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Zero-dimensional spaces in topology and asymptology.

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Note on lecturer

Professor Taras Banakh is working in many areas of mathmatics: general, categorical, geometric, and infinite-dimensional topologies, geometry of Banach spaces, measure theory, set theory and combinatorics, topological aglebra, asymptology, theory of groups and semigroups, etc. He is the author of more than 140 papers and has more than 60 coauthors. He graduated from Lviv University in Ukraine in 1989, he defended his Ph.D. in Lviv in 1993 and in 2000 got his habilitation. Currently he is a professor at Ivan Franko Lviv National University of Lviv (Ukriane) and Jan Kochanowski University in Kielce (Poland).

ZERO-DIMENSIONAL SPACES IN TOPOLOGY AND ASYMPTOTOLOGY

TARAS BANAKH

Abstract. We present a classification and charactrization of zero-dimensional spaces in various categories that naturally appear in topology and large scale geometry.

Geometric properties are characterized by their remaining invariant under the transformations of the principal group.
Felix Klein, Erlangen program, 1872.

In this famous Erlangen program Felix Klein suggested a uniform treatment of various geomeotries classifying them according to transformations groups that preserve given geometric properties. In these lecture notes we suggest a uniform approach to various geometric disciplines (topology, the theory of uniform spaces, asymptotic topology) suggesting a common language for those sciences. Everything will happen in a single category **PreU** of preuniform spaces and their multimaps. Geometric disciplines correspond to subcategories of this category, which differ by their morphisms. We shall consider five such subcategories: Top, Born, Micro, Macro, Bi. Each of them treats some specific properties of metric spaces: topological, bornological, micro-uniform, macro-uniform, bi-uniform. In each subcategory **C** of **PreU** we shall introduce the notions of isomorphism and equivalence and shall classify zero-dimensional homogeneous metric spaces up to the **C**-equivalence. Also we shall characterize some specific objects that appear in those classifications, like the Canto micro-cube 2^ω, the Cantor macro-cube $2^{<\mathbb{N}}$ or the Cantor bi-cube $2^{<\mathbb{Z}}$.

1. Preuniform spaces

Definition 1.1. A *preuniform space* is a pair $(X,\mathcal{U}_X)$ consisting of a set X and a preuniformity $\mathcal{U}_X$ on X. A *preuniformity* on X is any family $\mathcal{U}_X$ of subsets $U\subset X\times X$ which contain the diagonal $\Delta_X=\{(x,x):x\in X\}$ of X^2. Sets $U\in\mathcal{U}_X$ will be called *entourages* or *radii* of the preuniform space. For each entourage $U\in\mathcal{U}_X$ and a point $x\in X$ the set

$$B(x,U)=\{y\in X:(x,y)\in U\}$$

is called the ball of radius U centered at x.

There are many natural examples of preuniform spaces:

Example 1.2. (1) Each metric space (X,d) carries the *canonical preuniformity* consisting of the sets $U_\varepsilon=\{(x,y)\in X^2:d(x,y)<\varepsilon\}$ for $\varepsilon\in\mathbb{R}_+$ where $\mathbb{R}_+=(0,\infty)$ stands for the half-line.

(2) Any set X carries the *trivial* preuniformity $\{\Delta_X\}$ containing the diagonal $\Delta_X=\{(x,x):x\in X\}$ of X^2. Also X carries the *discrete* preuniformity $\{X\times X,\Delta_X\}$ which is generated by the discrete $\{0,1\}$-valued metric on X.

(3) Each group G carries the preuniformity consisting of the entourages $U_F=\{(x,y)\in G^2:x\in yF\}$ where $F\subset G$ is a finite subset consisting the neutral element of G.

(4) Each topological group G carries the preuniformity consisting of entourages $U_N = \{(x,y) \in G^2 : x \in yN\}$ where N is a neighborhood of the neutral element of G.
(5) Each uniform space is preuniform.
(6) Each coarse space (in the sense of J.Roe [Roe]) is preuniform.
(7) Each ball structure $\{B(x,r) : x \in X,\ r \in R\}$ (in the sense of I.Protasov [PP]) induces the preuniformity consisting of the entourages $U_r = \bigcup_{x\in X} \{x\} \times B(x,r)$ where $r \in R$ belongs to the set of radii R.
(8) Each topological space X carries the *canonical preuniformity* consisting of the sets $\bigcup_{x\in X} \{x\} \times O_x$, where each O_x is a neighborhood of x in X.
(9) More generally, each family $\mathcal{F}$ of subsets of X (in particular, each ideal in X) with $\cup\mathcal{F} = X$ generates the preuniformity on X that consists of the sets $\bigcup_{x\in X} \{x\} \times F_x$ where $x \in F_x \in \mathcal{F}$ for all $x \in X$.

Remark 1.3. It should be mentioned that our notion of a preuniform space is close to the notion of a ball structure introduced and studied by I.Protasov [PP], [PZ].

2. Morphisms of preuniform spaces

After considering preuniform spaces, which are objects of the category **PreU**, let us turn to the morphisms of this category. Those are multivalued maps called also multimaps.

A *multimap* between two sets X,Y is any subset $\Phi \subset X \times Y$ which can be thought as a multivalued map $\Phi : X \Rightarrow Y$ assigning to each point $x \in X$ the (possibly empty) subset $\Phi(x) = \{y \in Y : (x,y) \in \Phi\}$. For a subset $A \subset X$ and a multimap $\Phi : X \Rightarrow Y$ we put $\Phi(A) = \bigcup_{x\in A} \Phi(x)$. The sets $\Phi(X)$ and $\Phi^{-1}(Y)$ are called the *range* and *domain* of the multimap Φ.

Here $\Phi^{-1} = \{(y,x) : (x,y) \in \Phi\} \subset Y \times X$ denotes the multimap, inverse to Φ. The composition of two multimaps $\Phi : X \Rightarrow Y$ and $\Psi : Y \Rightarrow Z$ is defined as usual:

$$\Psi \circ \Phi = \{(x,z) \in X \times Z : \exists y \in Y\ (x,y) \in \Phi,\ (y,z) \in \Psi\}.$$

The multimap $\Psi \circ \Phi : X \Rightarrow Z$ assigns to each point $x \in X$ the subset $\Psi \circ \Phi(x) = \bigcup_{y\in\Phi(x)} \Psi(y)$.

A multimap $\Phi : X \Rightarrow Y$ is defined to be *single-valued* if $|\Phi(x)| \le 1$ for all $x \in X$.

Definition 2.1. By **PreU** we denote the category of preuniform spaces and their multimaps.

Now we consider five subcategories of **PreU** which distinguish by their morphisms.

Definition 2.2. A morphism $\Phi : X \Rightarrow Y$ between two preuniform spaces $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ is called

- *continuous* if $\forall x \in X\ \forall y \in \Phi(x)\ \forall \varepsilon \in \mathcal{U}_Y\ \exists \delta \in \mathcal{U}_X\quad \Phi(B(x,\delta)) \subset B(y,\varepsilon)$;
- *bornologous* if $\forall x \in X\ \forall y \in \Phi(x)\ \forall \delta \in \mathcal{U}_X\ \exists \varepsilon \in \mathcal{U}_Y\quad \Phi(B(x,\delta)) \subset B(y,\varepsilon)$;
- *micro-uniform* if $\forall \varepsilon \in \mathcal{U}_Y\ \exists \delta \in \mathcal{U}_X\ \forall x \in X\ \forall y \in \Phi(X)\quad \Phi(B(x,\delta)) \subset B(y,\varepsilon)$;
- *macro-uniform* if $\forall \delta \in \mathcal{U}_X\ \exists \varepsilon \in \mathcal{U}_Y\ \forall x \in X\ \forall y \in \Phi(X)\quad \Phi(B(x,\delta)) \subset B(y,\varepsilon)$;
- *bi-uniform* if Φ is micro-uniform and macro-uniform.

Exercise 2.3. Show that a multimap $\Phi : X \Rightarrow Y$ between metric spaces X,Y is bornologous if and only if for each bounded subset $B \subset X$ the image $\Phi(B)$ is bounded in Y.

Definition 2.4. By Bi (resp. Micro, Macro, Top, Born) we denote the subcategory of the category **PreU**, consisting of preuniform spaces and their bi-uniform (resp. micro-uniform, macro-uniform, continuous, bornologous) multimaps.

These five categories are linked by the identity functors:

$$\mathsf{Top} \longleftarrow \mathsf{Micro} \longleftarrow \mathsf{Bi} \longrightarrow \mathsf{Macro} \longrightarrow \mathsf{Born}$$

3. Isomorphisms and equivalences of preuniform spaces

Let $\mathbf{C}$ be a subcategory of the category $\mathbf{PreU}$. Morphisms of the category $\mathbf{C}$ will be called $\mathbf{C}$-*morphisms.*

Definition 3.1. A multi-map $\Phi : X \Rightarrow Y$ between preuniform spaces is called

- an *isomorphism* in $\mathbf{C}$ (or just a $\mathbf{C}$-*isomorphism*) if Φ, Φ^{-1} are $\mathbf{C}$-morphisms and $\Phi \circ \Phi^{-1} = \mathrm{Id}_Y$, $\Phi^{-1} \circ \Phi = \mathrm{Id}_X$;
- an *equivalence* in $\mathbf{C}$ (or just a $\mathbf{C}$-*equivalence*) if Φ, Φ^{-1} are $\mathbf{C}$-morphisms and $\Phi(X) = Y$, $\Phi^{-1}(Y) = X$.

Besides $\mathbf{C}$-isomorphisms and $\mathbf{C}$-equivalences we shall consider the related notions of $\mathbf{C}$-embedding, $\mathbf{C}$-immersion and $\mathbf{C}$-surjection.

Definition 3.2. A multi-map $\Phi : X \Rightarrow Y$ between preuniform spaces is called

- a $\mathbf{C}$-*immersion* if Φ, Φ^{-1} are $\mathbf{C}$-morphisms and $\Phi^{-1}(Y) = X$;
- a $\mathbf{C}$-*embedding* if Φ is a $\mathbf{C}$-immersion and both multimaps Φ, Φ^{-1} are single-valued;
- a $\mathbf{C}$-*surjection* if Φ is a $\mathbf{C}$-morphism and $\Phi(X) = Y$, $\Phi^{-1}(Y) = X$.

Definition 3.3. Two preuniform spaces X, Y are $\mathbf{C}$-*equivalent* (resp. $\mathbf{C}$-*isomorphic*) in the category $\mathbf{C}$ if there is a $\mathbf{C}$-equivalence (resp. $\mathbf{C}$-isomorphism) $\Phi : X \Rightarrow Y$.

In a similar way we can define $\mathbf{C}$-equivalent preuniformities.

Definition 3.4. Let $\mathbf{C}$ be a subcategory of $\mathbf{PreU}$ two preuniformities $\mathcal{U}, \mathcal{V}$ on a set X are called $\mathbf{C}$-*equivalent* if the identity map $\mathrm{id}_X : (X, \mathcal{U}) \to (X, \mathcal{V})$ is a $\mathbf{C}$-isomorphism.

It is clear that each $\mathbf{C}$-isomorphism is a $\mathbf{C}$-equivalence. For certain subcategories of $\mathbf{PreU}$ the converse also is true.

Definition 3.5. A preuniform space $(X, \mathcal{U}_X)$ is defined to be a T_1-*space* if $\{x\} = \bigcap_{\varepsilon \in \mathcal{U}_X} B(x, \varepsilon)$.

In particular, each topological T_1-space endowed with its canonical preuniformity (defined in Exercise 1.2(8)) is a preuniform T_1-space.

Proposition 3.6. *Let* $\mathbf{C}$ *be a subcategory of* $\mathbf{PreU}$ *such that each morphism* $\Phi : X \Rightarrow Y$ *in* $\mathbf{C}$ *is a continuous multi-map. Then each* $\mathbf{C}$*-morphism* $\Phi : X \Rightarrow Y$ *to a preuniform* T_1*-space* Y *is single-valued. Consequently, each* $\mathbf{C}$*-equivalence (resp.* $\mathbf{C}$*-immersion)* $\Phi : X \Rightarrow Y$ *between* T_1*-spaces is a* $\mathbf{C}$*-isomorphism (resp.* $\mathbf{C}$*-embedding).*

Corollary 3.7. *Each equivalence between preuniform* T_1*-spaces in the categories* Top*,* Micro*, or* Bi *is an isomorphism.*

In contrast, in the categories Macro and Born the equivalences are not necessarily isomorphisms.

Example 3.8. The multi-map $\Phi : \mathbb{Z} \Rightarrow \mathbb{R}$, $\Phi : x \mapsto [x, x+1)$, between the metric spaces $\mathbb{Z}$ and $\mathbb{R}$ is an equivalence (but not an isomorphism) in the category Macro. So, the metric spaces $\mathbb{Z}$ and $\mathbb{R}$ are equivalent but not isomorphic in the categories Macro or Born.

4. General Problems

In each subcategory $\mathbf{C}$ of the category $\mathbf{PreU}$ we can consider the following general problems.

Problem 4.1 (Classification)**.** Given a class $\mathbf{K}$ of "nice" objects of the category $\mathbf{C}$ compose a list $\mathbf{L}$ of preuniform spaces such that each space $X \in \mathbf{K}$ is $\mathbf{C}$-equivalent (or $\mathbf{C}$-isomorphic) to a space $Y \in \mathbf{L}$.

Problem 4.2 (Characterization)**.** Given a "nice" object X of the category $\mathbf{C}$ characterize preuniform spaces that are $\mathbf{C}$-equivalent (or $\mathbf{C}$-isomorphic) to X.

Problem 4.3 (Embedding, Immersion, Surjection)**.** Given two preuniform spaces X, Y find conditions under which there is a $\mathbf{C}$-embedding, $\mathbf{C}$-immersion, or $\mathbf{C}$-surjection $\Phi : X \Rightarrow Y$.

Problem 4.4 (Metrizability)**.** Characterize preuniform spaces which are $\mathbf{C}$-equivalent or $\mathbf{C}$-isomorphic to metric spaces.

Problem 4.5 (Dimension)**.** Introduce a reasonable dimension function

$$\mathbf{C}\text{-}\dim : \mathbf{C} \to \omega \cup \{\infty\}$$

such that $\mathbf{C}\text{-}\dim(\mathbb{R}^n) = n$ for all $n \in \mathbb{N}$ and $\mathbf{C}\text{-}\dim(X) = \mathbf{C}\text{-}\dim(Y)$ for any two $\mathbf{C}$-equivalent spaces X, Y.

5. Operations over preuniform spaces

In this section we describe some operations over preuniform spaces.

5.1. **Localization.** For a preuniform space $\mathbf{X} = (X, \mathcal{U}_X)$ its *localization* is the preuniform space $\dot{\mathbf{X}} = (X, \dot{\mathcal{U}}_X)$ endowed with the preuniformity $\dot{\mathcal{U}}_X$ that consists of the sets

$$\bigcup_{x \in X} \{x\} \times B(x, U_x)$$

where $U_x \in \mathcal{U}_X$ for all $x \in X$.

Exercise 5.1. Show that for a metric space X the localization $\dot{X}$ of its canonic preuniformity is Micro-equivalent (resp. Macro-equivalent) to the preuniformity generated by the family of open (resp. bounded) subsets of X, see Exercise 1.2(9).

Exercise 5.2. Show that a multimap $\Phi : X \Rightarrow Y$ between two preuniform spaces is continuous (resp. bornologous) if and only if $\Phi : \dot{X} \Rightarrow \dot{Y}$ is micro-uniform (resp. macro-uniform) as a multimap between the localizations of X, Y.

5.2. **Subspaces.** Let $(X, \mathcal{U}_X)$ be a preuniform space and $Y \subset X$ be a subset. The preuniformity $\mathcal{U}_X|Y = \{Y^2 \cap U : U \in \mathcal{U}_X\}$ is called the *induced preuniformity* on Y and $(Y, \mathcal{U}_X|Y)$ is called a *subspace* of $(X, \mathcal{U}_X)$.

Exercise 5.3. For a category $\mathbf{C} \in \{\mathsf{Top}, \mathsf{Micro}, \mathsf{Bi}, \mathsf{Macro}, \mathsf{Born}\}$ check that a multimap $\Phi : X \Rightarrow Y$ between preuniform spaces X, Y is a $\mathbf{C}$-embedding (resp. $\mathbf{C}$-immersion) if and only if $\Phi : X \to \Phi(X)$ is a $\mathbf{C}$-isomorphism (a $\mathbf{C}$-equivalence) between X and the subspace $\Phi(X)$ of the preuniform space Y.

5.3. **Finite products.** The product $\mathbf{X} \times \mathbf{Y}$ of two preuniform spaces $\mathbf{X} = (X, \mathcal{U}_X)$ and $\mathbf{Y} = (Y, \mathcal{U}_Y)$ is defined as the set $X \times Y$ endowed with the preuniformity $\mathcal{U}_{X \times Y}$ consisting of the entourages

$$U \cdot V = \{((x,y),(x',y')) : (x,x') \in U,\ (y,y') \in V\},$$

where $U \in \mathcal{U}_X$, $V \in \mathcal{U}_Y$.

Exercise 5.4. Show that for metric spaces X, Y the preuniformity of the product $X \times Y$ is Bi-equivalent to the preuniformity generated by the metric

$$d((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}$$

where d_X, d_Y are the metrics of the spaces X, Y.

5.4. **Tychonoff products.** Let X_α, $\alpha \in A$, be preuniform spaces. On their product $X = \prod_{\alpha \in A} X_\alpha$ consider the preuniformity $\mathcal{U}_X$ consisting of the sets

$$\lfloor U_\alpha \rfloor_{\alpha \in F} = \{((x_\alpha),(y_\alpha)) \in X^2 : \forall \alpha \in F\ \ (x_\alpha, y_\alpha) \in U_\alpha\}$$

where $F \subset A$ is a finite subset and $U_\alpha \in \mathcal{U}_{X_\alpha}$ for $\alpha \in F$. The obtained preuniform space $\prod_{\alpha \in A} X_\alpha$ is called the *Tychonoff product* of the preuniform spaces X_α, $\alpha \in A$.

If all preuniform spaces X_α, $\alpha \in A$, are equal to some fixed preuniform space X, then the Tychonoff product $\prod_{\alpha \in A} X_\alpha$ is denoted by X^A.

Two Tychonoff products will play an important role in our considerations:

- 2^ω, the *Cantor micro-cube* and
- ω^ω, the *Baire micro-space.*

Here the cardinals $2 = \{0,1\}$ and ω are endowed with the discrete preuniformities (induced by the 2-valued metrics).

Exercise 5.5. Prove that for a sequence $(X_n)_{n \in \omega}$ of metric spaces the preuniformity of their Tychonoff product $\prod_{n \in \omega} X_n$ is Micro-equivalent to the preuniformity generated by the metric

$$d((x_n)_{n \in \omega}, (y_n)_{n \in \omega}) = \max_{n \in \omega} \big(2^{-n} \min\{\delta_n(x_n,y_n), d_n(x_n,y_n)\}\big)$$

where d_n is the metric of X_n and δ_n is the discrete 2-valued metrics on X_n.

5.5. **Tychonoff coproducts.** In this section we shall describe the operation of a Tychonoff coproduct of pointed preuniform spaces. By a *pointed space* we understand a space X with a distinguished point $*_X \in X$.

By the *Tychonoff coproduct* $\coprod^{\alpha \in A} X_\alpha$ of a family of pointed preuniform spaces X_α, $\alpha \in A$, we understand the set

$$\textstyle\coprod^{\alpha \in A} X_\alpha = \{(x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha : |\{\alpha \in A : x_\alpha \ne *_{X_\alpha}\}| < \aleph_0\}$$

endowed with the preuniformity consisting of the entourages

$$\lceil U_\alpha \rceil^{\alpha \in F} = \big\{\big((x_\alpha),(y_\alpha)\big) \in \lfloor U_\alpha \rfloor_{\alpha \in F} : \alpha \notin F\ \Rightarrow\ x_\alpha = y_\alpha\big\}$$

where $F \subset A$ is a finite subset and $U_\alpha \in \mathcal{U}_{X_\alpha}$ for $\alpha \in F$.

If all preuniform spaces X_α, $\alpha \in A$, are equal to some fixed pointed preuniform space X, then the Tychonoff coproduct $\coprod^{\alpha \in A} X_\alpha$ will be denoted by $\coprod^A X$. For the index set $A = \mathbb{N}$ the Tychonoff coproduct $\coprod^A X$ will be denoted by $X^{<\mathbb{N}}$.

Among the spaces $X^{<\mathbb{N}}$ two are of special importance:

- $2^{<\mathbb{N}}$, the *Cantor macro-cube*;
- $\omega^{<\mathbb{N}}$, the *Baire macro-space.*

Exercise 5.6. Prove that for a sequence $(X_n)_{n\in\omega}$ of metric spaces the preuniformity of their Tychonoff coproduct $\coprod^{n\in\omega} X_n$ is Macro-equivalent to the preuniformity generated by the metric

$$d((x_n)_{n\in\omega},(y_n)_{n\in\omega}) = \max_{n\in\omega}(2^n \max\{\delta_n(x_n,y_n), d_n(x_n,y_n)\})$$

where d_n is the metric of X_n and δ_n is the discrete 2-valued metric on X_n.

For each function $f:\Pi\to\omega\cup\{\omega\}$ defined on the set Π of prime numbers, consider the Tychonoff coproduct

$$\coprod^f = \coprod^{p\in\Pi}\coprod^{f(p)} p.$$

Exercise 5.7. Prove that the preuniform space $\coprod^f$ is Bi-isomorphic to the Tychonoff coproduct $\coprod^{n\in\omega}\kappa_\omega$ of any sequence of cardinals $(\kappa_n)_{n\in\omega}$, $n\in\omega$ such that

- $\{\kappa_n : n\in\omega\}\subset\{1\}\cup\Pi$;
- $|\{n\in\omega:\kappa_n=p\}| = f(p)$ for all $p\in\Pi$.

5.6. **Tychonov bi-products.** Given a sequence of pointed preuniform spaces $(X_n)_{n\in\mathbb{Z}}$ indexed by integers, consider the preuniform space

$$\prod\!\!\coprod_{n\in\mathbb{Z}} X_n = \prod_{n\in-\mathbb{N}} X_n \times X_0 \times \coprod_{n\in\mathbb{N}} X_n$$

called the *Tychonoff bi-product* of the sequence $(X_n)_{n\in\mathbb{Z}}$. Here $-\mathbb{N}=\{n\in\mathbb{Z}:n<0\}$ stands for the set of negative integer numbers.

If all spaces X_n, $n\in\mathbb{Z}$, are equal to some fixed pointed preuniform space X, then the Tychonoff bi-product $\prod\!\!\coprod_{n\in\mathbb{Z}} X_n$ is denoted by $\prod\!\!\coprod^{\mathbb{Z}} X$ or just by $X^{<\mathbb{Z}}$.

Among the spaces $X^{<\mathbb{Z}}$ two are of special importance for us:

- $2^{<\mathbb{Z}}$, the *Cantor bi-cube*, and
- $\omega^{<\mathbb{Z}}$, the *Baire bi-space*.

6. The C-Metrizability of preuniform spaces

In this section we consider the problem of metrizablity of preuniform spaces in various subcategories **C** of **PreU**. We define a preuniform space to be **C**-*metrizable* if it is **C**-isomorphic to a metric space endowed with its canonical preuniformity.

First we consider the category Top of preuniform spaces and their continuous multimaps.

Theorem 6.1 (Top-metrization)**.** *A preuniform space X is* Top*-metrizable if and only if there is a sequence $\{U_n\}_{n\in\omega}\subset\mathcal{U}_X$ such that*

(1) $U_{n+1}\circ U_{n+1}^{-1}\subset U_n$ *for all* $n\in\omega$;
(2) $\Delta_X=\bigcap_{n\in\omega}U_n$;
(3) *for each* $x\in X$ *and* $U\in\mathcal{U}_X$ *there is* $n\in\omega$ *with* $B(x,U_n)\subset B(x,U)$.

A similar characterization holds for Born-metrizable spaces.

Theorem 6.2 (Born-metrization)**.** *A preuniform space X is* Born*-metrizable if and only if there is a sequence $\{U_n\}_{n\in\omega}\in\mathcal{U}_X$ such that*

(1) $U_n\circ U_n^{-1}\subset U_{n+1}$ *for all* $n\in\omega$,
(2) $X\times X=\bigcup_{n\in\omega}U_n$,
(3) *for each* $x\in X$ *and* $U\in\mathcal{U}_X$ *there is* $n\in\omega$ *with* $B(x,U)\subset B(x,U_n)$.

Next, we consider the metrization problem in the categories Micro, Macro and Bi.

Theorem 6.3 (Micro-metrization). *A preuniform space X is* Micro*-metrizable if and only if there is a sequence $\{U_n\}_{n\in\omega} \subset \mathcal{U}_X$ such that*

(1) $U_{n+1} \circ U_{n+1}^{-1} \subset U_n$ *for all* $n \in \omega$,
(2) $\Delta_X = \bigcap_{n\in\omega} U_n$;
(3) *for each* $U \in \mathcal{U}_X$ *there is* $n \in \omega$ *such that* $U_n \subset U$.

A similar characterization holds for the category Macro.

Theorem 6.4 (Macro-metrization). *A preuniform space X is* Macro*-metrizable if and only if there is a sequence $\{U_n\}_{n\in\omega} \subset \mathcal{U}_X$ such that*

(1) $U_n \circ U_n^{-1} \subset U_{n+1}$ *for all* $n \in \omega$,
(2) $X \times X = \bigcup_{n\in\omega} U_n$,
(3) *for each* $U \in \mathcal{U}_X$ *there is* $n \in \omega$ *such that* $U \subset U_n$.

Theorem 6.5 (Bi-metrization). *A preuniform space X is* Bi*-metrizable if and only if X is* Micro*-metrizable and* Macro*-metrizable if and only if there is a sequence $\{U_n\}_{n\in\mathbb{Z}} \subset \mathcal{U}_X$ such that*

(1) $U_n \circ U_n^{-1} \subset U_{n+1}$ *for all* $n \in \mathbb{Z}$,
(2) $\Delta_X = \bigcap_{n\in\mathbb{Z}} U_n$ *and* $\bigcup_{n\in\mathbb{Z}} U_n = X \times X$,
(3) *for each* $U \in \mathcal{U}_X$ *there are numbers* $n, m \in \mathbb{Z}$ *such that* $U_n \subset U \subset U_m$.

7. Uniformizations of a preuniform space

Definition 7.1. A preuniform space $(X, \mathcal{U}_X)$ is called

- Micro-*uniformizable* if $\forall \varepsilon \in \mathcal{U}_X\ \exists \delta \in \mathcal{U}_X \quad \delta \circ \delta^{-1} \subset \varepsilon$;
- Macro-*uniformizable* if $\delta \in \mathcal{U}_X\ \exists \varepsilon \in \mathcal{U}_X \quad \delta \circ \delta^{-1} \subset \varepsilon$;
- Bi-*uniformizable* if it is Micro-uniformizable and Macro-uniformizable.

It is easy to see that each metric space is Bi-uniformizable. Now given a category $\mathbf{C} \in \{\mathsf{Top}, \mathsf{Micro}, \mathsf{Macro}, \mathsf{Born}\}$, we define the uniformizable $\mathbf{C}$-(co)reflexion of a preuniform space.

Definition 7.2. The *uniformizable* $\mathbf{C}$*-reflexion* of a preuniform space $(X, \mathcal{U})$ is the preuniform space $X_{\mathbf{C}} = (X, \mathcal{U}_{\mathbf{C}})$ endowed with the maximal Bi-uniformizable preuniformity $\mathcal{U}_{\mathbf{C}}$ such that the identity map $\mathrm{id}_X : (X, \mathcal{U}) \to (X, \mathcal{U}_{\mathbf{C}})$ is a morphism of the category $\mathbf{C}$.

The dual notion is that of the uniformizable $\mathbf{C}$-coreflexion.

Definition 7.3. The *uniformizable* $\mathbf{C}$*-coreflexion* of a preuniform space $(X, \mathcal{U})$ is the preuniform space $X^{\mathbf{C}} = (X, \mathcal{U}^{\mathbf{C}})$ endowed with the maximal Bi-uniformizable preuniformity $\mathcal{U}^{\mathbf{C}}$ such that the identity map $\mathrm{id}_X : (X, \mathcal{U}^{\mathbf{C}}) \to (X, \mathcal{U})$ is a morphism of the category $\mathbf{C}$.

The constructions $X \mapsto X_{\mathbf{C}}$ and $X \mapsto X^{\mathbf{C}}$ determine the functors

$$(\cdot)_{\mathbf{C}}, (\cdot)^{\mathbf{C}} : \mathbf{C} \to \mathsf{Bi}.$$

In fact, the functors $(\cdot)_{\mathsf{Macro}}$, $(\cdot)_{\mathsf{Born}}$, $(\cdot)^{\mathsf{Micro}}$, $(\cdot)^{\mathsf{Top}}$ are not interesting.

Exercise 7.4. Show that for any preuniform space X the spaces X_{Macro}, X_{Born}, X^{Micro}, X^{Top} coincide with the preuniform space $(X, \mathcal{U})$ where $\mathcal{U}$ is the family of all subsets $U \subset X \times X$ that contain the diagonal Δ_X of X.

On the other hand, the Bi-uniformizable spaces X_{Top}, X_{Micro}, X^{Macro} and X^{Born} are not trivial and will play an important role in our considerations. We shall describe the preuniformities of these spaces with help of pseudometrics.

We recall that a *pseudometric* on a set X is a function $d : X \times X \to [0,)$ that satisfies three axioms:

- $d(x, x) = 0$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \le d(x, y) + d(y, z)$

for all points $x, y, z \in X$.

Each preudometric d on X induces the canonical preuniformity on X consisting of the entourages $U_\varepsilon = \{(x, y) \in X^2 : d(x, y) < \varepsilon\}$ where $\varepsilon \in \mathbb{R}_+$ and $\mathbb{R}_+ = (0, \infty)$.

Definition 7.5. A pseudometric d on a preuniform space $(X, \mathcal{U}_X)$ is defined to be

- *continuous* if the identity map $\mathrm{id}_X : (X, \mathcal{U}_X) \to (X, d)$ is continuous;
- *bornologous* if the identity map $\mathrm{id}_X : (X, d) \to (X, \mathcal{U}_X)$ is bornologous;
- *micro-uniform* if the identity map $\mathrm{id}_X : (X, \mathcal{U}_X) \to (X, d)$ is micro-uniform;
- *macro-uniform* if the identity map $\mathrm{id}_X : (X, d) \to (X, \mathcal{U}_X)$ is macro-uniform;
- *bi-uniform* if d is micro-uniform and macro-uniform.

Proposition 7.6. *For a preuniform space X the preuniformity of the space X_{Top} (resp. X_{Micro}) consists of all sets $U \subset X^2$ for which there is a continuous (resp. micro-uniform) pseudometric d on X such that $\{(x, y) \in X^2 : d(x, y) < 1\} \subset U$.*

A similar characterization holds for the uniformizations X^{Macro} and X^{Born}.

Proposition 7.7. *For a preuniform space X the preuniformity of the space X_{Born} (resp. X_{Macro}) consists of all sets $U \subset X^2$ for which there is a bornologous (resp. macro-uniform) pseudometric d on X such that $\Delta_X \subset U \subset \{(x, y) \in X^2 : d(x, y) < 1\}$.*

Exercise 7.8. Prove that for a paracompact topological space X the preuniformity of the Bi-uniformizable Top reflexion X_{Top} consists of all sets $U \subset X^2$ that contain a set of the form $\bigcup_{V \in \mathcal{V}} V \times V$ for some open cover $\mathcal{V}$ of X.

8. Completeness of preuniform spaces

Let $(X, \mathcal{U}_X)$ be a preuniform space. A sequence $(x_n)_{n\in\omega} \in X^\omega$ is defined to be

- *convergent* to a point $x \in X$ if $\forall U \in \mathcal{U}_X\ \exists n \in \omega\ \forall m \ge n\ \ x_m \in B(x, U)$;
- *Cauchy* if $\forall U \in \mathcal{U}_X\ \exists n \in \omega\ \forall m, k \ge n\ \ x_m \in B(x_k, U)$.

Exercise 8.1. Show that each convergent sequence in a preuniform space $(X, \mathcal{U}_X)$ is Cauchy in the preuniform space $(X, \mathcal{U}_X \circ \mathcal{U}_X^{-1})$ where $\mathcal{U}_X \circ \mathcal{U}_X^{-1} = \{U \circ U^{-1} : U \in \mathcal{U}_X\}$.

Definition 8.2. A preuniform space X is called *sequentially complete* if each Cauchy sequence (x_n) in X converges to some point $x \in X$.

It is clear that a metric space X is complete if and only if its canonical preuniformity is sequentially complete.

9. Covering numbers of preuniform spaces

Let $(X, \mathcal{U}_X)$ be a preuniform space and $A \subset X$ be a subset. For an entourage $\delta \in \mathcal{U}_X$ the cardinal

$$\mathrm{cov}_\delta(A) = \min\big\{|\mathcal{F}| : \mathcal{F} \subset \{B(a, \delta)\}_{a\in A},\ A \subset \cup\mathcal{F}\big\}$$

is called the *δ-covering number* of the set A.

For two entourages $\delta, \varepsilon \in \mathcal{U}_X$ and a point $x \in X$ let

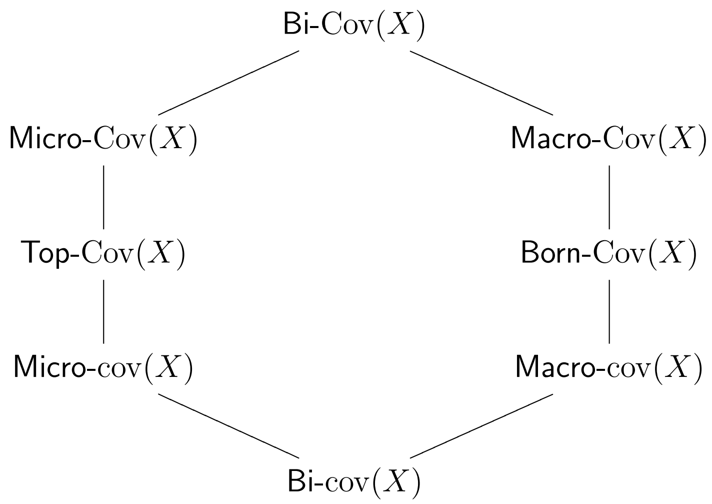
$$\mathrm{cov}_\delta^\varepsilon(x) = \mathrm{cov}_\delta(B(x,\varepsilon)).$$

For a cardinal κ by κ^+ we denote the cardinal-successor of κ.

The covering numbers $\mathrm{cov}_\delta^\varepsilon(x)$ will help us to define the following ten cardinals characteristics of a preuniform space X:

$$\begin{aligned}
&\mathsf{Micro}\text{-}\mathrm{cov}(X) = \min_{\varepsilon\in\mathcal{U}_X}\sup_{\delta\in\mathcal{U}_X}\big(\min_{x\in X}\mathrm{cov}_\delta^\varepsilon(x)\big)^+, && \mathsf{Macro}\text{-}\mathrm{cov}(X) = \min_{\delta\in\mathcal{U}_X}\sup_{\varepsilon\in\mathcal{U}_X}\big(\min_{x\in X}\mathrm{cov}_\delta^\varepsilon(x)\big)^+,\\
&\mathsf{Micro}\text{-}\mathrm{Cov}(X) = \min_{\varepsilon\in\mathcal{U}_X}\sup_{\delta\in\mathcal{U}_X}\big(\sup_{x\in X}\mathrm{cov}_\delta^\varepsilon(x)\big)^+, && \mathsf{Macro}\text{-}\mathrm{Cov}(X) = \min_{\delta\in\mathcal{U}_X}\sup_{\varepsilon\in\mathcal{U}_X}\big(\sup_{x\in X}\mathrm{cov}_\delta^\varepsilon(x)\big)^+,\\
&\mathsf{Top}\text{-}\mathrm{Cov}(X) = \min_{\varepsilon\in\mathcal{U}_X}\sup_{\delta\in\mathcal{U}_X}\sup_{x\in X}\big(\mathrm{cov}_\delta^\varepsilon(x)\big)^+, && \mathsf{Born}\text{-}\mathrm{Cov}(X) = \min_{\delta\in\mathcal{U}_X}\sup_{\varepsilon\in\mathcal{U}_X}\sup_{x\in X}\big(\mathrm{cov}_\delta^\varepsilon(x)\big)^+,\\
&\mathsf{Bi}\text{-}\mathrm{cov}(X) = \sup_{\delta,\varepsilon\in\mathcal{U}_X}\big(\min_{x\in X}\mathrm{cov}_\delta^\varepsilon(x)\big)^+, && \mathsf{Bi}\text{-}\mathrm{Cov}(X) = \sup_{\delta,\varepsilon\in\mathcal{U}_X}\big(\sup_{x\in X}\mathrm{cov}_\delta^\varepsilon(x)\big)^+,
\end{aligned}$$

For each preuniform space X these cardinal characteristics relate as follows:



As expected the **C**-covering numbers are invariant under **C**-equivalences.

Proposition 9.1. *If two preuniform spaces X, Y are:*

(1) Micro-*equivalent, then* $\mathsf{Micro}\text{-}\mathrm{Cov}(X) = \mathsf{Micro}\text{-}\mathrm{Cov}(Y)$, $\mathsf{Micro}\text{-}\mathrm{cov}(X) = \mathsf{Micro}\text{-}\mathrm{cov}(Y)$ *and* $\mathsf{Top}\text{-}\mathrm{Cov}(X) = \mathsf{Top}\text{-}\mathrm{Cov}(Y)$;
(2) Macro-*equivalent, then* $\mathsf{Macro}\text{-}\mathrm{Cov}(X) = \mathsf{Macro}\text{-}\mathrm{Cov}(Y)$, $\mathsf{Macro}\text{-}\mathrm{cov}(X) = \mathsf{Macro}\text{-}\mathrm{cov}(Y)$ *and* $\mathsf{Born}\text{-}\mathrm{Cov}(X) = \mathsf{Born}\text{-}\mathrm{Cov}(Y)$;
(3) Bi-*equivalent, then* $\mathsf{Bi}\text{-}\mathrm{Cov}(X) = \mathsf{Bi}\text{-}\mathrm{Cov}(Y)$ *and* $\mathsf{Bi}\text{-}\mathrm{cov}(X) = \mathsf{Bi}\text{-}\mathrm{cov}(Y)$.

Exercise 9.2. Show that the metric space $X = 2^\omega \times \omega$ and its Top-uniformization X_{Top} have the following covering numbers:

(1) $\mathsf{Micro}\text{-}\mathrm{cov}(X) = \mathsf{Top}\text{-}\mathrm{Cov}(X) = \mathsf{Micro}\text{-}\mathrm{Cov}(X) = \aleph_0$;
(2) $\mathsf{Macro}\text{-}\mathrm{cov}(X) = \mathsf{Born}\text{-}\mathrm{Cov}(X) = \mathsf{Macro}\text{-}\mathrm{Cov}(X) = 2$;
(3) $\mathsf{Bi}\text{-}\mathrm{cov}(X) = \mathsf{Bi}\text{-}\mathrm{Cov}(X) = \aleph_1$;
(4) $\mathsf{Micro}\text{-}\mathrm{cov}(X_{\mathsf{Top}}) = \mathsf{Top}\text{-}\mathrm{Cov}(X_{\mathsf{Top}}) = \aleph_0$, $\mathsf{Macro}\text{-}\mathrm{Cov}(X_{\mathsf{Top}}) = \aleph_1$;
(5) $\mathsf{Macro}\text{-}\mathrm{cov}(X_{\mathsf{Top}}) = \mathsf{Born}\text{-}\mathrm{Cov}(X_{\mathsf{Top}}) = \mathsf{Macro}\text{-}\mathrm{Cov}(X_{\mathsf{Top}}) = 2$;
(6) $\mathsf{Bi}\text{-}\mathrm{cov}(X_{\mathsf{Top}}) = \mathsf{Bi}\text{-}\mathrm{Cov}(X_{\mathsf{Top}}) = \aleph_1$.

Exercise 9.3. Show that the Cantor macro-cube $X = 2^{<\mathbb{N}}$ and its Born-uniformization X^{Born} have the following covering numbers:

(1) $\mathsf{Micro}\text{-}\mathrm{cov}(X) = \mathsf{Top}\text{-}\mathrm{Cov}(X) = \mathsf{Micro}\text{-}\mathrm{Cov}(X) = 2$;
(2) $\mathsf{Macro}\text{-}\mathrm{cov}(X) = \mathsf{Born}\text{-}\mathrm{Cov}(X) = \mathsf{Macro}\text{-}\mathrm{Cov}(X) = \aleph_0$;
(3) $\mathsf{Bi}\text{-}\mathrm{cov}(X) = \mathsf{Bi}\text{-}\mathrm{Cov}(X) = \aleph_0$;
(4) $\mathsf{Micro}\text{-}\mathrm{cov}(X^{\mathsf{Born}}) = \mathsf{Top}\text{-}\mathrm{Cov}(X^{\mathsf{Born}}) = \mathsf{Macro}\text{-}\mathrm{Cov}(X^{\mathsf{Born}}) = 2$,
(5) $\mathsf{Macro}\text{-}\mathrm{cov}(X^{\mathsf{Born}}) = \mathsf{Born}\text{-}\mathrm{Cov}(X^{\mathsf{Born}}) = \aleph_0$ and $\mathsf{Macro}\text{-}\mathrm{Cov}(X^{\mathsf{Born}}) = \aleph_1$;
(6) $\mathsf{Bi}\text{-}\mathrm{cov}(X^{\mathsf{Born}}) = \aleph_0$ and $\mathsf{Bi}\text{-}\mathrm{Cov}(X_{\mathsf{Top}}) = \aleph_1$.

Exercise 9.4. Given a sequence of non-zero cardinals $(\kappa_n)_{n\in\mathbb{Z}}$ endowed with the discrete preuniformities, calculate the covering numbers of the Tychonoff bi-product

$$X = \prod\!\!\coprod_{n\in\mathbb{Z}} \kappa_n = \prod_{n\in-\mathbb{N}} \kappa_n \times \kappa_0 \times \coprod_{n\in\mathbb{N}} \kappa_n$$

and the covering numbers of its Top- and Born-uniformizations X_{Top} and X^{Born}.

Definition 9.5. Let $\mathbf{C} \in \{\mathsf{Top}, \mathsf{Micro}, \mathsf{Bi}, \mathsf{Macro}, \mathsf{Born}\}$. A preuniform space $(X, \mathcal{U}_X)$ is defined to have $\mathbf{C}$-*bounded geometry* if $\mathbf{C}\text{-}\mathrm{Cov}(X) \le \aleph_0$.

Exercise 9.6. Prove that for each sequence $(\kappa_n)_{n\in\omega}$ of finite cardinals their Tychonoff product $\prod_{n\in\omega} \kappa_n$ and coproduct $\coprod^{n\in\omega} \kappa_n$ have Bi-bounded geometry.

Exercise 9.7. Prove that the preuniform space $2^\omega \times \omega \times 2^{<\mathbb{N}}$
(1) has Micro-bounded geometry;
(2) has Macro-bounded geometry;
(3) fails to have Bi-bounded geometry.

Exercise 9.8. Prove that a paracompact topological space X is locally compact if and only if its Top-uniformization X_{Top} has Top-bounded geometry.

10. Dimensions of preuniform spaces

Let $(X, \mathcal{U}_X)$ be a preuniform space and $\varepsilon \in \mathcal{U}_X$ be an entourage. By an ε-*chain* we understand any sequence $x_0, \dots, x_n \in X$ such that $x_i \in B(x_{i-1}, \varepsilon)$ for all positive $i \le n$.

By a *coloring* of a set X we understand any function $\chi : X \to Y$. Given a coloring $\chi : X \to Y$, we define a subset $A \subset X$ to be *monochrome* if $\chi(A) \subset \{y\}$ for some color $y \in Y$.

Definition 10.1. A preuniform space $(X, \mathcal{U}_X)$ has
(1) Micro-dimension $\mathsf{Micro}\text{-}\dim(X) \le n$ if for each $\varepsilon \in \mathcal{U}_X$ there is $\delta \in \mathcal{U}_X$ and a coloring $\chi : X \to n+1$ such that each monochome δ-chain $x_0, x_1, \dots, x_n$ lies in the ε-ball $B(x_0, \varepsilon)$;
(2) Macro-dimension $\mathsf{Macro}\text{-}\dim(X) \le n$ if for each $\delta \in \mathcal{U}_X$ there is $\varepsilon \in \mathcal{U}_X$ and a coloring $\chi : X \to n+1$ such that each monochome δ-chain $x_0, x_1, \dots, x_n$ lies in the ε-ball $B(x_0, \varepsilon)$.

Let

$$\mathsf{Micro}\text{-}\dim(X) = \min(\{\infty\} \cup \{n \in \mathbb{N} : \mathsf{Micro}\text{-}\dim(X) \le n\})$$

and

$$\mathsf{Macro}\text{-}\dim(X) = \min(\{\infty\} \cup \{n \in \mathbb{N} : \mathsf{Macro}\text{-}\dim(X) \le n\}).$$

Next, we define the Bi-, Top-, and Born-dimensions of a preuniform space.

Definition 10.2. For a preuniform space X let

- $\mathsf{Bi}\text{-}\dim(X) = \max\{\mathsf{Micro}\text{-}\dim(X), \mathsf{Macro}\text{-}\dim(X)\}$;
- $\mathsf{Top}\text{-}\dim(X) = \mathsf{Micro}\text{-}\dim(X_{\mathsf{Top}})$ and
- $\mathsf{Born}\text{-}\dim(X) = \mathsf{Macro}\text{-}\dim(X^{\mathsf{Born}})$.

Exercise 10.3. Prove that for a paracompact topological space X Top-$\dim(X)$ is equal to the usual covering topological dimension of X.

Proposition 10.4. *Let $\mathbf{C} \in \{\mathsf{Top}, \mathsf{Micro}, \mathsf{Bi}, \mathsf{Macro}, \mathsf{Born}\}$. If two preuniform spaces X and Y are $\mathbf{C}$-equivalent, then $\mathbf{C}$-$\dim(X) = \mathbf{C}$-$\dim(Y)$.*

Theorem 10.5. *For each category $\mathbf{C} \in \{\mathsf{Micro}, \mathsf{Macro}, \mathsf{Bi}, \mathsf{Top}\}$ and a number $n \in \mathbb{N}$ we get*

$$\mathbf{C}\text{-}\dim(\mathbb{R}^n) = n \quad \textit{and} \quad \mathsf{Born}\text{-}\dim(\mathbb{R}^n) = 1.$$

11. Preuniform spaces of dimension zero

In this section we charactrize preuniform spaces of dimension zero in various categories. Following I.Protasov, we define a preuniform space $(X, \mathcal{U}_X)$ to be *cellular* if each entourage $U \in \mathcal{U}_X$ is an equivalence relation, which means that $U = U^{-1}$ and $U \circ U = U$. It is clear that each cellular preuniformity is Micro-uniformizable and Macro-uniformizable.

Theorem 11.1. *For every category $\mathbf{C} \in \{\mathsf{Micro}, \mathsf{Bi}, \mathsf{Macro}\}$ a preuniform space X is $\mathbf{C}$-equivalent to a cellular preuniform space if and only if X is $\mathbf{C}$-uniformizable and has $\mathbf{C}$-$\dim(X) = 0$.*

Metric counterparts of cellular preuniform spaces are ultrametric spaces. We recall that a metric d on a set X is called an *ultrametric* if $d(x, z) \le \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$.

Theorem 11.2. *For every category $\mathbf{C} \in \{\mathsf{Top}, \mathsf{Micro}, \mathsf{Bi}, \mathsf{Macro}\}$ a preuniform space X is $\mathbf{C}$-equivalent to an ultrametric space if and only if X is $\mathbf{C}$-metrizable and has $\mathbf{C}$-$\dim(X) = 0$.*

12. The Born-classification of metric spaces

Theorem 12.1 (Classification and Characterization)**.** *A metric space X is Born-equivalent to:*

(1) $\emptyset$ *iff* $X = \emptyset$;
(2) $1 = \{0\}$ *iff* X *is bounded and not empty;*
(3) $\mathbb{N}$ *iff* X *is unbounded.*

Theorem 12.2 (Born-dimension)**.** *For a metric space X*

$$\mathsf{Born}\text{-}\dim(X) = \begin{cases} 0 & \text{if } X \text{ is bounded,} \\ 1 & \text{if } X \text{ is unbounded.} \end{cases}$$

Problem 12.3. Classify metric spaces up to the Born-isomorphism.

Unlike to the category Born, the problems of classification, characterization and dimension in the remaining 4 categories are non-trivial. We shall study these problems for homogeneous spaces.

13. Homogeneity

In this section we introduce several notions of homogeneity. We start with the isometric homogeneity of metric spaces.

Definition 13.1. A metric space X is *isometrically homogeneous* if for any points $x, y \in X$ there is a bijective isometry $f : X \to X$ with $f(x) = y$.

The homogeneity in the category Top is well-known:

Definition 13.2. A preuniform space X is Top-*homogeneous* if for any two points $x, y \in X$ there is a Top-equivalence $\Phi : X \Rightarrow X$ such that $y \in \Phi(x)$.

Next, we define the **C**-homogeneity for the categories $\mathbf{C} \in \{\mathsf{Micro}, \mathsf{Bi}, \mathsf{Macro}\}$.

For a multimap $\Phi : X \Rightarrow Y$ by

$$\Phi^2 : X^2 \to Y^2, \quad \Phi^2 : (x, x') \mapsto \Phi(x) \times \Phi(x')$$

we denote the square of Φ^2.

Definition 13.3. Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be two preuniform spaces and $s : \mathcal{U}_Y \to \mathcal{U}_X$, $S : \mathcal{U}_X \to \mathcal{U}_Y$ be two functions. A multimap $\Phi : X \Rightarrow Y$ is called

- Micro_s-*uniform* if $\forall \varepsilon \in \mathcal{U}_Y \;\; \Phi^2 \circ s(\varepsilon) \subset \varepsilon$;
- Macro^S-*uniform* if $\forall \delta \in \mathcal{U}_X \;\; \Phi^2(\delta) \subset S(\delta)$;
- Bi_s^S-*uniform* if Φ is Micro_s-uniform and Macro^S-uniform;
- a **C**-*equivalence* for $\mathbf{C} \in \{\mathsf{Micro}_s, \mathsf{Macro}^S, \mathsf{Bi}_s^S\}$ if $\Phi(X) = Y$, $\Phi^{-1}(Y) = X$ and both maps Φ and Φ^{-1} are **C**-uniform.

The functions s and S in the above definition are called *scale transforms*.

Exercise 13.4. Show that a multimap $\Phi : X \Rightarrow Y$ between two preuniform spaces X, Y is

- Micro-uniform iff Φ is Micro_s-uniform for some scale transform $s : \mathcal{U}_Y \to \mathcal{U}_X$;
- Micro-uniform iff Φ is Macro^S-uniform for some scale transform $S : \mathcal{U}_X \to \mathcal{U}_Y$.

Definition 13.5. A preuniform space $(X,\mathcal{U}_X)$ is defined to be

- Micro-*homogeneous* if there is a scale transform $s : \mathcal{U}_X \to \mathcal{U}_X$ such that such that for any points $x, y \in X$ there is a Micro_s-equivalence $\Phi : X \Rightarrow X$ with $y \in \Phi(x)$;
- Macro-*homogeneous* if there is a scale transform $S : \mathcal{U}_X \to \mathcal{U}_X$ such that such that for any points $x, y \in X$ there is a Macro^S-equivalence $\Phi : X \Rightarrow X$ with $y \in \Phi(x)$;
- Bi-*homogeneous* if there are scale transforms $s, S : \mathcal{U}_X \to \mathcal{U}_X$ such that such that for any points $x, y \in X$ there is a Bi_s^S-equivalence $\Phi : X \Rightarrow X$ with $y \in \Phi(x)$.

For a metric space those homogeneity properties relate as follows:

Macro-homogeneous
↑
isometrically homogeneous ⟶ Bi-homogeneous
↓
Micro-homogeneous
↓
Top-homogeneous

Exercise 13.6. Let $\mathbf{C} \in \{\mathsf{Top}, \mathsf{Micro}, \mathsf{Bi}, \mathsf{Macro}\}$. Prove that a preuniform space is **C**-homogeneous if and only if it is **C**-equivalent to a **C**-homogeneous preuniform space.

Exercise 13.7. Prove that each Micro-homogeneous compact metric space is Micro-equivalent to an isometrically homogeneous compact metric space.

Exercise 13.8. Prove that the Hilbert cube $[0, 1]^\omega$ is Top-homogeneous but not Micro-homogeneous.

Proposition 13.9. *If a preuniform space X is*

(1) Micro-*homogeneous, then* Micro-cov(X) = Top-Cov(X) = Macro-Cov(X)*;*
(2) Macro-*homogeneous, then* Macro-cov(X) = Born-Cov(X) = Macro-Cov(X)*;*
(3) Bi-*homogeneous, then* Bi-cov(X) = Bi-Cov(X).

14. Classification of isometrically homogeneous ultrametric spaces

In the following theorem, for any sequence of non-zero cardinals $(\kappa_n)_{n\in\omega}$ the Tychonoff product $\prod_{n\in\omega}\kappa_n$ and Tychonoff coproduct $\coprod^{n\in\omega}\kappa_n$ are endowed with the ultrametrics

$$d((x_n),(y_n)) = \max\{0, 2^{-n} : x_n \neq y_n\}$$

and

$$D((x_n),(y_n)) = \max\{0, 2^{n} : x_n \neq y_n\},$$

respectively.

Theorem 14.1 (**C**-classification)**.** *Assume that* **C** *is a subcategory of* **PreU** *whose morphisms contain all Lipschitz maps between ultrametric spaces.*

Each isometrically homogeneous complete ultrametric space X is **C***-isomorphic to the space* $\prod_{n\in\omega}\kappa_n \times \coprod^{n\in\omega}\lambda_n$ *for suitable sequences of cardinals* $(\kappa_n)_{n\in\omega}$ *and* $(\lambda_n)_{n\in\omega}$.

15. Classification and Characterizations in Top

Observation 15.1. *Two metric spaces X, Y are* Top*-equivalent iff X, Y are* Top*-isomorphic if and only if X, Y are homeomorphic.*

Below, the space ω of finite ordinals is considered as a metric space endowed with the discrete $\{0,1\}$-valued metric.

Theorem 15.2 (Top-Classification)**.** *Each infinite* Top*-homogeneous separable complete metric space X of* Top-dim$X = 0$ *is* Top*-equivalent to one of the spaces:* ω, 2^ω, $2^\omega \times \omega$, ω^ω.

The isometrically homogeneous spaces that appears in the Top-classification can be characterized as follows:

Theorem 15.3 (Top-characterizations)**.** *A separable complete metric space X of* Top-dim$X = 0$ *is* Top*-equivalent to:*

(1) ω *iff X is discrete and infinite;*
(2) 2^ω *iff X is compact without isolated points;*
(3) $2^\omega \times \omega$ *iff X is locally compact, non-compact, and has no isolated points;*
(4) ω^ω *iff X is nowhere locally compact.*

Next we give three general criteria of Top-equivalence, Top-embedding and Top-surjection.

Theorem 15.4 (Top-isomorphism)**.** *Two complete metric spaces X, Y are* Top*-isomorphic if*

(1) Top-dim(X) = Top-dim$(Y) = 0$*;*
(2) Top-Cov(X_{Top}) = Micro-cov(X_{Top}) = Top-Cov(Y_{Top}) = Micro-cov(Y_{Top})*;*
(3) Bi-Cov(X_{Top}) = Bi-cov(X_{Top}) = Bi-Cov(Y_{Top}) = Bi-cov(Y_{Top}).

Theorem 15.5 (Top-embedding)**.** *A metric space X admits a* Top*-embedding into a complete metric space Y if*

(1) Top-dim$(X) = 0$*;*
(2) Top-Cov$(X_{\mathsf{Top}}) \le$ Micro-cov(Y_{Top})*;*
(3) Bi-Cov$(X_{\mathsf{Top}}) \le$ Bi-cov(Y_{Top}).

Theorem 15.6 (Top-surjection)**.** *A complete metric space X admits a* Top*-surjection $f : X \to Y$ onto a complete metric space Y if*

(1) $\mathsf{Top}\text{-}\dim(X) = 0$;
(2) $\mathsf{Micro}\text{-}\mathrm{cov}(X_{\mathsf{Top}}) \ge \mathsf{Top}\text{-}\mathrm{Cov}(Y_{\mathsf{Top}})$;
(3) $\mathsf{Bi}\text{-}\mathrm{cov}(X_{\mathsf{Top}}) \ge \mathsf{Bi}\text{-}\mathrm{Cov}(Y_{\mathsf{Top}})$.

Exercise 15.7. Calculate the covering numbers of the model spaces: ω, 2^ω, $2^\omega \times \omega$, ω^ω.

16. Classification and Characterizations in Micro

Observation 16.1. *Two metric spaces X, Y are* Micro*-equivalent iff X, Y are* Micro*-isomorphic iff X, Y are uniformly homeomorphic.*

Theorem 16.2 (Micro-Classification)**.** *Each infinite* Micro*-homogeneous separable complete metric space X of* $\mathsf{Micro}\text{-}\dim(X) = 0$ *is* Micro*-equivalent to one of the spaces: ω, 2^ω, $2^\omega \times \omega$, ω^ω.*

Theorem 16.3 (Micro-Characterization of 2^ω)**.** *A preuniform space X is* Micro*-isomorphic to the Cantor micro-cube 2^ω if and only if:*

(1) *X is sequentially complete;*
(2) *X is* Micro*-metrizable;*
(3) $\mathsf{Micro}\text{-}\dim(X) = 0$;
(4) $\forall \varepsilon \in \mathcal{U}_X \;\; \mathrm{cov}_\varepsilon(X) < \aleph_0$;
(5) $\forall \varepsilon \in \mathcal{U}_X \; \exists \delta \in \mathcal{U}_X \; \forall x \in X \; \mathrm{cov}_\delta(B(x, \varepsilon)) \ge 2$.

Theorem 16.4 (Micro-Characterization of $2^\omega \times \omega$)**.** *A preuniform space X is* Micro*-isomorphic to $2^\omega \times \omega$ if and only if:*

(1) *X is sequentially complete;*
(2) *X is* Micro*-metrizable;*
(3) $\mathsf{Micro}\text{-}\dim(X) = 0$;
(4) $\forall \varepsilon \in \mathcal{U}_X \;\; \mathrm{cov}_\varepsilon(X) \le \aleph_0$;
(5) $\exists \varepsilon \in \mathcal{U}_X \;\; \mathrm{cov}_\varepsilon(X) \ge \aleph_0$;
(6) $\exists \varepsilon \in \mathcal{U}_X \; \forall \delta \in \mathcal{U}_X \; \exists m \in \mathbb{N} \; \forall x \in X \; \mathrm{cov}_\delta(B(x, \varepsilon)) \le m$;
(7) $\forall \varepsilon \in \mathcal{U}_X \; \exists \delta \in \mathcal{U}_X \; \forall x \in X \; \mathrm{cov}_\delta(B(x, \varepsilon)) \ge 2$.

Theorem 16.5 (Micro-Characterization of ω^ω)**.** *A metric space X is* Micro*-isomorphic to the Baire micro-space ω^ω if and only if:*

(1) *X is sequentially complete;*
(2) *X is* Micro*-metrizable;*
(3) $\mathsf{Micro}\text{-}\dim(X) = 0$;
(4) $\forall \varepsilon \in \mathcal{U}_X \; \mathrm{cov}_\varepsilon(X) \le \aleph_0$;
(5) $\forall \varepsilon \in \mathcal{U}_X \; \exists \delta \in \mathcal{U}_X \; \forall x \in X \; \mathrm{cov}_\delta(B(x, \varepsilon)) \ge \aleph_0$.

Next we give three general criteria of Micro-isomorphism, Micro-embedding and Micro-surjection.

Theorem 16.6 (Micro-isomorphism)**.** *Two complete metric spaces X, Y are* Micro*-isomorphic if*

(1) $\mathsf{Micro}\text{-}\dim(X) = \mathsf{Micro}\text{-}\dim(Y) = 0$;
(2) $\mathsf{Micro}\text{-}\mathrm{Cov}(X) = \mathsf{Micro}\text{-}\mathrm{cov}(X) = \mathsf{Micro}\text{-}\mathrm{Cov}(Y) = \mathsf{Micro}\text{-}\mathrm{cov}(Y)$;
(3) $\mathsf{Bi}\text{-}\mathrm{Cov}(X_{\mathsf{Micro}}) = \mathsf{Bi}\text{-}\mathrm{cov}(X_{\mathsf{Micro}}) = \mathsf{Bi}\text{-}\mathrm{Cov}(Y_{\mathsf{Micro}}) = \mathsf{Bi}\text{-}\mathrm{cov}(Y_{\mathsf{Micro}})$.

Theorem 16.7 (Micro-embedding)**.** *A metric space X admits a* Micro*-embedding into a complete metric space Y if*

(1) $\mathsf{Micro}\text{-}\dim(X) = 0$;
(2) $\mathsf{Micro}\text{-}\mathrm{Cov}(X) \le \mathsf{Micro}\text{-}\mathrm{cov}(Y)$;
(3) $\mathsf{Bi}\text{-}\mathrm{Cov}(X_{\mathsf{Micro}}) \le \mathsf{Bi}\text{-}\mathrm{cov}(Y_{\mathsf{Micro}})$.

Theorem 16.8 (Micro-surjection). *A complete metric space X admits a Micro-surjection $f : X \to Y$ onto a complete metric space Y if*

(1) $\mathsf{Micro}\text{-}\dim(X) = 0$;
(2) $\mathsf{Micro}\text{-}\mathrm{cov}(X) \ge \mathsf{Micro}\text{-}\mathrm{Cov}(Y)$;
(3) $\mathsf{Bi}\text{-}\mathrm{cov}(X_{\mathsf{Micro}}) \ge \mathsf{Bi}\text{-}\mathrm{Cov}(Y_{\mathsf{Micro}})$.

17. Classification and Characterizations in Macro

Observation 17.1. *Two metric spaces X, Y are Macro-equivalent if and only if they are coarsely equivalent in the sense of J. Roe* [Roe].

Theorem 17.2 (Macro-Classification). *Each non-empty Macro-homogeneous separable metric space X of $\mathsf{Macro}\text{-}\dim(X) = 0$ is Macro-equivalent to one of the spaces: 1, $2^{<\mathbb{N}}$, $\omega^{<\mathbb{N}}$.*

Theorem 17.3 (Macro-Characterization of $2^{<\mathbb{N}}$). *A preuniform space X is Macro-equivalent to the Cantor macro-cube $2^{<\mathbb{N}}$ if and only if:*

(1) *X is Macro-metrizable;*
(2) $\mathsf{Macro}\text{-}\dim(X) = 0$;
(3) $\exists \delta \in \mathcal{U}_X\ \forall \varepsilon \in \mathcal{U}_X\ \exists m \in \mathbb{N}\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \le m$;
(4) $\forall \delta \in \mathcal{U}_X\ \exists \varepsilon \in \mathcal{U}_X\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \ge 2$;

Theorem 17.4 (Macro-Characterization of $\omega^{<\mathbb{N}}$). *A preuniform space X is Macro-equivalent to the Baire macro-space $\omega^{<\mathbb{N}}$ if and only if:*

(1) *X is Macro-metrizable;*
(2) $\mathsf{Macro}\text{-}\dim(X) = 0$;
(3) $\exists \delta \in \mathcal{U}_X\ \forall \varepsilon \in \mathcal{U}_X\ \forall x \in X\ \mathrm{cov}_\delta(B(x,\varepsilon)) \le \aleph_0$;
(4) $\forall \delta \in \mathcal{U}_X\ \exists \varepsilon \in \mathcal{U}_X\ \forall x \in X\ \mathrm{cov}_\delta(B(x,\varepsilon)) \ge \aleph_0$.

Next we give three general criteria of Macro-equivalence, Macro-immersion and Macro-surjection.

Theorem 17.5 (Macro-equivalence). *Two metric spaces X, Y are Macro-equivalent if*

(1) $\mathsf{Macro}\text{-}\dim(X) = \mathsf{Macro}\text{-}\dim(Y) = 0$;
(2) $\mathsf{Macro}\text{-}\mathrm{Cov}(X) = \mathsf{Macro}\text{-}\mathrm{cov}(X) = \mathsf{Macro}\text{-}\mathrm{Cov}(Y) = \mathsf{Macro}\text{-}\mathrm{cov}(Y)$.

Theorem 17.6 (Macro-immersion). *A metric space X admits a Macro-immersion into a metric space Y if*

(1) $\mathsf{Macro}\text{-}\dim(X) = 0$;
(2) $\mathsf{Macro}\text{-}\mathrm{Cov}(X) \le \mathsf{Macro}\text{-}\mathrm{cov}(Y)$.

Theorem 17.7 (Macro-surjection). *A metric space X admits a Macro-surjection $f : X \to Y$ onto a metric space Y if*

(1) $\mathsf{Macro}\text{-}\dim(X) = 0$;
(2) $\mathsf{Macro}\text{-}\mathrm{cov}(X) \ge \mathsf{Macro}\text{-}\mathrm{Cov}(Y)$.

Exercise 17.8. Calculate the covering numbers of the Baire bi-space $X = \omega^{<\mathbb{N}}$ and its Born-uniformization X^{Born}.

18. The Π-function of a preuniform space

By Theorem 17.3, the ultrametric spaces $2^{<\mathbb{N}}$ and $3^{<\mathbb{N}}$ are Macro-equivalent. However, they are not Macro-isomorphic. To prove this fact, we introduce the notion of the Π-function π_X of a preuniform space X. This function is defined as follows.

Let $(X,\mathcal{U}_X)$ be a preuniform space. For a point $x \in X$ and an entourage $\varepsilon \in \mathcal{U}$ by $C^{\pm}_{\varepsilon}(x)$ we denote the $(\varepsilon \cup \varepsilon^{-1})$-connected component of x. This is the set of all points $y \in X$ that can be linked with x by a chain $x = x_0, x_1, \dots, x_n = y$ such that $(x_i, x_{i-1}) \in \varepsilon \cup \varepsilon^{-1}$ for all $i \le n$.

Let Π be the set of prime numbers and $\pi_X : \Pi \to \omega \cup \{\omega\}$ be the function assigning to each prime number $p \in \Pi$ the cardinal number

$$\pi_X(p) = \sup\{k \in \omega : \exists \varepsilon \in \mathcal{U}_X\ \forall x \in X\ \ p^k \text{ divides } |C^{\pm}_{\varepsilon}(x)|\}.$$

Here we assume that a cardinal n divides a cardinal m if $m = n \times k$ for some cardinal k.

Theorem 18.1. *If two preuniform spaces X, Y are* Macro-*isomorphic, then $\pi_X = \pi_Y$.*

Observe that for any $k \in \mathbb{N}$ and the ultrametric spaces $X = k^{<\mathbb{N}}$ we get

$$\pi_X(p) = \begin{cases} \omega & \text{if } p = k \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 18.2. *For any function $f : \Pi \to \omega + 1$ the metric space $X = \Pi^f$ has Π-function $\pi_X = f$.*

19. Classification and Characterizations in Bi

Theorem 19.1 (Bi-Classification)**.** *Each* Bi-*homogeneous separable complete metric space X of* Bi-$\dim(X) = 0$ *is* Bi-*equivalent to one of the spaces: ω, 2^ω, $2^\omega \times \omega$, ω^ω, $\omega \times 2^{<\mathbb{N}}$, $2^\omega \times 2^{<\mathbb{N}}$, $2^\omega \times \omega \times 2^{<\mathbb{N}}$, $\omega^\omega \times 2^{<\mathbb{N}}$, $\omega^{<\mathbb{N}}$, $2^\omega \times \omega^{<\mathbb{N}}$, $\omega^\omega \times \omega^{<\mathbb{N}}$, or $\coprod^f$ for a unique function $f : \Pi \to \omega \cup \{\omega\}$.*

Theorem 19.2 (Bi-Characterization of $2^\omega \times 2^{<\mathbb{N}}$)**.** *A preuniform space X is* Bi-*isomorphic to the space $2^\omega \times 2^{<\mathbb{N}}$ if and only if*

(1) *X is sequentially complete;*
(2) *X is* Bi-*metrizable;*
(3) Bi-$\dim(X) = 0$*;*
(4) $\forall \delta, \varepsilon \in \mathcal{U}_X\ \exists m \in \mathbb{N}\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \le m$*;*
(5) $\forall \varepsilon \in \mathcal{U}_X\ \exists \delta \in \mathcal{U}_X\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \ge 2$*;*
(6) $\forall \delta \in \mathcal{U}_X\ \exists \varepsilon \in \mathcal{U}_X\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \ge 2$*.*

Theorem 19.3 (Bi-Characterization of $\omega^\omega \times \omega^{<\mathbb{N}}$)**.** *A preuniform space X is* Bi-*isomorphic to the Baire bi-space $\omega^\omega \times \omega^{<\mathbb{N}}$ if and only if*

(1) *X is sequentially complete;*
(2) *X is* Bi-*metrizable;*
(3) Bi-$\dim(X) = 0$*;*
(4) $\forall \delta, \varepsilon \in \mathcal{U}_X\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \le \aleph_0$*;*
(5) $\forall \varepsilon \in \mathcal{U}_X\ \exists \delta \in \mathcal{U}_X\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \ge \aleph_0$*;*
(6) $\forall \delta \in \mathcal{U}_X\ \exists \varepsilon \in \mathcal{U}_X\ \forall x \in X\ \ \mathrm{cov}_\delta(B(x,\varepsilon)) \ge \aleph_0$*.*

Next we give three general criteria for the existence of a Bi-isomorphism, Bi-embedding and Bi-surjection.

Theorem 19.4 (Bi-isomorphism)**.** *Two complete metric spaces X, Y are* Bi-*isomorphic if*

(1) $\mathsf{Bi}\text{-}\dim(X) = \mathsf{Bi}\text{-}\dim(Y) = 0$;
(2) $\mathsf{Micro}\text{-}\mathrm{Cov}(X) = \mathsf{Micro}\text{-}\mathrm{cov}(X) = \mathsf{Micro}\text{-}\mathrm{Cov}(Y) = \mathsf{Micro}\text{-}\mathrm{cov}(Y)$;
(3) $\mathsf{Macro}\text{-}\mathrm{Cov}(X) = \mathsf{Macro}\text{-}\mathrm{cov}(X) = \mathsf{Macro}\text{-}\mathrm{Cov}(Y) = \mathsf{Macro}\text{-}\mathrm{cov}(Y)$;
(4) $\mathsf{Bi}\text{-}\mathrm{Cov}(X) = \mathsf{Bi}\text{-}\mathrm{cov}(X) = \mathsf{Bi}\text{-}\mathrm{Cov}(Y) = \mathsf{Bi}\text{-}\mathrm{cov}(Y)$ *and*
(5) $\min\{\mathsf{Bi}\text{-}\mathrm{cov}(X^{\mathsf{Macro}}), \mathsf{Bi}\text{-}\mathrm{cov}(Y^{\mathsf{Macro}})\} \ge \aleph_1$.

Theorem 19.5 (Bi-embedding). *A metric space X admits a* Bi*-embedding into a complete metric space Y if*

(1) $\mathsf{Bi}\text{-}\dim(X) = 0$;
(2) $\mathsf{Micro}\text{-}\mathrm{Cov}(X) \le \mathsf{Micro}\text{-}\mathrm{cov}(Y)$;
(3) $\mathsf{Macro}\text{-}\mathrm{Cov}(X) \le \mathsf{Macro}\text{-}\mathrm{cov}(Y)$;
(4) $\mathsf{Bi}\text{-}\mathrm{Cov}(X) \le \mathsf{Bi}\text{-}\mathrm{cov}(Y)$.

Theorem 19.6 (Bi-surjection). *A complete metric space X admits a* Bi*-surjection $f : X \to Y$ onto a complete metric space Y if*

(1) $\mathsf{Bi}\text{-}\dim(X) = 0$;
(2) $\mathsf{Micro}\text{-}\mathrm{cov}(X) \ge \mathsf{Micro}\text{-}\mathrm{Cov}(Y)$;
(3) $\mathsf{Macro}\text{-}\mathrm{cov}(X) \ge \mathsf{Macro}\text{-}\mathrm{Cov}(Y)$;
(4) $\mathsf{Bi}\text{-}\mathrm{cov}(X) \ge \mathsf{Bi}\text{-}\mathrm{Cov}(Y)$.

20. Classification Summary

The number of equivalence classes of infinite isometrically homogeneous complete ultrametric spaces in some subcategories of **PreU**:

$$\mathsf{Top}^{(4)} \longleftarrow \mathsf{Micro}^{(4)} \longleftarrow \mathsf{Bi}^{(11+\mathfrak{c})} \longrightarrow \mathsf{Macro}^{(2)} \longrightarrow \mathsf{Born}^{(1)}$$

21. Macro-classification of groups

Each group G will be considered as a preuniform space endowed with the preuniformity consisting of entourages $U_F = \{(x, y) \in G^2 : x \in yF\}$ where $F = F^{-1} \subset G$ is a finite symmetric subset containing the neutral element e of G.

In this section we consider the following (in general still open) problem.

Problem 21.1. Classify countable groups up to the Macro-equivalence or Macro-isomorphism.

We shall answer this problem the classes of locally finite and abelian groups. Let us recall that a group G is *locally finite* if each finite subset of G lies in a finite subgroup of G.

Theorem 21.2 (Smith, 2006). *A group G has* $\mathsf{Macro}\text{-}\dim(G) = 0$ *if and only if G is locally finite.*

Theorem 21.3. *Two countable locally finite groups G, H are:*

(1) Macro*-equivalent;*
(2) Macro*-isomorphic if and only if* $\pi_G = \pi_H$.

For an abelian group G by $r_0(G)$ we denote the *free rank* of G, which is equal to the maximal cardinality of a linearly independent subset $L \subset G$. The latter means that for any pairwise distinct points $x_1, \dots, x_n \in L$ and integer numbers $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ the equality

$$\lambda_1 x_1 + \dots + \lambda_n x_n = 0$$

implies $\lambda_1 = \cdots = \lambda_n = 0$.

A preuniform space $(X, \mathcal{U}_X)$ is called Macro-*connected* if there is an entourage $\varepsilon \in \mathcal{U}_X$ such that any two points $x, y \in X$ can be linked by an ε-chain $x = x_0, x_1, \dots, x_n = y$. In the opposite case the space $(X, \mathcal{U}_X)$ is called Macro-*disconnected.*

Theorem 21.4 (Banakh-Higes-Zarichnyi, 2010)**.** *For two countable abelian groups G, H the following conditions are equivalent:*

(1) *G, H are* Macro-*equivalent;*
(2) $\mathsf{Macro}\text{-}\dim(G) = \mathsf{Macro}\text{-}\dim(H)$ *and either both groups are* Macro-*connected or both are* Macro-*disconnected.*
(3) *$r_0(G) = r_0(H)$ and either both groups are finitely generated or both are infinitely generated.*

22. Some Problems

Given a subcategory $\mathbf{C}$ of $\mathbf{PreU}$, and two preuniform spaces X, Y, we write $X \le_{\mathbf{C}} Y$ if X admits a $\mathbf{C}$-immersion into Y.

Problem 22.1. Investigate the properties of the preorder $\le_{\mathbf{C}}$.

Definition 22.2. A sequence $(N_k)_{k\in\omega}$ of objects of $\mathbf{C}$ is called a $\mathbf{C}$-*number sequence* if $N_0 = \emptyset$ and for every $k \in \omega$ the following conditions hold:

(1) $N_k \le_{\mathbf{C}} N_{k+1}$ and $N_{k+1} \not\le_{\mathbf{C}} N_k$;
(2) for each metric space X in $\mathbf{C}$, $X \not\le_{\mathbf{C}} N_k$ implies $N_{k+1} \le_{\mathbf{C}} X$.

Exercise 22.3. The sequence $(k)_{k\in\omega}$ is a $\mathbf{C}$-number sequence in each category $\mathbf{C} \in \{\mathsf{Top}, \mathsf{Micro}, \mathsf{Bi}\}$.

Problem 22.4. Find a sequence of Macro-numbers.

Exercise 22.5. Show that no sequence of Born-numbers exists.

The proofs of the results announced in these lecture notes can be found in [BZ], [BHZ], [BDHM] and [Ba].

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