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Nierówności dla momentów L-statystyk i czasów pracy systemów

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(Wrocław, 22 – 23.10.2013)

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
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FUNDUSZ SPOŁECZNY

Inequalities for moments of L-statistics and lifetimes of reliability systems

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1.1 Order statistics and L-statistics

Let $X_1, \dots, X_n$ be random variables, and $X_{1:n}, \dots, X_{n:n}$ denote respective order statistics. If $X_1, \dots, X_n$ are i.i.d. with a common distribution function F, then $X_{j:n}$ has the distribution function

$$G_{j:n}(x) = \sum_{k=j}^{n} \binom{n}{k} F^k(x)[1 - F(x)]^{n-k} = F_{j:n}(F(x)).$$

If F has a density function f, then $X_{j:n}$ has the density function

$$g_{j:n}(x) = n\binom{n-1}{i-1} F^{j-1}(x)[1 - F(x)]^{n-i} f(x) = f_{j:n}(F(x))f(x).$$

Functions $F_{j:n}$ and $f_{j:n}$ are the distribution and density functions of jth order statistics $U_{j:n}$ for the standard uniform samples of size n. They are polynomials of degrees n and $n-1$, respectively. A useful formula for the expectation of jth order statistics is

$$\mathbb{E}X_{j:n} = \mathbb{E}F^{-1}(U_{j:n}) = \int_0^1 F^{-1}(x) f_{j:n}(x)dx.$$

Linear combinations of order statistics $\sum_{i=1}^{n} c_i X_{i:n}$ for arbitrarily fixed real coefficients $c_1, \dots, c_n$ are called L-statistics. They are commonly used in statistical inference. For instance, the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_{i:n}$, median $X_{\frac{n+1}{2}:n}$, midrange $\frac{1}{2}(X_{1:n} + X_{n:n})$, and trimmed means $\frac{1}{k-j+1}\sum_{i=j}^{k} X_{i:n}$ are useful in estimation of location parameter in various parametric and nonparametric models. The sample range $X_{n:n} - X_{1:n}$, inter-quartile distance $X_{\lfloor \frac{3}{4}n \rfloor:n} - X_{\lfloor \frac{1}{4}n \rfloor:n}$, and $\bar{X}_n - X_{1:n}$, are exemplary scale estimators. For calculating the expectation of L-statistic based on i.i.d. sample, a following formula is applied

$$\mathbb{E}\sum_{i=1}^{n} c_i X_{i:n} = \mathbb{E}\sum_{i=1}^{n} c_i F^{-1}(U_{j:n}) = \int_0^1 F^{-1}(x) \sum_{i=1}^{n} c_i f_{i:n}(x)dx.$$

1.2 Reliability systems

Informally speaking, a reliability system is a composition of elements which operates iff do so particular subsets of its elements. Formally, the rule is described with use of the system structure function. Let x_i denote the living status of ith component. It takes on two values 0 and 1, which means that the element already failed and is still working, respectively. The structure function $\phi\colon \{0,1\}^n \to \{0,1\}$ describes the living status of the system.

Definition 1 *(i) A system is called* monotone *if conditions $x_i \le y_i$, $i = 1, \ldots, n$, imply $\phi(x_1, \ldots, x_n) \le \phi(y_1, \ldots, y_n)$,*
(ii) semicoherent *if moreover $\phi(1, \ldots, 1) = 1$ and $\phi(0, \ldots, 0) = 0$,*
(iii) coherent *if moreover for every $i = 1, \ldots, n$ there exists $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \{0,1\}$ such that*

$$\phi(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - \phi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 1.$$

This means a failed monotone system cannot recover when some its working elements fail. A semicoherent system works (does not work) if do so all its components. A coherent system does not contain redundant elements which do not affect the system working status. Classic examples of coherent systems are the so called k-out-of-n systems, $k = 1, \ldots, n$, which work as long as do so at least k of its components. They have the structure functions

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i \ge k, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the parallel and series systems are 1-out-of-n and n-out-of-n, respectively. Another popular example of coherent system, called the bridge system, is described by means of Figure 1.

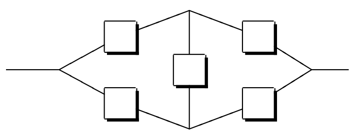

Figure 1: Bridge system

We assume that the components lifetimes are random, and denote them by $T_1, \ldots, T_n$. Then the system lifetime T, is a random variable that deterministically depends on on $T_1, \ldots, T_n$

$$T = \int_0^\infty \phi(\mathbf{1}_{[t,+\infty)}(T_1), \ldots, \mathbf{1}_{[t,+\infty)}(T_n))dt.$$

REMARKS: 1. If the system is semicoherent, then T coincides with either of $T_1, \ldots, T_n$. It is more convenient for us to note that T is equal to either of order statistics $T_{1:n}, \ldots, T_{n:n}$ of component lifetimes.
2. If ϕ is k-out-of-n, then $T = T_{n+1-k:n}$, the kth greatest order statistic.
3. If the system components are identical, it is natural to assume that $T_1, \ldots, T_n$ are exchangeable. Many researchers assume that $T_1, \ldots, T_n$ are i.i.d., but this model often does not describe well practical situations. Usually, when some components fail, the others have to bear an increased burden which shortens their further lifetimes.

Theorem 1 (Samaniego 1985, Navarro et al., 2008) *If ϕ is semicoherent, and $T_1, \ldots, T_n$ are exchangeable, then*

$$\mathbb{P}(T \le t) = \sum_{i=1}^{n} s_i \mathbb{P}(T_{i:n} \le t),$$

where the vector of non-negative combination coefficients $\mathbf{s} = (s_1, \ldots, s_n)$ summing up to 1, is called the Samaniego signature, and depends only on the system structure.

Obviously, the marginal distribution functions of order statistics $\mathbb{P}(T_{i:n} \le t)$ depends on the joint distribution of $(T_1, \ldots, T_n)$, but do not depend on the structure function ϕ.

SKETCH OF PROOF. For simplicity, we add first the assumption that $\mathbb{P}(T_i = T_j) = 0$ for $i \ne j$. Then

$$\mathbb{P}(T \le t) = \sum_{i=1}^{n} \mathbb{P}(T = T_{i:n} \le t) = \sum_{i=1}^{n} \mathbb{P}(T = T_{i:n} | T_{i:n} \le t) \mathbb{P}(T_{i:n} \le t).$$

Let Σ denote the set of all permutations of $\{1, \ldots, n\}$, and $\sigma \in \Sigma$. Note that if we know that $T_{\sigma(1)} < \ldots, T_{\sigma(n)}$ and the system structure, we are able to precisely determine the failure time of the system. If we know the order of failure times of components, we can say which element failure caused the system failure. Let $\Sigma(i) = \Sigma_\phi(i)$ denote the sets of permutations for which $T = T_{i:n}$. Under convention $T_{n+1:n} = +\infty$, the following relations hold true up to the sets of probability zero

$$\{T_{i:n} \le t\} = \bigcup_{j=i}^{n} \{T_{j:n} \le t < T_{j+1:n}\}$$

$$= \bigcup_{\sigma \in \Sigma} \bigcup_{j=i}^{n} \{T_{\sigma(1)} < \ldots < T_{\sigma(j)} \le t < T_{\sigma(j+1)} < \ldots < T_{\sigma(n)}\} = \bigcup_{\sigma \in \Sigma} \bigcup_{j=i}^{n} A_{\sigma,j,t},$$

say, and all the sets $A_{\sigma,j,t}$ are distinct. We also have

$$\{T_{i:n} \le t, T = T_{i:n}\} = \bigcup_{\sigma\in\Sigma(i)} \bigcup_{j=i}^{n} A_{\sigma,j,t}.$$

By the exchangeability property, $p(j,t) = \mathbb{P}(A_{\sigma,j,t})$ do not depend on σ. Therefore

$$\mathbb{P}(T = T_{i:n}|T_{i:n} \le t) = \frac{\sum_{\sigma\in\Sigma(i)} \sum_{j=i}^{n} p(j,t)}{\sum_{\sigma\in\Sigma} \sum_{j=i}^{n} p(j,t)} = \frac{|\Sigma(i)|}{n!} = \mathbb{P}(T = T_{i:n}) = s_i.$$

If $\mathbb{P}(T_i = T_j) > 0$ for some $i \ne j$, then we get the result by passage to the limit. $\square$

REMARKS: 4. If $\mathbb{P}(T_i = T_j) > 0$ for some $i \ne j$, then $\sum_{i=1}^{n} \mathbb{P}(T = T_{i:n}) > 1$, and $s_i \le \mathbb{P}(T = T_{i:n})$ do not have a natural probabilistic interpretation.
5. The formal definition of the Samaniego signature, based on the system structure function, is following

$$s_i = \frac{1}{\binom{n}{i-1}} \sum_{\sum_{j=1}^{n} x_j = n-i+1} \phi(x_1, \ldots, x_n) \; - \frac{1}{\binom{n}{i}} \sum_{\sum_{j=1}^{n} x_j = n-i} \phi(x_1, \ldots, x_n)$$

for $i = 1, \ldots, n$ (cf. Boland, 2001).
6. One can easily check that for the bridge system (see Figure 1), the signature amounts to $\left(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0\right)$.
7. A system with signature $\mathbf{s} = (s_1, \ldots, s_n)$ has the lifetime distribution identical with the distribution of randonly chosen k-out-of-n system, when the choice probability is s_{n+1-k}. In other words, $T \stackrel{d}{=} T_{I:n}$, where I is independent of $(T_1, \ldots, T_n)$, and equals to i with probability s_i, $i = 1, \ldots, n$. The observation provides a motivation for defining a mathematically convenient notion of mixed systems.

Definition 2 (Boland, Samaniego, 2004) *The* mixed *system with signature* $\mathbf{s} = (s_1, \ldots, s_n)$ *for arbitrary* $s_i \ge 0$ *summing up to* 1 *is the randomly chosen* k*-out-of-n system with choice probability is* s_{n+1-k}.

One can check that the lifetime distribution of every semicoherent system of size not greater than n is identical with the lifetime distribution of a mixed system of size n. Due to the Samaniego representation, we can represent lifetime moments of systems with exchangeable components as follows

$$\begin{aligned}
\mathbb{E}\,T &= \sum_{i=1}^{n} s_i\,\mathbb{E}T_{i:n}, \\
\mathbb{E}\,T^2 &= \sum_{i=1}^{n} s_i\,\mathbb{E}T_{i:n}^2, \\
\mathbb{V}ar\,T &= \sum_{i=1}^{n} s_i\,\mathbb{E}T_{i:n}^2 - \left(\sum_{i=1} n s_i\,\mathbb{E}T_{i:n}\right)^2 .
\end{aligned}$$

They all depend merely on the marginal distributions of order statistics. Also, the expectations of system lifetimes are identical with those of L-statistics, which coefficients are equal to those of the Samaniego signature vector. However, this is not true for the variances of L-statistics, because

$$\mathbb{V}ar\left(\sum_{i=1}^{n} c_i X_{i:n}\right) = \sum_{i,j=1}^{n} c_i\,c_j\,\mathbb{C}ov\,(X_{i:n}, X_{j:n})$$

depends on two-dimensional marginals as well.

1.3 Marginal distributions of order statistics of exchangeable samples

We present two characterization theorems.

Theorem 2 (Rychlik, 1993a) *Distribution functions $F_1, \dots, F_n$ are the distribution functions of consecutive order statistics from an identically distributed sample of size n with a common marginal F iff*

$$F_1 \geq \dots \geq F_n, \tag{1}$$

$$\sum_{i=1}^{n} F_i = nF. \tag{2}$$

The statement is also valid for the narrower class of exchangeable samples.

PROOF. $\Rightarrow$ The inequalities in (1) are obvious. The equality in (2) follows from the relations

$$\sum_{i=1}^{n} F_i(t) = \sum_{i=1}^{n} \mathbb{E}\mathbf{1}_{(-\infty,t]}(X_{i:n}) = \sum_{i=1}^{n} \mathbb{E}\mathbf{1}_{(-\infty,t]}(X_{i:n}) = nF(t).$$

$\Leftarrow$. The proof is constructive. We take random variable U uniformly distributed over $[0,1]$, and define $Y_i = F_i^{-1}(U)$, $i = 1,\dots,n$. Evidently, these random variables are ordered $Y_1 \le \dots \le Y_n$. Let $X_1,\dots,X_n$ denote a random permutation of $Y_1,\dots,Y_n$. Certainly, $X_1,\dots,X_n$ are exchangeable, and so identically distributed. Their order statistics $X_{i:n} = Y_i = F_i^{-1}(U)$, $i = 1,\dots,n$, have distribution functions F_i, $i = 1,\dots,n$. By definition, $\mathbb{P}(X_1 = Y_i) = \frac{1}{n}$. Therefore

$$\mathbb{P}(X_1 \le t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(Y_i = F_i^{-1}(U) \le t) = \sum_{i=1}^{n} F_i(t) = nF(t). \qquad \square$$

Theorem 3 (Rychlik, 1993a) *Let $H = \sum_{i=1}^{n} c_i F_i$ for some distribution functions satisfying (1) and (2) and arbitrarily fixed reals c_i, $i = 1,\dots,n$. Let $\underline{C}$, $\bar{C}\colon [0,1] \to \mathbb{R}$ be the greatest convex and smallest concave functions, respectively, satisfying $\underline{C}(0) = \bar{C}(0) = 0$ and*

$$\underline{C}\Big(\frac{j}{n}\Big) \le \sum_{i=1}^{j} c_i \le \bar{C}\Big(\frac{j}{n}\Big), \qquad j = 1,\dots,n.$$

Then

$$\underline{C} \circ F(x) \le \quad H(x) \quad \le \bar{C} \circ F(x), \qquad x \in \mathbb{R}. \tag{3}$$

The lower and upper equalities are attained uniformly in x by some $\underline{F}_1,\dots,\underline{F}_n$ and $\bar{F}_1,\dots,\bar{F}_n$, respectively, satisfying (1) and (2).

Remarks: 8. Functions $\underline{C}$ are $\bar{C}$ broken lines with the breaks at the points $\frac{j}{n}$ for some $1 \le j \le n-1$ (see, e.g., Figure 2). Therefore we can write

$$\begin{aligned}\underline{C}'(x) &= n\sum_{i=1}^{n} \underline{c}_i \mathbf{1}_{\left[\frac{i-1}{n},\frac{i}{n}\right)}(x),\\ \bar{C}'(x) &= n\sum_{i=1}^{n} \bar{c}_i \mathbf{1}_{\left[\frac{i-1}{n},\frac{i}{n}\right)}(x).\end{aligned}$$

One can show that $\underline{\mathbf{c}} = (\underline{c}_1,\dots,\underline{c}_n)$ and $\bar{\mathbf{c}} = (\bar{c}_1,\dots,\bar{c}_n)$ are the projections of $\mathbf{c} = (c_1,\dots,c_n)$ onto the convex cones of nondecreasing and nonincreasing sequence in the Euclidean space $\mathbb{R}^n$.

9. Sufficient conditions for attaining the lower bound in (3) are following. Let $0 = j_0 < j_1 < \dots < j_M = n$ for some $1 \le M \le n$ be all the integers such that $\underline{C}\left(\frac{j}{n}\right) = \sum_{i=1}^{j} c_i$. Then

$$F_{j_{m-1}+1}(x) = \dots = F_{j_m}(x) = \frac{nF(x) - j_{m-1}}{j_m - j_{m-1}}, \qquad F^{-1}\left(\frac{j_{m-1}}{n}\right) < x < F^{-1}\left(\frac{j_m}{n}\right),$$
$$m = 1,\dots,M,$$

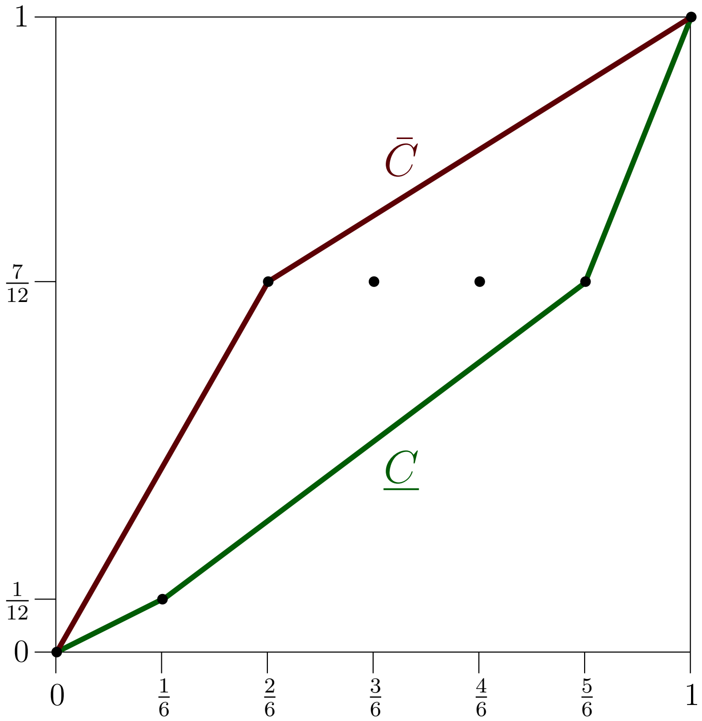

Figure 2: Functions $\underline{C}$, $\bar{C}$ for $\mathbf{c} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$

satisfy (1) and (2), and provide the equality in the first inequality of (3). The assumptions uniquely determine the marginal distributions of order statistics. If all $\frac{j_m}{n}$, $m = 0, \dots, M$, are the endpoints of the line segments in the broken line $\underline{C}$, then the conditions are necessary as well. If the endpoints $\left\{\frac{k_l}{n} : l = 0, \dots, L < M\right\}$ form a proper subset of $\left\{\frac{j_m}{n} : m = 0, \dots, M\right\}$, then the equality conditions are less stringent:

$$\begin{aligned} F_{j_{m-1}+1}(x) &= \dots = F_{j_m}(x), \quad m = 1, \dots, M, \\ \sum_{i=k_{l-1}+1}^{k_l} F_i(x) &= nF(x) - k_{l-1}, \quad F^{-1}\left(\frac{k_{l-1}}{n}\right) < x < F^{-1}\left(\frac{k_l}{n}\right), \quad l = 1, \dots, L. \end{aligned}$$

10. The conditions for attaining the upper bound in (3) are similar: it only suffices to replace $\underline{C}$ by $\bar{C}$ in Remark 9.

11. From Theorem 3 we immediately deduce the optimal bounds for the expectations of L-statistics based on identically distributed (exchangeable) sample with a common marginal F:

$$\int_0^1 F^{-1}(x)\bar{C}'(x)dx \le \mathbb{E}\sum_{i=1}^n c_i X_{i:n} \le \int_0^1 F^{-1}(x)\underline{C}'(x)dx.$$

The conditions for attaining the lower and upper expectation bounds are identical with the conditions for attaining the upper and lower distribution bounds, respectively, in inequalities (3).

EXAMPLE 1. Consider a single order statistic $X_{j:n}$. The coefficient vector $\mathbf{c} = (0, \dots, 0, 1, 0, \dots 0)$ has 1 at the position number j. The projection vectors $\underline{\mathbf{c}}$ and $\bar{\mathbf{c}}$ have coefficients $\underline{c}_i = 0$ for $i = 1, \dots, j-1$, and $\frac{1}{n+1-j}$ otherwise, and $\bar{c}_i = \frac{1}{j}$ for $i = 1, \dots, j$, and 0 otherwise. So we obtain the following sharp distribution and expectation bounds

$$\begin{aligned} \max\left\{0, \frac{nF(x)-j+1}{n-j+1}\right\} \le \quad F_j(x) & \quad \le \min\left\{\frac{nF(x)}{j}, 1\right\}, \\ \frac{n}{j}\int_0^{j/n} F^{-1}(x)dx \le \quad \mathbb{E}X_{j:n} & \quad \le \frac{n}{n-j+1}\int_{(j-1)/n}^{1} F^{-1}(x)dx. \end{aligned}$$

Establishing the equality conditions are left to the reader.

IDEA OF PROOF OF THEOREM 3: We fix $F(x) = u \in [0, 1]$, and maximize (minimize) the linear function

$$\sum_{i=1}^{n} c_i u_i$$

under the constraints

$$\begin{cases} 1 \ge u_1 \ge \dots \ge u_n \ge 0, \\ \sum_{i=1}^n u_i = nu. \end{cases}$$

The solutions of these linear programming problems $\underline{u}_1(u), \dots, \underline{u}_n(u)$ ($\bar{u}_1(u), \dots, \bar{u}_n(u)$, respectively) satisfy

$$\begin{aligned} \sum_{i=1}^{n} c_i \underline{u}_i(u) &= \underline{C}(u) \\ \Big(\sum_{i=1}^{n} c_i \bar{u}_i(u) &= \bar{C}(u), \qquad \text{respectively}\Big). \end{aligned}$$

As functions of argument $u \in [0, 1]$, they form continuous distribution functions on $[0, 1]$. $\square$

2 Expectation bounds

2.1 Mean-standard deviation bounds for L-statistics — dependent case

Theorem 4 (Rychlik, 1993b) *Let $X_1, \dots X_n$ be arbitrarily dependent identically distributed with mean μ and variance $0 < \sigma^2 < \infty$, and $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$. Put $\bar{c}_n = \frac{1}{n}\sum_{i=1}^n c_i$. Then*

$$\mathbb{E}\sum_{i=1}^n c_i \frac{X_{i:n} - \mu}{\sigma} \le \left[n \sum_{i=1}^n (\underline{c}_i - \bar{c}_n)^2 \right]^{1/2}. \tag{4}$$

If $\underline{\mathbf{c}} = (\underline{c}_1, \dots, \underline{c}_n)$ is nonconstant, then the equality in (4) *is attained if $X_1, \dots, X_n$ are the consecutive results of exhaustive drawing without replacement from the ordered population*

$$x_{i:n} = \mu + \sigma \frac{\underline{c}_i - \bar{c}}{\left[\frac{1}{n}\sum_{i=1}^n (\underline{c}_i - \bar{c}_n)^2\right]^{1/2}}, \qquad i = 1, \dots, n. \tag{5}$$

Remark 12. The lower bounds for $\mathbb{E}\sum_{i=1}^n c_i \frac{X_{i:n}-\mu}{\sigma}$ are identical with the negatives of the upper bounds for $\mathbb{E}\sum_{i=1}^n c_{n+1-i}\frac{X_{i:n}-\mu}{\sigma}$. Indeed, if we take $Y_i = -X_i$, $i = 1, \dots, n$, we get $\mathbb{E}Y_i = -\mu$, $\mathbb{V}ar\, Y_i = \sigma^2$, and $Y_{i:n} = -X_{n+1-i:n}$, $i = 1, \dots, n$. Accordingly,

$$\mathbb{E}\sum_{i=1}^n c_i \frac{Y_{i:n} + \mu}{\sigma} = -\mathbb{E}\sum_{i=1}^n c_i \frac{X_{n+1-i:n} - \mu}{\sigma} = -\mathbb{E}\sum_{i=1}^n c_{n+1-i} \frac{X_{i:n} - \mu}{\sigma}.$$

The minimal value of the first expectation coincides with the maximal value of the last one. The similar conclusions are valid if we consider the lower and upper bounds over the families of distributions that fulfil the property: if the distribution of $(X_1, \dots, X_n)$ belongs to the family, so does that of $(-X_1, \dots - X_n)$.

Proof of Theorem 4. We have

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n c_i \frac{X_{i:n} - \mu}{\sigma} &\le \int_{\mathbb{R}} \frac{x-\mu}{\sigma}\, \underline{C} \circ F(dx) = \int_0^1 \frac{F^{-1}(x) - \mu}{\sigma} [\underline{C}'(x) - n\bar{c}_n] dx \\
&\le \left\{ \int_0^1 \left[\frac{F^{-1}(x) - \mu}{\sigma} \right]^2 dx \cdot \int_0^1 [\underline{C}'(x) - n\bar{c}_n]^2 dx \right\}^{1/2} \\
&= 1 \cdot \left[n \sum_{i=1}^n (\underline{c}_i - \bar{c}_n)^2 \right]^{1/2}.
\end{aligned}$$

The equality in the latter inequality holds iff

$$\frac{F^{-1}(x)-\mu}{\sigma} = \frac{\underline{C}'(x)-n\bar{c}_n}{\left[n\sum_{i=1}^n(\underline{c}_i-\bar{c}_n)^2\right]^{1/2}}.$$

Note that the RHS integrates to 0, and has the unit L^2-norm, which is necessary for the LHS. In the urn model described in the theorem, the order statistics take on deterministic values

$$X_{j_{m-1}+1:n} = \ldots = X_{j_m:n} = \mu + \sigma\frac{\underline{c}_{j_m}-\bar{c}_n}{\left[\frac{1}{n}\sum_{i=1}^n \underline{c}_i-\bar{c}_n)^2\right]^{1/2}}, \qquad m = 1,\ldots,M.$$

Using Remark 9, we conclude that these are sufficient condition for attaining the equality in the first inequality. □

Remark 13. Replacing the Schwarz inequality by the Hölder one in the above proof, we can derive evaluations in different scale units $\sigma_p = (\mathbb{E}|X_1-\mu|^p)^{1/p}$ for various $p \geq 1$.

Theorem 4 implies sharp deterministic bounds on L-statistics.

Corollary 1 *Let* $\mathbf{c} = (c_1,\ldots,c_n) \in \mathbb{R}^n$ *be fixed, and* $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ *be arbitrary nonconstant. With the standard notation for the sample mean* $\bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i$ *and variance* $s_n^2 = \frac{1}{n}\sum_{i=1}^n(x_i-\bar{x}_n)^2$*, the following upper bound*

$$\sum_{i=1}^n c_i\frac{x_{i:n}-\bar{x}_n}{s_n} \leq \left[n\sum_{i=1}^n(\underline{c}_i-\bar{c}_n)^2\right]^{1/2}. \tag{6}$$

is sharp. The bound is attained by vectors $\mathbf{x}$ *with ordered values defined in* (5)*.*

For proving the inequality, it suffices to consider the model of exhaustive drawing without replacement from a population $\{x_1,\ldots,x_n\}$ satisfying $x_{1:n} < x_{n:n}$. The respective random vector of observations is exchangeable with $\mathbb{E}X_1 = \bar{x}_n$, $\mathbb{V}ar\,X_1 = s_n^2 > 0$, and $\mathbb{E}X_{i:n} = x_{i:n}$, $i = 1,\ldots,n$. So relation (4) can be rewritten as (6). In particular, for the single order statistics, yields

$$-\sqrt{\frac{n-j}{j}} \leq \frac{x_{j:n}-\bar{x}_n}{s_n} \leq \sqrt{\frac{j-1}{n+1-j}}.$$

Corollary 1 is useful in evaluating the expectations of standardized L-statistics coming from general (non-exhaustive) drawing without replacement model. Suppose that $X_1,\ldots,X_n$ are the outcomes of consecutive drawings without replacement from a numerical population $\pi = \{x_1,\ldots,x_N\}$ with $n < N$ and $x_{1:N} <$

$x_{N:N}$. Then $X_1, \dots, X_n$ are dependent and exchangeable with the common mean $\mathbb{E}X_1 = \bar{x}_N = \frac{1}{N}\sum_{i=1}^N x_i$ and variance $\mathbb{V}ar\, X_1 = s_N^2 = \frac{1}{N}\sum_{i=1}^N (x_i - \bar{x}_N)^2 > 0$. Consider the basic population $\pi_0 = \{1, \dots, N\}$ and transformation $g_\pi : \pi_0 \to \pi$ defined as $g_\pi(k) = x_{k:N}$. Let $U_1, \dots, U_n$ be the random results of consecutive drawings without replacement from π_0. One can easily check that

$$p_i(k) = \mathbb{P}(U_{i:n} = k) = \frac{\binom{k-1}{i-1}\binom{N-k}{n-i}}{\binom{N}{n}},$$

with the convention that $\binom{i}{j} = 0$ for $i < j$. For arbitrarily fixed $c_1, \dots, c_n$ we can write

$$\begin{aligned} \mathbb{E}\sum_{i=1}^n c_i \frac{X_{i:n} - \mu}{\sigma} &= \mathbb{E}\sum_{i=1}^n c_i \frac{g_\pi(U_{i:n}) - \mu}{\sigma} \\ &= \sum_{i=1}^n \sum_{k=1}^N c_i \frac{x_{k:n} - \bar{x}_N}{s_N} p_i(k) = \sum_{k=1}^N \left(\sum_{i=1}^n c_i p_i(k) \right) \frac{x_{k:n} - \bar{x}_N}{s_N}. \end{aligned}$$

So one can apply the results of Corollary 1 to the linear combination $\sum_{k=1}^N C_k x_{k:N}$ with $C_k = \sum_{i=1}^n c_i p_i(k)$, $k = 1, \dots, N$.

Analogously, we can evaluate the expectations of L-statistics from the drawing with replacement model. Assume now that $Y_1, \dots Y_n$ and $V_1, \dots, V_n$ are the results of drawings with replacement from populations π and π_0, respectively. We do not require here that $n \le N$. Both the samples are i.i.d., and the latter has the common marginal distribution function $F(x) = \frac{k}{N}$, $k \le x < k+1$, $0 \le x \le N$, with $\mathbb{E}Y_1 = \bar{x}_N$ and $\mathbb{V}ar\, Y_1 = s_N^2$. Also,

$$q_i(k) = \mathbb{P}(V_{i:n} = k) = F_{i:n}\left(\frac{k}{N}\right) - F_{i:n}\left(\frac{k-1}{N}\right), \qquad k = 1, \dots, N,$$

and, in consequence,

$$\mathbb{E}\sum_{i=1}^n c_i \frac{Y_{i:n} - \mu}{\sigma} = \sum_{k=1}^N \left(\sum_{i=1}^n c_i q_i(k) \right) \frac{x_{k:n} - \bar{x}_N}{s_N}.$$

This shows that we can again refer to the deterministic bounds of Corollary 1.

2.2 Mean-standard deviation bounds for L-statistics — independent case

We first present an auxiliary result.

Lemma 1 (Moriguti, 1953) *For a fixed function h integrable on an interval $[a,b]$, define $\underline{h}$ as the (right continuous, for definiteness) derivative of the greatest convex minorant $\underline{H}$ of the antiderivative $H(x) = \int_a^x h(t)\,dt$ of h. Then, for every nondecreasing function g on $[a,b]$, we have*

$$\int_a^b h(x)g(x)\,dx \le \int_a^b \underline{h}(x)g(x)\,dx,$$

(under the assumption that both the integrals exist). The equality holds iff g is constant on each interval contained in the open set $\{x : \underline{H}(x) < H(x)\}$.

Remark 14. It can be proved that if $h \in L^2([a,b],dx)$ then $\underline{h}$ is the projection of h onto the convex cone of nondecreasing elements of $L^2([a,b],dx)$.

Theorem 5 (Rychlik, 1998) *Let $X_1,\ldots,X_n$ be i.i.d. with a common mean μ and positive finite variance σ^2. For a fixed $\mathbf{c} = (c_1,\ldots,c_n) \in \mathbb{R}^n$, define $f_{\mathbf{c}:n} = \sum_{i=1}^n c_i f_{i:n}$. Let $\underline{f}_{\mathbf{c}:n}$ denote the derivative of the greatest convex minorant $\underline{F}_{\mathbf{c}:n}$ of the primitive $F_{\mathbf{c}:n} = \sum_{i=1}^n c_i F_{i:n}$ of $f_{\mathbf{c}:n}$. Then*

$$\mathbb{E}\sum_{i=1}^n c_i \frac{X_{i:n}-\mu}{\sigma} \le ||\underline{f}_{\mathbf{c}:n} - \sum_{i=1}^n c_i|| = \left[\int_0^1 \underline{f}^2_{\mathbf{c}:n}(x)dx - \left(\sum_{i=1}^n c_i\right)\right]^{1/2}.$$

If $\underline{f}_{\mathbf{c}:n}$ is non-constant, then the equality is attained iff the parent quantile function satisfies

$$\frac{F^{-1}(x)-\mu}{\sigma} = \frac{\underline{f}_{\mathbf{c}:n}(x) - \sum_{i=1}^n c_i}{||\underline{f}_{\mathbf{c}:n} - \sum_{i=1}^n c_i||}. \tag{7}$$

Proof. Applying the Moriguti and Schwarz inequalities, we obtain

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n c_i \frac{X_{i:n}-\mu}{\sigma} &= \int_0^1 \frac{F^{-1}(x)-\mu}{\sigma}\left[f_{\mathbf{c}:n}(x) - \sum_{i=1}^n c_i\right] dx \\
&\le \int_0^1 \frac{F^{-1}(x)-\mu}{\sigma}\left[\underline{f}_{\mathbf{c}:n}(x) - \sum_{i=1}^n c_i\right] dx \\
&\le \left\{\int_0^1 \left[\frac{F^{-1}(x)-\mu}{\sigma}\right]^2 dx \int_0^1 \left[\underline{f}_{\mathbf{c}:n}(x) - \sum_{i=1}^n c_i\right]^2 dx\right\}^{1/2}.
\end{aligned}$$

The first integral in the last line amounts to 1, and the square root of the latter one is the desired bound. The equality in the Schwarz inequality holds when (7) is satisfied. Note that the subtracted constant $\sum_{i=1}^n c_i$ and proportionality factor $1/||\underline{f}_{\mathbf{c}:n} - \sum_{i=1}^{n} c_i||$ are chosen so that the RHS has the integral equal to 0 and norm 1, as desired. We show now that (7) implies the equality in the Moriguti inequality as well. Note that $\underline{F}_{\mathbf{c}:n}$ is linear, and so $\underline{f}_{\mathbf{c}:n}$ is constant on every interval contained in the set $\{\underline{F}_{\mathbf{c}:n} < F_{\mathbf{c}:n}\}$. And the standardized quantile function is constant there. □

Example 2 (Moriguti, 1953). We specify the bounds for single order statistics. For $2 \le j \le n-1$, function $f_{j:n}$ is obviously positive on $[0,1]$, and first increasing on $\left[0, \frac{j-1}{n-1}\right]$, and then decreasing on $\left[\frac{j-1}{n-1}, 1\right]$. Therefore its primitive $F_{j:n}$ is convex increasing and concave increasing on the respective intervals. Hence its greatest convex minorant $\underline{F}_{j:n}$ is first identical with $F_{j:n}$, and then linear. The derivative $\underline{f}_{j:n}$ first coincides with $f_{j:n}$, and then it is constant. The change point $\alpha = \alpha(j,n)$ is the unique one among all $0 < x < 1$ such that the line passing through the points of the graph $(x, F_{j:n}(x))$ and $(1, F_{j:n}(1)) = (1,1)$ is tangent to the curve at x. So the point is determined by the equation $f_{j:n}(x)(1-x) = 1 - F_{j:n}(x)$. This can be reduced to a polynomial equation of degree $j-1$, and has to be solved numerically except for few small j's. Applying Theorem 5, we get

$$\mathbb{E}\,\frac{X_{j:n}-\mu}{\sigma} \le \left[\int_0^1 \underline{f}_{j:n}^2(x)dx - 1\right]^{1/2}$$
$$= \left[n\frac{\binom{2j-2}{j-1}\binom{2n-2j}{n-j}}{\binom{2n-1}{n}}F_{2j-1:2n-1}(\alpha) + (1-\alpha)f_{j:n}^2(\alpha) - 1\right]^{1/2}.$$

Since the projection $\underline{f}_{1:n}$ of $f_{1:n}$ is constant, and we trivially obtain $\mathbb{E}\frac{X_{1:n}-\mu}{\sigma} \le 0$. Since $\underline{f}_{n:n}$ and $f_{n:n}$ are identical, and we get a nice analytic formula

$$\mathbb{E}\,\frac{X_{n:n}-\mu}{\sigma} \le \frac{n-1}{\sqrt{2n-1}}.$$

If $j = 2 < n$, we get $\alpha = (n-1)^{-2}$, and in consequence we obtain a sophisticated formula

$$\mathbb{E}\,\frac{X_{2:n}-\mu}{\sigma} \le \left\{\frac{1}{(2n-1)(2n-3)}\left[n^2(n-1) - \frac{(n^2-2n)^{2n-1}}{(n-1)^{4n-5}}\right] - 1\right\}^{1/2}.$$

Remark 15. Using the Hölder inequality in Theorem 5 instead of the Schwarz one, we derive bounds expressed in different scale units.

2.3 Positive upper bounds — projection method

In the upper bounds on expected L-statistics from arbitrarily dependent and independent identically distributed samples, the projections onto convex cones of non-decreasing sequences and functions played an important role. It appears that the projections onto convex cones have much wider applications in evaluating the expectations of ordered random variables which was first observed by Gajek and Rychlik (1996). We recall a theorem useful in such applications.

Theorem 6 (Balakrishnan, 1981) *Let $\mathcal{C}$ be a closed convex cone in a real Hilbert space $\mathcal{H}$. Then for every $h \in \mathcal{H}$ the projection Ph onto $\mathcal{C}$ exists and it unique. Moreover, it is characterized by the following two relations*

$$\begin{aligned} (g,h) &\leq (g,Ph), \qquad\qquad g \in \mathcal{C}, \\ (g,Ph) &= (Ph,Ph). \end{aligned}$$

Corollary 2 *If $Ph \neq 0$, then*

$$\sup\{(g,h): \ g \in \mathcal{C},\ ||g|| = 1\} = ||Ph||,$$

and it is attained by $g = Ph/||Ph||$.

Proof. Indeed, applying the relations of the Theorem 6, we can write

$$\begin{aligned} (g,h) &\leq (g,Ph) \leq ||g|| \cdot ||Ph|| = ||Ph||, \\ \left(g, \frac{Ph}{||Ph||}\right) &= \frac{(Ph,Ph)}{||Ph||} = ||Ph||. \quad \square \end{aligned}$$

Below we present an exemplary application of the projection method for establishing sharp upper bound on the expectations of order statistics from the decreasing failure rate (DFR, for short) populations. We say that $F \in DFR$ if it has a density f and $(-\ln[1-F(x)])' = \frac{f(x)}{1-F(x)}$ is non-increasing. We prefer the definition: $F \in DFR$ iff $-\ln[1-F(x)]$ is concave, or equivalently, $F^{-1} \circ V(x) = F^{-1}(1-e^{-x})$ is convex. Then we admit distributions with atoms at the left end-points of their supports. The latter definition implies that the DFR family is closed. Suppose now that $X_1, \ldots, X_n$ are i.i.d. with some $F \in DFR$, $\mathbb{E}X_1 = \mu$, and $\mathbb{V}ar\, X_1 = \sigma^2 \in (0,\infty)$. We have

$$\begin{aligned} \mathbb{E} \sum_{i=1}^n c_i \frac{X_{i:n} - \mu}{\sigma} &= \int_0^1 \frac{F^{-1}(x) - \mu}{\sigma} \left[f_{\mathbf{c}:n}(x) - \sum_{i=1}^n c_i \right] dx \\ &= \int_0^\infty \frac{F^{-1}(1-e^{-x}) - \mu}{\sigma} \left[f_{\mathbf{c}:n}(1-e^{-x}) - \sum_{i=1}^n c_i \right] e^{-x}\, dx. \end{aligned}$$

Observe that functions $\frac{F^{-1}\circ V-\mu}{\sigma}$ form the convex cone

$\mathcal{C} = \{g \in L^2(\mathbb{R}_+, e^{-x}dx) : g$ is nondecresing, convex, and orthogonal to constants$\}$,

whereas $f_{\mathbf{c}:n} \circ V - \sum_{i=1}^n c_i$ is a fixed element of $L^2(\mathbb{R}_+, e^{-x}\,dx)$. It is orthogonal to constants, but usually (it depends on $\mathbf{c}$) neither non-decreasing nor convex. However, with use of Corollary 2, we obtain

$$\mathbb{E} \sum_{i=1}^n c_i \, \frac{X_{i:n} - \mu}{\sigma} \le \left|\left| P\left(f_{\mathbf{c}:n} \circ V - \sum_{i=1}^n c_i \right) \right|\right|,$$

and the equality holds for

$$\frac{F^{-1} \circ V - \mu}{\sigma} = \frac{P(f_{\mathbf{c}:n} \circ V - \sum_{i=1}^n c_i)}{||P(f_{\mathbf{c}:n} \circ V - \sum_{i=1}^n c_i)||},$$

when $P(f_{\mathbf{c}:n} \circ V - \sum_{i=1}^n c_i)$ is nonzero. So the main problem consists in determining the projection of $f_{\mathbf{c}:n} \circ V - \sum_{i=1}^n c_i$ onto the convex cone of non-decreasing convex functions in $L^2(\mathbb{R}_+, e^{-x}\,dx)$. We can ignore the orthogonality condition here, because $f_{\mathbf{c}:n} \circ V - \sum_{i=1}^n c_i$ itself is orthogonal. It follows from the easily verified fact that if the cone is translation invariant (and so is the family of non-decreasing, convex functions), then $(Ph, \mathbf{1}) = (h, \mathbf{1})$. If we get the projection, we can immediately calculate the bound, and establish attainability conditions. The trouble is, though, that there is not known a general algorithm of determining projections onto convex functions. So we should solve separately the projection problems for specific functions.

We present the solution to the problem for the particular functions $f_{j:n} \circ V - 1$, $1 \le j \le n < \infty$, which enables us to describe bounds for single order statistics from the DFR populations. We first exclude from the analysis decreasing functions $f_{1:n} \circ V - 1$, whose projections are evidently constant, and imply trivial zero bounds. Otherwise the functions share some shape properties. They are first convex increasing, then concave increasing, and finally decreasing (except for the case $j = n$). A natural candidate for projection is first identical with the original function, and then linear increasing. However, if the interval of convex increase is too short and steep (it happens for small j), then there is no enough space for the part identical with the original, and the projection is simply linear increasing. For very small j, the best linear approximation has a negative slope, and then the projection becomes constant. We come to the final solution in the following steps. We first assume that the projection has a left part identical with the original function. It has two parameters: the shape change point, and the slope of the linear component. If it is impossible to determine parameters which satisfy some necessary conditions for projection, we try to find the best linear approximation. If the line is decreasing, we arrive to the constant zero function. Basic steps of the projection procedure can be found in Gajek and Rychlik (1998), where the solution was used for more rough evaluations of expected order statistics in terms of square roots of second raw moments of the parent distribution. The mean-variance bounds are presented below. We first define

$$S(j,n) = \sum_{i=n+1-j}^{n} \frac{1}{i}, \qquad 1 \le j \le n.$$

Note that $S(j,n) = \mathbb{E}\, V_{j:n}$, when $V_1, \ldots, V_n$ are i.i.d. standard exponential.

Theorem 7 (Danielak, 2003) *Let $X_1, \ldots, X_n$ be i.i.d. with some parent distribution function $F \in DFR$, $\mathbb{E}X_1 = \mu$, and $\mathbb{V}ar\, X_1 = \sigma^2 \in (0, \infty)$.*
(i) If $S(j,n) < 1$, then $\mathbb{E}X_{j:n} \le \mu$.
(ii) If $1 < S(j,n) < 2$, then

$$\mathbb{E}\frac{X_{j:n} - \mu}{\sigma} \le S(j,n) - 1,$$

and the equality is attained by the parent exponential distribution with location $\mu - \sigma$ and scale σ.
(iii) If $S(j,n) > 2$, then

$$\mathbb{E}\frac{X_{j:n} - \mu}{\sigma} \le C = C(j,n),$$

where

$$C^2(j,n) = n\frac{\binom{2j-2}{j-1}\binom{2n-2j}{n-j}}{\binom{2n-1}{n}} F_{2j-1:2n-1}(z) + (1-z)[f_{j:n}^2(z) + 2\alpha f_{j:n}(z) + 2\alpha^2] - 1,$$

z is the smallest positive zero of the polynomial

$$\sum_{m=1}^{j-1} [2 - S(j-m+1, n-m+1)] f_{m:n+1}(x) - [n-j-1+S(1,n-j+1)] f_{j:n+1}(x),$$

and

$$\alpha = \alpha(z) = \frac{\sum_{m=1}^{j} S(j-m+1, n-m+1) f_{m:n+1}(z) - (n-j+1) f_{j:n+1}(z)}{2(n+1)(1-z)}.$$

The equality holds for the parent distribution function

$$F(x) = \begin{cases} 0, & \frac{x-\mu}{\sigma} < -\frac{1}{C}, \\ f_{j:n}^{-1}\left(\frac{x-\mu}{\sigma}C + 1\right), & -\frac{1}{C} \le \frac{x-\mu}{\sigma} < \frac{f_{j:n}(z)-1}{C}, \\ 1-(1-z)\exp\left(-\frac{x-\mu}{\sigma\alpha}C - \frac{f_{j:n}(z)-1}{\alpha}\right), & \frac{x-\mu}{\sigma} \ge \frac{f_{j:n}(z)-1}{C}. \end{cases}$$

REMARKS. 16. Using the integral approximation of the finite sequence $S(j,n) \approx \ln\frac{n+1}{n+1-j}$, we can write that $S(j,n) < 1$ if $\frac{j}{n+1} \lessapprox 1 - e^{-1} \approx 0.6321$, and $S(j,n) < 2$ if $\frac{j}{n+1} \lessapprox 1 - e^{-2} \approx 0.8647$.
17. We obtain analogous bounds for L-statistics from arbitrary dependent identically DFR-distributed samples, if we replace polynomial $f_{\mathbf{c}:n}$ by the stepwise function $\underline{C}'$ in the projection problem. In particular, we replace $f_{j:n}$ by $\frac{1}{n+1-j}\mathbf{1}_{\left[\frac{j-1}{n},1\right]}$, when we focus on single order statistics.
18. For the DFR populations, we do not obtain automatically lower bounds with use of the trick of Remark 12.

The projection method provides positive sharp upper mean-standard deviation bounds for the expectations of various statistical functions over various non-parametric families of distributions. For instance, it can be used evaluating the expectations of order statistics from independent and dependent samples, classic and kth record values, progressively censored order statistics, generalized order statistics introduced by Kamps (1995), their linear combinations and conditional expectations. Arbitrary, symmetric, symmetric unimodal and U-shaped, with monotone density and failure rate (also on the average), sub- and superadditive, NBU and NWU distributions are the exemplary families of marginal distributions that can be studied with use of the projection method. A comprehensive review of the projection method application can be found in Rychlik (2001), but a number of new results appeared later on.

2.4 Negative upper bounds - norm maximization method

We start with some preliminary remarks. Consider an abstract problem of maximizing (g,h) for fixed h and all g from a convex cone $\mathcal{C}$ with unit norm (cf. Corollary 2). If the upper bound is positive, it suffices to focus on the elements with positive inner product, and replace the original problem by the problem of *minimizing* norm over a convex set. Indeed, we can formally write

$$\sup_{g\in\mathcal{C},\,(g,h)>0,\,||g||=1}(g,h) = \frac{1}{\inf_{g\in\mathcal{C},\,(g,h)=1}||g||}.$$

If the upper bound is negative (or at least $(g,h)<0$ for all $g \in \mathcal{C}$), then the dual problem becomes the problem of *maximizing* norm over a convex set

$$\sup_{g\in\mathcal{C},\,||g||=1} (g,h) = \frac{-1}{\sup_{g\in\mathcal{C},\,(g,h)=-1}||g||}.$$

In the latter case, the mutual duality of the problems is useful in analysis. The norm functional is convex and satisfies

$$||\alpha g_1 + (1-\alpha)g_2|| \le \alpha||g_1|| + (1-\alpha)||g_2|| \le \max\{||g_1||, ||g_2||\}.$$

The convex norm functional attains its maximum over a convex set at some extreme points of its convex domain. This significantly simplifies the solution. It is worth pointing out that the norm does not need here to be defined by means of the inner product $||g|| = \sqrt{(g,g)}$. In the Banach spaces of l^p- and L^p-type, the representation of linear functionals are identical with those for l^2 and L^2.

Below we apply the norm maximization method for calculating negative sharp upper bounds on deterministic values of L-statistics, and on the expectations of order statistics with low ranks based on i.i.d. DFR-distributed samples. We first take into account the problem of maximizing

$$\sum_{i=1}^n c_i \frac{x_{i:n}-\bar{x}_n}{s_n^p}$$

over all $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ such that $x_{1:n} < x_{n:n}$. Here

$$s_p = \left(\frac{1}{n}\sum_{i=1}^n |x_i - \bar{x}_n|^p\right)^{1/p} > 0$$

for some fixed $p \ge 1$, and fixed $\mathbf{c} = (c_1,\ldots,c_n) \in \mathbb{R}^n$ is such that $\underline{c}_i = \bar{c}_n$, $i=1,\ldots,n$. The last condition is equivalent with

$$\bar{c}_n \le \min_{1\le j\le n-1} \frac{1}{j}\sum_{i=1}^j c_i. \tag{8}$$

Then we have the following projection inequality

$$\sum_{i=1}^n c_i \frac{x_{i:n}-\bar{x}_n}{s_n^p} \le \sum_{i=1}^n (\underline{c}_i - \bar{c}_n)\frac{x_{i;n}-\bar{x}_n}{s_n^p} = 0.$$

If

$$\bar{c}_n = \min_{1\le j\le n-1} \frac{1}{j}\sum_{i=1}^j c_i = \frac{1}{j_0}\sum_{i=1}^{j_0} c_i$$

for some $1 \leq j_0 \leq n-1$, then the equality in the projection inequality is attained when $x_{1:n} = \ldots = x_{j_0:n} < x_{j_0+1:n} = \ldots = x_{n:n}$, and the zero bound is optimal. Otherwise, the inequality is sharp, we can consider the dual norm maximization problem over

$$\left\{ \left(\frac{x_{1:n} - \bar{x}_n}{\sum_{i=1}^n (\underline{c}_i - \bar{c}_n)(x_{i:n} - \bar{x}_n)}, \ldots, \frac{x_{n:n} - \bar{x}_n}{\sum_{i=1}^n (\underline{c}_i - \bar{c}_n)(x_{i:n} - \bar{x}_n)} \right) : \, x_{1:n} < x_{n:n} \right\}$$
$$= \quad \{\mathbf{y} = (y_1, \ldots, y_n) : \, \mathbf{y} \text{ nondecreasing}, \sum_{i=1}^n y_i = 0, \sum_{i=1}^n (\underline{c}_i - \bar{c}_n) y_i = -1\}.$$

It is easy to verify that the set is convex and has $n-1$ extreme two-valued points: j repetitions of a negative one, and $n-j$ repetitions of a positive one for $j = 1, \ldots, n-1$. We see that the two-valued vectors are the candidates for solutions in both the subcases of (8). Therefore we combine them in the single theorem.

Theorem 8 (Goroncy, Rychlik, 2006) *Assume* (8) *and set*

$$U_p^{(j)}(\mathbf{c}) = \left[\frac{n^{p+1}}{j(n-j)^p + j^p(n-j)} \right]^{1/p} \sum_{i=1}^{j} (c_i - \bar{c}_n), \qquad j = 1, \ldots, n-1.$$

Suppose that $U_p^{(j_*)}(\mathbf{c}) = \min_{1 \leq j \leq n-1} U_p^{(j)}(\mathbf{c})$. *Then*

$$\sup_{x_{1:n} < x_{n:n}} \sum_{i=1}^n c_i \frac{x_{i:n} - \bar{x}_n}{s_p} = -U_p^{(j_*)}(\mathbf{c}).$$

and the supremum is attained by $\mathbf{x}^{(j_*)} \in \mathbb{R}^n$ *satisfying*

$$x_{i:n}^{(j_*)} = \begin{cases} \bar{x} - s_p(n-j_*) \left[\frac{n}{j_*(n-j_*)^p + j_*^p(n-j_*)} \right]^{1/p}, & i = 1, \ldots, j_*, \\ \bar{x} + s_p \, j_* \left[\frac{n}{j_*(n-j_*)^p + j_*^p(n-j_*)} \right]^{1/p}, & i = j_* + 1, \ldots, n. \end{cases}$$

Example 3. Applying Theorem 8, we write down the negative sharp upper deterministic bounds for the sample minimum:

$$\frac{x_{1:n} - \bar{x}_n}{s_p} \leq -\left[\frac{n}{(n-1)^p + n - 1} \right]^{1/p}, \qquad p \geq 1.$$

The inequality becomes equality if $\mathbf{x}$ has $n-1$ identical values, and one smaller than the others.

Now suppose that $X_1, \ldots, X_n$ are i.i.d. with a common DFR distribution, mean $\mathbb{E}X_1 = \mu$, and positive finite central absolute pth moment $\mathbb{E}|X_1 \mu|^p = \sigma_p^p$ for some $p \geq 1$. We aim at evaluating $\mathbb{E}\frac{X_{j:n}-\mu}{\sigma_p}$, for small j such that $S(j,n) < 1$ (cf Theorem 7). The dual problem is to maximize the norm of $L^p(\mathbb{R}_+, e^{-x}dx)$-space over the convex set

$$\left\{ \frac{F^{-1} \circ V - \mu}{-\int_0^\infty [F^{-1}(1-e^{-x}) - \mu][f_{j:n}(1-e^{-x}) - 1]e^{-x}\, dx} : \ F \in DFR \right\}$$
$$= \left\{ g \in L^p(\mathbb{R}_+, e^{-x}\, dx) : \ g \text{ is nondecreasing, convex}, \int_0^\infty g(x)e^{-x}dx = 0, \right.$$
$$\left. \int_0^\infty g(x)[f_{j:n}(1-e^{-x}) - 1]e^{-x}\, dx = -1 \right\}.$$

One can guess that the extreme points of the set are the broken lines:

$$g_\alpha(x) = a_\alpha + b_\alpha(x-\alpha)\mathbf{1}_{\mathbb{R}_+}(x-\alpha), \qquad \alpha > 0.$$

Two of three parameters are reduced here due to two integral constraints in the convex set definition. Relations

$$\frac{F^{-1} \circ V - \mu}{\sigma_p} = \frac{g_\alpha}{||g_\alpha||_p}$$

describe families of mixtures of the exponential distribitions with atoms of sizes $1-\exp(-\alpha)$ at their left-hand sides of supports. Precise formulation of the results is given below.

Theorem 9 (Rychlik, 2009) *Let $X_1, \ldots, X_n$ be i.i.d. random variables with a DFR distribution function F, finite mean μ, positive finite pth absolute central moment σ_p^p for some fixed $p \geq 1$, and let an integer $1 \leq j \leq n$ satisfy $S(j,n) < 1$.*
(i) If $p > 1$, then the trivial bound

$$\mathbb{E}_F \frac{X_{j:n} - \mu}{\sigma_p} \leq 0$$

is attained in the limit by sequences of parent DFR distributions which are mixtures of exponential distributions and atoms at their left end-points, as the atom probabilities tend to 1.
(ii) If $p = 1$, then

$$\mathbb{E}_F \frac{X_{j:n} - \mu}{\sigma_1} \leq \begin{cases} -\frac{1}{2}, & \text{if } 0 < S(j,n) < 1 - e^{-1}, \\ -\frac{e}{2}[1 - S(j,n)], & \text{if } 1 - e^{-1} < S(j,n) < 1. \end{cases}$$

In the first case, the equality is attained under the conditions described in point (i). In the latter one, the inequality becomes equality for the exponential distribution, without any discrete contribution.

REMARKS. 19. Due to the logarithmic approximation, $S(j,n) \approx 1 - e^{-1}$ when $\frac{j}{n+1} \approx 1 - \exp(-1 + e^{-1}) \approx 0.4685$.
20. *All the inequalities concerning the expectations of convex combinations of order statistics $\mathbb{E} \sum_{i=1}^{n} s_i X_{i:n}$, both from independent and arbitrarily dependent, identically distributed samples with marginal distributions from general and restricted families of distributions, hold true for the expected lifetimes $\mathbb{E}\,T$ of mixed systems with signatures $(s_1, \dots, s_n)$, composed of independent and exchangeable components, respectively, whose lifetime marginal distributions belong to the respective families.*

3 Variance bounds

3.1 Independent case

Theorem 10 (Papadatos, 1995) *Suppose that $X_1, \dots, X_n$ are i.i.d. and have a positive finite variance.*
(i) For $2 \le j \le n-1$, equation

$$u(1-u)f_{j:n}(u)[1 - 2F_{j:n}(u)] = (1-2u)F_{j:n}(u)[1 - F_{j:n}(u)]$$

has a unique solution $u_0 = u_0(j,n)$ in $(0,1)$.
(ii) The inequality

$$\frac{\mathbb{V}ar\, X_{j:n}}{\mathbb{V}ar\, X_1} \le \frac{F_{j:n}(u_0)[1 - F_{j:n}(u_0)]}{u_0(1-u_0)}$$

holds, and the equality is attained by two-point distributions such that the probability of the smaller point is u_0.
(ii) For $j = 1$ and n, we have

$$\frac{\mathbb{V}ar\, X_{j:n}}{\mathbb{V}ar\, X_1} \le n,$$

with the equality attained in the limit by two-point distributions, when the probability of the smaller one tends to 1 and 0, respectively.

PROOF. Using the Hoeffding representation of variance, we write

$$\mathbb{V}ar\, X_1 = \iint_{x \le y} F(x)[1 - F(y)]dxdy.$$

Similarly

$$\begin{aligned}
&\mathbb{V}ar\, X_{j:n} = \iint_{x\le y} F_{j:n}(F(x))[1-F_{j:n}(F(y))]\,dx\,dy\\
=&\ \iint_{0<F(x)\le F(y)<1} \frac{F_{j:n}(F(x))[1-F_{j:n}(F(y))]}{F(x)[1-F(y)]} F(x)[1-F(y)]\,dx\,dy\\
\le&\ \sup_{0<u=F(x)\le v=F(y)<1} \frac{F_{j:n}(u)}{u}\,\frac{1-F_{j:n}(v)}{1-v} \iint_{0<F(x)\le F(y)<1} F(x)[1-F(y)]\,dx\,dy\\
\le&\ \sup_{0<u\le v<1} \frac{F_{j:n}(u)}{u}\frac{1-F_{j:n}(v)}{1-v} \iint_{x\le y} F(x)[1-F(y)]\,dx\,dy, \qquad (9)
\end{aligned}$$

and the supremum is our bound. Below we determine it more precisely. Suppose first that $2\le j\le n-1$. It is easy to verify that function $(0,1)\ni v\to \frac{1-F_{j:n}(v)}{1-v}$ is increasing from 1 at 0 to a maximal value at some point $u_1<\frac{j-1}{n-1}$ satisfying $1-F_{j:n}(u_1)=(1-u_1)f_{j:n}(u_1)$ and then decreases to 0 at 1 (cf a reasoning from Example 2). We can similarly check that $(0,1)\ni u\to\frac{F_{j:n}(u)}{u}$ is increasing from 0 at 0 to a maximal value at some $u_2>\frac{j-1}{n-1}$ satisfying $F_{j:n}(u_2)=u_2 f_{j:n}(u_2)$ and then decreases to 1 at 1. Straightforward arguments lead to the conclusion that the maximal value of the product $\frac{F_{j:n}(u)}{u}\frac{1-F_{j:n}(v)}{1-v}$ over the triangle $\{0<u\le v<1\}$ can be found on a subsegment $\{(u,u):\,u_1\le u\le u_2\}$ of the hypotenuse. The product $\frac{F_{j:n}(u)}{u}\frac{1-F_{j:n}(u)}{1-u}$ is evidently increasing on $(0,u_1)$, and decreasing on $(u_2,1)$. Our purpose now is to prove that it is first increasing and then decreasing on (u_1,u_2). To this end, we show that functions $\ln\frac{F_{j:n}(u)}{u}$ and $\ln\frac{1-F_{j:n}(v)}{1-v}$ are concave there. We focus on the first one and calculate

$$\left[\ln\frac{F_{j:n}(u)}{u}\right]'' = -\frac{u^2f_{j:n}^2(u)-F_{j:n}^2(u)-u^2f'_{j:n}(u)F_{j:n}(u)}{u^2F_{j:n}^2(u)}.$$

It suffices to show that the numerator is positive for $0<u<u_2$. We have

$$\begin{aligned}
f_{j:n}(u) &= n\binom{n-1}{j-1}u^{j-1}(1-u)^{n-j},\\
f'_{j:n}(u) &= n\binom{n-1}{j-1}\left[(j-1)u^{j-2}(1-u)^{n-j}-(n-j)u^{j-1}(1-u)^{n-j-1}\right].
\end{aligned}$$

Therefore

$$u(1-u)f'_{j:n}(u) = (j-1)(1-u)f_{j:n}(u)-(n-j)uf_{j:n}(u) = [j-1-(n-1)u]f_{j:n}(u),$$

which we rewrite as

$$(1-u)f_{j:n}(u)+u(1-u)f'_{j:n}(u) = (j-nu)F_{j:n}(u) \qquad (10)$$

for convenience. Define

$$H(u) = u(1-u)f_{j:n}(u) - (j-nu)F_{j:n}(u).$$

Using (10) we get

$$\begin{aligned} H'(u) &= nF_{j:n}(u) - uf_{j:n}(u) \\ &= (n-j)\binom{n}{j}u^j(1-u)^{n-j} + \sum_{i=j+1}^{n}\binom{n}{i}u^i(1-u)^{n-i} > 0 \end{aligned}$$

for $0 < u < 1$. Hence $H(u) > 0$, $0 < u < 1$. Multiplying (10) by $-\frac{uF_{j:n}(u)}{1-u}$ and adding $u^2 f_{j:n}^2(u)$, we obtain

$$\begin{aligned} & u^2 f_{j:n}^2(u) - uf_{j:n}(u)F_{j:n}(u) - u^2 f'_{j:n}(u)F_{j:n}(u) \\ = \; & u^2 f_{j:n}^2(u) - \frac{u(j-nu)}{1-u}f_{j:n}(u)F_{j:n}(u) = \frac{uf_{j:n}(u)}{1-u}H(u) > 0. \end{aligned}$$

Since $uf_{j:n}(u) \geq F_{j:n}(u)$ for $u \leq u_2$,

$$\begin{aligned} & u^2 f_{j:n}^2(u) - F_{j:n}^2(u) - u^2 f'_{j:n}(u)F_{j:n}(u) \\ \geq \; & u^2 f_{j:n}^2(u) - uf_{j:n}(u)F_{j:n}(u) - u^2 f'_{j:n}(u)F_{j:n}(u) > 0, \end{aligned}$$

which is a desired claim. Since

$$\frac{1-F_{j:n}(v)}{1-v} = \frac{1-F_{j:n}(1-u)}{u} = \frac{F_{n+1-j:n}(u)}{u},$$

we analogously conclude that the function is log-concave on $(u_1, 1)$. It follows that the product function $\frac{F_{j:n}(u)}{u}\frac{1-F_{j:n}(u)}{1-u}$, $0 < u < 1$ has a unique local extremum on the hypotenuse, and this is the maximum point. The equation in statement (i) of the theorem is equivalent to the condition that the derivative of the product vanishes.

The conditions for attaining equality in (9) are

$$\{0 < F(x) \leq F(y) < 1\} = \{(F(x), F(y)) = (u_0, u_0)\}.$$

This means that the only possible value of $F(x)$, different from 0 and 1, is u_0. This is only possible for the two-point distribution such that the probability of the smaller point is u_0.

For $j = 1$ functions $\frac{F_{1:n}(u)}{u}$ and $\frac{1-F_{j:n}(v)}{1-v}$ decrease from n to 1, and from 1 to 0, respectively. Therefore

$$\sup_{0<u\leq v<1} \frac{F_{j:n}(u)[1-F_{j:n}(v)]}{u(1-v)} = \lim_{u \searrow 0} \frac{F_{j:n}(u)}{u}\frac{1-F_{j:n}(u)}{1-u} = n.$$

A similar conclusion holds for $j = n$. The attainability conditions are obvious then. $\square$

Jasiński i in. (2009) tried to extend the Papadatos result to the lifetimes of systems with independent identical components and arbitrary signature $\mathbf{s} = (s_1, \ldots, s_n)$. They replaced function $F_{j:n}$ by $F_{\mathbf{s}:n} = \sum_{i=1}^{n} s_i F_{i:n}$, and arrived to the inequality

$$\frac{\mathbb{V}ar\, T}{\mathbb{V}ar\, T_1} \leq \sup_{0<u\leq v<1} \frac{F_{\mathbf{s}:n}(u)[1 - F_{\mathbf{s}:n}(v)]}{u(1-v)}.$$

Other conclusions of the paper are pointed out below.

Remarks. 21. The supremum is attained either at a point (u_0, u_0) of the unit square diagonal or in the triangle interior point (u_0, v_0), $0 < u_0 < v_0 < 1$. In the latter case, both the numbers are the local maximum points of the factors of the product.

22. The bound for variance is attained (by two-point distributions, possibly in the limit) only in the former case, i.e. for the diagonal global maximum pair.

23. This happens when function $f_{\mathbf{s}:n}$ is unimodal. In particular, if the function is increasing and decreasing, then the bounds amount to ns_n and ns_1, respectively. Unimodality of the signature $\mathbf{s}$ implies unimodality of $f_{\mathbf{s}:n}$. The authors have not found an example of coherent system for which $f_{\mathbf{s}:n}$ is not unimodal (although they presented an example of bimodal $\mathbf{s}$, for which the respective $f_{\mathbf{s}:n}$ was still unimodal).

24. If $\mathbf{s}$ is symmetric. i.e. $s_i = s_{n+1-i}$, $i = 1, \ldots, n$, then $f_{\mathbf{s}:n}$ is symmetric about $\frac{1}{2}$, and in consequence

$$\frac{\mathbb{V}ar\, T}{\mathbb{V}ar\, T_1} \leq 1.$$

This holds, e.g., for the $\frac{n+1}{2}$-out-of-n system (sample median), and bridge system as well.

3.2 Exchangeable case

Theorem 11 (Miziuła and Rychlik, 2013) *Let $T_1, \ldots, T_n$ be nondegenerate exchangeable lifetimes of n components of a mixed (or semicoherent) system with signature $\mathbf{s} = (s_1, \ldots, s_n)$. Suppose that the common variance of $T_1, \ldots, T_n$ is positive and finite. Let T denote the lifetime of the system. Put $0 \leq \mathbb{m} = n \min\{\underline{s}_1, \overline{s}_n\} \leq 1$ and $1 \leq \mathbb{M} = n \max\{\overline{s}_1, \underline{s}_n\} \leq n$. Then inequalities*

$$\mathbb{m} \leq \frac{\mathbb{V}ar\, T}{\mathbb{V}ar\, T_1} \leq \mathbb{M}.$$

hold. These bounds are optimal for arbitrary system with signature $\mathbf{s}$ *and all possible exchangeable distributions of components such that* $0 < \mathbb{V}ar\, T_1 < +\infty$.

Note that $\mathbb{m}$ and $\mathbb{M}$ are the minimal and maximal values of the slopes of the greatest convex minorant $\underline{S}$ and smallest concave majorant $\bar{S}$ of the points $\left(\frac{j}{n}, \sum_{i=1}^{j} s_i\right)$, $j = 0, \ldots, n$ (cf. Theorem 3 and Figure 2).

Ideas of Proof. We only treat the upper bound, and start with proving that $\mathbb{E}(T_1 - \mu)^2 < \infty$ for some real μ implies

$$\frac{\mathbb{E}(T - \mu)^2}{\mathbb{E}(T_1 - \mu)^2} \leq \mathbb{M}.$$

Define functions

$$\begin{aligned} \underline{R}(x) &= \max\{\mathbb{M}(x-1) + 1, 0\} \leq \underline{S}(x), \\ \bar{R}(x) &= \min\{\mathbb{M}x, 1\} \geq \bar{S}(x). \end{aligned}$$

(see Figure 3). Due to Theorem 3,

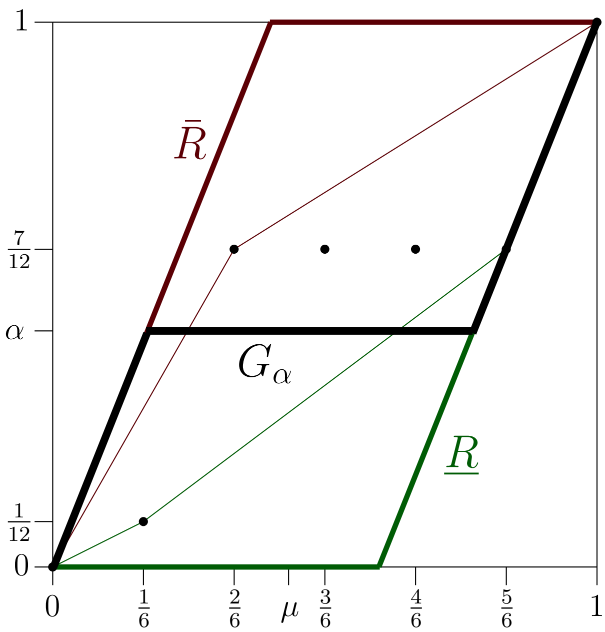

Figure 3: Functions $\underline{R}$, $\bar{R}$, G_α for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$ and $F \sim U(0, 1)$

$$\underline{R} \circ F \leq H = \sum_{i=1}^{n} s_i F_i \leq \bar{R} \circ F,$$

where F_i, $i = 1, \ldots, n$, denote the marginal distribution functions of order statistics. Suppose that $H(\mu) = \alpha \in [0, 1]$, and define

$$G_\alpha(x) = \begin{cases} \mathbb{M}F(x) = \bar{R} \circ F(x), & F(x) < \frac{\alpha}{\mathbb{M}} \\ \alpha, & \frac{\alpha}{\mathbb{M}} \leq F(x) < 1 - \frac{1-\alpha}{\mathbb{M}}, \\ \mathbb{M}\big[F(x) - 1\big] + 1 = \underline{R} \circ F(x), & F(x) > 1 - \frac{1-\alpha}{\mathbb{M}} \end{cases}$$

(see Figure 3). Therefore

$$\begin{aligned} \mathbb{E}(T-\mu)^2 = \int (x-\mu)^2 H(dx) &\leq \int (x-\mu)^2 G_\alpha(dx) \\ &= \mathbb{M} \int_{\{F(x) < \frac{\alpha}{\mathbb{M}}\} \cup \{F(x) \geq 1 - \frac{1-\alpha}{\mathbb{M}}\}} (x-\mu)^2 F(dx) \\ &\leq \mathbb{M} \int (x-\mu)^2 F(dx) = \mathbb{M}\, \mathbb{E}\, (T_1 - \mu)^2, \end{aligned}$$

as desired.

For the sharpness proof, we observe that

$$\begin{aligned} n\bar{s}_1 &= \max_{1 \leq j \leq n} \frac{n}{j} \sum_{i=1}^{j} s_i = \frac{n}{j_1} \sum_{i=1}^{j_1} s_i, \\ n\underline{s}_n &= \max_{1 \leq j \leq n} \frac{n}{j} \sum_{i=n+1-j}^{n} s_i = \frac{n}{j_2} \sum_{i=n+1-j_2}^{n} s_i, \end{aligned}$$

for some $1 \leq j_1, j_2 \leq n$. Therefore it suffices to take a random variable I, $\mathbb{P}(I = i) = s_i$, $i = 1, \ldots, n$, and for each $1 \leq j \leq n$ and $0 < \alpha < 1$ construct exchangeable vectors $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, with distributions dependent on j and α, such that

$$\begin{aligned} \lim_{\alpha \searrow 0} \frac{\mathbb{V}ar\, X_{I:n}}{\mathbb{V}ar\, X_1} &= \frac{n}{j} \sum_{i=1}^{j} s_i, \\ \lim_{\alpha \searrow 0} \frac{\mathbb{V}ar\, Y_{I:n}}{\mathbb{V}ar\, Y_1} &= \frac{n}{j} \sum_{i=n+1-j}^{n} s_i. \end{aligned}$$

We present here a construction of random variables attaining the first limit. Suppose that we have two urns. The first one contains j balls of value 0, and $n - j$ balls of value 1. Into the second urn, we put n balls with identical values 1. We choose one urn at random: the first one with probability α, and the other with probability $1 - \alpha$. Then we consecutively draw all the balls from the chosen urn, and denote the results of drawings by $X_1, \dots, X_n$. These random variables are exchangeable and take on two values 0 and 1. We have

$$\mathbb{P}(X_1 = 0) = \alpha \frac{j}{n},$$

and so

$$\mathbb{V}ar\, X_1 = \alpha \frac{j}{n} \left(1 - \alpha \frac{j}{n}\right).$$

For the order statistics,

$$\mathbb{P}(X_{i:n} = 0) = \begin{cases} \alpha, & i \leq j, \\ 0, & i > j. \end{cases}$$

Therefore

$$\begin{aligned} \mathbb{P}(X_{I:n} = 0) &= \alpha \sum_{i=1}^{j} s_i, \\ \mathbb{V}ar\, X_{I:n} &= \alpha \sum_{i=1}^{j} s_i \left(1 - \alpha \sum_{i=1}^{j} s_i\right). \end{aligned}$$

Accordingly,

$$\frac{\mathbb{V}ar\, X_{I:n}}{\mathbb{V}ar\, X_1} = \frac{\alpha \sum_{i=1}^{j} s_i \left(1 - \alpha \sum_{i=1}^{j} s_i\right)}{\alpha \frac{j}{n} \left(1 - \alpha \frac{j}{n}\right)} \longrightarrow \frac{n}{j} \sum_{i=1}^{j} s_i, \qquad \text{as } \alpha \searrow 0.$$

which completes our proof. $\square$

In the proof of the upper bound attainability, we use an urn construction of elementary probability theory. Miziuła and Rychlik (2013a) proposed another construction with Pareto marginal lifetime distribution which is more appealing in the reliability theory. We immediately check that for the k-out-of-n systems, we have

$$0 \leq \frac{\mathbb{V}ar\, T_{n+1-k:n}}{\mathbb{V}ar\, T_1} \leq \max\left\{\frac{n}{k}, \frac{n}{n+1-k}\right\}$$

(see Rychlik, 2008). For the bridge system,

$$0 \leq \frac{\mathbb{V}ar\, T}{\mathbb{V}ar\, T_1} = \frac{4}{3},$$

and this upper bound is significantly greater than the bound 1 in the i.i.d. case. The bounds are identical in the independent and exchangeable, if the signature vector is monotone. They are equal to ns_1 and ns_n, when the signature in nonincreasing and nondecreasing, respectively.

We finally note that the sharp inequalities of Theorem 11 remain valid if we replace the variance by far more general measures of dispersion (see Miziuła and Rychlik, 2013b). For instance, they hold for the median absolute deviations of system and component lifetimes.

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