt)\,dx = \int_A J(x)\,dx\,,$$

and

$$J(x) = \int_0^\infty g_t(x)\,\mu(dt) = \int_0^\infty g_t(x)\,\mu(t)\,dt\,.$$

Define $j : (0,\infty) \to (0,\infty)$ by the formula

$$j(r) = \int_0^\infty (4\pi)^{-d/2}\,t^{-d/2}\,e^{-r^2/4t}\mu(dt)$$

and note that $J(x) = j(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$. Since $x \to g_t(x)$ is continuous and radially decreasing, both G and J are continuous and radially decreasing.

Asymptotics of Green function and Lévy measure
Let ϕ be a CBF satisfying (H) and let $\alpha < d$, $\alpha \in (0,2)$. Then

$$G(x) \approx \frac{1}{|x|^d\,\phi(|x|^{-2})} \approx \frac{1}{|x|^{d-\alpha}\,l(|x|^{-2})}\,, \quad |x| \to 0\,.$$

Let ϕ be a CBF satisfying (H) and let $\alpha < d$, $\alpha \in (0,2)$. Then

$$J(x) \approx \frac{\phi(|x|^{-2})}{|x|^d} \approx \frac{l(|x|^{-2})}{|x|^{d+\alpha}}\,, \quad |x| \to 0\,.$$

We also need the following properties of the density j.
Let ϕ be a CBF satisfying (H) and let $\alpha < d$, $\alpha \in (0,2)$. Then for every $K > 0$ there exist $C_4 = C_4(K) > 1$ and C_5 such that

- (a) $j(r) \leqslant C_4\, j(2r)\,, \quad r \in (0,K)\,,$
- (b) $j(r) \leqslant C_5\, j(r+1)\,, \quad r > 1\,.$

In the sequel we indicate how to prove Harnack Inequality under the assumption (H). Before that, we need some technical tools.
First one is the **representation of the generator** L of the process X, in terms of the Lévy measure ν (for $f \in C_b^2((\mathbb{R}^d)^2)$):

$$Lf(z) = \int_{\mathbb{R}^d} [f(z+y) - f(z) - y \cdot \nabla f(z) \mathbf{1}_{\{|y| \leqslant 1\}}]\, d\nu(y)\,.$$

We also need

Dynkin's Formula:
For open $U \subseteq \mathbb{R}^d$ and $f \in \mathcal{D}_L$ we have

$$\mathbb{E}^x f(X_{\tau_U \wedge t}) - f(x) = \mathbb{E}^x \int_0^{\tau_U \wedge t} L\, f(X_s)\, ds\,.$$

and

Ikeda-Watanabe Theorem
Let D be a bounded domain with the exterior cone property. The distribution of $(X_{\tau_{D-}}, X_{\tau_D})$ has the density function $g^x(v,y)$ with respect to P^x, $x \in D$:

$$g^x(v,y) = G_D(x,v)\, J(v-y)\,, \quad (v,y) \in D \times D^c\,.$$

By integrating, we obtain the distribution of X_{τ_D}. G_D denotes here the Green function of the domain D.

The following formulas, referred to as "Lévy system" are coming from the theory of point processes and will be needed in the sequel.

For disjoint Borel sets A and B the first expression below is a martingale; so the second one follows

$$\sum_{s \leqslant t} \mathbf{1}_{\{X_{s-} \in A\}}\, \mathbf{1}_{\{X_s \in B\}} - \int_0^t \mathbf{1}_{\{X_s \in A\}} \int_B J(u - X_s)\, du\, ds\,,$$

$$\mathbb{E}^x[\sum_{s \leqslant t} \mathbf{1}_{\{X_{s-} \in A\}}\, \mathbf{1}_{\{X_s \in B\}}] = \mathbb{E}^x \int_0^t \mathbf{1}_{\{X_s \in A\}} \int_B J(u - X_s)\, du\, ds\,.$$

6.8 Harnack Inequality - estimate $P^x(\tau_{B(x,r)} \leqslant t)$ from above

$$P^x(\sup_{s \leqslant t} |X_s - X_t| > r) \leqslant c_3\, \phi(r^{-2})\, t$$

Proof. We assume $x = 0$. Let $f \in C_b^2((\mathbb{R}^d)^2)$, $0 \leqslant f \leqslant 1$, $f(0) = 0$, $f(y) = 1$ for $|y| \geqslant 1$. Then $|f(z+y) - f(z) - y \cdot \nabla f(z)| \leqslant (c_1/2)\, |y|^2$ with $c_1 = \sup_y \sum_{j,k} |\frac{\partial^2 f(y)}{\partial_i \partial_j}|$. If $f_r(y) = f(y/r)$, for $r \in (0,1)$, then

$$\begin{aligned} |Lf_r(z)| &\leqslant \int_{\mathbb{R}^d} |f_r(z+y) - f_r(z) - y \cdot \nabla f_r(z) \mathbf{1}_{\{|y| \leqslant r\}}| d\nu(y) \\ &\leqslant \frac{c_1}{2} \int_{\mathbb{R}^d} (\frac{|y|^2}{r^2} \mathbf{1}_{\{|y| \leqslant r\}} + |f_r(z+y) - f_r(z)| \mathbf{1}_{\{|y| \geqslant r\}}) J(y) dy \\ &\leqslant c_2 \int_{\mathbb{R}^d} (\frac{|y|^2}{r^2} \mathbf{1}_{\{|y| \leqslant r\}} + \mathbf{1}_{\{|y| \geqslant r\}}) J(y) dy \leqslant c_3\, \phi(r^{-2})\,. \end{aligned}$$

This and the application of Dynkin's Formula shows that $P^0(\tau_{B(0,r)} \leqslant t) \leqslant \mathbb{E}^0 f_r(X_{\tau_{B(0,r) \wedge t}}) \leqslant c_3 \phi(r^{-2}) \mathbb{E}^0[\tau_{B(0,r)} \wedge t] \leqslant c_3 \phi(r^{-2})\, t$.

6.9 Harnack Inequality - estimate of $\mathbb{E}^x[\tau_{B(x,r)}]$ from below

$$\inf_{z \in B(x,r/2)} \mathbb{E}^x[\tau_{B(x,r)}] \geqslant \frac{1}{4c_3\, \phi((r/2)^{-2})},$$

for any $r \in (0,1)$ and $x \in \mathbb{R}^d$.
Proof.

For $z \in B(x,r)$ we obtain $P^z(\tau_{B(x,r)} \leqslant t) \leqslant P^z(\tau_{B(z,r/2)} \leqslant t) = P^0(\tau_{B(0,r/2)} \leqslant t) \leqslant c_3\, \phi((r/2)^{-2})\, t$, by the previous lemma, with the same constant c_3. Furthermore

$$\begin{aligned} \mathbb{E}^x[\tau_{B(x,r)}] &= \int_0^\infty P^z(\tau_{B(x,r)} > t) \\ &\geqslant \int_0^{t_0} P^z(\tau_{B(x,r)} > t) \geqslant t_0 P^z(\tau_{B(x,r)} > t_0) \end{aligned}$$

and $\mathbb{E}^z[\tau_{B(x,r)}] \geqslant t_0 P^z(\tau_{B(x,r)} > t_0) \geqslant t_0(1 - c_3 \phi((r/2)^{-2}))\, t_0$. Choosing $t_0 = (2c_3 \phi((r/2)^{-2}))^{-1}$ we have $1 - c_3 \phi((r/2)^{-2}) = 1/2$ and we obtain the conclusion.

6.10 Harnack Inequality - estimate of $\mathbb{E}^x[\tau_{B(x,r)}]$ from above

$$\sup_{z \in B(x,r)} \mathbb{E}^z[\tau_{B(x,r)}] \leqslant \frac{c_9}{\phi((r)^{-2})},$$

for any $r \in (0,1)$ and $x \in \mathbb{R}^d$.
We obtain

$$\begin{aligned} 1 &\geqslant P^z(|X_{\tau_{B(x,r)}} - x| > r) = P^z(X_{\tau_{B(x,r)}} \in (\overline{B(x,r)})^c) \\ &\overset{Ik.-Wat.}{=} \int_{B(x,r)} \int_{(\overline{B(x,r)})^c} G_{B(x,r)}(z,y)\, j(|u-y|)\, du\, dy \end{aligned}$$

We estimate the inner integral for $y \in B(x,r)$:

$$\int_{(\overline{B(x,r)})^c} j(|u-y|)\,du = \int_{(\overline{B(x,r)})^c\cap B(x,r)} j(|u-y|)\,du$$
$$+ \int_{(\overline{B(x,r)})^c\cap B(x,r)^c} j(|u-y|)\,du$$

$$\begin{aligned}
&\geqslant \int_{(\overline{B(x,r)})^c\cap B(x,r)} j(2|u-x|)\,du + \int_{(\overline{B(x,r)})^c\cap B(x,r)^c} j(|u-x|+1)\,du\\
&\geqslant \int_{(\overline{B(x,r)})^c\cap B(x,r)} c^{-1}\,j(|u-x|)\,du + \int_{(\overline{B(x,r)})^c\cap B(x,r)^c} c^{-1}\,j(|u-x|)\,du\\
&= c^{-1}\int_{(\overline{B(x,r)})^c} j(|u-x|)\,du
\end{aligned}$$

By the property

$$\int_r^\infty s^{d-1}\,j(s) \approx \frac{l(r^{-2})}{r^\alpha} \approx \phi(r^{-\alpha})$$

and the previous calculations we obtain

$$\begin{aligned}
1 &\geqslant c^{-1}\int_{B(x,r)} G_{B(x,r)}(z,y)\,dy \int_{(\overline{B(x,r)})^c} j(|u-x|)\,du\\
&= \mathbb{E}^z[\tau_{B(x,r)}]\,c^{-1}\,c_1\int_r^\infty v^{d-1}\,j(v)\,dv = c_2\phi(r^{-2})\,\mathbb{E}^z[\tau_{B(x,r)}]
\end{aligned}$$

6.11 Harnack Inequality - Krylov-Safonov Estimate

There exists c_{10} such that $P^y[T_A < \tau_{B(x,3r)}] \geqslant c_{10}\frac{|A|}{|B(x,r)|}$, for any $r \in (0,1)$, all $x \in \mathbb{R}^d$ and $A \subset B(x,r)$, for all $y \in B(x,2r)$

Proof. We may assume that $P^y[T_A < \tau_{B(x,3r)}] < 1/4$. Otherwise, we would have $P^y[T_A < \tau_{B(x,3r)}] \geqslant 1/4 \geqslant |A|/(4|B(x,r)|)$. Set $\tau = \tau_{B(x,3r)}$. If $z \in B(x,3r)$ and $u \in A \subset B(x,r)$ then $|u-z| \leqslant 4r$. Since j is decreasing, we obtain $j(|u-z|) \geqslant j(4r)$. Thus, for a $t_0 > 0$ (to be specified later) we have

$$\begin{aligned}
&P^y(T_A < \tau) \geqslant \mathbb{E}^y \sum_{s\leqslant T_A\wedge\tau\wedge t_0} \mathbf{1}_{\{X_{s-}\neq X_s;\, X_s\in A\}}\\
&\overset{Levy\ syst.}{=} \mathbb{E}^y\int_0^{T_A\wedge\tau\wedge t_0}\int_A j(|u-X_s|)\,du\,ds\\
&\geqslant \mathbb{E}^y\int_0^{T_A\wedge\tau\wedge t_0}\int_A j(4r)\,du\,ds = j(4r)\,|A|\,\mathbb{E}^y[T_A\wedge\tau\wedge t_0]\,.
\end{aligned}$$

By the first lemma we obtain $P^y(\tau \leqslant t) \leqslant P^y(\tau_{B(y,r)} \leqslant t) \leqslant c_1\,\phi(r^{-2})\,t$. Choosing $t_0 = (4c_1\phi(r^{-2}))^{-1}$ we obtain $P^y(\tau \leqslant t) \leqslant 1/4$. Using this and $P^y[T_A < \tau_{B(x,3r)}] < 1/4$, we obtain

$$\begin{aligned}
&\mathbb{E}^y[T_A\wedge\tau\wedge t_0] \geqslant \mathbb{E}^y[t_0;\, T_A \geqslant \tau \geqslant t_0]\\
&= t_0\,P^y(T_A \geqslant \tau \geqslant t_0) \geqslant t_0\,[1 - P^y(T_A < \tau) - P^y(\tau < t_0)]\\
&\geqslant t_0/2 = \frac{1}{8c_1\phi(r^{-2})}\,.
\end{aligned}$$

Thus, we have obtained

$$P^y(T_A < \tau) \geqslant \frac{j(4r)\,|A|}{8c_1\phi(r^{-2})}\,.$$

Taking into account the property of the density of Lévy measure $j(r) \approx r^{-d}\,\phi(r^{-2})$, $J(x) = j(|x|)$ we finally obtain the conclusion.

6.12 Harnack Inequality - weak form

There exist constants c_{11}, c_{12} such that for bounded $H \geqslant 0$, supported in $B(x, 2r)^c$ we have

$$\mathbb{E}^z H(X_{\tau_{B(x,r)}}) \leqslant c_{11}\,\mathbb{E}^z[\tau_{B(x,r)}] \int H(y)\, j(|y-x|)\, dy\,,$$
$$\mathbb{E}^z H(X_{\tau_{B(x,r)}}) \geqslant c_{12}\,\mathbb{E}^z[\tau_{B(x,r)}] \int H(y)\, j(|y-x|)\, dy\,,$$

for all $x \in \mathbb{R}^d$, $z \in B(x,r)$ and every $r \in (0,1)$.
Proof. Let $y \in B(x,r)$ and $u \in B(x,2r)^c$. If $u \in B(x,2)$ we use the estimate

$$2^{-1}|u-x| \leqslant |u-y| \leqslant 2|u-x|\,,$$

while if $u \notin B(x,2)$ then we have

$$|u-x| - 1 \leqslant |u-y| \leqslant |u-x| + 1\,.$$

Let $B \subset B(x,2r)^c$. Uing Ikeda-Watanabe theorem, we obtain

$$\mathbb{E}^z[\mathbf{1}_B(X_{\tau_{B(x,r)}})] = \mathbb{E}^z \int_{B(x,r)} G_{B(x,r)}(z,w) \int_B j(|w-u|)\, dw\, du\,.$$

We estimate the inner integral as follows:

$$\begin{aligned}
\int_B j(|u-y|)\, du &= \int_{B\cap B(x,2)} j(|u-y|)\, du + \int_{B\cap B(x,2)^c} j(|u-y|)\, du \\
&\leqslant \int_{B\cap B(x,2)} j(2^{-1}|u-x|)\, du + \int_{B\cap B(x,2)^c} j(|u-x|-1)\, du \\
&\leqslant \int_{B\cap B(x,2)} c\, j(|u-x|)\, du + \int_{B\cap B(x,2)^c} c\, j(|u-x|)\, du \\
&= c\int_B j(|u-x|)\, du\,.
\end{aligned}$$

This shows the first inequality for $\mathbf{1}_B$, hence for H as above, by approximation by simple functions; the second one is proved analogously.

6.13 Harnack Inequality

Harnack Inequality
There exists $C > 0$ such that for any $r \in (0, 1/4)$, $x_0 \in \mathbb{R}^d$, any nonnegative function u, harmonic with respect to the process X in $B(x_0, 17r)$ we have

$$u(x) \leqslant C\, u(y) \quad \text{for all } x,\, y \in B(x_0, r)\,.$$

Proof of Harnack Inequality goes by contradiction; is long and complicated. The main ingredients: Krylov-Safonov type estimates and a weak form of Harnack Inequality. The main idea is that if there is a point $x \in B(x_0, r)$ such that $u(x) = K$, with $K > 0$ "too large", we can find a sequence of points in $B(x_0, 2r)$ on which u is unbounded.

6.14 Bibliographical notes

A good introduction to potential theory of subordinated Brownian motions is contained in the last chapter of the book [4]. A pioneering result on Harnack inequality is contained in [1] and further investigations were very much influenced by this paper. A more general context is presented in [6]; in our exposition we relied on this paper. Recently, much of attention i directed towards more general Lévy processes, see [5], also for more complete references on the subject.

6.14.1 Bibliography

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