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Diffusive processes and stochastic differential equations

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DIFFUSIVE PROCESSES AND STOCHASTIC DIFFERENTIAL EQUATIONS

Wrocław Lectures

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Contents

Chapter 1

Random walk and its parabolic rescaling limit

1.1 Stochastic processes and their finite-dimensional distributions

Let T be a subset of the real line and $(\Omega, \mathcal{F}, P)$ a probability space. A mapping

$$T \times \omega \ni (t, \omega) \longmapsto X_t(\omega) \in \mathbf{R}, \tag{1.1}$$

is called a stochastic process. So, on the one hand, stochastic process is a function

$$T \ni t \longmapsto X_t \in L_0(\Omega, \mathcal{F}, P; \mathbf{R}), \tag{1.2}$$

The values of this function are random variables with the 1-D probability distributions

$$\mu_t(B) = \boldsymbol{P}(X_t \in B), \qquad B \in \mathcal{B}, \tag{1.3}$$

On the other hand, the mapping

$$\Omega \ni \omega \longmapsto X_\cdot(\omega) \in T^{\mathbf{R}} \tag{1.4}$$

has as its values real-valued functions on T which are called the sample-paths (trajectories) of the process X. This duality will be explored in some detail.

1.2 Finite-dimensional distributions

More complete information about the process is given by its finite-dimensional distributions

$$\mu_{t_1,\ldots,t_n}(B) = \boldsymbol{P}((X_{t_1}, \ldots, X_{t_n}) \in B), \qquad B \in \mathcal{B}^n, \tag{1.5}$$

$n = 1, 2, \ldots; t_1, \ldots, t_n \in T$, and Kolmogorov's Consistency Theorem assures us that if we know all finite dimensional distribution than we can determine (at least theoretically) the full infinite dimensional probability distribution of the process on the space $T^{\mathbf{R}}$ of real functions on T (equipped with the natural sigma-field generated by the cylindrical sets).

1.3 Symmetric random walk; parabolic rescaling

Consider first a symmetric *random walk* on the one-dimensional lattice. Starting from the origin, the particle moves one step to the right, or left, with equal probability 1/2.

The consecutive steps are independent and are taken at the times

$$t_k = k\Delta t, \qquad k = 1, 2, \ldots,$$

with the step size (lattice distance) Δx, so that the set of possible positions of the particle is

$$x_k = k\Delta x, \qquad k = 1, 2, \ldots.$$

If $X(t, x)$ indicates whether the site x is occupied ($X = 1$), or unoccupied ($X = 0$) at time t, and we denote by

$$u(t_k, x_k) = \boldsymbol{P}\{X(t_k, x_k) = 1\} \tag{1.6}$$

the probability that at time t_k the particle is at site x_k then, clearly,

$$u(t_{k+1}, x_k) = \frac{1}{2}u(t_k, x_{k-1}) + \frac{1}{2}u(t_k, x_{k+1}). \tag{1.7}$$

This equation can be rewritten as a difference equation for function u

$$u(t_{k+1}, x_k) - u(t_k, x_k) = \frac{1}{2}\Big(u(t_k, x_{k-1}) + u(t_k, x_{x+1}) - 2u(t_k, x_k)\Big), \tag{1.8}$$

$k = 1, 2, \ldots$, with the initial condition $u(0, x) = \delta(x)$. Instead of directly solving the system (3) it is easier to notice that with *parabolic scaling*

$$\Delta t = (\Delta x)^2$$

we can rewrite (3) in the form

$$\frac{u(t_{k+1}, x_k) - u(t_k, x_k)}{\delta t} = \frac{1}{2}\frac{u(t_k, x_{k-1}) + u(t_k, x_{x+1}) - 2u(t_k, x_k)}{(\Delta x)^2}, \tag{1.9}$$

$k = 1, 2, \dots$, which, in the *hydrodynamic limit* $\Delta t = (\Delta x)^2 \to 0$, becomes the usual *linear diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}. \tag{1.10}$$

Solving via Fourier transform?

1.4 Brownian motion

The above passage from the difference to differential equation has its analogue in terms of the random walk itself approaching the continuous-time *Brownian motion*.

Consider a sequence $\xi_i, i = 1, 2, \dots$, of independent random variables with $\mathbf{P}\{\xi_i = \pm 1\} = 1/2$, so that $\mathbf{E}\xi_i = 0, \operatorname{Var}\xi_i = 1$. Then the position of the particle at time n is described by

$$X(n) = \xi_1 + \cdots + \xi_n, \tag{1.11}$$

with $\mathbf{E}X(n) = 0, \operatorname{Var} X(n) \nearrow \infty$ (see, Fig. 2.1.3). But

$$B_n = X(n)/\sqrt{n}$$

has variance 1 for all $n = 1, 2, \dots$, and its Fourier tranform

$$\phi_{B_n}(\lambda) = \mathbf{E}e^{i\lambda B_n} = \left(\mathbf{E}\exp\left[i\lambda\xi_1/\sqrt{n}\right]\right)^n = \cos^n(u/\sqrt{n}).$$

As $n \to \infty$, applying twice the l'Hospital rule to variable n,

$$\begin{aligned}
\lim_{n\to\infty} \log\phi_{B_n}(\lambda) &= \lim_{n\to\infty} n\log\cos\frac{\lambda}{\sqrt{n}} = \lim_{n\to\infty}\frac{\log\cos(\lambda/\sqrt{n})}{1/n} \\
&= \lim_{n\to\infty}\frac{-\sin(\lambda/\sqrt{n})(-\lambda/2n^{3/2})}{-(1/n^2)\cos(\lambda/\sqrt{n})} \\
&= \lim_{n\to\infty}\frac{-\cos(\lambda/\sqrt{n})(\lambda^2/2n^{3/2})}{1/n^{3/2}} = -\frac{\lambda^2}{2},
\end{aligned}$$

which means that $\lim_{n\to\infty}\phi_{B_n}(\lambda) = \exp[-\lambda^2/2]$, so that, in law, the random variables B_n converge to a standard Gaussian random variable, say B_∞, i.e., for all $x \in \mathbf{R}$,

$$\mathbf{P}\{B_\infty \le x\} = \int_{-\infty}^{x}\frac{\exp[-y^2/2]}{\sqrt{2\pi}}\,dy. \tag{1.12}$$

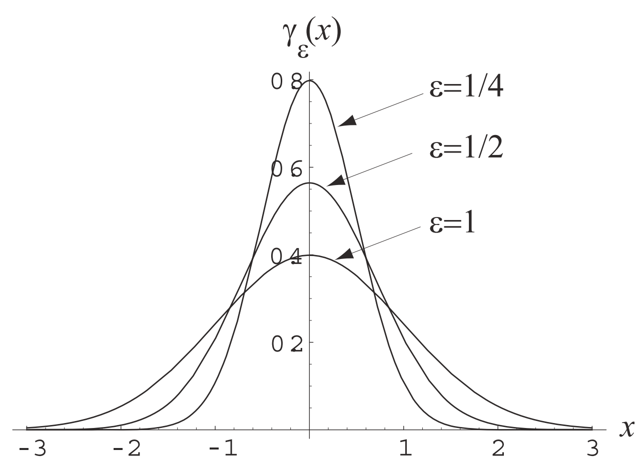
This is, of course, the elementary version of the *Central Limit Theorem.* Interpolating linearly the parabolically rescaled in time and space random walk we get

$$B_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i = \frac{X(\lfloor nt \rfloor)}{\sqrt{n}}, \tag{1.13}$$

where $\lfloor x \rfloor$ is the greatest integer $\le x$, and the finite-dimensional distributions of processes $B_n(t), t \in \mathbf{R}$, converge to the finite-dimensional distributions of the Brownian motion $B(t)$, i.e., the Gaussian process with independent and stationary increments, mean $\mathbf{E}B(t) = 0$, and variance $\operatorname{Var} B(t) = t$, so that

$$\operatorname{Cov}(B(t), B(s)) = t \wedge s \equiv \min\{t, s\}. \tag{1.14}$$

This is the celebrated *Invariance Principle* (see, e.g., Billingsley (1986), Theorem 37.8).



Chapter 2

Brownian motion as a measure on the space of continuous functions

2.1 Basic properties of Brownian motion

Several basic properties of the Brownian motion follow directly from the definition which only asserts that it is a mean-zero Gaussian process with the covariance function of the form $t \wedge s$. Indeed, we immediately get that,

(a) $B_0 = 0$,

(b) $\mathbf{E}B_t^2 = t$, and, consequently,

$$\mathbf{P}(X_t \le z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{z} e^{-x^2/(2t)}\, dx. \tag{2.1}$$

(c) B_t has orthogonal (uncorellated) increments over non-overlapping time intervals.

(d) Its finite-dimensional distributions are explicitly calculated:

$$\mathbf{P}(B_{t_1} \le z_1, \ldots, B_{t_n} \le z_n) = \int_{-\infty}^{z_1} \cdots \int_{-\infty}^{z_n} f_{t_1,\ldots,t_n}(x_1, \ldots .x_n)\, dx_1 \cdots dx_n, \tag{2.2}$$

with the pdf, for $0 < t_1 < \cdots < t_n$,

$$f_{t_1,\ldots,t_n}(x_1, \ldots .x_n) = \frac{e^{-x_1^2/(2t_1)}}{\sqrt{2\pi t_1}} \cdot \frac{e^{-(x_2-x_1)^2/(2(t_2-t_1))}}{\sqrt{2\pi(t_2-t_1)}} \cdots \frac{e^{-(x_n-x_{n-1})^2/(2(t_n-t_{n-1}))}}{\sqrt{2\pi(t_n-t_{n-1})}}, \tag{2.3}$$

which can be obtained from (b)and (c) and the n-dimensional change-of-variables formula in the appropriate integral.

(e) For any $t_0 > 0$, the process $\{X_t = B_t - B_{t_0}, t > 0\}$, is also a Brownian motion.

(f) Brownian motion has stationary increments

(g) Brownian motion has parabolic scaling, that is, for any $c > 0$, the process $\{X(t) = cB(t/c^2), t > 0\}$ is also a Brownian motion.

(h) Inversion in time: The process $\{X(t) = tB(1/t), t > 0\}$, is also a Brownian motion. Thus, the behavior of $B(t)$ at infinity determines its behavior at zero, and vice versa.

(i) Reflection in the time axis: The process $\{X(t) = -B(t), t > 0\}$, is also a Brownian motion.

2.2 Almost sure continuity of sample paths

Brownian motion (has a version that) has continuous trajectories with probability 1. This result was proved by Norbert Wiener in 1926. In the 1960s, Zbigniew Ciesielski found a very elegant proof of this fact using the orthonormal system of Haar wavelets which are defined on the unit interval which are defined by the formula:

$$h_{k2^{-n}}(t) = \begin{cases} 2^{(n-1)/2}, & \text{for } (k-1)2^{-n} < t \le k2^{-n}; \\ -2^{(n-1)/2}, & \text{for } k2^{-n} < t \le (k+1)2^{-n}; \\ 0, & \text{elsewhere.} \end{cases} \tag{2.4}$$

for $n = 1, 2, \ldots$, and odd $k < 2^n$, with $h_0 \equiv 1$.

Theorem: *The process defined by the infinite random series*

$$X_t = \gamma_0 \int_0^t h_0 + \sum_{n\ge1} \sum_{\text{odd } k<2^n} \gamma_{k2^{-n}} \int_0^t h_{k2^{-n}}, \quad t > 0, \tag{2.5}$$

where $\gamma_{k2^{-n}}$ are independent, zero-mean Gaussian random variables with variance 1, is a Brownian motion. The series converges uniformly with probability 1, so that X_t is continuous with probability 1.

A proof of this major results is illuminating and we will explore it below.

2.3 Nowhere differentiability of Brownian motion

The good news about the trajectories of the Brownian motion end at their continuity (actually a little more can be proved). Actually, the form of the covariance function immediately gives us a hint that $B(t)$ cannot be differentialble in the L_2 sense. But Dvoretzky, Erdös, and kakutani proved in 1961 that a much stronger result holds true.

Theorem: *Brownian motion is nowhere differentiable with probability 1. More precisely,*

$$\mathbf{P}(\dot{B}(t) \text{ exists for some } t) = 0 \tag{2.6}$$

We shall explore a subtle proof of this result.

Although, $W(t) = \dot{B}(t)$, which is called the white noise, does not exist in the classical sense, it is an object that is commonly used in physics and engineering. It can be rigorously introduced in the framework of the theory of distributions (generalized functions) and we will briefly describe below how this can be done.

Chapter 3

Poisson processes and their mixtures, Lévy processes

3.1 Why Poisson process?

Let us now move in the opposite direction from the continuous sample path Brownian motion model considered in the previous lecture, and consider a stochastic process $\{N_t, t \geq 0\}$, enjoying the following properties

(a) $N(t)$ takes values in the set of nonnegative integers, and $N(0) = 0$.

(b) $N(t)$ has stationary increments independent over non-overlapping intervals.

(c) $N(t)$ is nontrivial in the sense that, for each $t > 0$, we have $0 < \mathbf{P}(N(t) > 0) < 1$.

(d) $N(t)$ has only jumps of size 1.

The above set of properties seems like only a qualitative description of a process, but it turns out that they are restrictive enough to characterize the process completely in the quantitative sense.

Theorem: *If process $N(t), t > 0$, satisfies the above conditions (a)–(d), then it is a Poisson process, that is, there exists a constant $\mu > 0$, such that*

$$\mathbf{P}(N(t = k) = e^{-\mu t}\frac{\mu^k t^k}{k!}, \qquad k = 0, 1, 2, \ldots. \tag{3.1}$$

We will provide a proof of this result.

3.2 Finite dimensional distributions and covariance structure

Also, all the finite dimensional distributions can be explicitly calculated. Similarly, various conditional probabilities can be explicitly evaluated. In particular, order statistics of a uniform distribution play here an important role.

3.3 Waiting times and interjump times

The n-th waiting time is the random time when the process reaches the level n for the first time, i.e.,

$$W_n = \min\{t : N(t) = n\}. \tag{3.2}$$

Theorem: *The n-th waiting time W_n of the Poisson process with parameter μ has the Gamma pdf*

$$f_{W_n}(t) = e^{-\mu t}\frac{\mu^n t^{n-1}}{(n-1)!}, \qquad t \geq 0, \tag{3.3}$$

for $n = 1, 2, \ldots$.

The proof will be provided below.

On the other hand, the random interjump times

$$T_n = W_n - W_{n-1}, \qquad n = 1, 2, \ldots \tag{3.4}$$

have a simpler structure

Theorem: *Random variable $T_n, n = 1, 2, \ldots,$ are independent, identically distributed with the exponential pdf*

$$f_{T_n}(t) = \mu e^{-\mu t}, \qquad t \geq 0. \tag{3.5}$$

The proof will be provided below

Chapter 4

Lévy-Khinchne formula and infinitesimal generators of Lévy processes

4.1 From Poisson processes to Lévy processes

In this section we provide a brief, *ab ovo*, review of Lévy's model for a self-similar nonlocal hopping surface transport model that in other context was called anomalous diffusion. The role of the fractional Laplacian is explained and the model is compared to the traditional random walk/Brownian motion model.

Let us begin with an elementary one-dimensional example of the Poisson process where a particle is located on a 1-D lattice with unit spacing, starts at the origin, waits a random exponential time and then moves one unit to the right. Then the step is independently repeated. If we denote by $X(t)$ the position of the particle at time $t > 0$, then it is well known that this random quantity has the standard Poisson distribution, i.e., $Pr\{X(t) = k\} = e^{-t} \cdot t^k/k!,\ k = 0, 1, 2, \dots$. This distribution can be described in terms of its Fourier transform $\Phi(u)$ (characteristic function) as follows

$$\Phi(u) = Ee^{iuX(t)} = e^{-t}\sum_{k=0}^{\infty}(e^{iu})^k \cdot \frac{t^k}{k!} = \exp[t(e^{iu} - 1)] \tag{4.1}$$

If the jumps are of size a, i.e., $Pr\{X(t) = ak\} = e^{-t} \cdot t^k/k!,\ k = 0, 1, 2, \dots$. then the analogous calculation gives the Fourier transform

$$\Phi(u) = Ee^{iuX(t)} = e^{-t}\sum_{k=0}^{\infty}(e^{iu})^{ak} \cdot \frac{t^k}{k!} = \exp[t(e^{iua} - 1)] \tag{4.2}$$

Now consider a more complex model which is a composition of n independent simple Poisson processes, each with jump sizes $a_1, \dots, a_n$, respectively. For the resulting stochastic process $X(t)$, the Fourier transform

$$\Phi(u) = Ee^{iuX(t)} = \prod_{j=1}^{k} e^{-t} \sum_{k=0}^{\infty} (e^{iu})^{a_j k} \cdot \frac{t^k}{k!} = \exp\left(t \sum_{j=1}^{k} (e^{iua_j} - 1)\right). \quad (4.3)$$

If jumps sizes are continuously distributed, say with intensity $L(da)$, then the natural infinitesimal limiting procedure for the corresponding *Lévy process*, leads to the following representation of its Fourier transform

$$\Phi(u) = Ee^{iuX(t)} = \exp\left(t \int_{-\infty}^{\infty} (e^{iua} - 1)\, L(da)\right) \quad (4.4)$$

assuming that the time increments are stationary and independent in disjoint time intervals. Define Ψ so that, $\Phi(u) = \exp(-t\Psi(u))$ with $\Psi(u) = \int_{-\infty}^{\infty} (e^{iua} - 1)\, L(da)$. The intensity measure $L(da)$ is called the *Lévy measure* of the process $X(t)$ (see Bertoin (1996)).

When the Lévy measure has power scaling, i.e., $L(da) = da/|a|^{\alpha+1},\ 0 < \alpha < 2$, a simple calculation leads to the *α-stable* Fourier transform

$$\Phi(u) = e^{-ct|u|^\alpha} \qquad \textit{and} \qquad \Psi(u) = c|u|^\alpha, \quad c > 0, \quad (4.5)$$

and the corresponding *α-stable Lévy process.*

Remark 4.1.1 The special case of $\alpha = 2$ gives the Brownian motion with the Fourier transform of the form $\exp(-ctu^2)$ which can be inverted to yield the familiar Gaussian density of the form $\exp(-x^2/(2ct))$. For $\alpha = 1$, we obtain another familiar density, namely the Cauchy (Lorentz) density of the form $(\pi(1 + x^2))^{-1}$. In general, for other values of $0 < \alpha < 2$, the Fourier transform $\exp[-ct|u|^\alpha]$ cannot be inverted explicitly (see, e.g., Feller (1966), Bertoin (1996)).

Remark 4.1.2. In view of (A.5), increments of the Lévy α-stable process have distributions with the following self-similar scaling property: for any $t, c > 0$, $X(ct) \sim c^{1/\alpha} X(t)$, where $\sim$ stands for the equality of distributions of random quantities.

4.2 Infinitesimal generators of Lévy processes

Lévy processes are Markov processes with the associated Markov semigroup (i.e. $P_{t+s} = P_t P_s$) of convolution operators P_t acting on a bounded function

$f(x)$ via the formula

$$P_t f(x) = E^x(f(X(t)) = \int_R f(x+y)\, P(X(t) \in dy) \tag{4.6}$$

The *infinitesimal generator* $\mathcal{A}$ of such a semigroup is defined by the formula

$$\mathcal{A} = \lim_{h\to 0} \frac{P_h - P_0}{h} \tag{4.7}$$

and the family of functions (densities) $v(t,x) = P_t f(x)$ satisfies clearly the (generalized) Fokker-Planck evolution equation

$$v_t = \mathcal{A} v, \tag{4.8}$$

because

$$\lim_{h\to 0} \frac{P_{t+h} - P_t}{h} = \lim_{h\to 0} \frac{P_h - P_0}{h} P_t = \mathcal{A} P_t. \tag{4.9}$$

In the case of the usual Brownian motion the infinitesimal operator $\mathcal{A}$ is just the classical Laplacian Δ. In the case of general Lévy processes, we have the identity

$$\mathcal{F}(\mathcal{A}f)(u) = -\Psi(-u)\mathcal{F}f(u) \tag{4.10}$$

where $\mathcal{F}$ stands for the Fourier transform, because

$$\mathcal{F}(P_t f)(u) = E\left(\int_R e^{iux} f(X(t)+x)\, dx\right) = E\left(\int_R e^{iu(y-X(t))} f(y)\, dy\right)$$

$$= Ee^{-iuX(t)} \int_R e^{iuy} f(y)\, dy = \exp(-t\Psi(-u)\mathcal{F}f(u),$$

which implies that

$$\mathcal{F}(U^q f)(u) = (q + \Psi(-u))^{-1} \mathcal{F}f(u), q > 0$$

where

$$U^q f(x) = \int_0^\infty e^{-qt} P_t f(x)\, dt = E^x\left(\int_0^\infty e^{-qt} f(X(t))\, dt\right)$$

is the family of *resolvent operators* which satisfy the relation $U^q(q\mathbf{I} - \mathcal{A}) = \mathbf{I}$, for any $q > 0$.

Inverting the Fourier transform in (4.10) one gets the following representation for the infinitesimal operator of the Lévy process

$$\mathcal{A}f(x) = \int_R (f(x+y) - f(x))\, L(dy) \tag{4.11}$$

In the special case of the α-stable Lévy process, i.e., when $\Psi(u) = c|u|^\alpha$, the infinitesimal operator

$$\mathcal{A}f(x) = \int_R (f(x+y) - f(x))\, \frac{dy}{|y|^{\alpha+1}} \tag{4.12}$$

can be identified with the fractional power of the (negative) Laplacian $\mathcal{A} = -c(-\Delta)^{\alpha/2}$, since, in view of (B.7),

$$\mathcal{F}(\mathcal{A}f)(u) = -c(|u|^2)^{\alpha/2}\mathcal{F}f(u) \tag{4.13}$$

The above exposition only gives a sketch of the fractional Laplacian machinery. There are some mathematical details that have been omitted not to cloud the basic formal structure. Those details can be found in Feller(1966), Bertoin(1996), and Saichev and Woyczynski (1997).

Chapter 5

Selfsimilar Lévy processes and singular integrals

5.1 Selfsimilarity of Lévy processes

Brownian motion enjoyed the self-similarity property

$$B_{ct} =_d c^{1/2} B_t$$

Question: Can we find Lévy processes which are self-similar, perhaps with the parameter $\alpha \neq 2$? This turns out to be true if the Lévy measure in the Lévy-Khinchine formula has the self-similarity property itself. Indeed,

$$E \exp(iuX_t) = \exp\left(2t \int_0^\infty (\cos ux - 1) \frac{dx}{|x|^{\alpha+1}}\right) = e^{-ct|u|^\alpha}, \tag{5.1}$$

so that

$$Xct =_d c^{1/\alpha} X_t. \tag{5.2}$$

Such processes are called α-stable motions (processes).

5.2 Properties of α-stable motions

1) Moments
2) Tail properties of α-stable distributions
3) Trajectories of α-stable motions. Fractal properties.

5.3 Infinitesimal generators of α-stable processes

Fractional Laplacians and their properties as Fourier multiplier operators. Potential estimates for fractional laplacians.

Chapter 6

Stochastic integrals for Brownian motion and general Lévy processes

6.1 Wiener random integral

Random measure (orthogonally scattered) on $([0,1],\mathcal{B})$: A mapping

$$\mathcal{B} \ni A \longmapsto M(A) \in L^2(\Omega.\mathcal{F},\mathbf{P}) \tag{6.1}$$

such that

$$\mathbf{E}M(A) = 0$$

$$M(A\cup B) = M(A) + M(B), \qquad \text{if} \quad A\cap B = \emptyset,$$

and

$$\mathbf{E}M(A)\cdot M(B) = 0, \qquad \text{if} \quad A\cap B = \emptyset.$$

Brownian motion generates a random measure by the extension of the formula

$$M((a,b]) = B_b - B_a.$$

So does the symmetrization of the Poisson process.

Construction of the Wiener random integral. Start with simple functions

$$f(t) = \sum_i a_i I(A_i),$$

where A_i's are disjoint and define

$$\int f(t)M(dt) = \sum_i a_i M(A_i). \tag{6.2}$$

Isometry property,

$$\mathbf{E}\left|\int f(t)M(dt)\right|^2 = \int |f(t)|^2 m(dt) \tag{6.3}$$

where $m(A) = E|M(A)|^2$ is the control measure. The isometry permits extension of the random integral to all functions $f \in L^2([0,1], \mathcal{B}, m)$.

$$X_n = \int e^{int} M(dt),$$

gives a representation of second-order weakly stationary processes.

6.2 Itô stochastic integral for Brownian motion

Try to extend the definition of the Wiener integral to the case of stochastic integrands:

$$\int f(t,\omega) dB_t(\omega) \tag{6.4}$$

Again start with simple integrands, this time taking random values on disjoint intervals. Immediate problem trying to get the isometry property is the structure of statistical dependence between f_t and B_T. For nonanticipating integrands we also have isometry,

$$\mathbf{E}\left|\int f_t dB_t\right|^2 = \mathbf{E}\int |f_t|^2 dt. \tag{6.5}$$

Lebesgue measure is the control measure for Brownian motion.

Martingale structure of Itô integrals

$$\mathbf{E}\left(\int_0^s f_t dB_t \;\middle|\; \mathcal{F}_u\right) = \int_0^u f_t dB_t, \qquad u < s, \tag{6.6}$$

Maximal inequalities. Example of direct calculation:

$$\int_0^s B_t dB_t = \frac{1}{2}(B_s^2 - s)$$

Is there a more general formula behind ?

6.3 Itô stochastic integral for α-stable motion

What to do if the second moments are infinite? Define

$$\int f_s(\omega) dX_s(\omega)$$

relying on the knowledge of the characteristic function of the α-stable motion X_t.

Chapter 7

Itô stochastic differential equations

7.1 Itô's formula

Because the integrator in the Itô's Brownian integral has infinite variation , the standard rules of calculus do not apply. In particular , the change-of-variables formula (chain rule) takes a different form

Theorem: *Let*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s \tag{7.1}$$

be a stochastic integral process with nonanticipating processes u, and v, such that

$$\mathbf{P}\left(\int_0^t v^2(s)ds < \infty, \forall t > 0\right) = 1,$$

and

$$\mathbf{P}\left(\int_0^t |u(s)|ds < \infty, \forall t > 0\right) = 1.$$

If $g(t,x) t \geq 0, x \in \mathbf{R}$ *is twice differentiable then the process*

$$Y_t = g(t, X_T)$$

is also a stochastic integral and

$$Y_s = Y_0 + \int_0^s \frac{\partial g}{\partial t}(t,X_t)dt + \int_0^s \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\int_0^s \frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2, \tag{7.2}$$

where $(dX_t)^2$ is calculated according to the following rules:

$$dtdt = 0, \quad dtdB_t = 0, \quad dB_tdB_t = dt.$$

Proof:

Examples

7.2 Stochastic differential equations

Differential equations with noise

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)\dot{B}_t$$

where $\dot{B}_t$ is the so-called white noise, .i.e., the derivative of the Brownian motion. However, since Brownian motion is not differentiable, the rigorous approach call for interpretation of the above stochastic equation as Itô's stochastic integral equation

$$X_s = X_0 + \int_0^s b(t, X_t)dt + \int_0^s \sigma(t, X_t)dB_t, \tag{7.3}$$

which is traditional written in the form of the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{7.4}$$

1) Existence and uniqueness
2) Explicit solutions?
3) Properties of solutions.

Example 1: Population growth model. Ornstein-Uhlenbeck process.

Example 2. Bessel process

Example 3. Kalman-Bucy filter

Chapter 8

Asymmetric exclusion processes and their scaling limits

8.1 Asymmetric exclusion principles

Particles occupy integer lattice sites on the real line. Description: $X(k,t) = 1$, if site k is occupied at time t, and 0 otherwise. They obey the exclusion principle: two particles cannot occupy the same site at the same time.

This is the example considered by Kipnis (1986), Benassi, Fouque (1987), and Srinivasan (1991,1993). Another starting point is the observation that the queuing system consisting of an infinite series of queues can be interpreted in the language of the one-dimensional nearest neighbor simple exclusion process (see, e.g., Liggett(1985)). Indeed, if the lattice location of the i-th particle is denoted by x_i then, in view of the exclusion dynamics and nearest neighbor jumps, at time t

$$\cdots < x_{-1}(t) < x_0(t) < x_1(t) < \dots \tag{8.1}$$

Assume that the rate of this process is 1. If we denote by $\eta_i(t)$ the random variable equal to the number of empty sites between $x_i(t)$ and $x_{i+1}(t)$, then $\eta_i(t)$ can be considered as the length of the ith queue for an infinite queuing system with single servers in series, each with an exponential service time with intensity 1. Indeed, when the ith particle jumps to the right by one unit, then $\eta_i(t)$ changes into $\eta_i(t) - 1$ which means that the service for one customer was completed at the ith server, and η_{i+1} is changed to $\eta_{i+i}(t) + 1$, which means that a new customer was added to the queue at the $(i+1)$st server. In other words, the customer in the ith queue is served in exponential time with rate 1 and then joins the $(i-1)$st queue with probability p and $(i+1)$st queue with probability $1-p$.

Another way to code the asymmetric exclusion interacting particle system is by listing its states

$$X(t)=\{X(k,t):t\geq 0,k\in\mathbf{Z}\}\in\{0,1\}^{\mathbf{Z}}, \tag{8.2}$$

The set $\{k:X(k,t)=1\}\subset\mathbf{Z}$ is the set of occupied sites at time t. In the totally asymmetric case $p=1$, the infinitesimal generator for the Markov process $X(t)$ (which does exist, see, e.g., Liggett (1985))

$$\mathcal{L}f(X)=\sum_{k\in Z}X(k)\big(1-X(k+1)\big)\big[f(X^{k,k+1})-f(X)\big], \tag{8.3}$$

where the state $X^{k,k+1}$ is obtained from the state X by setting $X(k)=0, X(k+1)=1$ and keeping the other values fixed.

The above system's dynamics can also be encoded in the infinite system of ordinary stochastic differential equations

$$\begin{aligned} dX(t,k) \;=\;& X(t^-,k-1)\big[1-X(t^-,k)\big]\,dP(t,k-1)\\ &-X(t^-,k)\big[1-X(t^-,k+1)\big]\,dP(t,k)\\ &+X(t^-,k+1)\big[1-X(t^-,k)\big]\,dQ(t,k+1)\\ &-X(t^-,k)\big[1-X(t^-,k-1)\big]\,dQ(t,k) \end{aligned} \tag{8.4}$$

where $P(t,k)$ and $Q(t,k)$, $k\in\mathbf{Z}$, are independent Poisson processes with intensities p and $(1-p)$, representing jumps to the right and jumps to the left, respectively.

8.2 Scaling limit

Define the hyperbolic rescalings

$$X^h(t,x)=\sum_{k\in\mathbf{Z}}X\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x), \tag{8.5}$$

$$P^h(t,x)=\sum_{k\in\mathbf{Z}}P\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x), \tag{8.6}$$

$$Q^h(t,x)=\sum_{k\in\mathbf{Z}}Q\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x), \tag{8.7}$$

and introduce notation

$$F_{\pm h}u(x)=u(x)\big(1-u(x\pm h)\big), \tag{8.8}$$

$$D_{\pm h}u(x) = \pm\frac{u(x \pm h) - u(x)}{h}. \tag{8.9}$$

A direct verification shows that the system (4) can now be written in the form

$$\begin{aligned} dX^h(t,x) \;&=\; -D_{-h}\big[F_h\big(X^h(t^-,x)\big)\,d(hP^h(t,x))\big] \\ &\quad\; +D_h\big[F_{-h}\big(X^h(t^-,x)\big)\,d(hQ^h(t,x))\big]. \end{aligned} \tag{8.10}$$

THEOREM *(Benassi and Fouque (1987)) Let $p \neq 1/2$. As $h \to 0$, the solution $X^h(t,x)\,dx$ of (10) converges weakly to $u(t,x)\,dx$, where $u(t,x)$ is a decreasing and right continuous in the x-variable weak solution of the non-linear Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial B(u)}{\partial x} = 0, \tag{8.11}$$

$$u(0,x) = u_0(x) = b\mathbf{1}_{(-\infty,0]}(x) + a\mathbf{1}_{(0,\infty]}(x), \tag{8.12}$$

with some $0 \le a < b < \infty$ and

$$B(u) = u(1-u). \tag{8.13}$$

Moreover, for all t, x, we have $a \le u(t,x) \le b$.

Recall that the weak solution is understood in the following sense: For every smooth function $\phi : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ with compact support,

$$\int_{\mathbf{R}^+}\int_{\mathbf{R}} [u\phi_t + (2p-1)F(u)\phi_x]\;dx\,dt = -\int_{\mathbf{R}} u_0(x)\phi(0,x)\,dx. \tag{8.14}$$

Heuristically, the result is plausible, since, as $h \to 0$ in (10), $hP^h(t,x) \to pt$, $hQ^h(t,x) \to (1-p)t$, $D_{\pm h} \to \partial/\partial x$, and $F_{\pm h} \to F$.

8.3 Other queuing regimes related to non-nearest neighbor systems

Other queuing regimes related to non-nearest neighbor systems lead to scaling limits

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0, \tag{8.15}$$

with

$$H(u) = (1-u) - (1-u)^{N+1},$$

or

$$H(u) = u(1-u)^m,$$

where N, and m, are some integers.

Chapter 9

Nonlinear diffusion equations

9.1 Hyperbolic equations

The nonlinear hyperbolic equations describing the density profiles for the queuing networks in Chapter 8 are special cases of general conservation laws (see, e.g., Smoller (1994)) of the form

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0. \tag{9.1}$$

and in the case of initial conditions of the form

$$u(0,x) = u_0(x) = u_l \mathbf{1}_{(-\infty,0]}(x) + u_r \mathbf{1}_{(0,\infty]}(x), \tag{9.2}$$

where u_l and u_r are constants (so called Riemann problem), they can be solved explicitly under some extra conditions on function H.

Let us recall (see, e.g., Smoller (1994)) that a bounded and measurable function $u(t,x)$ is called a (weak) solution of the initial-value problem

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0, \qquad u(0,x) = u_0(x), \tag{9.3}$$

with bounded and measurable initial data u_0 if

$$\int_{t\geq 0} \int_{\mathbf{R}} \big(u\phi_t + H(u)\phi_x\big)\, dx\, dt + \int_{t=0} u_0 \phi \, dx = 0 \tag{9.4}$$

In general, solutions are not unique unless additional assumptions, such as the entropy condition mentioned below, are satisfied.

The solutions of the Riemann problem (1-2) are obviously invariant under hyperbolic rescaling, that is, for every constant $\lambda > 0$

$$u_\lambda(t,x) = u(\lambda t, \lambda x)$$

is a solution whenever u is. Thus one looks for the solutions of the form

$$u(t,x) = v(x/t) \tag{9.5}$$

This gives rise to three types of local behavior of the solutions of u:

- $u(t,x)$ is constant;
- $u(t,x)$ is a *shock wave* of the form

  $$u(t,x) = u_0 \mathbf{1}_{(-\infty,Vt)}(x) + u_1 \mathbf{1}_{[Vt,\infty)}(x), \tag{9.6}$$

  traveling with the velocity

  $$V = \frac{H(u_0) - H(u_1)}{u_0 - u_1}.$$

  For the sake of uniqueness one adds here the entropy condition $H'(u_0) > V > H'(u_1)$.
- $u(t,x)$ is a continuous *rarefaction wave* of the form (44) where v satisfies the ordinary differential equation

  $$v'(\xi)\big(H'(v(\xi)) - \xi\big) = 0. \tag{9.7}$$

9.2 Nonlinear diffusion approximations

For initial conditions not of Riemann type, in particular those with integrable data, or for more general random initial conditions, obtaining solutions of the conservations law is not a simple matter, even in approximate fashion. The usual approach then is to consider a parabolic regularization (the viscosity method) by considering the *nonlinear diffusion* equations

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = \epsilon \mathcal{L} u, \qquad u(0,x) = u_0(x), \tag{9.8}$$

where $\mathcal{L}$ is a dissipative operator of elliptic type, like e.g. the Laplacian. Then, of course, with the exception of the quadratic case giving rise to the

Burgers equation, one can not count on finding explicit solutions but two types of asymptotic results can be used as approximations.

The first kind provides the large time asymptotics of the regularized conservation laws and the second kind gives a Monte Carlo method of solving them via the interacting diffusions scheme (so-called propagation of chaos). We will briefly describe the two approaches.

Asymptotics for nonlinear diffusion equations. Not surprisingly, given the decay of their solution in time, the large time asymptotic behavior for parabolically regularized conservation laws is dictated by the asymptotic behavior of the nonlinearity $H(u)$ at points where the function is small. So, we have the following asymptotic results for regularized versions of the hyperbolic equations:

THEOREM 1. *Let $\epsilon > 0, m \geq 1$ and $u(t,x)$ be a positive weak solution of the Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial F(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0,x) = u_0(x) \geq 0, \tag{9.9}$$

with $F(u) = [u(1-u)^m]$. Then

(i) If $u_0 \in L^1(\mathbf{R})$ then u has the same large time asymptotics as the solution of the linear diffusion equation

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0,x) = u_0(x) \geq 0, \tag{9.10}$$

or more precisely

$$\|u(t,x) - U(t,x)\|_1 \to 0 \quad \text{as} \quad t \to \infty, \tag{9.11}$$

where $U(t,x) = (g * u_0)(t, x - (2p-1)t)$ and $g(t,x) = (4\pi t)^{-1/2} \exp(-|x|^2/(4t))$ is the standard Gaussian kernel.

(ii) If $1 - u_0 \in L^1(\mathbf{R})$ then:

In the case $m = 1$, u has the same large time asymptotics as the solution of the linear diffusion equation.

In the case $m = 2$, u has the same large time asymptotics as the selfsimilar source solution of the Burgers equations or more precisely, for each $p > 1$

$$t^{(1-1/p)/2}\|u(t,x) - U_M(t,x)\|_p \to 0 \quad \text{as} \quad t \to \infty, \tag{9.12}$$

where

$$U_M(t,x) = t^{-1/2}\exp(-x^2/(4t))\Big(K(M) + \frac{1}{2}\int_0^{x/(2\sqrt{t}}\exp(-\xi^2/4)\,d\xi\Big)^{-1}, \tag{9.13}$$

and $U_M(t,x) \to M\delta(x)$ *as* $t \to 0$ *with* $M = \|u_0\|_1$.

In the case $m \geq 3$, u *has the same large time asymptotics as the solution of the heat equations or more precisely, for each* $p > 1$ *there exists a constant* C *such that*

$$\|u(t,x) - U(t,x)\|_p \leq Ct^{-(1-1/p)/2}, \tag{9.14}$$

where $U(t,x) = (g * u_0)(t,x)$.

Sketch of the Proof. By the results of Escobedo and Zuazua (1991), Escobedo, Velazquez and Zuazua (1993) (see also Biler, Karch and Woyczynski (1999) for other regularizations of conservation laws) the asymptotic behavior of the solutions of the conservation laws depends on the asymptotic behavior of the nonlinearity H at its small values. So, for $H(u) = (2p-1)F(u) = (2p-1)u(1-u)^m$,

$$\lim_{u\to 0} \frac{(2p-1)F(u)}{u} = 2p - 1, \tag{9.15}$$

and

$$\lim_{u\to 1} \frac{(2p-1)F(u)}{(1-u)^m} = 2p - 1. \tag{9.16}$$

The first condition, together with the standard step removing the drift term in the linear diffusion equation gives (i), and the case $m = 1$ in the second condition gives the first part of (ii).

The critical case $m = 2$ yields the Burgers equation type asymptotics claimed in the second part of (ii), and the supercritical case $m \geq 3$ where the effect of the nonlinear convection term disappears in the limit.

Theorem 2. *Let* $\epsilon > 0, N \geq 1$ *and* $u(t,x)$ *be a positive weak solution of the Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial G(u)}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0,x) = u_0(x) \geq 0, \tag{9.17}$$

with $G(u) = (1-u) - (1-u)^{N+1}$. *Then if either* $u_0 \in L^1(\mathbf{R})$ *or* $1 - u_0 \in L^1(\mathbf{R})$ *then* u *has the same large time asymptotics as the solution of the linear diffusion equation*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0,x) = u_0(x) \geq 0, \tag{9.18}$$

or more precisely

$$\|u(t,x) - U(t,x)\|_1 \to 0 \quad \text{as} \quad t \to \infty, \tag{9.19}$$

where $U(t,x) = (g * u_0)(t, x - (2p-1)t)$ *and* $g(t,x) = (4\pi t)^{-1/2} \exp(-|x|^2/(4t))$ *is the standard Gaussian kernel.*

Sketch of the Proof: The proof of this result relies on the same asymptotics results that were employed in the proof of Theorem 1. But in this case $H(u) = (2p-1)[(1-u) - (1-u)^{N+1}]$ which has the linear asymptotics at both $u = 0$ and $u = 1$. So, the result follows by the usual reduction to the heat equation.

Chapter 10

Interacting diffusions approximations for nonlinear diffusion equations

10.1 Nonlinear processes

Interacting diffusions approximations for nonlinear diffusion equations. This section discusses a possibility of a Monte Carlo type approximation for solutions of nonlinear diffusion equations of the type that arise as parabolic regularizations of conservation laws of the encountered before. The idea is to use the following scheme known as the propagation of chaos result and depends on the the construction of the so-called nonlinear McKean process for our equations.

The basic observation is that if the regularizing operator $\mathcal{L}$ is the infinitesimal generator of a Lévy process then the parabolic equation of the previous chapter (say, $\epsilon = 1$) can be formally interpreted as a "Fokker–Planck–Kolmogorov equation" for a "nonlinear" diffusion process in the McKean's sense. Indeed, consider a Markov process $X(t)$, $t \geq 0$, which is a solution of the stochastic differential equation

$$\begin{aligned} dX(t) &= dS(t) - u^{-1}H(u(X(t),t))\,dt, \\ X(0) &\sim u_0(x)\,dx \quad \text{in law,} \end{aligned} \tag{10.1}$$

where $S(t)$ is the Lévy process with generator $-\mathcal{L}$. Assuming that $X(t)$ is a unique solution of (10.1), we see that the measure-valued function $v(dx,t) =$

$P(X(t) \in dx)$ satisfies the weak forward equation

$$\begin{aligned} \frac{d}{dt}\langle v(t), \eta\rangle &= \langle v(t), \widetilde{\mathcal{L}}_{u(t)}\eta\rangle, \ \eta \in \mathcal{S}(\mathbf{R}^n), \\ v(0) &= u(x,0)\,dx \end{aligned} \tag{10.2}$$

with $\widetilde{\mathcal{L}}_u = -\mathcal{L} + u^{-1}H(u)\cdot\nabla$. On the other hand $u(dx,t) = u(x,t)\,dx$ also solves (10.2) since

$$\frac{d}{dt}\langle u(t), \eta\rangle = \langle -\mathcal{L}u - \nabla\cdot H(u), \eta\rangle = \langle u, (-\mathcal{L} + u^{-1}H(u)\cdot\nabla)\eta\rangle$$

so that $v(dx,t) = u(dx,t)$ and, by uniqueness, u is the density of the solution of (1).

10.2 Interacting diffusions and Monte-Carlo methods

The above construction makes possible approximation of solutions of the parabolic equation equations via finite systems of interacting diffusions. To illustrate our point we will formulate this Monte Carlo algorithm in the special, and well known Burgers equation case where $\mathcal{L} = \Delta$, is the usual Laplacian and the nonlinearity $H(x) = x^2$ is quadratic. The more general results needed for the analysis of GS and multiserver queuing networks are under development (see Calderoni and Pulvirenti (1983), Sznitman (1991), Zhang (1995), Funaki and Woyczynski (1998), Woyczynski (1998), Biler, Funaki and Woyczynski (2000), Margolius, Subramanian and Woyczynski (2000), for more details on the subject).

For each $n \in \mathbf{N}$, let us introduce independent, symmetric, real-valued standard Brownian motion processes $\{S^i(t),\ i = 1, 2, \ldots, n\}$, and let $\delta_\epsilon(x) := (2\pi\epsilon)^{-1/2}\exp\left[-x^2/2\epsilon\right], \epsilon > 0$, be a regularizing kernel. Consider a system of n interacting particles with positions $\{X^i(t)\}_{i=1,\ldots,n} \equiv \{X^{i,n,\epsilon}(t)\}_{i=1,\ldots,n}$, and the corresponding measure-valued process (empirical distribution) $\bar{X}^n(t) \equiv \bar{X}^{n,\epsilon}(t) := \frac{1}{n}\sum_{i=1}^n \delta(X^{i,n,\epsilon}(t))$, with the dynamics provided by the system of regularized singular stochastic differential equations

$$dX^i(t) = dS^i(t) + \frac{1}{n}\sum_{j\neq i}\delta_\epsilon(X^i(t) - X^j(t))\,dt, \quad i = 1, \ldots, n, \tag{10.3}$$

and the initial conditions $X^i(0) \sim u_0(x)$ (in distribution, thus, $u_0 \in L_1$ here). Then, for each $\epsilon > 0$, the empirical process $\bar{X}^{n,\epsilon}(t) \Longrightarrow$

$u^\epsilon(x,t)\,dx$, in probability, as $n \to \infty$, where $\Rightarrow$ denotes the weak convergence of measures, and the limit density $u^\epsilon \equiv u^\epsilon(x,t),\ t > 0,\ x \in \mathbf{R}$, satisfies the regularized Burgers equation $u_t^\epsilon + (\frac{1}{2}(\delta_\epsilon * u^\epsilon) \cdot u^\epsilon)_x = \Delta u^\epsilon$. with the initial condition $u(0,x) = u_0(x)$. The speed of convergence is controlled (see Bossy and Talay (1996)). Moreover, under some additional technical conditions, for a class of test functions ϕ, $E|\langle \bar{X}^{n,\epsilon(n)}(t) - u(t), \phi\rangle| \longrightarrow 0$, as $n \to \infty, \epsilon(n) \to 0$, where $u(t) = u(x,t)$ is a solution of the nonregularized Burgers equation $u_t + (u^2)_x = \Delta u$ with the initial condition $u(0,x) = u_0(x)$.