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Introduction to linear dynamics

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A linear dynamical system is given by a (continuous linear) operator T on a topological vector space X; in most cases of interest, X is a Banach space or a Fréchet space. Important concepts in linear dynamics are that of a hypercyclic operator (which demands the existence of a dense orbit) and that of a chaotic operator (which demands, in addition, the existence of a dense set of periodic points). Apart from being interesting in its own right, the study of linear dynamical systems blends nicely methods from topological dynamics, functional analysis, operator theory and classical complex analysis. For introductions to the theory of linear dynamical systems we refer to [1] and [2].

The course will be based on [2], with additional material taken from [1]. There will be 15 lectures of 90 minutes. Below we will give an outline of these lectures. Depending on the interests of the audience and the progress made there may be some last-minute modifications. A certificate can be obtained upon the successful completion of some exercises taken from the two books.

The course will be divided into two parts.

In the first part (roughly one week) we will provide an introduction to the main concepts of linear dynamics. We will introduce and discuss the notions of hypercyclicity and linear chaos and provide criteria for deciding if a given operator is hypercyclic or chaotic. We will study in detail some important classes of operators like (weighted) backward shifts or composition operators. Necessary conditions for hypercyclicity will be based essentially on spectral theory. We will also study, in a parallel investigation, the dynamics of semigroups $(T_t)_{t\geqslant 0}$ of operators, which has interesting applications.

In the second part (roughly one week) we will discuss, depending on time, various advanced topics in linear dynamics. We will first derive some central results from linear dynamics: Ansari's theorem that any power of a hypercyclic operator is hypercyclic, the Bourdon–Feldman theorem that any somewhere dense orbit is everywhere dense, and the León–Müller theorem that any rotation of a hypercyclic operator is hypercyclic. We next study the existence of hypercyclic and chaotic operators on arbitrary Fréchet spaces. We then turn to some advanced topics in linear dynamics: the relatively new notion of a frequently hypercyclic operator, the existence of common hypercyclic vectors for a family of operators, and the existence of large subspaces of hypercyclic vectors for a given operator.

References

[1] F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge University Press, Cambridge, 2009.

[2] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear chaos*, Springer-Verlag, London, 2011.

Lecture 1. Topological dynamical systems

In this lecture we provide a brief introduction into the theory of topological (i.e. non-linear) dynamical systems.

A (*topological*) *dynamical system* simply consists of a continuous map $T\colon X \to X$ on a metric space X. Its dynamics is determined by the behaviour of its orbits. For any $x \in X$

$$\operatorname{orb}(x, T) = \{x, Tx, T^2x, \dots\}$$

is called the *orbit* of x under T. Here,

$$T^0 = I,$$

the identity map, and

$$T^n = T \circ \cdots \circ T \quad (n-\text{fold})$$

is the nth iterate of T, $n \geqslant 1$. At the center of our interest throughout the course will be mappings that possess a *dense orbit.* While it is far from obvious if a mapping has a dense orbit, there is an extremely useful and practicable characterization in most situations of interest.

Theorem (Birkhoff transitivity theorem)**.** *Let T be a continuous map on a separable complete metric space X without isolated points. Then the following assertions are equivalent*:

(i) *T is topologically transitive;*
(ii) *there exists some $x \in X$ such that* $\operatorname{orb}(x, T)$ *is dense in X.*

If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

Here, T is called *topologically transitive* if, for any pair U, V of non-empty open subsets of X, there exists some $n \geqslant 0$ such that $T^n(U) \cap V \neq \varnothing$.

Our second major concern in this course is the notion of chaos. The following has by now been generally accepted as the right mathematical definition of chaos; it was proposed by Devaney in 1986.

Definition (Devaney chaos)**.** A dynamical system $T\colon X \to X$ is said to be *chaotic* (*in the sense of Devaney*) if it satisfies the following conditions:

(i) T is topologically transitive;
(ii) T has a dense set of periodic points.

Here, a point $x \in X$ is called *periodic* if there exists some $n \geqslant 1$ such that $T^n x = x$.

The notion of topological transitivity suggests further related notions of great interest. A dynamical system $T\colon X \to X$ is called *weakly mixing* if $T \times T$ is topologically transitive as a mapping on $X \times X$, and it is called *mixing* if, for any pair U, V of non-empty open subsets of X, there exists some $N \geqslant 0$ such that $T^n(U) \cap V \neq \varnothing$ for all $n \geqslant N$. We will, in particular, study the notion of weak mixing in greater detail.

Lecture 2. Hypercyclic and chaotic operators

From this lecture on we will study linear dynamics, that is, the dynamics of (continuous linear) operators on Banach spaces, or more generally on Fréchet spaces.

We are interested in the Fréchet space case mainly for the sake of several interesting examples that require this generality. We will briefly recall the notion of a Fréchet space. We will also define an *F-norm* and show that the topology of a Fréchet space can always be induced by an F-norm. In this way many arguments can be carried out in a Fréchet space as if one was working in a Banach space.

Thus, let T be an operator on a Fréchet space X. The operators with a dense orbit are called *hypercyclic*. Any vector with a dense orbit is called a *hypercyclic vector*. The set of hypercyclic vectors is therefore

$$HC(T) = \{x \in X : \overline{\mathrm{orb}(x, T)} = X\}.$$

We will then show by an application of the Birkhoff transitivity theorem the following classical examples.

Example (Birkhoff 1929)**.** Let $X = H(\mathbb{C})$ be the (Fréchet) space of entire functions, endowed with the usual topology of locally uniform convergence. Let $a \in \mathbb{C}$, $a \neq 0$. Then the translation operators $T_a \colon X \to X$ given by

$$T_a f(z) = f(z + a)$$

are hypercyclic.

Example (MacLane 1952)**.** Let again $X = H(\mathbb{C})$ be the space of entire functions. Then the differentiation operator $D \colon X \to X$,

$$Df = f'$$

is hypercyclic.

Example (Rolewicz 1969)**.** Let X be one of the sequence spaces ℓ^p, $1 < p < \infty$, or c_0. Let $B \colon X \to X$ be the backward shift given by $B(x_n) = (x_{n+1})$. Then the multiples

$$\lambda B, \quad |\lambda| > 1,$$

are hypercyclic.

In 1991, Godefroy and Shapiro suggested to use Devaney's definition also in the case of linear operators. Thus, an operator is *chaotic* if it is hypercyclic and it has a dense set of periodic points. In complex spaces, the second condition simplifies due to the following useful, if elementary observation.

Proposition. *Let T be a linear map on a complex vector space X. Then the set* $\mathrm{Per}(T)$ *of periodic points of T is given by*

$$\mathrm{Per}(T) = \mathrm{span}\{x \in X : \ Tx = e^{\alpha\pi i}x \textit{ for some } \alpha \in \mathbb{Q}\}.$$

This shows that we need only look for a large supply of eigenvectors to suitable unimodular eigenvalues. It turns out that the three classical hypercyclic operators of Birkhoff, MacLane and Rolewicz are all even chaotic.

We will also study the set $HC(T)$ of hypercyclic vectors in greater detail. By the Birkhoff transitivity theorem, $HC(T)$ is large (in fact residual) once it is non-empty. We will show that, surprisingly, it is also large in an algebraic sense.

Theorem (Herrero–Bourdon)**.** *Any hypercyclic operator has a dense invariant subspace consisting, except for zero, of hypercyclic vectors.*

For complex spaces, the proof follows easily from the fact that the adjoint of a hypercyclic operator cannot have an eigenvalue.

Lecture 3. The Hypercyclicity Criterion

We will start by revisiting (weakly) mixing operators in the light of linearity. We will then study sufficient conditions for an operator to be hypercyclic or chaotic.

Theorem (Hypercyclicity Criterion)**.** *Let T be an operator on a separable Fréchet space. If there are dense subsets $X_0, Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and maps $S_{n_k} : Y_0 \to X$, $k \geqslant 1$, such that, for any $x \in X_0$, $y \in Y_0$,*

(i) $T^{n_k} x \to 0,$
(ii) $S_{n_k} y \to 0,$
(iii) $T^{n_k} S_{n_k} y \to y,$

then T is weakly mixing, and in particular hypercyclic.

We know from Lecture 1 that the second condition in the definition of chaos is satisfied if the operator has 'many' eigenvectors to suitable eigenvalues. A related condition also leads to hypercyclicity, and therefore to chaos.

Theorem (Godefroy–Shapiro eigenvalue criterion)**.** *Let T be an operator on a separable Fréchet space X. Suppose that the subspaces*

$$X_0 := \operatorname{span}\{x \in X : Tx = \lambda x \textit{ for some } \lambda \in \mathbb{K} \textit{ with } |\lambda| < 1\},$$
$$Y_0 := \operatorname{span}\{x \in X : Tx = \lambda x \textit{ for some } \lambda \in \mathbb{K} \textit{ with } |\lambda| > 1\}$$

are dense in X. Then T is mixing, and in particular hypercyclic.

If, moreover, X is a complex space and also the subspace

$$Z_0 := \operatorname{span}\{x \in X : Tx = e^{\alpha\pi i} x \textit{ for some } \alpha \in \mathbb{Q}\}$$

is dense in X, then T is chaotic.

It had been an open problem for some time if every hypercyclic operator satisfies the Hypercyclicity Criterion. Light was first shed onto this question by the following remarkable finding.

Theorem (Bès–Peris). *Let T be an operator on a separable Fréchet space. Then the following assertions are equivalent:*

(i) *T satisfies the Hypercyclicity Criterion;*
(ii) *T is weakly mixing.*

This reduced the above question to a problem that had previously been posed by Herrero (1992): Is every hypercyclic operator weakly mixing? This problem was solved in the negative by De la Rosa and Read (2009); subsequently Bayart and Matheron showed that counter-examples can be constructed in many classical Banach space, including the Hilbert space ℓ^2. These results, however, are too deep to be covered in these lectures.

Given now that there is a difference between weak mixing and hypercyclicity it is interesting to know which additional assumptions make a hypercyclic operator weakly mixing. We show in particular that every chaotic operator is weakly mixing.

1. Lectures 4 and 5. Classes of hypercyclic and chaotic operators

The aim of these two lectures is to provide large classes of hypercyclic and chaotic operators. We will begin with weighted backward shift operators, the 'favorite testing ground of operator-theorists' (Salas). Given a weight w, that is, a sequence $w = (w_n)_n$ of non-zero numbers, the corresponding weighted backward shift B_w is defined by

$$B_w(x_1, x_2, x_3, \ldots) = (w_2x_2, w_3x_3, w_4x_4, \ldots).$$

These operators will be considered on Fréchet sequence spaces, that is, a Fréchet space of sequences so that convergence in the space implies coordinatewise convergence. Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ denote the nth unit sequence. We start by looking at the unweighted shift B.

Theorem. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent*:

(i) *B is hypercyclic*;
(ii) *B is weakly mixing*;
(iii) *there is an increasing sequence $(n_k)_k$ of positive integers such that $e_{n_k} \to 0$ in X as $k \to \infty$.*

In a similar way one can characterize when B is mixing or chaotic. When using a suitable conjugacy, these results lead to a full characterization of hypercyclicity and chaos for weighted backward shifts.

Theorem. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the weighted shift B_w is an operator on X.*

(a) *The following assertions are equivalent:*
  (i) *B_w is hypercyclic*;
  (ii) *B_w is weakly mixing*;

(iii) *there is an increasing sequence* $(n_k)_k$ *of positive integers such that*

$$\Big(\prod_{\nu=1}^{n_k} w_\nu\Big)^{-1} e_{n_k} \to 0$$

in X *as* $k \to \infty$.

(b) *Suppose that the basis* $(e_n)_n$ *is unconditional. Then the following assertions are equivalent:*

(i) B_w *is chaotic*;

(ii) *the series*

$$\sum_{n=1}^{\infty} \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$

converges in X;

(iii) *the sequence*

$$\Big(\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1}\Big)_n$$

belongs to X;

(iv) B_w *has a non-trivial periodic point.*

We next study infinite-order differential operators on the space $H(\mathbb{C})$ of entire functions. Given an entire function of exponential type

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

we will show that

$$\varphi(D)f = \sum_{n=0}^{\infty} a_n D^n f$$

converges in $H(\mathbb{C})$ for every entire function f and defines an operator on $H(\mathbb{C})$. This contains not only the differentiation operator D itself as a special case but also the translation operators $T_a\colon f(\cdot) \to f(\cdot + a)$ studied by Birkhoff: in that case it suffices to take $\varphi(z) = e^{az}$.

The operators $\varphi(D)$ have a simple characterizing property.

Proposition. *Let T be an operator on $H(\mathbb{C})$. Then the following assertions are equivalent*:

(i) $T = \varphi(D)$ *for some entire function* φ *of exponential type*;

(ii) T *commutes with* D;

(iii) T *commutes with each* $T_a, a \in \mathbb{C}$.

We will then derive the following surprisingly general result.

Theorem (Godefroy–Shapiro)**.** *Suppose that* $T\colon H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, *is an operator that commutes with* D, *that is,*

$$TD = DT.$$

Then T *is mixing and chaotic.*

Birkhoff's operators T_a can also be understood as composition operators $f \to f \circ \varphi$ with $\varphi(z) = z + a$. This suggests to study general composition operators

$$C_\varphi f = f \circ \varphi$$

on spaces $H(\Omega)$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}$, where $\varphi\colon \Omega \to \Omega$ is an automorphism. It turns out that the so-called run-away property characterizes hypercyclicity.

Definition. Let Ω be a domain in $\mathbb{C}$ and $\varphi\colon \Omega \to \Omega$, a holomorphic map. Then the sequence $(\varphi^n)_n$ is called a *run-away sequence* if, for any compact subset $K \subset \Omega$, there is some $n \in \mathbb{N}$ such that

$$\varphi^n(K) \cap K = \varnothing.$$

We will then derive the following.

Theorem (Bernal–Montes)**.** *Let Ω be a simply connected domain in $\mathbb{C}$. Let $\varphi\colon \Omega \to \Omega$ be an automorphism. Then C_φ is hypercyclic if and only if $(\varphi^n)_n$ is a run-away sequence.*

Motivated by Rolewicz's example and the Godefroy–Shapiro theorem we will finally consider functions of the backward shift.

Theorem. *Let X be one of the complex sequence spaces ℓ^p, $1 \leqslant p < \infty$, or c_0. Furthermore, let $\varphi(z) = \sum_{n=0}^\infty a_n z^n$ be a non-constant holomorphic function on a neighbourhood of $\overline{\mathbb{D}}$. Let $\varphi(B) = \sum_{n=0}^\infty a_n B^n$. Then the following assertions are equivalent*:

(i) $\varphi(B)$ *is chaotic*;
(ii) $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$;
(iii) $\varphi(B)$ *has a non-trivial periodic point.*

Lecture 6. Necessary conditions for hypercyclicity and chaos

After having stated various sufficient conditions for hypercyclicity and chaos we are now looking for necessary conditions in order to show that some large classes of operators do not contain any hypercyclic or any chaotic operator. The first restriction goes back to Rolewicz.

Theorem (Rolewicz)**.** *There are no hypercyclic operators on finite-dimensional spaces.*

In particular, linear chaos is an infinite-dimensional phenomenon.

Most of the other known necessary conditions have to do, in one way or another, with the spectrum of the operator. We will therefore assume now that X is a complex Banach space. Using the Riesz decomposition theorem we will first show that, for any hypercyclic operator T, the spectrum of T must intersect the unit circle $\mathbb{T}$. This result can subsequently be improved to the following.

Theorem (Kitai)**.** *Let T be a hypercyclic operator. Then every connected component of $\sigma(T)$ meets the unit circle.*

For a chaotic operator one can show that, in addition, $\sigma(T)$ has no isolated points and it contains infinitely many roots of unity.

In view of the Riesz theory of compact operators one easily deduces the following.

Theorem. *No compact operator is hypercyclic.*

In particular, finite-rank operators cannot be hypercyclic. The same is also true for any finite-rank perturbation of a multiple of the identity. On the other hand, we will see in Lecture 10 that compact perturbations of a multiple of the identity can be hypercyclic. Chaos, however, is again excluded.

Theorem. *No compact perturbation of a multiple of the identity can be chaotic.*

By a celebrated recent result of Argyros and Haydon (2011) there are infinite-dimensional separable complex Banach spaces on which *every* operator is a compact perturbation of a multiple of the identity. Thus there are infinite-dimensional separable Banach spaces that do not support any chaotic operator. We will return to the corresponding question for hypercyclicity in Lecture 10.

We will also look at Hilbert space operators. The notion of a normal operator is familiar. A more general notion is that of a hyponormal operator, which in turn was generalized to that of a paranormal operator. This latter can even be formulated in Banach spaces.

Definition. An operator T on a Banach space X is called *paranormal* if, for all $x \in X$,

$$\|Tx\|^2 \leqslant \|T^2x\|\|x\|.$$

We will then show the following.

Theorem. *No paranormal operator on a Banach space is hypercyclic.*

Lectures 7 and 8. Hypercyclic and chaotic C_0-semigroups

In these lectures we study continuous-time linear dynamical systems. The corresponding mathematical model is that of a C_0-semigroup. A one-parameter family $(T_t)_{t\geqslant 0}$ of operators on a Banach space X is called a *C_0-semigroup* if the following three conditions are satisfied:

(i) $T_0 = I$;
(ii) $T_{t+s} = T_tT_s$ for all $s, t \geqslant 0$;
(iii) $\lim_{s\to t} T_sx = T_tx$ for all $x \in X$ and $t \geqslant 0$.

C_0-semigroups arise naturally in applications in the following way. First, the *infinitesimal generator* of a C_0-semigroups is defined by

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_tx - x),$$

which exists for a dense set $D(A)$ of vectors x. If $x \in D(A)$ then

$$u(t) = T_tx$$

is a solution of the *abstract Cauchy problem*

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) \quad \text{for } t \geqslant 0, \\ u(0) = x. \end{cases} \tag{ACP}$$

Conversely, given an abstract Cauchy problem (ACP), the solutions are (often) given by the C_0-semigroup with infinitesimal generator A. Thus, dynamical properties of C_0-semigroups can be interpreted as dynamical properties of the solutions of abstract Cauchy problems.

Now, for an arbitrary C_0-semigroup $(T_t)_t$, dynamical notions are easily transferred from the discrete to the continuous case. For any $x \in X$ we call

$$\text{orb}(x, (T_t)) = \{T_t x : t \geqslant 0\}$$

the *orbit* of x under $(T_t)_{t\geqslant 0}$. The semigroup is called *hypercyclic* if there is some $x \in X$ whose orbit under $(T_t)_{t\geqslant 0}$ is dense in X. In such a case, x is called a *hypercyclic vector* for $(T_t)_{t\geqslant 0}$. Moreover, the semigroup is said to be *chaotic* if it is hypercyclic and its set of periodic points is dense in X. Here, a point $x \in X$ is called periodic if $T_t x = x$ for some $t > 0$. The notions of topological transitivity and (weak) mixing are transferred just as easily.

As in the discrete case we have useful criteria for hypercyclicity and chaos.

Theorem (Hypercyclicity Criterion for semigroups). *Let $(T_t)_{t\geqslant 0}$ be a C_0-semigroup on X. If there are dense subsets $X_0, Y_0 \subset X$, a sequence $(t_n)_n$ in $\mathbb{R}_+$ with $t_n \to \infty$, and maps $S_{t_n}\colon Y_0 \to X$, $n \in \mathbb{N}$, such that, for any $x \in X_0$, $y \in Y_0$,*

(i) $T_{t_n} x \to 0$,
(ii) $S_{t_n} y \to 0$,
(iii) $T_{t_n} S_{t_n} y \to y$,

then $(T_t)_{t\geqslant 0}$ is weakly mixing and, in particular, hypercyclic.

There is also a Godefroy–Shapiro type eigenvalue criterion that involves the eigenvectors of the infinitesimal generator A of T.

A simple but instructive example is that of the *translation semigroup* given by

$$T_t f(x) = f(x+t), \quad t, x \geqslant 0$$

on the weighted function space

$$X = L^p_v(\mathbb{R}_+) = \{f\colon \mathbb{R}_+ \to \mathbb{K} : f \text{ is measurable and } \textstyle\int_0^\infty |f(x)|^p v(x)dx < \infty\},$$

where $1 \leqslant p < \infty$ and the weight $v\colon \mathbb{R}_+ \to \mathbb{R}$ is a strictly positive locally integrable function. Under a suitable condition on the weight v, $(T_t)_t$ is a C_0-semigroup on $L^p(v)$.

We will characterize when this semigroup is hypercyclic, (weakly) mixing or chaotic.

There is an obvious link between discrete-time and continuous-time dynamics. If $(T_t)_t$ is a C_0-semigroup and $t_0 > 0$ is fixed then the sub-orbit

$$T_{nt_0}x, \quad n \geqslant 0$$

is nothing but the orbit of x under the operator T_{t_0}. One calls $(T_{nt_0})_n$ a *discretization* of $(T_t)_t$. Obviously, if some T_{t_0} is a hypercyclic (chaotic) operator then $(T_t)_t$ is a hypercyclic (chaotic) semigroup. By a deep result, the converse is also true for hypercyclicity.

Theorem (Conejero–Müller–Peris). *If $(T_t)_{t\geqslant 0}$ is a hypercyclic C_0-semigroup then every operator T_{t_0}, $t_0 > 0$, is hypercyclic.*

The corresponding result for chaos fails, however.

We will then discuss some particular applications. We consider, for example, the following first-order abstract Cauchy problem on the space $X = L^1(\mathbb{R}_+)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + \dfrac{2x}{1+x^2}u, \\ u(0,x) = \varphi(x), \quad x \in \mathbb{R}_+. \end{cases} \tag{1}$$

The solution C_0-semigroup is given by

$$T_t\varphi(x) = \frac{1+(x+t)^2}{1+x^2}\varphi(x+t), \quad x,t \in \mathbb{R}_+.$$

We will show that this semigroup is chaotic.

We will also consider the following infinite system of ODEs:

$$\begin{cases} \dfrac{df_n}{dt} = -\alpha_n f_n + \beta_n f_{n+1}, \quad n \geqslant 1, \\ f_n(0) = a_n, \quad n \geqslant 1, \end{cases} \tag{2}$$

where $(\alpha_n)_n, (\beta_n)_n$ are positive sequences and $(a_n)_n \in \ell^1$ is a real sequence. This can be understood as an abstract Cauchy problem on ℓ^1 with A given by

$$Af = (-\alpha_n f_n + \beta_n f_{n+1})_n \ \text{ for } f = (f_n)_n.$$

We will find conditions for $(\alpha_n)_n$ and $(\beta_n)_n$ under which the solution semigroup is chaotic.

Lecture 9. Some fundamental results in linear dynamics

In this lecture we will prove or at least state several fundamental results of linear dynamics which have one thing in common: their proofs all rely, in one way or another, on a connectedness argument.

We will start with the following.

Theorem (Ansari)**.** *Let T be an operator on a Fréchet space. If T is hypercyclic then so is every power T^p, $p \in \mathbb{N}$. In fact, the operators have the same set of hypercyclic vectors: $HC(T) = HC(T^p)$.*

The proof uses in a crucial way the fact that the set $HC(T)$ of hypercyclic vectors of T is always a connected set, which is a consequence of the Herrero–Bourdon theorem (Lecture 2).

It is very easy to give an example of the failure of Ansari's theorem for non-linear mappings (take $Tx = -x$ on $\{-1, 1\}$). This is simply due to the fact that the set of points with dense orbit is no longer connected. The following *separation theorem* is a substitute for Ansari's theorem in the case of an arbitrary continuous mapping $T : X \to X$ on a metric space X without isolated points:

If $x \in X$ has dense orbit under T but not under T^p, $p > 1$, then there are a divisor $m > 1$ of p and a partition $D_0, \dots, D_{m-1}$ of $D = \{x \in X : \operatorname{orb}(x, T) \text{ is dense in } X\}$ into closed (and open) subsets with the following properties:

(i) $T(D_j) \subset D_{j+1 (\operatorname{mod} m)}$, $j = 0, \dots, m-1$;
(ii) for $j = 0, \dots, m-1$, the orbit of $T^j x$ under T^p is contained and dense in D_j.

Here is a simple application of Ansari's theorem (we are back in the linear situation). If T is hypercyclic then so is $-T$ (simply because they have the same square). In the same way, over the complex scalars, one has that λT is hypercyclic whenever T is, provided that λ is of the form $\lambda = \exp(2\pi i k/2^n)$. This has prompted the question if the same is true whenever $|\lambda| = 1$. This is indeed so.

Theorem (León–Müller)**.** *Let T be an operator on a complex Fréchet space X. If T is hypercyclic then so is every multiple λT, $|\lambda| = 1$. In fact, the operators have the same set of hypercyclic vectors:*

$$HC(T) = HC(\lambda T).$$

We will use a variant of the proof of León–Müller to prove Ansari's theorem.

Ansari's theorem can also be proved in a different way. Since

$$\operatorname{orb}(x, T) = \bigcup_{k=0}^{p-1} \operatorname{orb}(T^k x, T^p),$$

the orbit $\operatorname{orb}(x, T)$ can only be dense if some orbit $\operatorname{orb}(T^k x, T^p)$, $k = 0, \dots, p-1$, is somewhere dense. If now any somewhere dense orbit was already dense one could easily deduce Ansari's theorem. The latter, quite surprisingly, is indeed true.

Theorem (Bourdon–Feldman)**.** *Let T be an operator on a Fréchet space X and $x \in X$. If $\operatorname{orb}(x, T)$ is somewhere dense in X, then it is dense in X.*

We mention that the Conejero–Müller–Peris theorem (see Lectures 7 and 8) can also be proved by the León–Müller technique.

Lecture 10. Existence of hypercyclic and chaotic operators

Having met a wide variety of hypercyclic and chaotic operators in Lectures 4 and 5 one might wonder if every infinite-dimensional separable Banach space supports a hypercyclic operator. This is indeed the case.

Theorem (Ansari–Bernal)**.** *Every infinite-dimensional separable Banach space supports a mixing, and therefore hypercyclic, operator.*

Bonet and Peris have subsequently shown that the result extends to Fréchet spaces; we will concentrate on the Banach space setting because the proof is less technical there.

We need too ingredients. First, a result of Salas provides another large and very flexible class of hypercyclic operators on the classical sequence spaces.

Theorem (Salas)**.** *Let X be one of the spaces ℓ^p, $1 \leqslant p < \infty$, or c_0. Let B_w be a weighted backward shift on X. Then the operator $T := I + B_w$ is mixing on X.*

Incidentally, if $w_n \to 0$ then B_w is a compact operator. Thus a hypercyclic operator can be a compact perturbation of the identity, as announced in Lecture 6.

We now have to transport Salas' operators to an arbitrary infinite-dimensional separable Banach space. To define an operator similar to $I+B_w$ there we need some kind of basis in the space. While it is known that not every separable Banach space has a basis, a biorthogonal system with some additional properties will suffice for our purpose, leading to a proof of the Ansari–Bernal theorem.

As for chaos we have already noted in Lecture 6 that there are infinite-dimensional separable Banach spaces that do not support any chaotic operator. On a positive side one can show that every such Banach space supports a hypercyclic operator with an infinite-dimensional subspace of periodic points.

Next one might wonder 'how many' operators on a given Banach space are hypercyclic. Since no hypercyclic operator can have norm $\|T\| \leqslant 1$, the set of hypercyclic operators is not dense in the space $L(X)$ of all operators when endowed with the usual operator norm topology. However, when we pass to a natural, weaker topology, the strong operator topology (SOT), we obtain a positive answer.

Theorem. *Let X be an infinite-dimensional separable Fréchet space. Then the set of hypercyclic operators on X is SOT-dense in $L(X)$.*

One may also show the abundance of the set of hypercyclic operators in a different way. It is easily shown that any dense orbit is a linearly independent set. On may therefore wonder if, conversely, any dense linearly independent countable set in an arbitrary Banach space is the orbit of some vector under a suitable operator. This is indeed the case, as Grivaux showed.

Theorem. *Let X be a Banach space, and let $\{x_n : n \in \mathbb{N}\}$ be a dense linearly independent set in X. Then there exists an operator T on X, necessarily hypercyclic, such that* $\mathrm{orb}(x_1, T) = \{x_n : n \in \mathbb{N}\}$.

This result has a surprising consequence: any countable set of hypercyclic operators on a given Banach space possesses a dense subspace of common hypercyclic vectors, except zero.

Lectures 11 and 12. Frequently hypercyclic operators

In these lectures we will introduce and discuss an interesting strong form of hypercyclicity that was introduced by Bayart and Grivaux in 2004. Recall that the *lower density* a subset $A \subset \mathbb{N}_0$ is defined as

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{\text{card}\{0 \leqslant n \leqslant N : n \in A\}}{N+1}.$$

Definition. An operator T on a Fréchet space X is called *frequently hypercyclic* if there is some $x \in X$ such that, for any non-empty open subset U of X,

$$\underline{\text{dens}}\,\{n \in \mathbb{N}_0 : T^n x \in U\} > 0.$$

In this case, x is called a *frequently hypercyclic vector* for T. The set of frequently hypercyclic vectors for T is denoted by $FHC(T)$.

Obviously any frequently hypercyclic vector is hypercyclic. In fact, a vector $x \in X$ is frequently hypercyclic if and only if, for any non-empty open subset U of X, there is a strictly increasing sequence $(n_k)_k$ of positive integers such that

$$T^{n_k} x \in U \quad \text{for all } k \in \mathbb{N}, \quad \text{and } n_k = O(k).$$

We will briefly discuss the motivation behind this definition. In fact, Bayart and Grivaux arrived at their definition by applying ergodic theory to linear operators. If we suppose that the space X supports a non-degenerate probability measure μ for which T is ergodic then the Birkhoff ergodic theorem implies that T is frequently hypercyclic. We will, however, not pursue this line any further here.

Now, are there operators that have this stronger dynamical property? Surprisingly, it turns out that the three classical operators of Birkhoff (translation operator on $H(\mathbb{C})$), MacLane (differentiation operator on $H(\mathbb{C})$) and Rolewicz (multiple of the backward shift) are all even frequently hypercyclic. For the translation operator this can be shown rather directly by the Runge approximation theorem; for the other two examples it is a consequence of the following criterion.

Theorem (Frequent Hypercyclicity Criterion)**.** *Let T be an operator on a separable Fréchet space X. If there is a dense subset X_0 of X and a map $S\colon X_0 \to X_0$ such that, for any $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(ii) $\sum_{n=0}^{\infty} S^n x$ *converges unconditionally,*

(iii) $TSx = x$,

then T is frequently hypercyclic (and mixing and chaotic).

As an application we obtain a sufficient condition for a weighted backward shift to be frequently hypercyclic.

Theorem. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the weighted backward shift B_w is an operator on X. If*

$$\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$

converges unconditionally in X then B_w is frequently hypercyclic (and mixing and chaotic).

On the other hand, it is not difficult to give (easy) examples of hypercyclic operators that are not frequently hypercyclic. It is also known that some frequently hypercyclic operators are not chaotic, but the examples are rather complicated. In contrast the following is an open problem.

Problem. Is every chaotic operator frequently hypercyclic?

Another implication has turned out to be correct.

Theorem. *Every frequently hypercyclic operator on a Fréchet space is weakly mixing.*

Next, does every infinite-dimensional separable Banach space support a frequently hypercyclic operator? The answer is negative, as it was for chaotic operators. The space constructed by Argyros and Haydon provides again a counter-example, see Lecture 6. In fact, the non-existence follows from the following result of Shkarin.

Theorem. *Let T be a frequently hypercyclic operator on a complex Banach space. Then its spectrum $\sigma(T)$ has no isolated points.*

The proof of this result applies the theory of entire functions in a beautiful way. Some other properties are still undecided; for example, is the inverse of an invertible frequently hypercyclic operator itself frequently hypercyclic?

In marked contrast to the case of hypercyclicity the set $FHC(T)$ of frequently hypercyclic vectors is usually not residual. As a special case of a more general, technical result we have the following.

Theorem. *Let T be an operator on a separable Fréchet space X. Suppose that there exists a dense subset $X_0 \subset X$ such that $T^n x \to 0$ for all $x \in X_0$. Then $FHC(T)$ is of first Baire category.*

It is an open problem if $FHC(T)$ can ever be other than of first Baire category. In the case of hypercyclic operators on a separable Fréchet space X we have that $HC(T)$ is always residual. As an immediate consequence we have that every vector in X is the sum of two hypercyclic vectors for T:

$$X = HC(T) + HC(T).$$

While this argument breaks down for frequent hypercyclicity one can still ask if the corresponding result remains true. This, indeed, depends on the operator. Under mild assumptions we have that

$$X \neq FHC(T) + FHC(T).$$

This is, for example, so for the differentiation operator on $H(\mathbb{C})$ and the multiples of the backward shift on ℓ^p, say. However, there are operators for which

$$X = FHC(T) + FHC(T).$$

This is the case, for example, for the translation operator $f(\cdot) \to f(\cdot + a)$, $a \neq 0$, on the space $H(\mathbb{C})$ of entire functions. It is not known if such a result can also hold for a Banach space operator.

Lectures 13 and 14. Common hypercyclic vectors

It is an immediate consequence of the Baire category theorem that whenever we have two, or even countably many hypercyclic operators on a given Fréchet space then there exists a vector that is simultaneously hypercyclic for each of these operators; such a vector is called a *common hypercyclic vector*. This is a purely structural property and has nothing to do with the nature of the operators under consideration.

For example, the differentiation operator and any translation operator on the space $H(\mathbb{C})$ of entire functions have common hypercyclic vectors.

However, the problem comes to life again when we allow uncountable families of operators. In 1999, Salas posed the problem if the multiples λB, $|\lambda| > 1$, on ℓ^p, say, share a common hypercyclic vector. This was answered in the positive by Abakumov and Gordon in 2003. Their explicit construction was subsequently turned into a general existence theorem, which we will describe in these lectures.

The general problem is the following. Let $(T_\lambda)_{\lambda\in\Lambda}$ be a family of hypercyclic operators on a common separable Fréchet space X. Is then

$$\bigcap_{\lambda\in\Lambda} HC(T_\lambda) \neq \varnothing?$$

The answer is not always positive.

Example. Let B be the (unweighted) backward shift. Then the family of operators

$$\lambda B \oplus \mu B, \quad \lambda, \mu > 1$$

on ℓ^2 does not have a common hypercyclic vector.

As a general rule it seems to be difficult for a two-parameter family of operators to have a common hypercyclic vector. One might object that the Abakumov–Gordon result gives such a family with a *real* two-parameter family (when the λ is complex). But that is only apparent because by the León–Müller theorem (see Lecture 9) $\lambda_1 B$ and $\lambda_2 B$ have the same hypercyclic vectors if $|\lambda_1| = |\lambda_2|$. Thus the real content of the Abakumov–Gordon result is that the operators λB, $\lambda > 1$, share a common hypercyclic vector.

So we will in the sequel concentrate on the case when the parameter set Λ is an interval in $\mathbb{R}$. We will also suppose that the family depends continuously on λ, that is, that, for any $x \in X$,

$$\Lambda \to X, \quad \lambda \to T_\lambda x$$

is continuous. We then call $(T_\lambda)_{\lambda\in\Lambda}$ a *continuous family*.

Next, by our introductory remark it seems that the Baire category theorem loses its power when we consider uncountable families of operators. This is only partially true. The following is a first step towards obtaining common hypercyclic vectors, and its simple proof is based on the Baire category theorem.

Theorem. *Let $(T_\lambda)_{\lambda\in\Lambda}$ be a continuous family on a separable Fréchet space X. Then the set of common hypercyclic vectors is a dense G_δ-set if and only if, for any interval $K = [a, b] \subset \Lambda$ and for any pair U, V of non-empty open subsets of X,*

$$\exists x \in U\ \forall\lambda \in K\ \exists n \in \mathbb{N}_0 : T_\lambda^n x \in V.$$

This result shows us the way. As in topological transitivity it suffices to verify a condition for two non-empty open sets; but here the task is much more difficult in view of the fact that the same $x \in U$ has to work for all $\lambda \in K$.

The following result of Costakis and Sambarino can then be proved. For notational convenience we write $T^n(\lambda) := T_\lambda^n$.

Theorem (Common Hypercyclicity Criterion)**.** *Let $\Lambda \subset \mathbb{R}$ be an interval and $(T_\lambda)_{\lambda\in\Lambda} = (T(\lambda))_{\lambda\in\Lambda}$ a continuous family of operators on a separable Fréchet space X. Suppose that, for any compact subinterval $K \subset \Lambda$, there is a dense subset X_0 of X and maps $S_n(\lambda)\colon X_0 \to X$, $n \geqslant 0$, $\lambda \in \Lambda$, such that, for any $x \in X_0$,*

(i) $\sum_{n=0}^{m} T^m(\lambda)S_{m-n}(\mu_n)x$ *converges unconditionally, uniformly for $m \geqslant 0$ and $\lambda \geqslant \mu_0 \geqslant \ldots \geqslant \mu_m$ from K;*

(ii) $\sum_{n=0}^{\infty} T^m(\lambda)S_{m+n}(\mu_n)x$ *converges unconditionally, uniformly for $m \geqslant 0$ and $\lambda \leqslant \mu_0 \leqslant \mu_1 \leqslant \ldots$ from K;*

(iii) *for any $\varepsilon > 0$ there is some $\delta > 0$ such that, for any $n \geqslant 1$, $\lambda, \mu \in K$,*

$$\text{if} \quad 0 \leqslant \mu - \lambda < \tfrac{\delta}{n} \quad \text{then} \quad \|T^n(\lambda)S_n(\mu)x - x\| < \varepsilon;$$

(iv) $T^n(\lambda)x \to 0$ *as $n \to \infty$, uniformly for $\lambda \in K$.*

Then the set of common hypercyclic vectors of the family $(T_\lambda)_{\lambda\in\Lambda}$ is a dense G_δ-set, and in particular non-empty.

Here, $\|\cdot\|$ denotes an F-norm inducing the topology of X. And a family of series $\sum_{n=1}^\infty x_{\lambda,n}$, $\lambda \in \Lambda$, is said to *converge unconditionally, uniformly in* λ if for any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for any $\varepsilon_n \in \{0,1\}$ and any $\lambda \in \Lambda$ we have

$$\Big\| \sum_{n=N}^{\infty} \varepsilon_n x_{\lambda,n} \Big\| < \varepsilon.$$

Unfortunately, the conditions in the theorem are very strong. They imply in particular that every operator T_λ is even frequently hypercyclic and mixing.

We will then apply the result in various situations. It is most useful when considering a family of multiples of an operator. The following contains the result of Abakumov and Gordon.

Theorem. *Let T be an operator on a separable Fréchet space X. Suppose that there is a dense set $X_0 \subset X$, a mapping $S\colon X_0 \to X$ and some $\lambda_0 \geqslant 0$ such that, for any $x \in X_0$,*

(i) *there is some $n \in \mathbb{N}_0$ such that $T^n x = 0$,*
(ii) $\frac{1}{\lambda^n} S^n x \to 0$ *for any* $\lambda > \lambda_0$,
(iii) $TSx = x$.

Then the set of common hypercyclic vectors for the operators λT, $|\lambda| > \lambda_0$, is a dense G_δ-set, and in particular non-empty.

This result also implies, for example, that the multiples λD, $\lambda \neq 0$, of the differentiation operator on $H(\mathbb{C})$, have a common hypercyclic vector.

We will report on additional results on common hypercyclicity. In particular, the following result of Shkarin shows the problems one faces when looking at more complicated parameter sets.

Theorem. *Let T be an operator on a complex Fréchet space and $\Lambda \subset \mathbb{C} \times \mathbb{R}_+$ be such that the family $(\lambda I + \mu T)_{(\lambda,\mu)\in\Lambda}$ has a common hypercyclic vector. Then Λ has three-dimensional Lebesgue measure 0.*

Lecture 15. Hypercyclic subspaces

We will end the course with a brief discussion of another important topic in linear dynamics. By the Herrero–Bourdon theorem (see Lecture 2), every hypercyclic operator admits a dense subspace in which every non-zero vector is hypercyclic. We ask here for a large space of hypercyclic vectors in a different sense: does a given hypercyclic operator admit a closed and infinite-dimensional subspace in which every non-zero vector is hypercyclic? Such a subspace is called a *hypercyclic subspace.*

We will see that in many cases the answer is positive. However, in contrast to the case of dense subspaces of hypercyclic vectors, we will also find counter-examples. This makes the notion considered here particularly interesting.

We point out that the two meanings of largeness are almost incompatible. The only dense and closed subspace is the whole space itself. And while there do exist operators for which every non-zero vector is hypercyclic, such examples are extremely rare and difficult to construct. Indeed, the commonly known hypercyclic operators all have large supplies of non-hypercyclic vectors.

We begin with negative results. Montes showed in 1996 that the multiples of the backward shift, λB, $|\lambda| > 1$, do not possess hypercyclic subspaces, the reason being that the operator is bounded below by a large constant on finite-codimensional spaces. Generalizing this observation Montes arrived at the following result.

Theorem. *Let T be an operator on a Banach space X. Suppose that there are closed subspaces $M_n \subset X$, $n \geqslant 1$, of finite codimension and positive numbers C_n, $n \geqslant 1$, with $C_n \to \infty$ as $n \to \infty$ such that*

$$\|T^n x\| \geqslant C_n \|x\| \quad \text{for any } x \in M_n,\ n \geqslant 1.$$

Then T does not possess any hypercyclic subspace.

The proof uses an orthogonality argument in the case of Hilbert spaces. For Banach space one has to replace the orthogonality by a suitable substitute. There is also a Fréchet space version of the result that is more technical.

On the positive side we have the following.

Theorem (Montes)**.** *Let X be a separable Fréchet space with a continuous norm, and let T be an operator on X. Suppose that there exists an increasing sequence $(n_k)_k$ of positive integers such that*

(i) *T satisfies the Hypercyclicity Criterion for $(n_k)_k$,*
(ii) *there exists an infinite-dimensional closed subspace M_0 of X such that $T^{n_k}x \to 0$ for all $x \in M_0$.*

Then T has a hypercyclic subspace.

We will use this result to show that the differentiation operator and the translation operators on the space of entire functions have hypercyclic subspaces.

We will only prove the result for Banach spaces. Montes' original proof was by a direct but technical construction, using the notion of basic sequences. We will outline an alternative, more structural proof due to Chan. Any operator T induces a left-multiplication operator $L_T S = TS$ on the space $L(X)$ of operators on X. We will see that the Hypercyclicity Criterion for T induces some kind of hypercyclicity on L_T. Now if S is a hypercyclic vector for L_T of norm less than 1 then $(I + S)(M_0)$ turns out to be a hypercyclic subspace for T.

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