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Characteristic classes with Applications to Geometry, Topology and Number Theory

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Contents

Abstract

The goal of this lecture notes is to introduce to Characteristic Classes. This is an important tool of the contemporary mathematics, indispensable to work in geometry and topology, and also useful in number theory. Classical roots of characteristic classes overlap: the Euler characteristic, indices of vector fields and the Poincaré–Hopf theorem, Plücker formulas for plane curves, the Euler characteristic of the Milnor fibre, Riemann–Roch and Riemann–Hurwitz theorems for curves and Schubert calculus. The present approach to the characteristic classes treats them as elements of the cohomology rings and their analogues. We shall discuss the Chern classes of complex vector bundles, characteristic classes of real vector bundles, various characteristic classes of singular analytic varieties. The fundamental theorems on characteristic classes will be proven, in particular, the Grothendieck–Hirzebruch–Riemann–Roch theorem. Characteristic classes mark out the place where many domains of the contemporary mathematics meet: geometry, topology, singularities, representation theory, algebra and combinatorics. As for what concerns these last three domains, we shall discuss the basic properties of Schur functions.

To the memory of my Father (1928–2012)

1. Introduction

We will give no precise references in this introduction, postponing them to the forthcoming sections, where these issues will be discussed in detail. In case, we do not plan, however, to spend more time on some topics, some references will be given.

In mathematics, a characteristic class is a way of associating to each principal bundle on a topological space X a cohomology class of X. We shall work mainly with vector bundles. Characteristic classes are usually located in cohomology rings, Chow rings, De Rham cohomology, arithmetic Chow groups...

The cohomology class measures the extent to which the bundle is "twisted" – particularly, whether it admits sections or not. In other words, characteristic classes are global invariants which measure the deviation of a local product structure from a global product structure. They are one of the unifying geometric concepts in algebraic topology, differential geometry and algebraic geometry.

It is important not to deal with single characteristic classes, but with "formations" of them; in particular we need to use polynomials in characteristic classes.

People contributing to characteristic classes:

- Hopf, Stiefel, Whitney, Chern, Pontrjagin (Milnor, Stasheff) – they invented Euler class, Stiefel–Whitney classes and Pontrjagin classes to compute the Euler–Poincaré characteristic, to solve imbedding problems in topology, to investigate which differentiable manifolds admit complex structures,...
- Hirzebruch, Borel, Grothendieck, Bott – They did a link between characteristic classes and representation theory. They extended the classical Riemann–Roch theorem from curves and surfaces to all varieties, and even for morphisms.
- Fulton, Lascoux – they developed algebraic and combinatorial tools to compute with characteristic classes. In fact, many of this tools was already used by Grassmann, Schubert and old Italian geometers: Segre, Pieri, Giambelli. Classical mathematical objects like Schur functions and Schubert polynomials appeared to be of fundamental importance for geometry.
- Totaro – he gave an interesting insight to the problem: which characteristic classes are interesting for singular varieties?

We shall now discuss classical roots of characteristic classes.

1.1. Euler characteristic

The Euler characteristic was classically defined for the surfaces S of polyhedra, according to the formula

$$\chi(S) = V - E + F\,,$$

where V, E, and F are respectively the numbers of vertices, edges and faces in the given polyhedron. Any convex polyhedron's surface has Euler characteristic

$$\chi(S) = V - E + F = 2\,.$$

This result is known as Euler's polyhedron formula or theorem. It corresponds to the Euler characteristic of the sphere (i.e. $\chi = 2$), and applies to all spherical polyhedra.

Proof. The proof goes via "undressing" S. Remove one face from S, and deform the remaining part of the surface by flattening it on the plane. For this "plane polyhedron" T, V and E are the same, but instead of F we have F-1. So we want to show that for such a T, we have $V - E + F = 1$. Pick a triangulation of T: in any polygon which is not a triangle we add an edge. We repeat it until all faces become triangles. This does not affect $V - E + F$. Then from a border face, we remove its edge which does not belong to another face. This does not change $V - E + F$. For a triangle, $V - E + F = 1$. □

Then we have surfaces with holes, or, equivalently spheres with handles. The number g of holes (or handles) is called the genus of S and denoted by g. For a surface S of genus g, we have

$$\chi(S) = V - E + F = 2 - 2g\,.$$

Proof. We triangulate a sphere S with g handles in such a way that the borders of handles (2 circles: A and B) are edges of the triangulation. We cut the surfaces along the B-circles. Each handle will get a new boundary edge, denote it by A^*. The number of faces rests the same. The new vertices will be recompensated by the new edges. We deform the surface further, by flattening the handles. We get a sphere with $2g$ holes. We know from the above that for a sphere without holes, $V - E + F = 2$. Each hole makes the number of faces smaller by 1. The additional vertices are recompensated by additional edges. In sum, we have $V - E + F = 2 - 2g$. □

The polyhedral surfaces discussed above are, in modern language, two-dimensional finite CW-complexes. (When only triangular faces are used, they are two-dimensional finite simplicial complexes.) In general, for any finite CW-complex X, the Euler characteristic can be defined as the alternating sum

$$\chi(X) = n_0 - n_1 + n_2 - n_3 + \cdots\,,$$

where $n_i = n_i(X)$ denotes the number of simplexes of dimension i in the complex.

More generally, for any topological space X, we can define the ith Betti number $b_i = b_i(X)$ as the rank of the ith singular homology group. The Euler characteristic can then be defined as the alternating sum

$$\chi(X) = b_0 - b_1 + b_2 - b_3 + \cdots\,.$$

This quantity is well-defined if the Betti numbers are all finite, and if they are zero beyond a certain index i_0. For simplicial complexes, this is not the same definition as in the previous paragraph but a homology computation shows that the two definitions will give the same value for $\chi(X)$.

The names: Euler number and Euler–Poincaré characteristic are also used.

We state now two basic properties of the Euler characteristic.

Lemma 1. (i) *If X and Y are subcomplexes of the simplicial complex $X \cup Y$, then*

$$\chi(X \cup Y = \chi(X) + \chi(Y) - \chi(X \cap Y)\,.$$

(ii) *If $X \to Y$ is an m-fold unbranched covering, then*

$$\chi(X) = m \cdot \chi(Y)\,.$$

Proof. (i) The assertion follows from the equality

$$n_i(X \cup Y) = n_i(X) + n_i(Y) - n_i(X \cap Y)\,.$$

(ii) A sufficiently fine triangulation of Y may be lifted to triangulation of X. For each i, we have

$$n_i(X) = m \cdot n_i(Y)\,,$$

which implies the assertion. □

We often try to arrange the computations of Euler characteristics in such a way that the contribution of $X \cap Y$ is 0. Note that we have the following result.

Proposition 2. *If Y is a closed algebraic subset of a complex variety X, then*

$$\chi(X) = \chi(Y) + \chi(X - Y)\,.$$

The proof of this simple result requires some advanced techniques and is omitted here.

1.2. Vector fields and the Poincaré–Hopf Theorem

Consider first an open subset $U \subset \mathbf{R}^m$ and a smooth vector field $v\colon U \to \mathbf{R}^m$ with an isolated zero at the point $P \in U$. The function

$$\overline{v}(Q) = \frac{v(Q)}{||(v(Q)||}$$

maps a small sphere centered at P into unit sphere. The degree of this map is called the index of v at P, and denoted by $i_v(P)$. One can prove that this definition of index is invariant under diffeomorphism of U. Thus we have a well-defined notion of the index of an isolated zero of a vector field on an oriented manifold. We also have

Lemma 3. *Let v and w be smooth vector fields on a compact oriented manifold with isolated zeros. Then*

$$\sum_{v(P)=0} i_v(P) = \sum_{w(Q)=0} i_w(Q)\,.$$

For a proof, see [34].

Theorem 4. *Let M be a compact oriented manifold. Let v be a smooth vector field on M with isolated zeros. If M has boundary, then v is required to point outward at all boundary points. Then*

$$\sum_{v(P)=0} i_v(P) = \chi(M)\,.$$

[For $m = 2$, Poincaré (1885); in general, Hopf (1926).] The idea of a proof is to use the lemma and study some special vector field. Namely, we triangulate the manifold, and consider an appropriate smooth vector field v having exactly one zero in the center of each simplex. Moreover, the index of v at such center is $(-1)^d$, where d is the dimension of the simplex. The assertion now follows from the expression: $\chi(M) = \sum(-1)^i n_i$. For details, see [34].

1.3. Gauss–Bonnet formulas

The Gauss–Bonnet formula for surfaces connects their geometry (in the sense of curvature) to their topology (in the sense of the Euler characteristic).

Theorem 5. *For a compact oriented Riemannian 2-dimensional manifold M,*

$$\int\int_M K dA = 2\pi\chi(M)\,,$$

where K is the Gauss curvature of M, and dA is the element of area.

This result admits a generalization to higher dimensions, due to Chern [9] and others.

Theorem 6. *Let M be a compact oriented $2n$-dimensional Riemannian manifold, and let Ω be the curvature form of the Levi–Civita connection. This means that Ω is an $\mathfrak{so}(2n)$-valued 2-form on M. So Ω can be regarded as a skew-symmetric $2n \times 2n$ matrix whose entries are 2-forms. One may therefore take the Pfaffian of Ω, $Pf(\Omega)$, which turns out to be a $2n$-form. Then we have*

$$\int_M Pf(\Omega) = (2\pi)^n\chi(M)\,.$$

For details, see [34].

If $\dim(M) = 4$, we have

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(|Rm|^2 - 4|Rc|^2 + R^2\right) d\mu\,,$$

where Rm is the full Riemann curvature tensor, Rc is the Ricci curvature tensor, and R is the scalar curvature.

1.4. Why characteristic CLASSES?

These are numbers which are most important in counting. Perhaps numbers are sufficient for geometers? We use $\chi(X)$, $i_v(P)$, $\deg(X)$, $g(X)$ etc.

- In many geometric problems, we must, however, compute the fundamental class $[Z]$ of a subscheme $Z \subset X$ in the intersection ring of X. And usually the rank of the group, where $[Z]$ is situated, is large. It is much easier and handy to work with classes of cycles than with large sequences of integers.
- And even, if we agree that the numbers are most important, having in disposal the intersection ring with cycles which can by multiplied, we get numbers from the monomials $\alpha_1 \cdot \alpha_2 \cdots \alpha_r$, where $\sum \dim(\alpha_i) = \dim(X)$. These α's can be some characteristic classes. For instance, the Riemann-Roch expressions, giving some numbers, are built in this way (see below).

1.5. Euler class

We could continue the list of manifestations of the Euler characteristic in mathematics. An explanation for the ubiquity of this notion is the fact that this is a characteristic class, usually called Euler class (topology) or top Chern class (algebraic geometry). According to the beginning of this section, we must specify the bundle that we use: this is the tangent bundle. Let us switch to complex algebraic geometry to simplify the discussion. By the rank of a vector bundle we understand the dimension d of its fibers. For a vector bundle E of rank d on X (not only for the tangent one), there are Chern classes $c_i(E) \in H^{2i}(X, \mathbf{Z})$, $i = 0, 1, \ldots, d$ such that $c_1(\mathcal{O}(D)) = [D]$, and

– for a morphism $f \colon Y \to X$, $c_i(f^*E) = f^*c_i(E)$ (naturality);
– for vector bundles E', E'', $c_h(E_1 \oplus E_2) = \sum_{i+j=h} c_i(E')c_j(E'')$ (Whitney sum formula). In fact, let

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of vector bundles. Then we have

$$c_h(E) = \sum_{i+j=h} c_i(E')c_j(E'') .$$

Chern classes are a universal tool overlapping many geometric concepts. The basic notion of degree of a projective scheme is spelled in the following way: the degree of $Z \subset \mathbf{P}^n$ is $[Z] \cdot c_1(\mathcal{O}(1))^{n-d}$, where $d = \dim(Z)$. If Z is nonsingular, then this degree is $c_1(\mathcal{O}_Z(1))^d$, where $\mathcal{O}_Z(1) = \mathcal{O}(1)|_Z$. It is important to use polynomials in Chern classes, and not single characteristic classes. The Giambelli formula for the class of a Schubert subvariety on the Grassmannian, strongly supports this claim. Let $G_r(V)$ be the set of all r-dimensional planes in an n-dimensional complex vector space V. It is an algebraic variety of dimension $r(n-r)$, equipped with the tautological bundle R of rank r. Consider a flag

$$V_0 \subset V_1 \subset \cdots \subset V_n = V$$

of vector spaces with $\dim(V_i) = i$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \leqslant n-r$, we define

$$\Omega_\lambda = \{L \subset V : \dim(L \cap V_{n-r+i-\lambda_i}) \geqslant i,\ 1 \leqslant i \leqslant r\} .$$

The class $[\Omega_\lambda]$ of this Schubert variety does not depend on the flag and is called a Schubert class. The Giambelli formula asserts that

$$[\Omega_\lambda] = s_\lambda(R^*) ,$$

the Schur function of R^*, which admits an algebraic expression as a polynomial in the Chern classes of R. These functions play a role in representation theory. In fact, there are many connections between characteristic classes and representation theory.

An expression of the Euler number in terms of characteristic classes is: for a nonsingular variety X,

$$\chi(X) = c_{\dim X}(TX) . \tag{1}$$

(Since $H^{\dim X}(X, \mathbf{Z})$ has a canonical generator $[X]$, the RHS of the above formula is understood to be the (integer) coefficient of $[X]$.) With the Chern classes in differential

geometry, defined with the help of curvature, the equation (1) is a generalized Gauss–
–Bonnet theorem.

1.6. The number of lines on a nonsingular cubic surface

Let $\mathbf{P}^n = \mathbf{P}(E)$, E vector space of dimension $n+1$. Let $G = G_d(\mathbf{P}^n) = G_{d+1}(E)$, R the universal subbundle of E_G.

A hypersurface $X \subset \mathbf{P}^n$ of degree m is given by zeros of a section of $\mathrm{Sym}^m(E^*)$ on $\mathbf{P}^n$. Since R^* is a quotient of E_G^*, this determines a section of $\mathrm{Sym}^m(R^*)$ on G, whose zero scheme is the Fano scheme F of d-planes in X. If codimension of F in G is espected, then

$$[F] = c_{\mathrm{top}}(\mathrm{Sym}^m(R^*))\,.$$

In case $m = 3$, $d = 1$, $n = 3$, i.e., we are looking for the number of lines on a nonsingular cubic surface,

$$[F] = c_4(\mathrm{Sym}^3(R^*)) = 18c_1^2c_2 + 9c_2^2\,,$$

where $c_i = c_i(R^*)$. Both $c_1^2c_2$ and c_2^2 contribute with multiplicity one to the class of the point. Thus, the looked for number is: $18 + 9 = 27$.

1.7. Singular varieties

A natural question emerges: what about singularities? Do we need the tangent bundle to have characteristic classes $c_i(X)$ of a singular variety X with $c_{\dim X}(X) = \chi(X)$?

The answer is: we do not need the tangent bundle. One can construct such characteristic classes for singular varieties! The starting point was done by such classics as Plücker, Weierstrass, Clebsch, M. Noether. They investigated the failure of the formulas for plane curves in nonsingular case, caused by the appearance of singularities. For example, let C be an irreducible plane curve having only cusps and nodes as singularities. We have

$$g(C) = \frac{1}{2}(d-1)(d-2) - \delta - \kappa\,,$$

where δ is the number of nodes, and κ is the number of cusps. We shall study Plücker formulas, and then generalize them to higher dimensional varieties. In the case of singular hypersurfaces (in particular, for singular curves on nonsingular surfaces), the Milnor fibration will be a very useful tool.

In fact, the existence of functorial Chern classes for singular varieties was a conjecture of Grothendieck and Deligne; it was solved positively by R. MacPherson in 1974. (Note that 6 years before the conjecture was formulated, M-H. Schwartz [45] constructed such classes topologically though with no reflection on their functoriality.)

1.8. Riemann–Roch

The original problem motivating the work on this topic can be formulated as follows: given a connected nonsingular projective variety X and a vector bundle E over X, calculate the dimension $\dim H^0(X,E)$ of the space of global sections of E. The great intuition of

Serre told him that the problem should be reformulated using higher cohomology groups as well. Namely, Serre conjectured that the number

$$\chi(X,E)=\sum(-1)^i \dim H^i(X,E)$$

could be expressed in terms of topological invariants related to X and E. Naturally, Serre's point of departure was the Riemann–Roch for a curve X:

$$\chi(X,E)=c_1(E)+\operatorname{rank}(E)(1-g(X))\,,$$

and for a surface X,

$$\chi(E,X)=\frac{1}{2}\big(c_1(E)^2-2c_2(E)+c_1(E)c_1(X)\big)+\frac{1}{12}\operatorname{rank}(E)\big(c_1(X)^2+c_2(X)\big)\,.$$

The conjecture was proved in 1953 by F. Hirzebruch, inspired by earlier ingenious calculations of J.A. Todd. This was an absolute breakthrough for the computations with Riemann–Roch. Here is the formula discovered by Hirzebruch for an n-dimensional variety X:

$$\chi(X,E)=\deg(\operatorname{ch}(E)\operatorname{td}X)_{2n}\,,$$

where $(-)_{2n}$ denotes the degree $2n$ component of an element of the cohomology ring $H^*(X,\mathbf{Z})$, and

$$\operatorname{ch}(E)=\sum e^{a_i},\quad \operatorname{td}X=\prod\frac{x_j}{1-e^{-x_j}}$$

(where the a_i are the Chern roots of E: these are the classes of the line bundles which formally split E, and the x_j are the Chern roots of the tangent bundle TX).

Now the *relative* Hirzebruch–Riemann–Roch theorem, discovered by Grothendieck asserts the commutativity of the diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{f_!} & K(Y) \\ \Big\downarrow{\scriptstyle \operatorname{ch}_X(-)\operatorname{td}X} & & \Big\downarrow{\scriptstyle \operatorname{ch}_Y(-)\operatorname{td}Y} \\ H^*(X,\mathbf{Z}) & \xrightarrow{f_*} & H(Y,\mathbf{Z})\,. \end{array}$$

Though the Hirzebruch–RR has many applications, the essential contribution of the Grothendieck–RR has no really spectacular applications up to now!

1.9. Smoothing algebraic cycles

This is an interesting application of "Riemann–Roch without denominators" of Jouanolou by Kleiman. Let X be a nonsingular scheme over an algebraically closed field, $f\colon X\hookrightarrow \mathbf{P}^m$ a locally closed imbedding, $h=c_1f^*\mathcal{O}(1)$. For any $\alpha\in CH^p(X)$, there is a bundle E on X and an integer n such that

$$(p-1)!\alpha=c_p(E)-nh^p\,.$$

Kleiman uses this result together with the geometry of Schubert cycles to show that $(p-1)!\alpha$ is smoothable, that is, rationally equivalent to a cycle $\sum n_i[V_i]$ with all V_i nonsingular, provided $p > (\dim(X)-2)/2$.

Earlier, in characteristic zero, Hironaka had shown that α itself can by smoothed if $\dim(\alpha) \leqslant 3$ and $p > (\dim(X)+2)/2$. The impossibility of smoothing cycles in general was shown by Hartshorne, Rees, Thomas. The question of smoothing with rational coefficients remains open.

1.10. Two currents of the lectures

This lecture notes will consist of two currents:

I. Intersection ring of nonsingular variety, Chern classes, GHRR and applications.
II. Plücker formulas, Chern classes of singular varieties, examples and applications.

2. Some intersection theory

The main reference here is the Fulton's monograph on intersection theory [13], as well as its summary [14]. An earlier paper [15] can serve as an introduction to the ideas developed here.

Our principal task here is to study characteristic classes. So, in principle, we assume the knowledge of intersection rings, where the characteristic classes are located (for example, cohomology rings). We shall, however, describe the construction of an intersection ring in algebraic geometry following Fulton and MacPherson. Their construction does not use a moving lemma.

2.1. Rational equivalence

Let X be a variety over a field k. Given a (pure dimensional) subscheme Z of X, by

$$[Z] = \sum m_i[Z_i]$$

we denote its fundamental class, where Z_i are irreducible components of Z and $m_i = l(\mathcal{O}_{Z,Z_i})$ are their geometric multiplicities.

For any variety X over any field, let Z_rX be the group of r-cycles $\sum n_i[V_i]$ on X, that is the free abelian group on the r-dimensional subvarieties of X. Two r-cycles are rationally equivalent if they differ by a sum of cycles of the form

$$\sum[\mathrm{div}(f_i)]\,,$$

where $f_i \in k(W_i)^*$, with W_i subvarieties of X of dimension $r+1$. The group of r-cycles modulo rational equivalence on X is denoted by $CH_r(X)$, and we write

$$CH_*X = \oplus CH_r(X) = Z_*X/\sim\,,$$

where $\sim$ denotes rational equivalence. These groups are often called the Chow groups of X (this is a justification of the notation).

Remark 7. This definition is equivalent to the following classical one. Two r-cycles are rationally equivalent if they differ by a sum

$$\sum n_i([V_i(0)] - [V_i(\infty)])$$

with $n_i \in \mathbf{Z}$, V_i subvarieties in $X \times \mathbf{P}^1$ whose projections to $\mathbf{P}^1$ are dominant, and $V_i(0)$ and $V_i(\infty)$ are the scheme-theoretic fibres of V_i over 0 and ∞.

If $f\colon X \to Y$ is a proper morphism, the formula

$$f_*[V] = \deg(V/f(V))[f(V)]$$

determines a homomorphism $f_*\colon Z_rX \to Z_rY$.

Theorem 8. *If $f\colon X \to Y$ is proper, and α and β are rationally equivalent cycles on X, then $f_*(\alpha)$ and $f_*(\beta)$ are rationally equivalent cycles on Y.*

For a proof, see [13].

Thus have an induced homomorphism

$$f_*\colon CH_rX \to CH_rY\ ,$$

(the push-forward) making CH_r a covariant functor for proper morphisms.

In particular, if X is complete, that is the projection $p\colon X \to \operatorname{Spec}(k)$ is proper, the degree of a zero cycle is well defined on rational equivalence classes. We set

$$\int_X \alpha = \deg(\alpha) = p_*(\alpha),$$

identifying $CH_0(\operatorname{Spec}(k))$ with $\mathbf{Z}$. For an arbitrary cycle α on X, we write $\int_X \alpha$ for the degree of the 0th component of α.

We shall now describe a class of morphisms $f\colon X \to Y$ for which there is $f^*\colon CH_rY \to CH_{r+n}X$, where n is the relative dimension of f. For any r-dimensional subvariety V of Y, $f^{-1}(V)$ will be a subscheme of X of pure dimension $r+n$, and we will define

$$f^*[V] = [f^{-1}(V)].$$

This class of morphisms includes:

(1) projections $p\colon Y \times F \to Y$, F an n-dimensional variety; here $p^*[V] = [V \times F]$;
(2) projections $p\colon E \to Y$ (resp. $\mathbf{P}(E) \to Y$) from a bundle to its base; here $p^*[V] = [E|_V]$ (resp. $[\mathbf{P}(E|_V)]$);
(3) open imbeddings $j\colon U \to X$, with $n = 0$; here $j^*[V] = [V \cap U]$;
(4) any dominant (nonconstant) morphism from an $n+1$-dimensional variety to a nonsingular curve.

A class of maps including these for which this pullback is defined on rational equivalence classes is the class of flat morphisms; this includes all smooth morphisms.

2.2. A short exact sequence

There is a useful short exact sequence for rational equivalence.

Proposition 9. *Let Y be a closed subscheme of a scheme X, and let $U = X - Y$. Let $i\colon Y \hookrightarrow X$ and $j\colon U \to X$ be the inclusions. Then the sequence*

$$CH_r(Y) \to CH_r(X) \to CH_r(U) \to 0$$

is exact for any r.

Proof. Since any subvariety V of U extends to a subvariety $\overline{V}$ of X, the map j^* is surjective.

If $\alpha \in Z_r$ is such that $j^*\alpha \sim 0$ then

$$j^*\alpha = \sum [\mathrm{div}(f_i)]$$

where $f_i \in k(W_i)^*$, W_i subvarieties of U. Since $k(W_i) = k(\overline{W}_i)$, f_i correspond to a function $\overline{f}_i \in k(\overline{W}_i)^*$, and in Z_rU

$$j^*(\alpha - \sum [\mathrm{div}(\overline{f}_i)] = 0$$

Hence there is $\beta \in Z_rY$ such that

$$i_*\beta = \alpha - \sum [\mathrm{div}(\overline{f}_i)]\,.$$

The proposition is proved. □

2.3. Chow groups of vector bundles

Proposition 10. *Let E be a vector bundle of rank d on X, $p = p_E\colon E \to X$ the projection. Then the pull-back homomorphisms*

$$p^*\colon CH_rX \to CH_{r+d}E$$

are all isomorphisms.

Proof. We shall show that $p^*\colon CH_rX \to CH_{r+d}E$ is surjective for all r. Pick a closed scheme Y of X, so that $U = X - Y$ is an affine open set over which E is trivial. We have a commutative diagram

$$\begin{array}{ccccccc} CH_*(Y) & \longrightarrow & CH_*(X) & \longrightarrow & CH_*(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ CH_*(p^{-1}Y) & \longrightarrow & CH_*(E) & \longrightarrow & CH_*(p^{-1}U) & \longrightarrow & 0 \end{array}$$

where the rows are exact, and the vertical maps are flat pull-backs. By a diagram chase it suffices to prove the assertion for the restrictions of E to U and to Y. By Noetherian induction, that is, by repeating the process on Y, it suffices to prove it for $X = U$. Thus we can assume $E = X \times k^d$. The projection factors

$$X \times k^d \to X \times k^{d-1} \to X\,,$$

so we may assume $r = 1$.

For any $(r+1)$-dimensional subvariety V of E, we must show that $[V] \in p^* CH_r(X)$. We may replace X by the closure of $p(V)$, thus assuming that X is a variety and p maps V dominantly to X. Let R be the coordinate ring of X, $K = k(X)$ the quotient field of R, and let $\mathfrak{p}$ be the prime ideal in $R[t]$ corresponding to $[V]$. If $\dim X = r$, then $V = E$, so $[V] = p^*[X]$. Thus we can assume $\dim X = r + 1$. Since V dominates X and $V \neq E$, the prime ideal $\mathfrak{p}K[t]$ is nontrivial. Let $f \in K(t)$ generate $\mathfrak{p}K[t]$. Then

$$[V] - \operatorname{div}(f) = \sum n_i [V_i]$$

for some $(r+1)$-dimensional subvarieties V_i of E whose projections to X are not dominant. Hence $V_i = p^{-1}(W_i)$ with $W_i = p(V_i)$, so

$$[V] = \sum n_i p^*[W_i] + \operatorname{div}(f)\,,$$

as needed.

For the proof of injectivity, see [13]. □

Let s_E denote the zero section of a vector bundle E. Using the proposition, we define Gysin homomorphisms

$$s_E^* \colon CH_r(E) \to CH_{r-d}(X),$$

$d = \operatorname{rank}(E)$, by the formula

$$s_E^*(\alpha) = (p^*)^{-1}(\alpha)\,.$$

2.4. Intersecting with divisors

We discuss now intersecting with divisors. If D is a Cartier divisor on X, and α a r-cycle on X, we define an intersection class

$$D \cdot \alpha \in CH_{r-1}(Z)\,,$$

where Z is the intersection of the support of D and the support of α. By linearity it suffices to define $D \cdot [V]$ if V is a subvariety of X. Let $i \colon V \hookrightarrow X$ be the inclusion. There are two cases:

(i) If V is not contained in the support of D, then by restricting local equations, D determines a Cartier divisor, denoted i^*D, on V. In this case, set

$$D \cdot [V] = [i^*D],$$

the associated Weil divisor of i^*D on V. In this case $D \cdot [V]$ is a well-defined cycle.

(ii) If $V \subset \mathrm{Supp}(D)$, then the line bundle $\mathcal{O}_X(D)$ restricts to a line bundle $i^*\mathcal{O}_X(D)$ on V. Pick a Cartier divisor C on V whose line bundle is isomorphic to this line bundle: $\mathcal{O}_V(C) \simeq i^*\mathcal{O}_X(D)$ and set

$$D \cdot [V] = [C],$$

the associated Weil divisor of C. Since C is well defined up to a principal divisor on V, $[C]$ is well defined in $CH_{r-1}(V)$.

In general because of the ambiguity in case (ii), $D \cdot \alpha$ is only defined up to rational equivalence.

We have: if $\alpha \sim \beta$, then $D \cdot \alpha = D \cdot \beta \in CH_*(\mathrm{Supp}\, D)$.

If D is an effective divisor on X and $f\colon D \hookrightarrow X$ is the inclusion, it follows from this property that $[V] \mapsto D \cdot [V]$ determines a Gysin homomorphism

$$f^*\colon CH_r(X) \to CH_{r-1}(D).$$

2.5. Normal cones

We shall need normal cones. This notion is a generalization of the notion of a normal bundle. If $X \hookrightarrow Y$ is an imbedding of nonsingular varieties, then the normal bundle

$$N_X Y = (TY|_X)\ /TX$$

can be presented as $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{a}/\mathfrak{a}^2))$, where $\mathfrak{a}$ is the ideal (or sheaf of ideals) defining X in Y.

Similarly, if X is a subscheme of an affine variety Y, defined by an ideal $\mathfrak{a}$ in the coordinate ring R of Y, then the normal cone $C = C_X Y$ to X in Y is defined to be

$$C = \mathrm{Spec}(\oplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}).$$

The isomorphism of the coordinate ring of X with $R/\mathfrak{a} = \mathfrak{a}^0/\mathfrak{a}^1$ determines a projection $p_C\colon C \to X$, and an imbedding $s_C\colon X \hookrightarrow$C, called the zero section. We have

$$p_C \circ s_C = id_X\,.$$

If $\mathfrak{a}$ is generated by a regular sequence, then

$$\oplus \mathfrak{a}^n/\mathfrak{a}^{n+1} \simeq \mathrm{Sym}(\mathfrak{a}/\mathfrak{a}^2)\,,$$

and the normal cone $C_X Y$ is a vector bundle, the normal bundle to X in Y.

In spite of the marvelous brevity of this algebraic definition, its geometry is not simple. For example if $X = P$ is a point, then $C_P Y$ is the tangent cone at P to Y. This concept is closely related to the blow-up of a variety Y along the subscheme X. The projective normal cone $\mathbf{P}(C) = \mathbf{P}(C_X Y)$ is defined by

$$\mathbf{P}(C) = \mathrm{Proj}(\oplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1})\,.$$

The blow-up of Y along X is the variety $\tilde{Y} = \mathrm{Bl}_X Y$ together with a proper morphism $\pi\colon \tilde{Y} \to Y$ such that:

(i) The inverse image scheme $E = \pi^{-1}(X)$ is a Cartier divisor on $\tilde{Y}$, called exceptional divisor.
(ii) E is isomorphic to $\mathbf{P}(C_X Y)$, and the map from E to X induced by π is the projection from $\mathbf{P}(C)$ to X.
(iii) The induced map from $\tilde{Y} - E$ to $Y - X$ is an isomorphism.

We have

$$\mathrm{Bl}_X Y = \mathrm{Proj}(\oplus_{n=0}^{\infty} \mathfrak{a}^n)\,.$$

The above constructions globalize in an obvious way to the case of an arbitrary closed subscheme X of an arbitrary variety Y.

We shall need the following two results about the blow-up. The first one is well-known:

Proposition 11. *If $X \subset Y' \subset Y$ are closed imbeddings, there is a canonical imbedding of $\mathrm{Bl}_X Y'$ in $\mathrm{Bl}_X Y$ such that the exceptional divisor of $\mathrm{Bl}_X Y$ restricts to the exceptional divisor of $\mathrm{Bl}_X Y'$.*

Lemma 12. *If a morphism $Y \to Z$, where Z is a nonsingular curve, is flat, then the composite morphism: $\mathrm{Bl}_X Y \to Y \to Z$ is also flat, for any subscheme $X \subset Y$.*

(See, e.g., [13], B.6.7. p. 436.)

Example 13. Suppose that $C_1 = D_1 \cup E$, $C_2 = D_2 \cup E$ are plane curves such that D_1 and D_2 meet properly. Let $W = C_1 \cap C_2$. Then the normal cone $C = C_W(\mathbf{C}^2)$ has components over each component of E, and over each point in $D_1 \cap D_2$, including those points which are in E. To be more precise, let f_1, f_2 and g be polynomials in $k[x, y]$ defining D_1, D_2 and E. Then the ideal $\mathfrak{a}$ of W is $(f_1 g, f_2 g)$. Consider the homomorphism:

$$R/\mathfrak{a}[z_1, z_2] \to \oplus \mathfrak{a}^n/\mathfrak{a}^{n+1}\,,$$

such that $z_i \mapsto f_i g$ modulo $\mathfrak{a}^2$. One can prove that the kernel of this homomorphism is generated by $f_2 z_1 - f_1 z_2$. Hence $C_W(\mathbf{C}^2)$ is the subscheme of $W \times \mathbf{C}^2$ defined by the equation: $f_2 z_1 - f_1 z_2 = 0$. From this description, one can obtain information about the components of C.

We shall also need the projective completion of the cone $C = C_X Y$. Consider the cone:

$$C \oplus 1 = \mathrm{Spec}\big(\oplus_n \mathfrak{a}^n/\mathfrak{a}^{n+1} \otimes_{R/\mathfrak{a}} R/\mathfrak{a}[t]\big)\,,$$

where the imbedding $X \hookrightarrow Y$ is defined by the ideal (or sheaf of ideals) $\mathfrak{a}$. This is the normal cone to $X \times \{\infty\}$ in $Y \times \mathbf{P}^1$. Its projectivization

$$\mathbf{P}(C \oplus 1) = \mathrm{Proj}\big(\oplus_n \mathfrak{a}^n/\mathfrak{a}^{n+1} \otimes_{R/\mathfrak{a}} R/\mathfrak{a}[t]\big)$$

is called the projective completion of C. The element t determines a regular section of $\mathcal{O}_{C\oplus 1}(1)$ on $\mathbf{P}(C \oplus 1)$, whose zero scheme is isomorphic to $\mathbf{P}(C)$. The difference

$\mathbf{P}(C \oplus 1) - \mathbf{P}(C)$ is isomorphic to C, and $\mathbf{P}(C) \subset \mathbf{P}(C \oplus 1)$ is called the hyperplane at infinity in $\mathbf{P}(C \oplus 1)$.

2.6. Deformation to normal cone

We shall now discuss deformation to the normal cone. Let X be a closed subscheme of a scheme Y, and let $C = C_X Y$ be the normal cone to X in Y. We shall deform the imbedding $X \hookrightarrow Y$ (as a general fibre) to the zero section imbedding $X \hookrightarrow C_X Y$ (the special fibre).

More precisely, we shall construct a scheme $M = M_X Y$, together with an imbedding of $X \times \mathbf{P}^1$ in M, and a flat morphism $\rho \colon M \to \mathbf{P}^1$ so that

$$\begin{array}{ccc} X \times \mathbf{P}^1 & \hookrightarrow & M \\ & {\scriptstyle pr} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathbf{P}^1 & \end{array}$$

commutes, and such that:

(i) Over $\mathbf{P}^1 - \{\infty\} = \mathbf{A}^1$, $\rho^{-1}(\mathbf{A}^1) = Y \times \mathbf{A}^1$, and the embedding is trivial imbedding

$$X \times \mathbf{A}^1 \hookrightarrow Y \times \mathbf{A}^1 .$$

(ii) Over ∞, the divisor $M_\infty = \rho^{-1}(\infty)$ is the sum of two effective Cartier divisors:

$$M_\infty = \mathbf{P}(C \oplus 1) + \tilde{Y} \,,$$

where $\tilde{Y}$ is the blow-up of Y along X. The imbedding of $X = X \times \{\infty\}$ in M_∞ is the zero-section imbedding of X in C, followed by the canonical open imbedding of C in $\mathbf{P}(C \oplus 1)$. The divisors $\tilde{Y}$ and $\mathbf{P}(C \oplus 1)$ intersect in the scheme $\mathbf{P}(C)$, which is imbedded as the hyperplane at infinity in $\mathbf{P}(C \oplus 1)$ and as the exceptional divisor in $\tilde{Y}$.

In particular, the image of X in M_∞ is disjoint from $\tilde{Y}$. Letting $M^\circ = M_X Y^\circ$ be the complement of $\tilde{Y}$ in M, one has a family of imbeddings of X:

$$\begin{array}{ccc} X \times \mathbf{P}^1 & \hookrightarrow & M^\circ \\ & {\scriptstyle pr} \searrow \quad \swarrow {\scriptstyle \rho^\circ} & \\ & \mathbf{P}^1 & \end{array}$$

which deforms the given imbedding of X in Y to the zero-section imbedding of X in $C_X Y$.

Proof. Let M be the blow-up of $Y \times \mathbf{P}^1$ along the subscheme $X \times \{\infty\}$.

We first identify imbeddings of $X \times \mathbf{P}^1$, $\mathbf{P}(C \oplus 1)$ and $\tilde{Y}$ into M under this blow-up. Since $X \times \mathbf{P}^1$ is such that

$$X \times \{\infty\} \hookrightarrow X \times \mathbf{P}^1 \hookrightarrow Y \times \mathbf{P}^1 \,,$$

we see by Proposition 11 that the blow-up of $X \times \mathbf{P}^1$ along $X \times \{\infty\}$ is a subscheme of M. But $X \times \{\infty\}$ is a Cartier divisor on $X \times \mathbf{P}^1$. Hence the blow-up of $X \times \mathbf{P}^1$ along $X \times \{\infty\}$ is isomorphic to $X \times \mathbf{P}^1$ itself.

Since

$$C_{X\times\{\infty\}}(Y \times \mathbf{P}^1) = C \oplus 1\,,$$

the exceptional divisor of the blow-up is $\mathbf{P}(C \oplus 1)$, which thus is canonically imbedded in M.

Since $Y \times \{\infty\}$ is such that

$$X = X \times \{\infty\} \hookrightarrow Y \times \{\infty\} \hookrightarrow Y \times \mathbf{P}^1\,,$$

we see by Proposition 11 that $\tilde{Y}$ is imbedded as a subscheme of M.

The flatness of ρ follows from Lemma 12.

Assertion (i) holds because $M \to Y \times \mathbf{P}^1$ is an isomorphism outside of $X \times \{\infty\}$, so also outside $Y \times \{\infty\}$.

Assertion (ii) about $\mathbf{P}(C \oplus 1)$ and $\tilde{Y}$ will follow from their algebraic local descriptions (since both schemes are globally imbedded in M, it suffices to prove this assertion locally).

Let $Y = \operatorname{Spec}(R)$ and let X be defined by the ideal $\mathfrak{a}$ in R. We want to investigate M_∞. We identify $\mathbf{P}^1 - \{0\}$ with the affine line $\mathbf{A}^1 = \operatorname{Spec}(k[t])$ (so that the point ∞ becomes now the point 0). The blow-up of $Y \times \mathbf{A}^1$ along $X \times \{0\}$ is equal to $\operatorname{Proj}(S)$, where $S = \oplus S_n$ is a graded algebra with

$$S_n = \mathfrak{a}^n + \mathfrak{a}^{n-1}t + \cdots + Rt^n + Rt^{n+1} + \cdots\,.$$

The scheme $\operatorname{Proj}(S)$ has open affine covering by the sets $\operatorname{Spec}(S_{(r)})$, where

$$S_{(r)} = \{\frac{s}{r^n} : s \in S_n\}\,,$$

and r runs over the set of generators of the ideal generated by $\mathfrak{a}$ and t in $R[t]$. For $r \in \mathfrak{a}$, $\mathbf{P}(C \oplus 1)$ is defined in $\operatorname{Spec}(S_{(r)})$ by the equation: $r/1$, with $r \in S_0$, and $\tilde{Y}$ is defined by the equation: t/r. By virtue of $t = (r/1)\cdot(t/r)$, we thus get

$$M_\infty = \mathbf{P}(C \oplus 1) + \tilde{Y}\,.$$

It remains to identify the complement of $\tilde{Y}$ in the fibre over 0 of the composite of the blown-down to $Y \times \mathbf{A}^1$ followed by the projection to $\mathbf{A}^1$. The complement of $\tilde{Y}$ in the blow-up of $Y \times \mathbf{A}^1$ along $X \times \{0\}$ is $\operatorname{Spec}(S_{(t)})$. Since the fibre over 0 is given by $t = 0$, the looked at complement is $\operatorname{Spec}(S_{(t)}/tS_{(t)})$. We have

$$S_{(t)} = \cdots \oplus \mathfrak{a}^n t^{-n} \oplus \cdots \oplus \mathfrak{a}t^{-1} \oplus R \oplus Rt \oplus \cdots \oplus Rt^n \oplus \cdots\,,$$

that is, $S_{(t)} = \oplus_{n=-\infty}^{\infty} \mathfrak{a}^n t^{-n}$, with $\mathfrak{a}^n = R$ for $n \leqslant 0$. Since

$$tS_{(t)} = \cdots \oplus \mathfrak{a}^{n+1}t^{-n} \oplus \cdots \oplus \mathfrak{a}^2 t^{-1} \oplus \mathfrak{a} \oplus Rt \oplus Rt^2 \oplus \cdots \oplus Rt^{n+1} \oplus \cdots\,,$$

we obtain

$$S_{(t)}/tS_{(t)} = \oplus_{n=0}^{\infty}\mathfrak{a}^n/\mathfrak{a}^{n+1}\,.$$

Hence the complement of $\tilde{Y}$ in the fibre over 0 is

$$\mathrm{Spec}(\oplus_{n=0}^{\infty}\mathfrak{a}^n/\mathfrak{a}^{n+1}) = C\,. \;\square$$

2.7. Gysin morphism for regular imbeddings

We now discuss specialization to the normal cone. Let X be a closed subscheme of a scheme Y, and Let $C = C_X Y$ be the normal cone to X in Y. Define the specialization homomorphisms

$$\sigma\colon Z_r(Y) \to Z_r(C)$$

by the formula

$$\sigma([V]) = [C_{V\cap X}V]$$

for any r-dimensional subvariety V of Y, and extending linearly to all r-cycles.

Proposition 14. *If a cycle $\alpha \in Z_r Y$ is rationally equivalent to zero on Y, then $\sigma(\alpha)$ is rationally equivalent to zero on C.*

Therefore σ passes to rational equivalence, defining the specialization homomorphisms

$$\sigma\colon CH_r(Y) \to CH_r(C).$$

Proof. Let $M^\circ = M_X^\circ Y$ be the deformation space. We have

$$M^\circ - C = Y \times \mathbf{A}^1\,.$$

Denote by $i\colon C \hookrightarrow M^\circ$ and $j\colon Y \times \mathbf{A}^1 \to M^\circ$ the inclusions. Consider the diagram

$$\begin{array}{ccccccc} CH_{r+1}(C) & \xrightarrow{i_*} & CH_{r+1}(M^\circ) & \xrightarrow{j^*} & CH_{r+1}(Y\times\mathbf{A}^1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i^*} & & \uparrow{\scriptstyle pr^*} & & \\ & & CH_r(C) & \xleftarrow{\ \sigma\ } & CH_r(Y) & & \end{array}$$

where pr^* is the flat pullback associated with the projection $pr\colon Y \times \mathbf{A}^1 \to Y$. The row is an exact sequence by Proposition 61.

The map i^* is the Gysin map for divisors. Exercise: Show that $i^* i_* = 0$. (Hint: use the fact that the normal bundle to C in M° is trivial.) Hence there is a well defined induced morphism from $CH_{r+1}(Y \times \mathbf{A}^1)$ to $CH_r(C)$. This morphism sends $\alpha \in CH_{r+1}(Y \times \mathbf{A}^1)$ to $i^*(\beta)$, where $\beta \in CH_{r+1}(M^\circ)$ is such that $j^*(\beta) = \alpha$. We thus also have a morphism

$$\sigma\colon CH_r(Y) \to CH_r(C)\,,$$

obtained by composing the preceding morphism with pr^*.

To prove the proposition it suffices to verify that

$$\sigma([V]) = [C_W V]\,,$$

where $W = V \cap X$. We have

$$pr^*([V]) = [V \times \mathbf{A}^1]\,.$$

The variety $M^\circ_W V$ is a closed subvariety of $M^\circ_X Y$ which restricts under j^* to $V \times \mathbf{A}^1$:

$$j^*([M^\circ_W V]) = [V \times \mathbf{A}^1]\,.$$

The Cartier divisor $C = M^\circ \cap \rho^{-1}(\infty)$ intersects $M^\circ_W V$ in $C_W V$, so we have

$$i^*([M^\circ_W V]) = [C_W V].$$

This completes the proof. □

We come now to Gysin homomorphism for an arbitrary regular imbedding $i\colon X \hookrightarrow Y$ of codimension d. Let $N = N_X Y$ be normal bundle. We define the Gysin homomorphism

$$i^*\colon CH_r Y \to CH_{r-d} X$$

to be the composite

$$i^* = s_N^* \circ \sigma\,,$$

where s_N^* is the defined earlier Gysin morphism associated with the zero section

$$s_N\colon X \hookrightarrow N\,.$$

Suppose that $f\colon X \hookrightarrow Y$ and $g\colon Y \hookrightarrow Z$ are regular imbeddings of codimensions d and d'. Then the composite $g \circ f\colon X \hookrightarrow Z$ is a regular imbedding of codimension $d + d'$.

Theorem 15. *In this situation, for $\alpha \in CH_r(Z)$, we have the following equality in $CH_{r-d-d'}(X)$:*

$$(gf)^*\alpha = f^*(g^*\alpha)\,.$$

We shall now use these Gysin morphisms to define the multiplication in the Chow group of a nonsingular variety, without using the moving lemma.

2.8. Intersection ring of a nonsingular variety

We are working in algebraic category over an arbitrary field k.

We shall use the following intersection ring $CH^*(X)$. If X is an n-dimensional nonsingular variety (i.e., smooth over the base field k), then the diagonal imbedding δ of X in $X \times X$ is a regular imbedding of codimension n. Given $\alpha \in CH_a X$ and $\beta \in CH_b X$, a product $\alpha \cdot \beta \in CH_m X$, $m = a + b - n$, is defined by

$$\alpha \cdot \beta = \delta^*(\alpha \times \beta)\,.$$

We note that if $\alpha \in CH_a V$ and $\beta \in CH_b W$, where V, W are closed subschemes of X, then the product $\alpha \cdot \beta$ has a natural refinement in $CH_m(V \cap W)$ defined by the refined Gysin homomorphism (cf. [13]) constructed from the fibre square:

$$\begin{array}{ccc} U \cap W & \hookrightarrow & W \\ \downarrow & & \downarrow \\ U & \hookrightarrow & X \end{array}$$

Set

$$CH^p X = CH_r X$$

for $r = n - p$.

If $f\colon Y \to X$ is morphism of nonsingular varieties, then the graph morphism

$$\gamma_f\colon Y \to Y \times X$$

is a regular imbedding of codimension n. Define $f^*\colon CH^p X \to CH^p Y$ by the formula

$$f^*\alpha = \gamma_f^*(\alpha \times [X])\,.$$

Theorem 16. *For X nonsingular, the above product makes CH^*X into an associative, commutative ring with unit $1 = [X]$. For a morphism $f\colon Y \to X$ of nonsingular varieties, the homomorphism $f^*\colon CH^*X \to CH^*Y$ is a ring homomorphism. If also $g\colon Z \to Y$, with Z nonsingular, then $(fg)^* = g^* f^*$.*

The commutativity property is obvious, and the contravariant functoriality property follows from Theorem 15. As for what concerns the associativity property, consider the following commutative diagram:

$$\begin{array}{ccc} X & \overset{\delta}{\hookrightarrow} & X \times X \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta\times 1} \\ X \times X & \underset{1\times\delta}{\hookrightarrow} & X \times X \times X \end{array}$$

For cycles α, β, γ on X, the wanted equality

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma),$$

that is,

$$\delta^*(\delta \times 1)^*(\alpha \times \beta \times \gamma) = \delta^*(1 \times \delta)^*(\alpha \times \beta \times \gamma)$$

follows by application of Theorem 15 to the compositions: $(\delta \times 1) \circ \delta$ and $(1 \times \delta) \circ \delta$.

The intersection ring $CH^*(X)$ is often called the Chow ring of X.

Remark 17. The geometry of intersecting cycles in the above construction is hidden in the brevity of deformation to the normal cone, and it is not easy to see directly. We refer the reader to [15] and [16] for discussion of geometrical aspects of this construction, and for some explicit computation.

2.9. The Chow ring of a projective bundle

Lemma 18. *Let E be a vector bundle on X of rank $d+1$, and $\mathbf{P}(E)$ a projective bundle with the projection p to X. Let L be the dual of the tautological line subbundle of p^*E, and $\xi = \xi_E$ the class of the divisor associated to L. Then each $\beta \in CH^q(\mathbf{P}(E))$, $q = 0, \ldots, d$, has a unique presentation of the form:*

$$\beta = \sum_{i=0}^{d} (p^*\alpha_i) \cdot \xi^i \,,$$

where $\alpha_i \in CH^{q-i}(X)$.

Proof. We want to show that the map:

$$\bigoplus_{i=0}^{d} CH_*(X) \to CH_*(\mathbf{P}(E))$$

such that

$$(\alpha_0, \alpha_1, \ldots, \alpha_d) \mapsto \sum_{i=0}^{d} (p^*\alpha_i) \cdot \xi^i \,,$$

is an isomorphism.

By the following diagram:

$$\begin{array}{ccccccc}
\bigoplus_{i=0}^d CH_*(Y) & \longrightarrow & \bigoplus_{i=0}^d CH_*(X) & \longrightarrow & \bigoplus_{i=0}^d CH_*(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
CH_*(p^{-1}Y) & \longrightarrow & CH_*(\mathbf{P}(E)) & \longrightarrow & CH_*(p^{-1}U) & \longrightarrow & 0
\end{array}$$

we reduce the surjectivity to Y and U instead of X. Then by Noetherian induction, we reduce the surjectivity to U, where we can assume that $E|_U$ is trivial. Then $\mathbf{P}(E) = X \times \mathbf{P}(k^{d+1}) = X \times \mathbf{P}^d$, and the assertion follows from an elementary fact that $\xi^i = [\mathbf{P}^{d-i}] \in CH^*(\mathbf{P}^d)$.

Since $\alpha_i = \beta \cdot \xi^{d-i}$, we see that the expression in the lemma is unique. □

Grothendieck in [21] gave the following result modelled on the Leray–Hirsch theorem in topology.

Theorem 19. *Let X, P be nonsingular varieties, $f \colon P \to X$ a morphism such that the tangent map is surjective at each point, and S a subset of $CH^*(P)$. Suppose that for any locally closed, irreducible and nonsingular subset Y of X, $f^{-1}(Y)$ is nonsingular, and that for any $x \in CH^*(f^{-1}(Y))$, there is an open subset $U \subset Y$ such that $x|_{f^{-1}(U)}$ belongs to the $CH^*(U)$-module generated by $j^*(S)$, where $j \colon f^{-1}(U) \to P$ is the inclusion. Then S generates $CH^*(P)$ as a $CH^*(X)$-module.*

The proof also uses the short exact sequence for rational equivalence (Proposition 61).

2.10. Computing the Chow rings

We shall soon construct Chern classes of vector bundles. For a line bundle L, the choise what should be its Chern class is obvious: $c_1(L)$ is the class of the associated divisor.

We record without proof the following lemma.

Lemma 20. *Let E be a vector bundle of rank d on a nonsingular variety X, s a section of E. Suppose that E admits a filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E\,,$$

where $\operatorname{rank}(E_i) = i$. *Let $Y_i = \{x \in X : s(x) \in E_i\}$. Suppose that Y_i is nonsingular for all i. Let s_i be a section of $(E_i/E_{i-1})|_{Y_i}$ induced by s, and assume that s_i is transverse to the zero section for all i. Then we have*

$$[Y_d] = \prod_{i=1}^{d} c_1(E_i/E_{i-1})\,.$$

Corollary 21. *With the assumptions of the lemma, if s vanishes nowhere, then*

$$\prod_{i=1}^{d} c_1(E_i/E_{i-1}) = 0\,.$$

The following result is useful to compute the Chow rings in numerous situations.

Proposition 22. *Let $f\colon Z \to X$ be a morphism of nonsingular varieties. Assume that there is a filtration*

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z$$

with nonsingular varieties Z_r, $r = 0, \ldots, n$. Let

$$i_r\colon Z_r \hookrightarrow Z \quad \text{and} \quad j_r\colon Z_r - Z_{r-1} \to Z$$

denote the inclusions. Suppose that for any r, there is a subset $S_r \subset CH^*(Z_r)$ such that $j_r^*(S_r)$ generates the $CH^*(X)$-module $CH^*(Z_r - Z_{r-1})$ (the structure of $CH^*(X)$-module is defined by $(f \circ j_r)^*$). Then

$$S = \bigcup (i_r)_*(S_r)$$

generates the $CH^*(X)$-module $CH^*(Z)$.

Proof. Use the short exact sequence for rational equivalence (Proposition 61). □

3. Chern classes

We follow here the "undressing" approach of Grothendieck, see [22]. Other references are [13] and [5].

For a vector bundle E of rank d on X, we shall construct the Chern classes $c_i(E) \in CH^i(X)$, $i = 0, 1, \ldots, d$. They will satisfy the following conditions.

(i) (Normalization) Let D be a divisor on X. Then $c_1(\mathcal{O}(D)) = [D]$.
(ii) (Naturality) Let $f\colon Y \to X$ be a morphism. Then $c_i(f^*E) = f^*c_i(E)$ for any i.
(iii) (Additivity) Let
$$0 \to E' \to E \to E'' \to 0$$
be an exact sequence. Then we have
$$c_h(E) = \sum_{i+j=h} c_i(E')c_j(E'')\,.$$

Theorem 23. *Such Chern classes exist, and they are unique.*

Introducing the total Chern class: $c(E) = 1+c_1(E)+\cdots+c_d(E) \in CH^*(X)$, the conditions characterizing Chern classes are rephrased as follows:

(i) For a divisor D, $c(\mathcal{O}(D)) = 1 + [D]$.
(ii) If $f\colon Y \to X$ is a morphism, $c(f^*E) = f^*c(E)$.
(iii) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then
$$c(E) = c(E')c(E'')\,.$$

3.1. Construction

To construct such Chern classes, we proceed as follows. Let E be a vector bundle on X of rank d, and let $\mathbf{P}(E)$ be the projective bundle with the projection p onto X. Let $L_E = \mathcal{O}_E(1)$ be the dual of the tautological line subbundle of p^*E. Let $\xi = \xi_E$ be the class of the associated divisor.

We define the Chern classes $c_1(E),\dots,c_d(E)$ to be the (unique) coefficients in the relation expressing ξ_E^d as a $CH^*(X)$-combination of $1,\xi_E,\dots,\xi_E^{d-1}$:
$$\xi_E^d + \sum_{i=1}^{d} p^*(c_i(E))\cdot \xi_E^{d-i} = 0\,,$$
and set $c_0(E) = 1$, $c_i(E) = 0$ for $i > d$.

Let us check the normalization condition (ii). Suppose that E is of rank 1. Then $\mathbf{P}(E) = X$ and we have
$$\xi_E + c_1(E) = 0\,,$$
so that $c_1(E) = -\xi_E$, which fits with $L_E = E^*$.

We use the following construction of "undressing" (or splitting) a vector bundle.

Let $E^{(1)} = p^*E/L_E^*$ be a bundle of rank $d-1$ on $X^{(1)} = \mathbf{P}(E)$. Similarly, we set $X^{(2)} = \mathbf{P}(E^{(1)})$ and $E^{(2)} = (E^{(1)})^{(1)}$ a bundle of rank $d-2$ on $X^{(2)}$. Proceeding in this way, we construct $X^{(i)}$ and a bundle $E^{(i)}$ of rank $d-i$ on $X^{(i)}$, $1 \leqslant i \leqslant d$. We call a flag of length i in a vector space V an increasing sequence $(V_j)_{0\leqslant j\leqslant i}$ of subspaces V_j with $\dim V_j = j$. Thus $X^{(i)}$ is a fibration over X whose fibre over x consists of all flags of length i in E_x. In particular, $X^{(d)}$ is a variety of complete flags (of maximal length d) in E, denoted by $Fl(E)$. On $X^{(i)}$, we set

$$E_i = \mathrm{Ker}(E \twoheadrightarrow E^{(i)})\,.$$

Taking pullbacks of E_i to $Fl(E)$, we get the filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E\,.$$

The line bundles E_i/E_{i-1} are called the factors of the splitting and their first Chern classes are often called the Chern roots of E.

3.2. Unicity

We show that Chern classes satisfying (i), (ii), (iii) are unique. Let E be a vector bundle of rank d on X. Let $X' = Fl(E)$ be the flag variety, equipped with the canonical projection $f\colon X' \to X$. The induced homomorphism $f^*\colon CH^*(X') \to CH^*(X)$ is injective. So it is sufficient to work with $f^*c(E)$. By naturality, we have $f^*c(E) = c(f^*(E))$. Then additivity and normalization give

$$c(f^*E) = c(E_1)c(E_2/E_1)\cdots c(E_d/E_{d-1}) = \prod_{i=1}^{d}(1 + c_1(E_i/E_{i-1}))$$

In other words, the ith Chern class of E must be the i-th elementary symmetric function in the Chern roots of E.

3.3. Naturality and additivity

We show the naturality. Suppose we have a vector bundle E of rank d on Y and a morphism $f\colon X \to Y$. The splitting construction is functorial. Let $F = f^*E$. We get $\overline{f}\colon \mathbf{P}(F) \to \mathbf{P}(E)$. We have $L_F = \overline{f}^* L_E$ and thus $\xi_F = \overline{f}^*(\xi_E)$. Let $p\colon \mathbf{P}(F) \to X$ and $q\colon \mathbf{P}(E) \to Y$ be the canonical projections. We have from the definition of the Chern classes

$$\sum_{i=0}^{d} q^*(c_i(E)) \cdot \xi_E^{d-i} = 0\,.$$

After applying $\overline{f}^*$, and using $\overline{f}^* q^* = p^* f^*$, we get

$$\sum_{i=0}^{d} p^*(f^* c_i(E)) \cdot \xi_F^{d-i} = 0.$$

Hence $c_i(F) = f^*(c_i(E))$ for any i.

We show additivity. Let $P = Fl(E') \times_X Fl(E'')$ be the fibre product of two flag varieties and $g\colon P \to X$ the canonical projection. Since $g^*\colon CH^*X \to CH^*P$ is a monomorphism, it suffices to show that

$$g^*c(E) = g^*c(E')g^*c(E'') = c(g^*E')c(g^*E'')\,.$$

The bundles g^*E' and g^*E'' are splitted. Taking the splitting of g^*E' and the preimage, under the map $g^*E \to g^*E''$, of the splitting of g^*E'', we obtain a splitting of g^*E. Therefore to prove additivity, it is enough to show that for a completely splitted bundle

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E\,,$$

on X, we have in $CH^*(X)$,

$$c(E) = \prod_{i=1}^{d} (1 + c_1(E_i/E_{i-1}))\,.$$

Let $X' = \mathbf{P}(E)$, $f\colon X' \to X$ the projection, $E'_i = f^*(E_i)$, $L = L_E$, $\xi_i = c_1(E_i/E_{i-1})$, $\xi'_i = f^*(\xi_i)$. The line bundle L^* is a subbundle of E'; we treat the injection $L^* \hookrightarrow E'$ as a section s of $L \otimes E' =: F'$, which vanishes nowhere. Setting $F'_i = L \otimes E'_i$, we get the splitting of F', whose factors are

$$F'_i/F'_{i-1} = L \otimes (E'_i/E'_{i-1}),$$

so we have

$$c_1(F'_i/F'_{i-1}) = \xi_E + \xi'_i\,.$$

We set

$$Y_i = \{x' \in X' : s(x') \in F'_i\}\,.$$

This is also the set of the points x' such that $L^*_{x'} \subset E'_{x'}$, i.e., the set Y_i identifies with $\mathbf{P}(E_i)$. Thus Y_i is nonsingular, closed subset of X'.

Let s_i be a section of $F'_i/F'_{i-1}|_{Y_i}$ obtained from $s|_{Y_i}$. We claim that this section is transverse to the zero section of F'_i/F'_{i-1}. This is a local question, so we can assume that

$$E = X \times k^d\,, \quad E_i = X \times k^i \quad \text{so that} \quad Y_i = X \times \mathbf{P}(k^i)\,.$$

Then $L|_{Y_i}$ is the pull-back of the tautological line bundle $\mathcal{O}(1)$ on $\mathbf{P}(k^i)$. But $\mathcal{O}(1)$ is the line bundle whose every section is transverse to the zero section with the set of zeros being the hyperplane. This shows our claim.

Since the section s of $L \otimes E'$ is transverse to the zero section and vanishes nowhere, we obtain by Lemma 20 that

$$(\xi_E + \xi'_1)(\xi_E + \xi'_2) \cdots (\xi_E + \xi'_d) = 0\,.$$

Therefore $c_i(E)$ is indeed the ith elementary symmetric function in $\xi_1, \ldots, \xi_d$.

The theorem has been proved. □

Corollary 24. In the course of the proof, we established the "splitting principle", that is, we showed that there exists a nonsingular variety $Fl(E)$ and a morphism $f\colon FL(E) \to X$ such that $f^*\colon CH^*(X) \to CH^*(Fl(E))$ is a monomorphism, and the pullback of E admits a filtration with successive quotients being line bundles. Then the $c_i(E)$ is the elementary symmetric function of degree i in the first Chern classes of these quotients. Thus to prove an identity involving the Chern classes of a vector bundle, we can assume that it splits in a direct sum of line bundles.

Remark 25. Due to the naturality property of Chern classes, we shall often skip writing pullback of functions, or pullback indices.

Chern classes are defined on the Grothendieck group $K(X)$, i.e., on virtual bundles $E - F$, cf. 6.4. Since we wish to have

$$1 = c(0) = c(E - E) = c(E)c(-E)\,,$$

hence $c(-E) = c(E)^{-1}$, and we define

$$c(E - F) = 1 + c_1(E - F) + c_2(E - F) + \ldots = c(E)c(-F) = c(E)c(F)^{-1}\,.$$

3.4. Segre classes

We introduce Segre classes: $s_i(E) = (-1)^i c_i(-E)$ for $i \leqslant 0$, and $s_i(E) = 0$ for $i < 0$. So we have

$$(1 + c_1(E) + c_2(E) + c_3(E) + \cdots)(1 - s_1(E) + s_2(E) - s_3(E) + \cdots) = 1\,.$$

The element $s_i(E)$ is the ith complete symmetric function in the Chern roots of E. The Segre classes satisfy the following properties:

(i) For a divisor D, $s_i(\mathcal{O}(D)) = [D]^i$.

(ii) If $f\colon Y \to X$ is a morphism, then $s_i(f^*E)) = f^*s_i(E)$ for every i.

(iii) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then for any h

$$s_h(E) = \sum_{i+j=h} s_i(E')s_j(E'')\,.$$

If we introduce the total Segre class

$$s(E) = 1 + s_1(E) + s_2(E) + \cdots$$

then these properties can be rewritten as follows:

(i) For a divisor D,
$$s(\mathcal{O}(D)) = \frac{1}{1 - [D]}\,.$$

(ii) If $f\colon Y \to X$ is a morphism, then $s(f^*E)) = f^*s(E)$.

(iii) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then

$$s(E) = s(E')s(E'')\,.$$

Moreover, we have the following push-forward formula. Let $q\colon G^1(E) \to X$ be Grassmann bundle parametrizing the quotients of rank 1 of E. Let $\mathcal{O}(1)$ be the tautological quotient line bundle on $G^1(E)$.

Lemma 26. *We have for any $j \geqslant 0$,*

$$s_j(E) = q_*(c_1(\mathcal{O}(1))^{j+n-1}),$$

where $n = \operatorname{rank}(E)$.

This may be seen by interpreting q_* as a symmetrizing operator (see, e.g., [42]).

3.5. The top Chern class

The following basic result translates geometry of schemes into algebra of Chern classes.

Theorem 27. *Let s be a section of a vector bundle E of rank d on X, with the scheme of zeros Z. Suppose that $\operatorname{codim}_X Z = d$. Then $[Z] = c_d(E)$.*

Proof. Suppose first that s is transverse to the zero section s_E of E.

Let $F = E \oplus \mathbf{1}$. Then $\mathbf{P}(F)$ contains $\mathbf{P}(E)$ as a closed subset, whose open complement is E. We claim that in $CH^*(\mathbf{P}(F))$ the following formula holds:

$$[s_E(X)] = \sum_{i=0}^{d} c_i(E) \cdot \xi_F^{d-i}. \tag{2}$$

It is sufficient to prove this identity after splitting the bundle E. Let

$$f \colon X' = FL(E) \to X$$

be a morphism such that $f^* \colon CH^*(X) \to CH^*(X')$ is a monomorphism, and the bundle $E' = f^*E$ is completely splitted. Note that f induces a morphism $f_E \colon E' \to E$, and then a morphism

$$f_P \colon \mathbf{P}(E' \oplus \mathbf{1}) \to \mathbf{P}(E \oplus \mathbf{1}).$$

Since ${f_P}^*$ is a monomorphism, a knowledge of $[s_{E'}(X')]$ implies the knowledge $[s_E(X)]$. Since f_P is transverse to $s_E(X)$, we have

$$[s_{E'}(X')] = [f_P^{-1}(s_E(X)] = f_P^*[s_E(X)].$$

Thus for $F' = E' \oplus \mathbf{1}$, it is enough to show that in $CH^*(\mathbf{P}(E' \oplus \mathbf{1}))$

$$[s'_E(X')] = \sum_{i=0}^{d} c_i(E') \cdot {\xi'_F}^{d-i}, \tag{3}$$

i.e., that the pullback by f_P^* of (2) holds true.

So, without loss of generality, we reduced our considerations to the case where E on X admits a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E,$$

where $\operatorname{rank}(E_i) = i$. Consider the following filtration of F:

$$\mathbf{1} = E_0 \oplus 1 \subset E_1 \oplus \mathbf{1} \subset E_2 \oplus \mathbf{1} \subset \cdots \subset E_{d-1} \oplus \mathbf{1} \subset E_d \oplus \mathbf{1} = E \oplus \mathbf{1} = F \,.$$

Let $\pi\colon \mathbf{P}(F) \to X$ be the canonical projection. Set $F' = \pi^* F$. Then the rank $d+1$ bundle $L_F \otimes F'$ on $\mathbf{P}(F)$ admits a splitting with factors:

$$L_F \otimes \pi^*(E_i/E_{i-1}),\ 1 \leqslant i \leqslant d, \quad \text{and} \quad L_F \otimes \pi^*(\mathbf{1}) = L_F \,.$$

Moreover, the inclusion $L_F^* \hookrightarrow F'$ induces a section s of $L_F \otimes F'$. By the proof of Theorem 23 we know that the assumptions of Lemma 20 are satisfied. We obtain the following formula for the class od the scheme of zeros of s:

$$[Y_d] = \prod_{i=1}^{d} c_1(L_F \otimes \pi^*(E_i/E_{i-1}))\,.$$

But Y_d also admits another interpretation. It is the set of points in $\mathbf{P}(F)$ corresponding to lines in $\mathbf{P}(F)$ which are contained in the subbundle $\mathbf{1}$, that is, the set $s_E(X)$, where E is identified with an open subset of $\mathbf{P}(F) = \mathbf{P}(E \oplus \mathbf{1})$. Since

$$c_1(L_F \otimes \pi^*(E_i/E_{i-1})) = \xi_F + \pi^*(\xi_i)\,,$$

we get the equation (3).

We note that

$$Z(s) = s^{-1}(s_E(X))\,.$$

Therefore by (2), we obtain

$$[Z(s)] = s^*[s_E(X)] = s^*(\sum_{i=0}^{d} c_i(E) \cdot \xi_F^{d-i}) = c_d(E)\,.$$

Indeed, since $L_F|_E$ is trivial, $s^* L_F$ is trivial on X and thus $s^*(\xi_F) = 0$.

In the proof, we showed that for any section s of E,

$$s^*([s_E(X)]) = c_d(E)\,. \tag{4}$$

This equality concludes the proof of the theorem. □

A particular case of (4) asserts that

$$s_E^*(s_E)_*([X]) = c_d(E)\,.$$

More generally, we have

Theorem 28. *Let $i\colon X \hookrightarrow Y$ be a closed imbedding of nonsingular varieties, $N = N_X Y$ the normal bundle. Then for $\alpha \in CH^* X$, we have*

$$i^*(i_*\alpha) = c_{\text{top}} N \cdot \alpha\,.$$

For a proof, see [31], where a certain variant of the deformation to the normal bundle is used.

In particular, by the projection formula applied to i_*, we have

$$[X]^2 = i^*(c_{\mathrm{top}} N) \,.$$

We apply this self-intersection formula to the diagonal imbedding $\delta\colon X \to X \times X$, with $n = \dim(X)$. Since

$$N = N_X(X \times X) \simeq TX \,,$$

we have

$$[\Delta]^2 = c_n(N) = c_n(X) = \chi(X) \,,$$

which is the Lefschetz formula for the Euler characteristic of X.

We add the following related formula for the blow-up (loc.cit.). Consider the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ j\ } & \tilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\ i\ ]{} & Y \end{array}$$

where i is a closed imbedding of nonsingular varieties, and f the blow-up of Y along X. Let $N = N_X Y$. The exceptional divisor $\tilde{X}$ identifies with $\mathbf{P}(N)$. Set $Q = g^*N/L_N^*$. Then for $\alpha \in CH^*(X)$, we have

$$f^* i_*(\alpha) = j_*(c_{\mathrm{top}} Q \cdot g^*\alpha) \,.$$

3.6. Examples of Chern classes

If X is a nonsingular variety, the classes $c_i(TX) \in H^*(X, \mathbf{Z})$ will be called the Chern classes of X. For example, let us consider the complex projective space $X = \mathbf{P}^n$. It follows from the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)} \to T\mathbf{P}^n \to 0$$

(the so called Euler sequence) and from the additivity property that

$$c(T\mathbf{P}^n) = (1+h)^{n+1} \,,$$

where $h = [H]$ is the (cohomology) class of a hyperplane H. Therefore

$$c_i(T\mathbf{P}^n) = \binom{n+1}{i} h^i \,.$$

In particular, we have $c_n(T\mathbf{P}^n) = (n+1)h^n$, which implies $\chi(\mathbf{P}^n) = n+1$ since $\deg(H^n) = 1$.

In [29], the following formula for the Grassmannian $\mathcal{G}$ of all m-dimensional subspaces of $\mathbf{C}^{n+1}$ is given:

$$c(T\mathcal{G}) = |c_{j-i}|_{i,j} \cdot \Big|\binom{h}{j}\Big|_{j,h} \cdot |c_{n+1+p-h}|_{h,p} \,,$$

where $0 \leqslant i,p \leqslant m$, $0 \leqslant j,h \leqslant n+1$ and $c_i = c_i(Q)$ with Q the tautological quotient bundle on $\mathcal{G}$. The problem of expressing $c(\mathcal{G})$ in the additive basis of Schur functions $s_\lambda(Q)$ (see below) remains open.

The paper contains also a certain formula for a flag variety in the form of a quadratic expression in the Schubert classes.

Let $i\colon X \hookrightarrow Y$ be imbedding of nonsingular varieties. Let $N = N_X Y$ be the normal bundle. By the additivity and naturality properties, we have in $CH(X)$

$$c(TX) = i^*(c(TY))/c(N)\,. \tag{5}$$

Suppose now that X is the complete intersection of r divisors, $X = D_1 \cap \cdots \cap D_r$. Then

$$N = i^*(\mathcal{O}(D_1) \oplus \ldots \oplus \mathcal{O}(D_r))$$

and thus

$$c(N) = i^*((1+[D_1])\cdots(1+[D_r])). \tag{6}$$

We substitute Eq.(6) to the RHS of Eq.(5). Then we apply i_* to the RHS and use the projection formula. We get

$$i_*(c(TX)) = c(TY)\prod \frac{[D_j]}{1+[D_j]}\,. \tag{7}$$

Note that

$$\frac{[D_j]}{1+[D_j]} = [D_j] - [D_j]^2 + \cdots\,.$$

If $Y = \mathbf{P}^n$ and $\deg(D_j) = d_j$, then (7) gives

$$c_1(TX) = (n+1-\sum d_j)h\,.$$

Let E be a vector bundle on a complex manifold X and $s\colon X \to E$ its section. Suppose that s is transverse to the zero section of E. Let Z be the scheme of zeros of s. Then the normal bundle $N_Z X$ of Z in X is canonically isomorphic to $E|_Z$.

We want to apply the formula asserting that for a compact complex manifold X, the Euler characteristic $\chi(X)$ of X is

$$\chi(X) = \int_X c(TX) \cap [X]\,.$$

We have in the Grothendieck group $K(Z)$ (cf. Section 6)

$$[TZ] = [TX|_Z] - [E|_Z]\,,$$

and thus we get

$$\chi(Z) = \int_Z c(TX|_Z)c(-E|_Z) = \int_Z \sum_{i+j=\dim Z} (-1)^j c_i(TX|_Z)s_j(E|_Z)\,.$$

It is often more convenient to carry the computation on X. By the projection formula, we obtain

$$\chi(Z) = \int_X c_n(X)\Big(\sum_{i+j=\dim Z} (-1)^j c_i(X) s_j(E)\Big)\,.$$

For example, if $X = \mathbf{P}^N_{\mathbf{C}}$ and $E = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n)$, $d_i \geqslant 1$, then the set of zeros Z of a section transverse to the zero section is the nonsingular intersection of n hypersurfaces of degrees $d_1, \dots, d_n$, and the above formula for its Euler characteristic reads, with $d = \dim Z = N - n$,

$$\chi(Z) = \left(\prod_{i=1}^n d_i\right) \left[\sum_{i=0}^d (-1)^{d-i} \binom{N+1}{i} s_{d-i}(d_1, \dots, d_n)\right],$$

where $s_{d-i}(d_1, \dots, d_n)$ is the complete symmetric function in $d_1, \dots, d_n$ of degree $d - i$.

4. Polynomials in Chern classes

In this section, the main references are [30] and [18].

4.1. Schur functions

A reference here is [32].

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition, that is a weakly decreasing sequence of natural numbers. The partition λ is often displayed pictorially by its (Ferrers') diagram with λ_1 boxes in the first row, λ_2 boxes in the second row etc.

Assume that $\mathbb{A} = (a_1, \dots, a_n)$ be an alphabet, i.e., a finite set of indeterminates $a_1, \dots, a_n$. Sometimes we shall denote the alphabet $\{a_1, \dots, a_i\}$ by $\mathbb{A}_i$ to mark its cardinality.

We define

$$s_\lambda(\mathbb{A}) = \begin{vmatrix} s_{\lambda_1}(\mathbb{A}) & s_{\lambda_1+1}(\mathbb{A}) & \dots & s_{\lambda_1+n-1}(\mathbb{A}) \\ s_{\lambda_2-1}(\mathbb{A}) & s_{\lambda_2}(\mathbb{A}) & \dots & s_{\lambda_2+n-2}(\mathbb{A}) \\ \dots & \dots & \dots & \dots \\ s_{\lambda_n-(n-1)}(\mathbb{A}) & s_{\lambda_n-(n-2)}(\mathbb{A}) & \dots & s_{\lambda_n}(\mathbb{A}) \end{vmatrix}. \tag{8}$$

These functions are algebraic tools of Schubert calculus. Schubert calculus is a way of describing the Chow ring of a Grassmannian, which combines geometry, algebra and combinatorics. This variety admits a cellular decomposition into Schubert cells (labelled by partitions). Their closures are called Schubert varieties. The classes of Schubert varieties, called Schubert classes, form a **Z**-basis of the Chow ring ("Basis theorem"). There are here some "special" Schubert classes: the Chern classes of the tautological quotient bundle on the Grassmannian. The special Schubert classes generate the Chow ring multiplicatively. The "Pieri formula" expresses the product of a general Schubert class by a special Schubert class as a combination of Schubert classes. The "Giambelli formula" presents a general Schubert class as a Schur function in the special Schubert classes. There is also the "Littlewood–Richardson rule", which expresses the product of two general Schubert classes as a combination of Schubert classes. For an introduction to Schubert calculus, see [28].

We shall below prove the Giambelli and Pieri formulas as particular instances of more general statements.

We extend the definition (8) to arbitrary sequences of integers. Each such function, if not zero, is modulo a sign equal to the Schur function indexed by a suitable partition (exercise.)

Let E and F be vector bundles of rank m and n on some variety. We want to express the Chern classes of $E \otimes F$ in terms of the Chern classes of E and F. If $a_1, \dots, a_m$ are the Chern roots of E, and $b_1, \dots, b_n$ are the Chern roots of F, then $\{a_i + b_j\}$ are the Chern roots of $E \otimes F$. We thus have

$$c(E \otimes F) = \prod (1 + a_i + b_j)\,.$$

This formula is, however, not sufficient to study problems of algebraic geometry. We need to "separate" the variables a_i and b_j. Also, we need a formula with more geometric flavour.

For partitions μ and λ, we shall write $\mu \subset \lambda$ if $\mu_i \leqslant \lambda_i$ for any i, or, equivalently if the diagram of μ is contained in the diagram of λ.

Let $\mu = (\mu_1, \dots, \mu_n) \subset \lambda = (\lambda_1, \dots, \lambda_n)$ be partitions. We set

$$d_{\lambda,\mu} = \Big| \binom{\lambda_i + n - i}{\mu_j + n - j} \Big|_{1 \leqslant i,j \leqslant n}\,.$$

Set $\mathbb{B} = (b_1, \dots, b_n)$.

Lemma 29. *We have*

$$s_\lambda(b_1 + 1, \dots, b_n + 1) = \sum_{\mu \subset \lambda} d_{\lambda,\mu} s_\mu(\mathbb{B})\,.$$

Proof. Hint. Use the following Jacobi–Trudi formula (and prove it): For any alphabet $\mathbb{A} = \{a_1, \dots, a_n\}$, we have

$$s_\lambda(\mathbb{A}) = \frac{|a_j^{\lambda_i + n - i}|}{|a_j^{n-i}|}\,.$$

It turns out that with the help of Schur functions, we can "separate" the variables in $\prod_{i,j}(a_i + b_j)$. To this end, we need the notion of the "dual partition". Namely, for a partition λ, let $\tilde{\lambda}$ be the partition whose diagram has the rows equal to the columns of the diagram of λ. The following is a variant of a Cauchy formula.

Proposition 30. *We have*

$$\prod_{i,j} (a_i + b_j) = \sum_{\lambda} s_\lambda(\mathbb{A}) s_{C\tilde{\lambda}}(\mathbb{B})\,,$$

where the sum is over all partitions $\lambda \subset (n^m)$, *and* $C\tilde{\lambda} = (m - \tilde{\lambda}_n, \dots, m - \tilde{\lambda}_1)$.

For a proof, see [32].

Combining these results, we obtain

Theorem 31. *We have*

$$c(E\otimes F)=\sum_{\lambda,\mu} d_{\lambda,\mu}s_\mu(E)s_{C\tilde{\lambda}}(F)\,,$$

summed over pairs of partitions λ,μ *such that* $\mu\subset\lambda\subset(n^m)$.

This is a good solution to the question of finding a useful formula for the Chern classes of tensor product of vector bundles.

4.2. Supersymmetric Schur functions

Let $\mathbb{B}$ be another alphabet. We set

$$s_\lambda(\mathbb{A}-\mathbb{B})=\begin{vmatrix} s_{\lambda_1}(\mathbb{A}-\mathbb{B}) & s_{\lambda_1+1}(\mathbb{A}-\mathbb{B}) & \dots & s_{\lambda_1+n-1}(\mathbb{A}-\mathbb{B})\\ s_{\lambda_2-1}(\mathbb{A}-\mathbb{B}) & s_{\lambda_2}(\mathbb{A}-\mathbb{B}) & \dots & s_{\lambda_2+n-2}(\mathbb{A}-\mathbb{B})\\ \dots & \dots & \dots & \dots\\ s_{\lambda_n-(n-1)}(\mathbb{A}-\mathbb{B}) & s_{\lambda_n-(n-2)}(\mathbb{A}-\mathbb{B}) & \dots & s_{\lambda_n}(\mathbb{A}-\mathbb{B})\end{vmatrix}.$$

This is the supersymmetric Schur function associated with λ.

We state now useful results on Gysin pushforwards, cf. [41]. Let E be a vector bundle on X of rank n. Let $\pi\colon \mathcal{G}=G^q(E)\to X$ be the Grassmann bundle parametrizing the quotients of rank q of E. Setting $r=n-q$, we also can treat $\mathcal{G}$ as the Grassmann bundle parametrizing the subbundles of rank r of E. Let R denote the tautological subbundle of rank r on G and Q the tautological quotient bundle on G.

Proposition 32. *Let F be a vector bundle on X.*
(i) *Let* $\lambda=(\lambda_1,\dots,\lambda_q)$, $\mu=(\mu_1,\dots,\mu_r)$ *be two partitions. Then*

$$\pi_*\big(s_\lambda(Q-F_{\mathcal{G}})\cdot s_\mu(R-F_{\mathcal{G}})\big)=s_{\lambda_1-r,\dots,\lambda_q-r,\mu_1,\dots,\mu_r}(E-F).$$

(ii) *Let* $\lambda=(\lambda_1,\dots,\lambda_q)$ *be a partition such that* $\lambda_q\geqslant \operatorname{rank}(F)$ *and* $\mu=(\mu_1,\mu_2,\dots)$ *an arbitrary partition. Then*

$$\pi_*\big(s_\lambda(Q-F_{\mathcal{G}})\cdot s_\mu(R-F_{\mathcal{G}})\big)=s_{\lambda_1-r,\dots,\lambda_q-r,\mu}(E-F).$$

Proof. We prove (i). Let $q=1$ and $\xi=c_1(Q)$. It follows from the identity

$$s_i(R-F_{\mathcal{G}})=s_i(E_{\mathcal{G}}-F_{\mathcal{G}})-\xi s_{i-1}(E_{\mathcal{G}}-F_{\mathcal{G}})$$

that for $\mu=(\mu_1,\dots,\mu_{n-1})$, where $n=\operatorname{rank}(E)$,

$$s_\mu(R-F_{\mathcal{G}})=\begin{vmatrix} 1 & \xi & \dots & \xi^{n-1}\\ s_{\mu_1-1}(E_{\mathcal{G}}-F_{\mathcal{G}}) & s_{\mu_1}(E_{\mathcal{G}}-F_{\mathcal{G}}) & \dots & s_{\mu_1+n-2}(E_{\mathcal{G}}-F_{\mathcal{G}})\\ s_{\mu_2-2}(E_{\mathcal{G}}-F_{\mathcal{G}}) & s_{\mu_2-1}(E_{\mathcal{G}}-F_{\mathcal{G}}) & \dots & s_{\mu_2+n-3}(E_{\mathcal{G}}-F_{\mathcal{G}})\\ \vdots & \vdots & \ddots & \vdots\end{vmatrix}.$$

(Use the elementary operations on the columns of the determinant.)

For $i \geqslant 0$ and $j \geqslant 0$, we have

$$\pi_*(\xi^i s_j(Q - F_{\mathcal{G}})) = \\ = \pi_*(\xi^{i+j} - \xi^{i+j-1}c_1(F_{\mathcal{G}}) + \xi^{i+j-2}c_2(F_{\mathcal{G}}) - \ldots + (-1)^j \xi^i c_j(F_{\mathcal{G}})).$$

If $i + j < n - 1$, then this expression is zero; if $i + j = n - 1$, it equals 1. Assuming additionally that $i \leqslant n - 1$ (which is our case), we get, for $i + j > n - 1$, that this expression is equal to

$$s_{i+j-n+1}(E) - s_{i+j-n}(E)c_1(F) + \ldots + (-1)^{i+j-n+1}c_{i+j-n+1}(F) = s_{i+j-n+1}(E - F).$$

Hence we infer

$$\pi_*(s_l(Q - F_{\mathcal{G}}) \cdot s_{\mu_1,\ldots,\mu_{n-1}}(R - F_{\mathcal{G}})) \\ = \begin{vmatrix} s_{l-n+1}(E-F) & s_{l-n+2}(E-F) & \ldots & s_l(E-F) \\ s_{\mu_1-1}(E-F) & s_{\mu_1}(E-F) & \ldots & s_{\mu_1+n-2}(E-F) \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix} \\ = s_{l-n+1,\mu_1,\ldots,\mu_{n-1}}(E - F).$$

To make the induction step from $q - 1$ to q, we use the following commutative diagram of Grassmann bundles:

$$\begin{array}{ccc} G^{q-1}(S') = Fl^{q,1}E = G^1(Q) & \xrightarrow{\pi_3} & G^q E) \\ \downarrow \pi_1 & & \downarrow \pi \\ G^1(E) & \xrightarrow[\pi_2]{} & X \end{array}$$

Here, $Fl^{q,1}E$ is the flag variety parametrizing flags of quotients of E of ranks q and 1. The following tautological sequences correspond to the Grassmannian extensions:

$$\begin{aligned} \pi\colon &\quad R \rightarrowtail E \twoheadrightarrow Q\,; \\ \pi_1\colon &\quad R \rightarrowtail R' \twoheadrightarrow P\,; \\ \pi_2\colon &\quad R' \rightarrowtail E \twoheadrightarrow \mathcal{O}(1)\,; \\ \pi_3\colon &\quad P \rightarrowtail Q \twoheadrightarrow \mathcal{O}(1)\,. \end{aligned}$$

We have

$$\begin{aligned} &\pi_*(s_{\lambda_1,\ldots,\lambda_q}(Q - F) \cdot s_{\mu_1,\ldots,\mu_r}(R - F)) \\ &\quad = \pi_*(\pi_3)_*\big[s_{\lambda_1+q-1}(\mathcal{O}(1) - F) \cdot s_{\lambda_2,\ldots,\lambda_q}(P - F) \cdot s_{\mu_1,\ldots,\mu_r}(R - F)\big] \\ &\quad = (\pi_2)_*\Big[s_{\lambda_1+q-1}(\mathcal{O}(1) - F) \cdot (\pi_1)_*\big(s_{\lambda_2,\ldots,\lambda_q}(P - F) \cdot s_{\mu_1,\ldots,\mu_r}(R - F)\big)\Big] \\ &\quad = (\pi_2)_*\big[s_{\lambda_1+q-1}(\mathcal{O}(1) - F) \cdot s_{\lambda_2-r,\ldots,\lambda_q-r,\mu_1,\ldots,\mu_r}(R' - F)\big] \\ &\quad = s_{\lambda_1-r,\lambda_2-r,\ldots,\lambda_q-r,\mu_1,\ldots,\mu_r}(E - F). \end{aligned}$$

Here, the first and fourth equalities follow from the case $q = 1$, and the third equality follows from the induction assumption. This proves (i).

The proof of (ii) can be found in [18], pp.125-126. □

4.3. Thom–Porteous formula

For a morphism $\varphi\colon F \to E$ of vector bundles of ranks m and n on X, let $D_r(\varphi)$ be the set of points $x \in X$, where $\operatorname{rank}(\varphi_x) \leqslant r$. This is a closed algebraic subset of X of codimension less than or equal to $(m-r)(n-r)$.

Theorem 33. *Let $\varphi\colon F \to E$ be a morphism of vector bundles on a nonsingular variety X. Suppose that* $\operatorname{codim}_X D_r(\varphi) = (m-r)(n-r)$. *Then*

$$[D_r(\varphi)] = s_{m-r,\ldots,m-r}(E - F)\,,$$

where $m-r$ is repeated $n-r$ times.

Proof. Let $\mathcal{G} = G_r(E)$ be the Grassmann bundle parametrizing all subbundles of rank r of E. Let $\pi\colon \mathcal{G} \to X$ denote the canonical projection. Denote by Q the tautological quotient bundle of rank $n-r$ on $\mathcal{G}$. We define inside $\mathcal{G}$ the subscheme Z as the zero scheme of the composite map

$$F_{\mathcal{G}} \to E_{\mathcal{G}} \to Q\,,$$

where the first map is the pullback of φ to $\mathcal{G}$, and the second one is the canonical surjection. By the construction, the projection π restricted to Z factorizes through $D_r(\varphi)$.

We claim that $\pi_{|Z}\colon Z \to D_r(\varphi)$ is birational. Indeed, over each point x in the set $D_r(\varphi) - D_{r-1}(\varphi)$ we get the unique point in Z, namely $\operatorname{Im}(\varphi_x)$, which is a vector space of dimension r. We thus have

$$\pi_*([Z]) = [D_r(\varphi)]\,.$$

By virtue of $\dim Z = \dim D_r(\varphi)$ and the assumption on the codimension of $D_r(\varphi)$, the codimension of Z in $\mathcal{G}$ is equal to $\operatorname{rank}(F^*_{\mathcal{G}} \otimes Q)$. We thus have

$$[D_r(\varphi)] = \pi_*(c_{m(n-r)}(F^*_{\mathcal{G}} \otimes Q))\,.$$

But

$$c_{m(n-r)}(F^*_{\mathcal{G}} \otimes Q) = s_{(m,\ldots,m)}(Q - F_{\mathcal{G}})\,,$$

where m is repeated $n-r$ times.

Finally, by Proposition 32(i) we get

$$[D_r(\varphi)] = \pi_* s_{(m,\ldots,m)}(Q - F_{\mathcal{G}}) = s_{m-r,\ldots,m-r}(E - F)\,,$$

where $m-r$ is repeated $n-r$ times. □

4.4. Multi–Schur functions

Let $\mathbb{A}^1, \ldots, \mathbb{A}^n$ and $\mathbb{B}^1, \ldots, \mathbb{B}^n$ be alphabets. We define

$$s_\lambda(\mathbb{A}^1 - \mathbb{B}^1, \mathbb{A}^2 - \mathbb{B}^2, \ldots, \mathbb{A}^n - \mathbb{B}^n)$$

as

$$\begin{vmatrix} s_{\lambda_1}(\mathbb{A}^1 - \mathbb{B}^1) & s_{\lambda_1+1}(\mathbb{A}^1 - \mathbb{B}^1) & \ldots & s_{\lambda_1+n-1}(\mathbb{A}^1 - \mathbb{B}^1) \\ s_{\lambda_2-1}(\mathbb{A}^2 - \mathbb{B}^2) & s_{\lambda_2}(\mathbb{A}^2 - \mathbb{B}^2) & \ldots & s_{\lambda_2+n-2}(\mathbb{A}^2 - \mathbb{B}^2) \\ \ldots & \ldots & \ldots & \ldots \\ s_{\lambda_n-(n-1)}(\mathbb{A}^n - \mathbb{B}^n) & s_{\lambda_n-(n-2)}(\mathbb{A}^n - \mathbb{B}^n) & \ldots & s_{\lambda_n}(\mathbb{A}^n - \mathbb{B}^n) \end{vmatrix}.$$

Lemma 34. (Lascoux's lemma) *Let $\mathbb{D}$ be an alphabet of cardinality $m \leqslant n$. If in the last $n-m$ columns of the determinant*

$$s_\lambda(\mathbb{A}^1 - \mathbb{B}^1, \mathbb{A}^2 - \mathbb{B}^2, \ldots, \mathbb{A}^n - \mathbb{B}^n)\,,$$

we change the argument from $\mathbb{A}^\bullet - \mathbb{B}^\bullet$ to $\mathbb{A}^\bullet - \mathbb{B}^\bullet - \mathbb{D}$, then the value of the determinant remains the same.

Proof. We get the assertion, by adding to any column (from the last $n-m$ columns) a linear combination of the first m columns, using the identity

$$s_i(\mathbb{A}^\bullet - \mathbb{B}^\bullet - \mathbb{D}) = s_i(\mathbb{A}^\bullet - \mathbb{B}^\bullet) + s_{i-1}(\mathbb{A}^\bullet - \mathbb{B}^\bullet)s_1(-\mathbb{D}) + \cdots + s_{i-m}(\mathbb{A}^\bullet - \mathbb{B}^\bullet)s_m(-\mathbb{D})\,. \; \square$$

We need some more notation. We shall add partitions component-wise, and by (m^n) we mean the partition with n parts equal to m.

Lemma 35. (Factorization property) *Let $\mathbb{A}$ and $\mathbb{B}$ be alphabets of cardinalities n and m. Then for a partition $\mu = (\mu_1, \mu_2, \ldots)$ with $\mu_1 \leqslant m$, and partition $\nu = (\nu_1, \ldots, \nu_n)$, we have*

$$s_{((m^n)+\nu),\mu}(\mathbb{A} - \mathbb{B}) = s_\nu(\mathbb{A}) s_\mu(-\mathbb{B}) s_{(m^n)}(\mathbb{A} - \mathbb{B})\,.$$

Proof. We use twice the Lascoux lemma. Let $\lambda = ((m^n)+\nu), \mu$. First, in the determinant representing $s_\lambda(\mathbb{A} - \mathbb{B})$, we subtract $\mathbb{A}$ from the arguments in the last $l(\lambda) - n$ columns. We get the following factorization:

$$s_\lambda(\mathbb{A} - \mathbb{B}) = s_\mu(\mathbb{A} - \mathbb{B} - \mathbb{A}) s_{(m^n)+\nu)}(\mathbb{A} - \mathbb{B}) = s_\mu(-\mathbb{B}) s_{(m^n)+\nu)}(\mathbb{A} - \mathbb{B})\,.$$

Let now $\kappa = (m^n) + \nu$. Second, in the determinant representing $s_{\tilde{\kappa}}(\mathbb{B} - \mathbb{A})$, we subtract $\mathbb{B}$ from the arguments in the last $(m + \nu_1 - m = \nu_1)$ columns. We get the factorization

$$s_{\tilde{\kappa}}(\mathbb{B} - \mathbb{A}) = s_\nu(\mathbb{A}) s_{(m^n)}(\mathbb{A} - \mathbb{B})\,.$$

The assertion now follows by combining these two factorizations. $\square$

4.5. Class of Schubert variety

Suppose that V is a vector bundle of rank N on a nonsingular variety X. For $0 < n < N$, let $G^n(V)$ be the Grassmann bundle parametrizing all quotient bundles of V of rank n. The Grassmannian $G^n(V) \to X$ is equipped with the tautological quotient bundle Q_n of rank n.

Consider a flag

$$V_\bullet : V \supset V_1 \supset V_2 \supset \cdots \supset V_n$$

of vector bundles on X with $\operatorname{rank}(V_i) = v_i$, $i = 1, \ldots, n$.

We define the subvariety $\Omega(V_\bullet) \subset G^n(V)$ by the following rank conditions:

$$\operatorname{rank}\operatorname{Coker}(V_1 \to V \to Q_n) \geqslant 1\,, \quad \ldots \quad , \operatorname{rank}\operatorname{Coker}(V_n \to V \to Q_n) \geqslant n\,.$$

Set $\lambda_i = v_i - n + i$. We get a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n)$.

Theorem 36. (The Giambelli formula) *We have in $CH^*(G^n(V))$*

$$[\Omega(V_\bullet)] = s_\lambda(Q_n - V_1, Q_n - V_2, \ldots, Q_n - V_n)\,.$$

We shall need two preliminary results.

Claim Let L be a line bundle with $c_1(L) = a$ and let F be a vector bundle on the same variety with the Chern roots $b_1, \ldots, b_m$. Then

$$c_m(\operatorname{Hom}(F, L)) = c_m(F^* \otimes L) = \prod(a - b_i) = s_m(L - F)\,. \tag{9}$$

This is a simple consequence of the splitting principle.

To state the next lemma we need a short exact sequence of vector bundles on X:

$$0 \to L \to Q \to Q' \to 0\,,$$

where $\operatorname{rank}(L) = 1$.

Lemma 37. *For any sequence of vector bundles $V_1, V_2, \ldots, V_n$ and a sequence of numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ with* $\operatorname{rank} V_n \leqslant \lambda_n$ *, we have*

$$s_{\lambda_1+1,\lambda_2+1,\ldots,\lambda_{n-1}+1}(Q' - V_1, \ldots, Q' - V_{n-1})s_{\lambda_n}(L - V_n) \\ = (-1)^{n+1}s_{\lambda_1,\ldots,\lambda_{n-1},\lambda_n+n-1}(Q - V_1, \ldots, Q - V_{n-1}, L - V_n).$$

Proof. The determinant for

$$s_{\lambda_1,\lambda_2,\ldots,\lambda_n+n-1}(Q - V_1, Q - V_2, \ldots, Q - V_{n-1}, L - V_n)$$

is

$$\begin{vmatrix} s_{\lambda_1}(Q-V_1) & s_{\lambda_1+1}(Q-V_1) & \dots & s_{\lambda_1+n-1}(Q-V_1) \\ s_{\lambda_2-1}(Q-V_2) & s_{\lambda_2}(Q-V_2) & \dots & s_{\lambda_2+n-2}(Q-V_2) \\ \dots & \dots & \dots & \dots \\ s_{\lambda_{n-1}-n+2}(Q-V_{n-1}) & s_{\lambda_{n-1}-n+3}(Q-V_{n-1}) & \dots & s_{\lambda_{n-1}+1}(Q-V_{n-1}) \\ s_{\lambda_n}(L-V_n) & s_{\lambda_n+1}(L-V_n) & \dots & s_{\lambda_n+n-1}(L-V_n) \end{vmatrix}.$$

We note the following relations:

$$s_i(Q-V_j) = s_i(Q'-V_j) + s_{i-1}(Q'-V_j)s_1(L)$$

and

$$s_i(L-V_n) = s_i(-V_n) + s_{i-1}(-V_n)S_1(-V_n).$$

Using them we subtract L from the arguments in the last $(n-1)$ columns of the determinant. We obtain

$$\begin{vmatrix} s_{\lambda_1}(Q-V_1) & s_{\lambda_1+1}(Q'-V_1) & \dots & s_{\lambda_1+n-1}(Q'-V_1) \\ s_{\lambda_2-1}(Q-V_2) & s_{\lambda_2}(Q'-V_2) & \dots & s_{\lambda_2+n-2}(Q'-V_2) \\ \dots & \dots & \dots & \dots \\ s_{\lambda_{n-1}-n+2}(Q-V_{n-1}) & s_{\lambda_{n-1}-n+3}(Q'-V_{n-1}) & \dots & s_{\lambda_{n-1}+1}(Q'-V_{n-1}) \\ s_{\lambda_n}(L-V_n) & s_{\lambda_n+1}(-V_n) & \dots & s_{\lambda_n+n-1}(-V_n) \end{vmatrix}.$$

The last row consists of zeros except the first entry. By the Laplace expansion along the last row, the assertion of the lemma follows. □

Proof of the theorem. The proof goes by induction on n. If $n = 1$, then $\Omega(V_\bullet)$ is the scheme of zeros of the map $V_1 \to Q_1$. Hence

$$[\Omega(V_\bullet)] = c_{v_1}(\mathrm{Hom}(V_1, Q_1)) = s_{v_1}(Q_1 - V_1)\,.$$

Assume that the theorem holds for $n-1$. Consider the following commutative diagram of Grassmannian extensions:

$$\begin{array}{ccc} Fl^{n,n-1}(V) & \xrightarrow{p} & G^{n-1}(V) \\ \downarrow{\scriptstyle q} & & \downarrow \\ G^n(V) & \longrightarrow & X \end{array}$$

where $Fl^{n,n-1}(V)$ is the flag variety parametrizing the quotients of ranks n and $n-1$ of V. Let $Z \subset G^{n-1}(V)$ be the subvariety defined by the rank conditions:

$$\text{rank Coker}(V_1 \to V \to Q_{n-1}) \geqslant 1\,,\ \dots\ ,\text{rank Coker}(V_{n-1} \to V \to Q_{n-1}) \geqslant n-1\,.$$

By the inductive assumption, the class of Z in $CH(G^{n-1}(V))$ is equal to

$$s_{\lambda_1+1,\dots,\lambda_{n-1}+1}(Q_{n-1}-V_1, Q_{n-1}-V_2, \dots, Q_{n-1}-V_{n-1})\,. \qquad (10)$$

The class of the cycle $p^{-1}(Z)$ is equal to $p^*([Z])$, and so is given by the expression (10) in $CH(Fl^{n,n-1}(V))$. Let $W \subset p^{-1}(Z)$ be the subvariety defined by the condition

$$\text{the map } V_n \to Q_n \text{ vanishes}. \tag{11}$$

We claim that the restriction to W of the projection q

$$q|_W : W \to \Omega(V_\bullet)$$

is a birational map. Indeed, over the set where

$$\dim \operatorname{Coker}(V_{n-1} \to Q_n) = n-1$$

the quotient space Q_{n-1} can be reconstructed:

$$Q_{n-1} = \operatorname{Coker}(V_{n-1} \to Q_n).$$

Hence we have in $H^*(G^n(V))$

$$[\Omega(V_\bullet)] = q_*([W]). \tag{12}$$

To compute $[W]$, note that along $p^{-1}(Z)$ the map $V_n \to Q_n$ factorizes through

$$L = \operatorname{Ker}(Q_n \to Q_{n-1})$$

since $V_n \subset V_{n-1}$ and $V_{n-1} \to Q_{n-1}$ vanishes. Therefore the condition (11) is equivalent to

$$\text{the map } V_n \to L \text{ vanishes}. \tag{13}$$

We thus get

$$[W] = p^*([Z]) \cdot c_{v_n}(\operatorname{Hom}(V_n, L)).$$

By the formula (9), Lemma 37, and using the expression (10), we obtain

$$[W] = (-1)^{n+1} s_{\lambda_1,\dots,\lambda_{n-1},\lambda_n+n-1}(Q_n - V_1, \dots, Q_n - V_{n-1}, L - V_n). \tag{14}$$

To push forward the class $[W]$ to $CH(G^n(V))$ by q_*, we note that q is the bundle of $(n-1)$-dimensional quotients of Q_n. For simplicity of arguments, we pass to the dual bundles: Q_n^* and V_i^*. The role of q is played now by the projective bundle $\mathbf{P}(Q_n^*) \to G^n(V)$, and $\mathcal{O}(1)$ is equal to L^*. By Proposition 32(i), we have

$$q_*(s_j(L^* - V_n^*)) = s_{j-(n-1)}(Q_n^* - V_n^*),$$

so that

$$q_*(s_j(L - V_n)) = (-1)^{n-1} s_{j-(n-1)}(Q_n - V_n).$$

By the projection formula and the Laplace expansion along the last row, we get

$$q_* s_{\lambda_1,\dots,\lambda_{n-1},\lambda_n+n-1}(Q_n - V_1, \dots, Q_n - V_{n-1}, L - V_n) = (-1)^{n-1} s_\lambda(Q_n - V_1, \dots, Q_n - V_n).$$

The assertion of the theorem now follows from the last identity and formulas (12), (14). □

Remark 38. This proof is based on unpublished notes of Lascoux (following his lectures at Roma II, 1984), and was worked out jointly with A. Weber. A parallel result was obtained independently by Kempf and Laksov in [26].

Remark 39. There exists a far reaching generalization of such degeneracy loci formulas due to Fulton. It uses correspondences in flag bundles. For a sketch, see [18, Chapter I and II].

5. Computing in the Chow ring of G/P

The material exposed in this section stems from [43].

One of the main problems of Schubert calculus on flag varieties (or generalized flag varieties G/B) is to give expressions for the structure constants for the cohomological multiplication of Schubert classes. The main problem is to describe these structure constants as the cardinalities of some sets, but also "closed" formulas for the structure constants are of some interest.

The first main tool that we use is the action of the Weyl group on $CH^*(G/B)_{\mathbf{Q}}$ expressed in terms of Schubert classes. The second main tool is the theory of BGG-operators acting as skew derivations on $CH^*(G/B)_{\mathbf{Q}}$. These fundamental tools were developed mainly by Bernstein–Gelfand–Gelfand and Demazure in the 70's, as a continuation of the work of Borel.

We illustrate our method with short proof the Chevalley formula (Theorem 46), and transparent, purely algebro-combinatorial proof of the classical Pieri formula (Theorem 51).

The references here are: [2, 3, 6, 10, 11, 12].

5.1. Characteristic map and BGG-operators

Let G be a semisimple algebraic group and $B \subset G$ a Borel subgroup.

Let X be a variety on which B acts freely (from the right). Suppose that the quotient X/B exists so that $p\colon X \to X/B$ is a principal B-bundle. On the other hand, let $\mu\colon B \to GL(V)$ be a linear representation. We denote by L_μ the rank $\dim V$ vector bundle $X \times^B V$, that is, the quotient of $X \times V$ by the equivalence relation

$$(x, v) \sim (xb, \mu(b)^{-1}v),$$

where $x \in X$, $b \in B$, and $v \in V$. Equivalently, if U is an open subset of X/B, then $\Gamma(U, L_\mu)$ is the set of morphisms $\varphi\colon p^{-1}(U) \to V$ such that $\varphi(xb) = \mu(b)^{-1}\varphi(x)$.

In particular, with any character χ of B (that is, a homomorphism of B into the multiplicative group) there is associated a line bundle L_χ; this induces a homomorphism of groups $X^*(B) \to \mathrm{Pic}(X/B)$, where $X^*(B)$ denotes the group of characters of B.

Composing this homomorphism with the homomorphism of the first Chern class from $\mathrm{Pic}(X/B)$ to $CH^1(X/B)$, one gets a homomorphism from $X^*(B)$ to $CH^1(X/B)$, which extends to a homomorphism of graded rings

$$c\colon \mathrm{Sym}(X^*(B)) \to CH^*(X/B)$$

from the symmetric algebra of the $\mathbf{Z}$-module $X^*(B)$ to the cohomology ring of X/B; this homomorphism is called the characteristic map of the fibre bundle $p\colon X \to X/B$. In this section, $S = \oplus S^k$ will denote the symmetric algebra $\mathrm{Sym}(X^*(B)) = \oplus \,\mathrm{Sym}^k(X^*(B))$.

Choose a maximal torus $T \subset B$ with the Weyl group $W = N_G(T)/T$ of (G,T). Then W acts on the group of characters $X^*(T)$ of T, and since $X^*(B) = X^*(T)$, this induces an action of W on S.

The root system of (G,T) is denoted by R; the set R^+ of positive roots consists in the opposites of roots of (B,T). Let $\Delta \subset R^+$ be the associated basis of R. The Weyl group W is generated by simple reflections, i.e. by the reflections associated with the elements of Δ. For any root $\alpha \in R$, we denote by s_α the reflection associated with α. The reflection s_α can be realized as a linear endomorphism of the Euclidean space $X^*(T)\otimes \mathbf{R}$, equipped with a W-invariant inner product $(\ ,\ )$. We have

$$s_\alpha(\lambda) = \lambda - (\alpha^\vee, \lambda)\alpha\,,$$

where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$.

By a reduced decomposition of an element $w \in W$ we understand a presentation $w = s_{\alpha_1}\cdots s_{\alpha_l}$ where all $\alpha_p \in \Delta$, and l is the smallest number occurring in such a presentation, called the length of w and denoted $l(w)$.

By w_0 we denote the longest element of W, the unique element of W with length equal to the cardinality of R^+.

We shall need the following "BGG-operators" A_w, $w \in W$, acting on the ring S.

Definition 40. *Given a root α and $f \in S$, we set*

$$A_\alpha(f) := \frac{f - s_\alpha(f)}{\alpha}\,.$$

The operator A_α is a well defined (group) endomorphism on S lowering the degree by 1. Note that $A_\alpha(f) = (\alpha^\vee, f)$ for $f \in S^1$; this will be used in the proof of Theorem 46.

We now record (cf. [2], Theorem 3.4 and [11], Théorème 1).

Lemma 41. *If $\alpha_1,\ldots,\alpha_k$ and $\beta_1,\ldots,\beta_k$ are simple roots such that*

$$s_{\alpha_1}\cdots s_{\alpha_k} = s_{\beta_1}\cdots s_{\beta_k}$$

are two reduced decompositions, then

$$A_{\alpha_1}\cdots A_{\alpha_k} = A_{\beta_1}\cdots A_{\beta_k}\,.$$

Thus for $w \in W$, given its reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_k}$, the operator

$$A_w := A_{\alpha_1} \cdots A_{\alpha_k}$$

is well-defined (i.e. doesn't depend on a reduced decomposition of w).

The following result says how the BGG-operators act on products (cf., e.g., [11] Eq. (6), p. 289):

Lemma 42. *We have for $f, g \in S$ and a simple root α,*

$$A_\alpha(f \cdot g) = A_\alpha(f) \cdot g + s_\alpha(f) \cdot A_\alpha(g)\,. \tag{15}$$

Geometric interpretations of BGG-operators are related to correspondences in flag bundles.

5.2. Structure constants for Schubert classes

In the geometry of flag manifolds G/B a large role is played by the Schubert cells BwB/B and their closures called Schubert varieties. We set

$$X^w := \overline{Bw_0wB/B}\,.$$

The cohomology class $[X^w]$ of X^w lies in $CH^{l(w)}(G/B)$. The Schubert cells form a cellular decomposition of G/B, so the classes $[X^w]$ form an additive basis for the cohomology.

Our goal, in this section, is to give a closed formula for the constants c^u_{wv}, appearing in the decomposition of the product

$$[X^w] \cdot [X^v] = \sum_u c^u_{wv} [X^u] \tag{16}$$

of Schubert classes.

We shall need a couple of tools that we describe now.

The characteristic map $c \colon S \to CH^*(G/B)$ of the fibration $G \to G/B$ is usually called the Borel characteristic map. Its kernel is generated by positive degree W-invariants, and $c \otimes \mathbf{Q}$ is surjective (cf. [11]), so that the Chow ring $CH^*(G/B)_{\mathbf{Q}}$ is identified with the quotient of $S \otimes \mathbf{Q}$ modulo the ideal generated by positive degree W-invariants. By combining this last property with Lemma 42, we infer that the BGG-operators induce – via the characteristic map – operators A_w on $CH^*(G/B)_{\mathbf{Q}}$ lowering the degree by $l(w)$.

In particular, for $a, b \in CH^*(G/B)_Q$ and a simple root α, we have

$$A_\alpha(a \cdot b) = A_\alpha(a) \cdot b + s_\alpha(a) \cdot A_\alpha(b)\,. \tag{17}$$

Iterations of this equation will play an important role in the present section and the next one.

Note also that the action of W on S induces – via the characteristic map – an action of W on $CH^*(G/B)_{\mathbf{Q}}$. (This action will be described below in terms of Schubert classes – cf. Lemma 44.)

We record the following equation relating: the characteristic map, BGG-operators, and Schubert classes (cf. BGG, Section 4 and [12], Section 4): for $f \in S^k$, in $CH^*(G/B)$ we have

$$c(f) = \sum_{l(w)=k} A_w(f)[X^w]\,. \tag{18}$$

This equation is closely related to the question of finding polynomial representatives of Schubert classes – a problem that we do not address here (cf. [18] for a discussion of this issue).

The next result says how the operators A_w act on Schubert classes (cf. [2], Theorem 3.14 (i)):

Lemma 43. *For $l(vw^{-1}) = l(v) - l(w)$, we have*

$$A_w([X^v]) = [X^{vw^{-1}}]\,, \tag{19}$$

and in the opposite case, $A_w([X^v]) = 0$.

We have also the following formula for the action of a simple reflection on a Schubert class (cf. [2], Theorem 3.12 (iv) and [12], Proposition 3):

Lemma 44. *For a simple root α and $w \in W$,*

$$s_\alpha([X^w]) = [X^w] \qquad \textit{if} \;\; l(ws_\alpha) = l(w) + 1\,; \tag{20}$$

$$s_\alpha([X^w]) = -[X^w] - \sum (\beta^\vee, \alpha)[X^{ws_\alpha s_\beta}] \qquad \textit{if} \;\; l(ws_\alpha) = l(w) - 1\,, \tag{21}$$

where the sum is over all positive roots $\beta \neq \alpha$ such that $l(ws_\alpha s_\beta) = l(w)$.

We now proceed towards computing the structure constants c^u_{wv}. By combining Equations (16) and (19), we can express the coefficient c^v_{wv} as follows:

$$c^u_{wv} = A_u([X^w] \cdot [X^v])\,. \tag{22}$$

Suppose that $l(w) = k$ and $l(v) = l$. Take a reduced decomposition of u:

$$u = s_{\alpha_1} \cdots s_{\alpha_{k+l}}\,.$$

Iterating (17) we obtain

$$c^u_{wv} = A_{\alpha_1} \cdots A_{\alpha_{k+l}}([X^w] \cdot [X^v]) = \sum A_I([X^w]) \cdot A^I_\alpha([X^v])\,,$$

where the sum is over all subsequences $I = (i_1 < \cdots < i_k) \subset \{1, 2, \ldots, k+l\}$, $A_I := A_{\alpha_{i_1}} \cdots A_{\alpha_{i_k}}$, and A^I_α is obtained by replacing in $A_{\alpha_1} \cdots A_{\alpha_{k+l}}$ each A_{α_i} by s_{α_i} for $i \in I$. By Lemma 43 we infer the following result.

Theorem 45. *With the above notation,*

$$c^u_{wv} = \sum A^I_\alpha([X^v])\,, \tag{23}$$

where the sum runs over all I such that $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$ is a reduced decomposition of w.

Applying successively to the summands in (23) the formulas (19), (20), and (21), we get an expression for the constants c^u_{wv}.

Recall the following formula from [10] for multiplication by the classes of Schubert divisors in $CH^*(G/B)$:

Theorem 46. (Chevalley) *For $w \in W$, and a simple root α,*

$$[X^w] \cdot [X^{s_\alpha}] = \sum (\beta^\vee, \omega_\alpha)[X^{ws_\beta}]\,, \tag{24}$$

where β runs over positive roots such that $l(ws_\beta) = l(w) + 1$ and ω_α denotes the fundamental weight associated with α.

Proof. We prove Equation (24) using Theorem 45. By the definition of a fundamental weight, we have for $\gamma \in \Delta$,

$$(\omega_\alpha, \gamma^\vee) = \delta_{\alpha\gamma}\,,$$

the Kronecker delta. This implies that

$$A_\gamma(\omega_\alpha) = \delta_{\alpha\gamma}\,,$$

and using Equation (18) we get

$$c(\omega_\alpha) = [X^{s_\alpha}]\,.$$

Fix $w \in W$ and pick $f \in S \otimes \mathbf{Q}$ such that

$$(c \otimes \mathbf{Q})(f) = [X^w]\,.$$

Then in $CH^*(G/B, \mathbf{Q})$ we have

$$[X^{s_\alpha}] \cdot [X^w] = (c \otimes \mathbf{Q})(\omega_\alpha \cdot f)\,, \tag{25}$$

and by Theorem 1 we obtain that the coefficient of the Schubert class $[X^u]$ in the expansion of (25) can be evaluated as the sum (23) with $[X^v]$ replaced by ω_α.

Take a reduced decomposition $u = s_{\alpha_1} \cdots s_{\alpha_h}$. By the familiar "Exchange Condition", a reduced decomposition for w can be gotten from the one for u by omitting one simple reflection if $u = ws_\beta$ for some (positive) root β. Conversely, if $w = s_{\alpha_1} \cdots s_{\alpha_{p-1}} s_{\alpha_{p+1}} \cdots s_{\alpha_h}$, then

$$w^{-1}u = s_{\alpha_h} \cdots s_{\alpha_p} \cdots s_{\alpha_h} = s_\beta$$

for $\beta = s_{\alpha_h} \cdots s_{\alpha_{p+1}}(\alpha_p)$. The root β is positive because $s_{\alpha_h} \cdots s_{\alpha_1}$ is reduced.

Since the omitted simple reflection is unique, the looked at sum (23) has exactly one summand

$$s_{\alpha_1} \cdots s_{\alpha_{p-1}} A_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_h}(\omega_\alpha) = A_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_h}(\omega_\alpha)\,.$$

The latter expression equals $(\beta^\vee, \omega_\alpha)$ because $A_{\alpha_p}(g) = (\alpha_p^\vee, g)$ for $g \in S^1$, the inner product $(\ ,\ )$ is W-invariant, and $s_{\alpha_h} \cdots s_{\alpha_{p+1}}(\alpha_p^\vee) = \beta^\vee$. This proves the theorem. □

The same method works for all spaces G/P, where P is a parabolic subgroup of G. Let θ be a subset of Δ and let W_θ be the subgroup of W generated by $\{s_\alpha\}_{\alpha\in\theta}$. We set $P_\theta := BW_\theta B$. Denote by W^θ the set

$$W^\theta := \{w \in W :\ l(ws_\alpha) = l(w) + 1 \quad \forall \alpha \in \theta\}.$$

This last set is the set of minimal length left coset representatives of W_θ in W.

The projection $G/B \to G/P_\theta$ induces an injection

$$CH^*(G/P) \hookrightarrow CH^*(G/B)\,,$$

which additively identifies $CH^*(G/P_\theta)$ with $\bigoplus_{w\in W^\theta} \mathbf{Z}[X^w]$. Multiplicatively $CH^*(G/P_\theta)_{\mathbf{Q}}$ is identified with the ring of invariants $CH^*(G/B)_{\mathbf{Q}}{}^{W_\theta}$. We refer for details to [2], Sect. 5.

The restriction $c\colon S^{W_\theta} \to CH^*(G/P_\theta)$ of the Borel characteristic map satisfies, for any W_θ-invariant f from S^k, the following equation in $CH^*(G/P_\theta)$:

$$c(f) = \sum_{\substack{w\in W^\theta\\ l(w)=k}} A_w(f)[X^w]\,. \tag{26}$$

Remark 47. Equation (17) is often called the "Leibniz-type formula".

5.3. A proof of the Pieri formula

In this section, we give a proof of the classical Pieri formula for the Grassmannian $G_n(m)$ of n-dimensional subspaces in $\mathbf{C}^m$ via the above method. In fact, there are two Pieri formulas: for multiplication by the Chern classes of the tautological subbundle on $G_n(m)$, and for multiplication by the Chern classes of the tautological quotient bundle on $G_n(m)$. The latter version appears more often mainly because the Chern classes of the tautological quotient bundle enjoy a simple interpretation in terms of the classical "Schubert conditions": the kth Chern class is represented by the locus of all n-planes in $\mathbf{C}^m$ which have positive dimensional intersection with a fixed $(m-n-k+1)$-plane in $\mathbf{C}^m$. By passing to the dual Grassmannian, we see that both formulas are, in fact, equivalent. We shall treat in detail the latter case. We also make a link with the ring of symmetric functions, known since Giambelli.

For the remainder of this paragraph, we set $q := m - n$.

In the following, I, J will denote *strict* partitions contained in the partition $(m, m-1, \ldots, q+1)$ with exactly n parts [1]. (We identify partitions with their Young diagrams, as is customary.) Note that such partitions contain the "upper-left triangle"

$$\delta = (n, n-1, \ldots, 1)\,.$$

On the other hand, λ, μ will denote "ordinary" partitions contained in (q^n). In fact, there is a bijection between these two sets: with I, we associate λ defined by $\lambda_p = i_p - n + p - 1$ for $p = 1, \ldots, n$.

[1] In other words, $I = (i_1, \ldots, i_n)$ where $m \geqslant i_1 > \cdots > i_n \geqslant 1$.

Also, we associate with I the following permutation w_I in the symmetric group S_m (which is the Weyl group of type A_{m-1}):

$$w_I = \cdots (s_{q-\lambda_3+3} \cdots s_{q+1} s_{q+2})(s_{q-\lambda_2+2} \cdots s_q s_{q+1})(s_{q-\lambda_1+1} \cdots s_{q-1} s_q) . \tag{27}$$

It is easy to see, that the right-hand side of (27) gives a reduced decomposition of w_I.

Take for example $m = 7$, $n = 3$, and $I = (6,4,3)$. Then $\lambda = (3,2,2)$ and $w_I = s_5 s_6 s_4 s_5 s_2 s_3 s_4$ which is the permutation $[1,3,6,7,2,4,5]$ (we display a permutation as the sequence of its consecutive values).

In general, for $I = (m \geqslant i_1 > \cdots > i_n \geqslant 1)$, we have in S_m,

$$w_I = [j_1 < \cdots < j_q \, , m+1-i_n < \cdots < m+1-i_1] \, ,$$

where $j_1, \ldots, j_q$ are uniquely determined by I.

Let $B \subset SL_m$ be the Borel group of lower triangular matrices. Using the notation of the previous section, we set $P = P_\theta$, where θ is obtained by omitting the simple root $\varepsilon_n \; - \; \varepsilon_{n+1}$ in the basis $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-1} - \varepsilon_m$ of the root system of type (A_{m-1}):

$$\{\varepsilon_i - \varepsilon_j \mid i \neq j\} \subset \oplus_{i=1}^m \mathbf{R}\varepsilon_i \, .$$

We have an identification $SL_m/P = G_n(m)$. We set

$$X^I := \overline{B w_0 w_I P/P} \, ,$$

where $w_0 = [m, m-1, \ldots, 1]$, and $X^\lambda := X^I$ for λ associated with I as above. Note that $[X^\lambda] \in CH^{|\lambda|}(G_n(m))$, where $|\lambda|$ denotes the sum of the parts of λ.

Denote by $(k+)$ the strict partition $(k+n, n-1, \ldots, 1)$, so that its associated λ is a one-row partition (k).

We want to compute the coefficients c_J in the expansion:

$$[X^I] \cdot [X^{(k+)}] = \sum_J c_J [X^J] \, .$$

Set $x_i := -\varepsilon_{m+1-i}$ for $i = 1, \ldots, m$, so that $c(x_1), \ldots, c(x_q)$ are the Chern roots of the tautological quotient bundle on $G_n(m)$. The Borel characteristic map allows us to treat $CH^*(G_n(m))$ as a quotient of the ring S' of polynomials symmetric in $\mathbb{X}_q$ and in $\mathbb{X}_m \setminus \mathbb{X}_q$. (Recall that for type (A_{m-1}), the characteristic map is surjective without tensoring by $\mathbf{Q}$.) The operators s_α and A_α indexed by the simple roots corresponding to P are induced by the following operators s_i and A_i, $i = 1, \ldots, q-1, q+1, \ldots, m-1$, on S'. The operator s_i interchanges x_i with x_{i+1}, leaving other variables invariant, and A_i is the ith simple (Newton's) divided difference ∂_i: for $f \in S'$,

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} \, .$$

The operator A_w on S', in this case ($w \in S_m$), will be denoted by ∂_w, as is customary.

Let $e_k = e_k(\mathbb{X}_q)$ be the kth elementary symmetric polynomial in $\mathbb{X}_q$. We now record:

Lemma 48. *For any $k = 1, \ldots, q$, the following equation holds in $CH^*(G_n(m))$:*

$$c(e_k) = [X^{(k)}]\,.$$

Proof. By virtue of Equation (26), it suffices to show that

$$\partial_w(e_k) = 0 \quad \text{unless} \quad w = w_{(k+)}\,, \quad \text{and} \quad \partial_{w_{(k+)}}(e_k) = 1\,.$$

Note that $w_{(k+)} = s_{q-k+1} \cdots s_{q-1} s_q$. The displayed assertion follows by induction on the number of variables, by invoking the following properties of ∂_i:

$$\partial_i(e_h(\mathbb{X}_j)) \neq 0 \quad \text{only if} \quad j = i\,,$$

$$\partial_i(e_h(\mathbb{X}_i)) = e_{h-1}(\mathbb{X}_{i-1})\,.$$

The lemma is proved. □

This lemma says that $X^{(k)}$ represents the kth Chern class of the tautological rank q quotient bundle on $G_n(m)$.

Number the successive columns of J from left to right with $m, m-1, \ldots, 1$, the successive rows from top to bottom with $1, \ldots, n$, and use the matrix coordinates for boxes in J.

Let J^* be the effect of subtracting the triangle δ from J. In the following, D will denote a subset of J^*.

Definition 49. *Read J^* row by row from left to right and from top to bottom. Every box from D (resp. from $J^* \setminus D$) in column i gives us s_i (resp. ∂_i). Then ∂_J^D is the composition of the resulting s_i's and ∂_i's (the composition written from right to left), and r_D is the word obtained by erasing all the ∂_i's from ∂_J^D.*

In particular, r_{J^*} is the reduced decomposition (27) of w_J, and $\partial_J^\emptyset = \partial_{w_J}$.

Take for example $m = 8$, $n = 3$, and $J = (8, 6, 5)$. In the following picture, "$*$" depicts a box in D and "$\circ$" stands for a box in $J^* \setminus D$. Moreover, the row-numbers and column-numbers are displayed.

$$\begin{array}{ccccccccc}
 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
1 & \times & \times & \times & * & * & * & * & * \\
2 & \times & \times & * & \circ & \circ & \circ & & \\
3 & \times & * & \circ & * & * & & &
\end{array}$$

Then we have

$$\partial_J^D = s_4 s_5 \partial_6 s_7 \partial_3 \partial_4 \partial_5 s_6 s_1 s_2 s_3 s_4 s_5 \qquad \text{and} \qquad r_D = s_4 s_5 s_7 s_6 s_1 s_2 s_3 s_4 s_5\,.$$

If r_D is a reduced decomposition of w_I, then D is a disjoint union of the following "p-ribbons". For fixed $p = 1, \ldots, n$, the p-ribbon consists of all boxes of D giving rise to those s_i (in r_D) which "transport" the item "$m+1-i_p$" from its position in $[1,2,\ldots,m]$ to its position in the sequence w_I.

In the above example, for $I=(7,5,2)$, the 1-ribbon consists of the asterisks in the first row, the 2-ribbon is $\{(3,4),(3,5),(2,6)\}$, and the 3-ribbon is $\{(3,7)\}$.

It can happen that some p-ribbon is empty. Suppose that p is such that the p-ribbon is not empty (this is equivalent to the fact that the box $(p,n+p-1)$ belongs to the p-ribbon). Then the column-numbers of boxes in the p-ribbon are $m+1-i_p,\ldots,n+p-2,n+p-1$, and their row-numbers weakly increase while reading D from left to right and from top to bottom.

By Theorem 45 and Lemma 48, we have

$$c_J=\sum \partial_J^D(e_k)\,, \tag{28}$$

where the sum is over all subsets $D\subset J^*$ such that r_D is a reduced decomposition of w_I.

We need the following lemma.

Lemma 50. *Suppose that there exist i and j such that (i,j) and $(i-1,j-1)$ are in $J^*\setminus D$. Then $\partial_J^D(e_k)=0$ for any k.*

Proof. We set $E:=\prod_{h=1}^n(1+x_h)$ and we shall prove that $\partial_J^D(E)=0$. To compute with compositions of the s_h's and ∂_h's in ∂_J^D , it is handy to introduce the following more general functions. For $\mathbf{a}=(a_1,a_2,\ldots,a_m)\in\{0,1\}^m$, we set

$$E_{\mathbf{a}}:=\prod_{h=1}^m(1+a_hx_h)\,,$$

so that $E=E_{(1,\ldots,1,0,\ldots,0)}$ with q 1's. We have:

$$s_h(E_{\mathbf{a}})=E_{\mathbf{a}'}\quad\text{where}\ \ \mathbf{a}'=(a_1,\ldots,a_{h-1},a_{h+1},a_h,a_{h+2},\ldots,a_m)\,; \tag{29}$$

$$\partial_i(E_{\mathbf{a}})=d\cdot E_{\mathbf{a}'}\quad\text{if}\ a_{h+1}=a_h+d\,, \tag{30}$$

where $\mathbf{a}'=(a_1,\ldots,0,0,\ldots,a_n)$ is $\mathbf{a}$ with a_h,a_{h-1} replaced by zeros.

Since, by the assumption, the box $(i-1,j-1)$ belongs to $J^*\setminus D$, using (29) and (30) we see that the operator ∂_j in ∂_J^D corresponding to the box (i,j) "kills" the function $E_{\mathbf{a}}$ which has been obtained by applying the previous operators s_h and ∂_h (in ∂_J^D) to E. This proves the lemma. □

First, it follows from this lemma that there is at most one $D\subset J^*$ such that r_D is a reduced decomposition of w_I and $\partial_I^D(e_k)\neq 0$, namely $D=I^*$. (Indeed, the p-ribbon must exactly coincide with the pth row of I^*.) In other words, the sum in (28) has at most one summand.

Second, applying Lemma 50 again, we see that $D=I^*$ gives a non-zero contribution to the sum in (28) iff $J\setminus I$ is a horizontal strip with pairwise separated rows[2]. In this case, using (29) and (30), we obtain $\partial_J^{I^*}(e_k)=1$.

[2] Recall that a horizontal strip is a skew diagram with at most one box in each column, and a *vertical strip* is a skew diagram with at most one box in each row.

We rewrite the outcome of the above considerations in terms of Schubert classes $[X^\lambda] \in CH^*(G_n(m))$ in part (i) of the following theorem. Part (ii) follows from part (i) by passing to the dual Grassmannian.

Theorem 51. (Pieri) (i) *For any partition* $\lambda \subset (q^n)$ *and* $k = 1, \dots, q$,

$$[X^\lambda] \cdot [X^{(k)}] = \sum_\mu [X^\mu]\,, \tag{31}$$

where $|\mu| = |\lambda| + k$ *and* $\mu \setminus \lambda$ *is a horizontal strip.*

(ii) *For any partition* $\lambda \subset (q^n)$ *and* $k = 1, \dots, n$,

$$[X^\lambda] \cdot [X^{(1,\dots,1)}] = \sum_\mu [X^\mu]\,, \tag{32}$$

where 1 *appears* k *times,* $|\mu| = |\lambda| + k$ *and* $\mu \setminus \lambda$ *is a vertical strip.*

For example, we have in $H^*(G_3(8), \mathbf{Z})$:

$$[X^{(4,2)}] \cdot [X^{(3)}] = [X^{(5,4)}] + [X^{(5,3,1)}] + [X^{(5,2,2)}] + [X^{(4,4,1)}] + [X^{(4,3,2)}]\,.$$

Remark 52. There are several (really) different proofs of the Pieri formula. We do not attempt to make a survey here. The proof that appears most often in monographs is based on studying the triple intersection of general translates of Schubert varieties.

Remark 53. The Schubert classes $[X^{(k)}]$ and $[X^{(1,\dots,1)}]$ are often called "special". These classes satisfy the following property: the corresponding $w \in W$ has a unique reduced decomposition. This seems to be a proper group-theoretic characterization of a "special Schubert class", and was also noticed by Kirillov and Maeno.

6. Riemann–Roch

The original problem motivating the work on this topic can be formulated as follows: given a connected nonsingular projective variety X and a vector bundle E over X, calculate the dimension $\dim H^0(X, E)$ of the space of global sections of E. The great intuition of Serre told him that the problem should be reformulated using higher cohomology groups as well. Namely, Serre conjectured that the number

$$\chi(X, E) = \sum (-1)^i \dim H^i(X, E)$$

could be expressed in terms of topological invariants related to X and E. Naturally, Serre's point of departure was a reformulation of the classical Riemann–Roch theorem for a curve X: given a divisor D and its associated line bundle $\mathcal{O}(D)$,

$$\chi(X, \mathcal{O}(D)) = \deg D + \frac{1}{2}\chi(X)\,.$$

(An analogous formula for surfaces was also known.)

The conjecture was proved in 1953 by F. Hirzebruch, inspired by earlier ingenious calculations of J.A. Todd.

6.1. Classical Riemann–Roch theorem for curves

Let X be a nonsingular projective curve over an algebraically closed field k. Let $D = \sum n_P P$ be a divisor on X, and $\mathcal{O}(D)$ denote the associated line bundle. Set

$$L(D) = \{f \in R(X) : \mathrm{ord}_P f \geqslant -n_P \text{ for any } P \in X\},$$

or, equivalently $L(D) = \Gamma(X, \mathcal{O}(D))$. Let $l(D) = \dim_k L(D)$. We have $l(D) < \infty$. The problem is to compute the number $l(D)$.

Lemma 54. *(i) If* $l(D) \neq 0$*, then* $\deg D \geqslant 0$.
(ii) If $l(D) \neq 0$ *and* $\deg D = 0$*, then* D *is linearly equivalent to* 0*, so* $\mathcal{O}_X(D) = \mathcal{O}_X$.

Proof. If $l(D) \neq 0$, then the linear system $|D|$ of effective divisors linearly equivalent to D is nonempty. So D is linearly equivalent to some effective divisor, and thus $\deg D \geqslant 0$.

If $\deg D = 0$ then D is linearly equivalent to an effective divisor of degree 0, which must be the zero divisor. □

Riemann proved in 1857 that

$$l(D) \geqslant \deg D + 1 - g\,, \tag{33}$$

where $g = \dim \Gamma(X, \Omega^1_X)$ is the genus of X.

Let $K = K_X$ be the canonical divisor on X. Roch in 1864 computed the difference between the LHS and RHS of (33):

$$l(D) - l(K - D) = \deg D + 1 - g\,. \tag{34}$$

This is a classical Riemann–Roch theorem for curves.

Example 55. Since $\deg K = 2g - 2$, we have $l(K) = g$. If $\deg D > 2g - 2$, then by the lemma, $l(K - D) = 0$, and the Riemann–Roch theorem gives $l(D)$ as a function of the degree and genus.

The Riemann–Roch theorem can serve to classify varieties.

Example 56. Suppose that $g(X) = 0$. We shall show that X is isomorphic to $\mathbf{P}^1$. Let P, Q be two different points on X. Set $D = P - Q$. We have $\deg(K - D) = -2$. Hence, by the lemma, $l(K - D) = 0$. Thus, using the Riemann–Roch theorem, we obtain $l(D) = 1$. Since $\deg D = 0$, then D is linearly equivalent to zero. This implies that P is linearly equivalent to Q, so that X is isomorphic to $\mathbf{P}^1$. □

Example 57. Let X be a nonsingular elliptic curve; so $g = 1$, $\deg K_X = 0$, $l(K) = 1$, and thus K is linearly equivalent to 0. Let $P_0 \in X$.

We claim that the assignment $P \mapsto \mathcal{O}_X(P - P_0)$, establishes $1 - 1$ correspondence between the points on X and elements of a group scheme $\mathrm{Pic}^0(X)$. We shall show that for any divisor $D \in \mathrm{Pic}^0 X$ there exists a unique $P \in X$ such that $D \sim P - P_0$. Using the Riemann–Roch theorem for $D + P_0$, we obtain

$$l(D + P_0) - l(K - D - P_0) = 0\,.$$

Since

$$\deg(K - D - P_0) = -1$$

we have $l(K - D - P_0) = 0$. Hence also $l(D + P_0) = 0$. Consequently $\dim |D + P_0| = 0$, and thus there exists a unique effective divisor linearly equivalent to $D + P_0$. Since the latter divisor is of degree 1, the former effective divisor must be a single point P. We have $P \sim D + P_0$, hence $D \sim P - P_0$.

6.2. A brief tour on cohomology of sheaves

We follow here the exposition in [20], where one can find the proofs lacking here.

We start with the following problem of Mittag–Leffler. Let X be a Riemannn surface, $P \in X$, and z a local coordinate of X around P. For a given Laurent series $\sum a_n z^n$, by the principal part at P, we shall mean

$$\sum_{k=1}^{n} a_k z^{-k}\,.$$

Problem Suppose that a discrete set of points $\{P_m\}$ on X is given. Assume that a principal part is attached to any point P_m. Does it exists a meromorphic function on X, which is holomorphic away of $\{P_m\}$ whose principal part at each point P_m is equal to the given one?

To solve this problem, we pick a covering $U = \{U_\alpha\}$ of X by open sets such that any U_α contains at most one point P_m. Suppose that a meromorphic function f_α on U_α is a solution to the problem on U_α. Set

$$f_{\alpha\beta} = f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)\,.$$

In $U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0\,.$$

A global solution of the problem is equivalent to find for all α a function $g_\alpha \in \mathcal{O}(\mathcal{U}_\alpha)$ such that $f_{\alpha\beta} = g_\beta - g_\alpha$ on $U_\alpha \cap U_\beta$. For, having such g_α's, we define

$$f = f_\alpha + g_\beta\,.$$

This is a globally defined function satisfying the needed conditions. (Note that we can take $g_\alpha = f_\alpha$, which leads to the trivial solution.)

In terms of Cech cohomology, we consider the groups

$$\{\{f_{\alpha\beta}\} : f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0\} = Z^1(U, \mathcal{O})\,,$$

$$\{\{f_\alpha\beta\} : f_{\alpha\beta} = g_\beta - g_\alpha \text{ for some}\{g_\alpha\}\} = \delta C^0(U, \mathcal{O}),$$

and look at the group

$$H^1(U, \mathcal{O}) = \frac{Z^1(U, \mathcal{O})}{B^1(U, \mathcal{O})}.$$

This group is the obstruction to solving the problem in general.

We shall work with presheaves, sheaves and morphisms between sheaves (see [23, 20] for their definitions and examples). We shall need the kernels, cokernels, complexes and exact sequences of morphisms of sheaves. For a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ we consider the sheaf $\operatorname{Ker}\varphi$, given by

$$\operatorname{Ker}\varphi(U) = \operatorname{Ker}(\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)).$$

Warning: $\operatorname{Coker}(\varphi)(U) = \mathcal{G}(U)/\varphi_U(\mathcal{F}(U)))$ is not a sheaf. For example, consider

$$\exp : \mathcal{O} \to \mathcal{O}^* \ \text{ on } \ \mathbf{C} \setminus \{0\} \ \text{ such that } \ \ f \in \mathcal{O}(U) \mapsto \exp(2\pi i f) \in \mathcal{O}^*(U)$$

is not in the image of $\mathcal{O}(\mathbf{C} \setminus \{0\})$ by exp, but its restriction to every contractible open subset $U \subset \mathbf{C} \setminus \{0\}$ is in the image of $\mathcal{O}_U$.

A sequence of sheaves

$$0 \to \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \to 0$$

is exact if $\mathcal{E} = \ker\psi$ and $\mathcal{G}$ is the sheaf associated to the presheaf $\operatorname{Coker}\varphi$ (we write $\mathcal{G} = \mathcal{F}/\mathcal{E}$).

Warning This does not imply that the sequence

$$0 \to \mathcal{E}(U) \xrightarrow{\varphi_U} \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{G}(U) \to 0$$

is exact at $\mathcal{G}(U)$ for any U. It implies that for any $\sigma \in \mathcal{G}(U)$ and any $P \in U$ there exists an open set V such that $P \in V \subset U$ and $\sigma|_V \in \operatorname{Im}\psi_V$.

Example 58. (i) If $X \hookrightarrow Y$ is an imbedding, we have an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

where $\mathcal{I}_X$ is the ideal sheaf defining X in Y.

(ii) The sequence of sheaves

$$0 \to \mathbf{Z} \to \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \to 0 \ \text{ such that } \ \ \exp(f) = e^{2\pi i f}$$

is exact.

A sequences of morphisms of sheaves

$$\cdots \to \mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{F}_{n+1} \xrightarrow{\varphi_{n+1}} \mathcal{F}_{n+2} \to \cdots$$

is exact if for any n, the sequence

$$0 \to \operatorname{Ker}(\varphi_n) \to \mathcal{F}_n \to \operatorname{Ker}(\varphi_{n+1}) \to 0$$

is exact.

We shall now define the cohomology of sheaves $H^i(X, \mathcal{F})$. Let $\mathcal{F}$ be a sheaf on X, and $U = \{U_\alpha\}$ a locally finite covering. We set

$$C^0(U, \mathcal{F}) = \prod_\alpha \mathcal{F}(U)$$

$$C^1(U, \mathcal{F}) = \prod_{\alpha<\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

$$\cdots$$

$$C^p(U, \mathcal{F}) = \prod_{\alpha_0<\alpha_1<\cdots<\alpha_p=\alpha} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p})$$

$$\cdots$$

We shall call an element

$$\sigma = \{\sigma_\alpha \in \mathcal{F}(\bigcap_{h=0}^{p} U_{\alpha_h})\}_\alpha \in C^p(U, \mathcal{F})$$

a p-cochain of $\mathcal{F}$.

We define a coboundary operator

$$\delta \colon C^p(U, \mathcal{F}) \to C^{p+1}(U, \mathcal{F})$$

by

$$(\delta\sigma)_{\alpha_0,\alpha_1,\dots,\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_{p+1}}|_{U_{\alpha_0}\cap\ldots\cap U_{\alpha_{p+1}}}\,.$$

Example 59. If $\sigma = \{\sigma_U\} \in C^0(U, \mathcal{F})$, then

$$(\delta\sigma)_{U,V} = -\sigma_U + \sigma_V|_{U\cap V}\,.$$

If $\sigma = \{\sigma_{U,V}\} \in C^1(U, \mathcal{F})$, then

$$(\delta\sigma)_{U,V,W} = \sigma_{U,V} + \sigma_{V,W} - \sigma_{U,W}|_{U\cap V\cap W}\,.$$

We shall call $\sigma \in C^p(U, \mathcal{F})$ a cocycle if $\delta\sigma = 0$. We shall call $\sigma \in C^p(U, \mathcal{F})$ a coboundary if $\sigma = \delta\tau$ for some $\tau \in C^{p-1}(U, \mathcal{F})$.

Lemma 60. *We have $\delta^2 = 0$.*

This is an exercise.

We write

$$Z^p(U, \mathcal{F}) = \operatorname{Ker}\delta$$

and
$$H^p(U,\mathcal{F}) = \frac{Z^p(U,\mathcal{F})}{\delta C^{p-1}(U,\mathcal{F})}\,.$$

Let $U = \{U_\alpha\}_{\alpha\in I}$ and $U' = \{U'_\beta\}_{\beta\in I'}$ be two open locally finite coverings of X. We say that U' is an refinement of U iff for any $\beta \in I'$ there is $\alpha \in I$ such that $U'_\beta \subset U_\alpha$. If U' refines U, then there is a map $\eta\colon I' \to I$ such that $U'_\beta \subset U_{\eta\beta}$ for any β. We obtain a map
$$\rho_\eta\colon C^p(U,\mathcal{F}) \to C^p(U',\mathcal{F})$$
such that
$$(\rho_\eta\sigma)_{\beta_0,\dots,\beta_p} = \sigma_{\eta\beta_0,\dots,\eta\beta_p}|_{U_{\beta_0}\cap\cdots\cap U_{\beta_p}}\,.$$
We have
$$\delta\circ\rho_\eta = \rho_\eta\circ\delta\,,$$
so we have an induced map
$$\rho\colon H^p(U,\mathcal{F}) \to H^p(U',\mathcal{F})\,.$$

Lemma 61. *The map ρ does not depend on η. For another $\theta\colon I' \to I$ such that $U'_\beta \subset U_{\theta\beta}$, the map ρ_η is a map of complexes homothopic to ρ_θ.*

We define $H^p(X,\mathcal{F})$ to be the direct limit of $H^p(U,\mathcal{F})$, taken on more and more refined open locally finite coverings U of X.

For example, the space of global sections of $\mathcal{F}$, $H^0(X,\mathcal{F})$ is equal to $H^0(U,\mathcal{F})$ for any open covering U of X.

We say that an open covering U is acyclic for $\mathcal{F}$ if
$$H^p(U_{\alpha_1}\cap\cdots\cap U_{\alpha_h},\mathcal{F}) = 0$$
for any $p > 0$, any $h \geqslant 1$ and any $\alpha_1,\dots,\alpha_h$.

We record

Theorem 62. *If a covering U is acyclic for $\mathcal{F}$, then*
$$H^p(U,\mathcal{F}) = H^p(X,\mathcal{F})\,.$$

Let
$$0 \to \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \to 0 \tag{35}$$
be an exact sequence of sheaves on X. For any covering U of X, the maps φ and ψ induce maps
$$\varphi\colon C^p(U,\mathcal{E}) \to C^p(U,\mathcal{F}) \quad\text{and}\quad \psi\colon C^p(U,\mathcal{F}) \to c^p(U,\mathcal{G})\,.$$
Since φ,ψ commute with δ, we get the induced maps
$$\varphi\colon H^p(X,\mathcal{E}) \to H^p(X,\mathcal{F}) \quad\text{and}\quad \psi\colon H^p(X,\mathcal{F}) \to H^p(X,\mathcal{G})\,.$$

We next define the coboundary map

$$\delta^*\colon H^p(X,\mathcal{G}) \to H^{p+1}(X,\mathcal{E})\,.$$

Let $\sigma \in C^p(U,\mathcal{G})$ and $\delta(\sigma)=0$. After passing to some refinement U' of U, we can find $\tau \in C^p(U',\mathcal{F})$ such that $\psi(\tau)=\rho(\sigma)$. Then

$$\psi\delta(\tau) = \delta\psi(\tau) = \delta\rho(\sigma) = 0\,.$$

Hence, if we pass to a further refinement U'', we can find $\mu \in C^{p+1}(U'',\mathcal{E})$ such that $\varphi(\mu)=\delta(\tau)$. We have

$$\varphi\delta(\mu) = \delta\varphi(\mu) = \delta^2(\tau) = 0\,.$$

Since α is injective, we thus obtain $\delta(\mu)=0$. Hence $\mu \in Z^{p+1}(U',\mathcal{E})$, and we set $\delta^*(\sigma) = \mu \in H^{p+1}(X,\mathcal{E})$. We record

Theorem 63. *The long sequence*

$$\cdots \to H^{p-1}(X,\mathcal{G}) \to H^p(X,\mathcal{E}) \to H^p(X,\mathcal{F}) \to H^p(X,\mathcal{G}) \to H^{p+1}(X,\mathcal{E}) \to \cdots$$

of cohomology groups induced by (35) *is exact.*

Remark 64. Consider the sequence (35). Let $\sigma \in \mathcal{G}(X)$. We may ask: Is σ the image of some element from $\mathcal{F}(X)$? The answer is YES if $\delta^*(\sigma)=0$. This last vanishing holds, e.g., if $H^1(X,\mathcal{E})=0$.

Example 65. We now revisit the Mittag–Leffler problem. Let X be a Riemann surface. Consider the exact sequence

$$0 \to \mathcal{O} \to \mathcal{M} \xrightarrow{\varphi} \mathcal{PP} \to 0$$

of sheaves, where $\mathcal{O}$ denotes the sheaf of holomorphic functions and $\mathcal{M}$ of the meromorphic ones. For an open subset U, we set

$$\mathcal{PP}(U) = \{(P_m, f_m)\colon \{P_m\} \text{ is discrete in } U \text{ and } f_m \in \mathcal{M}_{P_m}/\mathcal{O}_{P_m}\}\,,$$

which gives rise to the sheaf $\mathcal{PP}$.

The datum of the Mittag–Leffler problem is: $g \in \mathcal{PP}(X)$. We ask when $g=\varphi(f)$ for some global meromorphic function f? The answer to this question is: If and only if $\delta^*(g)=0$ in $H^1(X,\mathcal{O})$; e.g., when $H^1(X,\mathcal{O})=0$.

Let E be a vector bundle on a projective variety over an algebraically closed field of arbitrary characteristic. Then $H^p(X,E) = H^p(X,\mathcal{O}(E))$ is a finite dimensional vector space (see, e.g., [23]).

We invoke Serre duality. For the canonical bundle $K_X = \Omega^n_X = \Lambda^n\Omega^1_X$, where $\Omega^1_X = (TX)^*$ and $n=\dim X$, we have

$$H^p(X,E) = H^{n-p}(X,E^*\otimes K_X)^*\,,$$

for $p = 0, 1 \ldots, n$. This duality allows us to understand better the summand in the LHS of the Riemann–Roch equation that was found by Roch.

Example 66. *Let X be a nonsingular projective curve. For a divisor D on X, we have*

$$H^0(X, \mathcal{O}(-D) \otimes K)^* \simeq H^1(X, \mathcal{O}(D)),$$

so that

$$l(K - D) = \dim H^1(X, \mathcal{O}(D)).$$

For a projective curve X and a coherent sheaf $\mathcal{F}$ on X, we set

$$\chi(X, \mathcal{F}) = \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F}).$$

Then by the virtue of the example, the classical Riemann–Roch theorem says:

$$\chi(X, \mathcal{O}(D)) = \deg D + 1 - g. \tag{36}$$

With cohomology of sheaves, a proof of (36) is natural and straightforward. For $D = 0$,

$$\chi(X, \mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) = 1 - \dim H^0(X, \Omega^1_X)$$

by Serre duality. Since the RHS of (36) equals $0 + 1 - g$, the equality (36) follows. Let now D be an arbitrary divisor on X and $P \in X$ be any point. Let us tensorize the exact sequence of sheaves on X

$$0 \to \mathcal{O}(-P) \to \mathcal{O} \to k(P) \to 0$$

by $\mathcal{O}(D + P)$. We get an exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D + P) \to k(P) \to 0.$$

We note

Lemma 67. *For the exact sequence (35), we have*

$$\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G}).$$

The lemma implies

$$\chi(\mathcal{O}(D + P)) = \chi(\mathcal{O}(D)) + 1.$$

Since we also have

$$\deg(D + P) = \deg D + 1,$$

we obtain that (36) for D and $D + P$ hold or fail simultaneously. (This also holds true for $D - P$ and D, of course.) Since any divisor can be reached from 0 in a finite number of steps by adding or subtracting a point each time, this shows that (36) holds for any D. □

6.3. The Hirzebruch–Riemann–Roch theorem

The main reference here is [24].

For a coherent sheaf $\mathcal{F}$, we shall investigate the Euler characteristic

$$\chi(X, \mathcal{F}) = \chi(\mathcal{F}) = \sum (-1)^i \dim H^i(X, \mathcal{F}) \,,$$

satisfying the additivity property

$$\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G}) \,.$$

for an exact sequence (35).

Let E be a vector bundle of rank d on a nonsingular variety X. Let $x_1, \ldots, x_d \in CH^*(Fl(E))$ be the Chern roots of E. We set

$$ch(E) = \sum exp(x_i) \quad \text{with} \quad exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \,.$$

We have

$$ch(E) = d + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 - 2c_2^2 - 4c_4) + \cdots \,.$$

We record

Lemma 68. *If*

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles on a variety, then

$$ch(E) = ch(E) + ch(E'') \,.$$

If E, F are vector bundles on a variety, then

$$ch(E \otimes F) = ch(E) \cdot ch(F) \,.$$

Let

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum (-1)^{i-1} \frac{B_i}{(2i)!} x^{2i} \,,$$

where B_i are the Bernoulli numbers (see, e.g., [24]). We define

$$td(E) = \prod Q(x_i) \,.$$

We have

$$td(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \ldots \,.$$

Given a nonsingular variety X, we shall write $td(X)$ for $td(TX)$.

We record without proof the following property

Lemma 69. *If*

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles on a variety, then

$$td(E) = td(E') \cdot td(E'') .$$

Here comes the fundamental relation between the Chern character ch, Todd genus td and the top Chern class.

Proposition 70. *If E is a rank d vector bundle, then*

$$\sum_{p=0}^{d} (-1)^p ch(\Lambda^p E^*) = \frac{c_d(E)}{td(E)} .$$

Proof. We have, using the splitting principle and the Chern roots $x_1, \ldots, x_d$ of E,

$$\begin{aligned} LHS = \sum_{p=0}^{d} (-1)^p \sum_{i_1 < \cdots < i_p} exp(-x_{i_1} - \cdots - x_{i_p}) \\ = \prod (1 - exp(-x_i)) \\ = \prod x_i \prod (1 - exp\frac{-x_i}{x_i}) = RHS. \ \square \end{aligned}$$

Theorem 71. (Hirzebruch–Riemann–Roch) *If E is a vector bundle on a projective variety X of dimension n, then*

$$\chi(E) = \deg(ch(E) \cdot td(X))_n = \int_X ch(E) td(X), . \tag{37}$$

Example 72. Let E be a vector bundle of rank d on a nonsingular curve X. Then

$$\chi(E) = \deg(d + c_1(E))(1 + \frac{1}{2} c_1(TX))_1 = c_1(E) + \frac{1}{2} d c_1(TX) = c_1(E) + d(1-g) .$$

If $E = \mathcal{O}_X(D)$, then we get

$$\chi(\mathcal{O}_X(D)) = \deg D + 1 - g,$$

in agreement with the formula from the previous subsection.

Example 73. If E is rank d vector bundle on a nonsingular surface X, then $\chi(E)$ is equal to

$$\deg[(d + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E))(1 + \frac{1}{2} c_1(TX) + \frac{1}{12}(c_1(TX)^2 + c_2(TX))]_2 ,$$

so that with $e_i = c_i(E)$ and $c_i = c_i(TX)$, we have

$$\chi(E) = \int_X \frac{1}{2}(e_1^2 - 2e_2) + \frac{1}{2} e_1 c_1 + \frac{1}{12} d(c_1^2 + c_2) .$$

We have

$$\chi(\mathcal{O}_X) = \frac{1}{12}\int_X c_1^2 + c_2 = \frac{1}{12}(K_X^2 + \chi)\,,$$

where χ is the topological Euler characteristic. For $E = \mathcal{O}(D)$, we have

$$\chi(\mathcal{O}(D)) = \frac{1}{2}((D^2) - (K_X \cdot D)) + \chi(X, \mathcal{O}_X)\,.$$

Example 74. If E is rank d vector bundle on a nonsingular threefold X, then

$$\chi(E) = \int_X \frac{1}{6}(e_1^3 - 3e_1e_2 + 3e_3) + \frac{1}{4}c_1(e_1^2 - 2e_2) + \frac{1}{12}(c_1^2 + c_2)e_1 + \frac{d}{24}c_1c_2\,,$$

where $e_i = c_i(E)$ and $c_i = c_i(TX)$. If $E = \mathcal{O}_X(D)$ for a divisor D, then

$$\chi(\mathcal{O}_X(D)) = \int_X \frac{1}{6}D^3 + \frac{1}{4}c_1 \cdot D^2 + \frac{1}{12}(c_1^2 + c_2)\cdot D + \frac{1}{24}c_1c_2\,.$$

Example 75. If TX is trivial (e.g. X is a nonsingular elliptic curve, or more generally, an abelian variety), then for any vector bundle on X,

$$\chi(X, E) = \int_X ch_n(E)\,,$$

where $n = \dim X$. In, particular, if $E = L$ is a line bundle, then

$$\chi(L) = \frac{1}{n!}\int_X c_1(L)^n\,,$$

so that for any divisor D, $\int_X D^n$ is divisible by $n!$.

The Hirzebruch–Riemann–Roch formula implies many interesting divisibility properties of characteristic numbers.

Example 76. The following result shows how useful the Hirzebruch–Riemann–Roch can be. In [44], the so called diagonal property was studied: "There exists a vector bundle E of rank equal to $\dim X$ on $X \times X$ and a section s of E such that the diagonal Δ is the zero scheme of s". For example, projective spaces, Grassmannians and flag manifolds SL_n/P have this property.

A nonsingular quadric $Q_3 \subset \mathbf{P}^4$ over an algebraically closed field fails to have the diagonal property. To this end, we use the following Corollary 6 (loc.cit). Let X be a nonsingular and proper scheme over an algebraically closed field with finitely generated $\mathrm{Pic}(X)$. If for any cohomologically trivial line bundle L on X [3], either the L^{-1}-point property fails, or the $L \otimes \omega_X^{-1}$-point property fails, then X does not admit the diagonal property.

There are two cohomologically trivial line bundles on Q_3: $L_1 = \mathcal{O}_{Q_3}(-1)$ and $L_2 = \mathcal{O}_{Q_3}(-2)$. Since $\omega_{Q_3} = \mathcal{O}_{Q_3}(-3)$, we have

$$L_1^{-1} = L_2 \otimes \omega_{Q_3}^{-1}\,,$$

[3] This means that for each $x \in X$ there exists a vector bundle on X of rank equal to $\dim X$ with determinant equal to L and a section vanishing at x with multiplicity 1.

so, by the above result, it suffices to show that the L_1^{-1}-point property fails.

We use a standard presentation of the Chow ring $CH^*(Q_3)$ as $\oplus_{i=0}^3 \mathbf{Z}$ with a suitable ring structure. Let $[Q_2]$, $[L]$, and $[P]$ (quadric surface, line, and point) be the generators of $CH^1(Q_3)$, $CH^2(Q_3)$, and $CH^3(Q_3)$ respectively. There are the following relationships:

$$[Q_2]^2 = 2[L] \quad \text{and} \quad [Q_2] \cdot [L] = [P]\,. \tag{38}$$

If E is a vector bundle on Q_3, the total Chern class of E is of the form

$$1 + d_1(E)[Q_2] + d_2(E)[L] + d_3(E)[P]\,,$$

where $d_i(E) \in \mathbf{Z}$.

The argument now boils down to showing that there is no rank 3 vector bundle E on Q_3 with $d_3(E) = 1$ and $d_1(E) = 1$.

Suppose – *ad absurdum* – that such a bundle exists. We use the Hirzebruch–Riemann–Roch theorem. In fact, we use the following explicit version of the formula for nonsingular quadric 3-folds: for a vector bundle E on Q_3,

$$\chi(Q_3, E) = \frac{1}{6}(2d_1^3 - 3d_1 d_2 + 3d_3) + \frac{3}{2}(d_1^2 - d_2) + \frac{13}{6} d_1 + \operatorname{rank}(E)\,, \tag{39}$$

where $d_i = d_i(E)$ are the above numbers. Substituting the present values, we get

$$\chi(Q_3, E) = \frac{15}{2} - 2d_2\,.$$

This contradicts the fact that $\chi(Q_3, E)$ is integer.

6.4. The Grothendieck–Riemann–Roch theorem

We start with recalling the following result.

If X is a complete variety and $\mathcal{F}$ is a coherent sheaf on X, then $\dim H^j(X, \mathcal{F}) < \infty$.

And this is its relative version:

If $f\colon X \to Y$ is a proper morphism, and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{R}^j f_* \mathcal{F}$ is coherent on Y.

To formulate a relative version of the Hirzebruch theorem, let a proper morphism $f\colon X \to Y$ between nonsingular varieties be given. We want to understand the relationship between

$$\operatorname{ch}_X(-)\operatorname{td}(X) \quad \text{and} \quad \operatorname{ch}_Y(-)\operatorname{td}(Y),$$

"induced" by f. In the case of $f\colon X \to Y = \textit{point}$, we should obtain the Hirzebruch–Riemann–Roch theorem. The relativization of the right-hand side of (37) is easy: there exists a well defined additive map of the Chow rings $f_*\colon CH^*(X) \to CH^*(Y)$, and $\deg(z)_n$ corresponds to $f_*(z)$ for $z \in CH^*(X)$.

What about the left-hand side of (37)? The relative version of the $H^j(X, \mathcal{F})$ are the coherent modules $\mathcal{R}^j f_* \mathcal{F}$, vanishing for $j > 0$. In order to construct a relative version of the alternating sum, Grothendieck defines the following group $K(Y)$ (now called the

Grothendieck group) It is the quotient group of a "very large" free abelian group $\bigoplus \mathbf{Z}[E]$ generated by the isomorphism classes $[E]$ of vector bundles on Y, modulo the relation

$$[E] = [E'] + [E'']$$

for each exact sequence

$$0 \to E' \to E \to E'' \to 0\,. \tag{40}$$

The group $K(Y)$ has the following universal property: every map φ from $\bigoplus \mathbb{Z}[E]$ to an abelian group, satisfying

$$\varphi([E]) = \varphi([E']) + \varphi([E''])\,, \tag{41}$$

factors through $K(Y)$. In our situation, we define

$$\varphi([E]) := \sum (-1)^j [\mathcal{R}^j f_* E] \in K(Y)\,.$$

Observe that (41) follows from the long exact sequence of derived functors

$$\cdots \longrightarrow \mathcal{R}^j f_* E' \longrightarrow \mathcal{R}^j f_* E \longrightarrow \mathcal{R}^j f_* E \longrightarrow \mathcal{R}^{j+1} f_* E \longrightarrow \cdots\,,$$

associated with the short exact sequence (40) (see, e.g., [23]) Thus, we obtain an additive map

$$f_! \colon K(X) \to K(Y).$$

Now the *relative* Hirzebruch–Riemann–Roch theorem, discovered by Grothendieck ([102], [4]) asserts the commutativity of the diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\quad f_! \quad} & K(Y) \\ {\scriptstyle \mathrm{ch}_X(-)\mathrm{td}(X)} \downarrow & & \downarrow {\scriptstyle \mathrm{ch}_Y(-)\mathrm{td}(Y)} \\ CH^*(X) & \xrightarrow{\quad f_* \quad} & CH^*(Y)\,. \end{array}$$

(Note that due to its additivity, the Chern character $\mathrm{ch}(-)$ is well defined in K-theory.)

Proof. The original proof of Grothendieck was published in the article [4] by Borel and Serre, in which the authors developed sheaf-theoretic methods in algebraic geometry. The beauty of mathematics in this paper is absolutely remarkable.

The present proof was worked out by Fulton and MacPherson [13]. This proof will use in an essential way the extension of the Chern classes to coherent sheaves on a nonsingular variety. We invoke

Theorem 77. (Hilbert syzygy theorem) *For a coherent sheaf $\mathcal{F}$ on a nonsingular variety X, there exists an exact finite complex*

$$0 \to E_m \to \cdots \to E_1 \to E_0 \to \mathcal{F} \to 0\,,$$

where E_i are finitely generated locally free sheaves.

Let F be a vector bundle of rank d on a nonsingular variety X. Consider the map

$$\bar{f}\colon X \to F \to \mathbf{P}(F \oplus \mathbf{1}) = Y\,, \tag{42}$$

where the first map is the zero section of F and the second map is the imbedding onto the complement of the $\mathbf{P}(F)$ in $\mathbf{P}(F \oplus \mathbf{1})$. Let $p\colon Y \to X$ be the canonical projection, and Q the tautological quotient rank d bundle on Y.

For the section $s\colon \mathbf{1}_Y \hookrightarrow p^*(F \oplus \mathbf{1}_Y) \twoheadrightarrow Q$ its zero scheme is

$$Z(s) = \bar{f}(X) \simeq X\,.$$

Consider now the Koszul complex associated with s:

$$0 \to \Lambda^d Q^* \to \cdots \to \Lambda^2 Q^* \to Q^* \to \mathcal{O}_Y \to \bar{f}_* \mathcal{O}_X \to 0$$

where the map $Q^* \to \mathcal{O}_Y$ is the cosection s^*. This complex is exact on Y.

For a vector bundle E on X, we consider

$$0 \to \Lambda^d Q^* \otimes p^*E \to \cdots \to \Lambda^2 Q^* \otimes p^*E \to Q^* \otimes p^*E \to p^*E \to \bar{f}_* E \to 0\,.$$

This complex is exact on Y, i.e. it is a resolution of $\bar{f}_*E$. We thus have

$$ch \bar{f}_*[E] = \sum_{i=0}^{d} ch(\Lambda^i Q^* \otimes p^*E) = \sum_{i=0}^{d} ch(\Lambda^i Q^*) ch(p^*E)) = c_d(Q) td(Q)^{-1} ch(p^*E)$$

by the multiplicativity of ch and Proposition 70. By the projection formula for $\bar{f}$, we have

$$\bar{f}_*(\bar{f}^*\alpha) = \alpha \cdot \bar{f}_*[X] = c_d(Q) \cdot \alpha\,,$$

so that we rewrite the last expression as

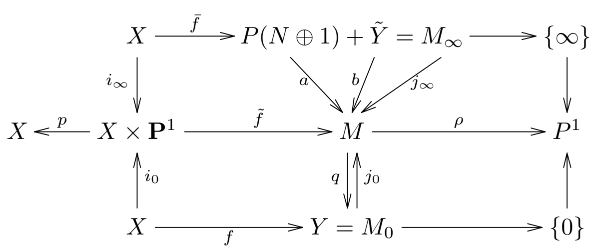
$$\bar{f}_* \bar{f}^*(td(Q)^{-1} ch(p^*E)) = \bar{f}_*(td(F) ch(E))$$

since $\bar{f}^*Q = F$ and $\bar{f}^*p^*E = E$. So the Grothendieck–Riemann–Roch theorem holds for the imbedding (42).

Let now $f\colon X \hookrightarrow Y$ denote an arbitrary closed imbedding of nonsingular varieties. Let $N = N_Y X$ be the normal bundle. We shall deform f to the embedding (42) for $F = N$:

$$\bar{f}\colon X \to N \to \mathbf{P}(N \oplus \mathbf{1}) = Y\,.$$

We invoke now the deformation to the normal bundle from Section (2.6).



Let E be a vector bundle on X. We want to show that

$$ch(f_*E) = f_*(td(N)^{-1}ch(E)) .$$

Let $\tilde{E} = p^*E$. Using the Hilbert syzygy theorem, pick a finite resolution of $\tilde{f}_*\tilde{E}$ by vector bundles on M (which is nonsingular)

$$G\colon 0 \to G_n \to \cdots \to G_0 \to \tilde{f}_*(\tilde{E}) \to 0 . \qquad (43)$$

Since M is flat over $\mathbf{P}^1$, the restrictions of the exact sequence (43) on M to the fibers M_0 and M_∞ are exact. More precisely,

(i) the complex j_0^*G is a resolution of $j_0^*\tilde{f}_*(\tilde{E}) = f_*E$ on $Y = M_0$;
(ii) the complex j_∞^*G is resolution of $j_\infty^*\tilde{f}_*(\tilde{E}) = \bar{f}_*E$ on $Y = M_\infty$.

We have

$$\bar{f}(X) \cap \tilde{Y} = \emptyset .$$

Therefore
(iii) the complex a^*G is a resolution of $\bar{f}_*E$ on $\mathbf{P}(N \oplus \mathbf{1})$;

(iv) the complex b^*G is acyclic on $\tilde{Y}$.

For a complex

$$\mathcal{F}_\bullet\colon 0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

of coherent sheaves on X we shall write $ch(\mathcal{F}_\bullet)$ for $\sum(-1)^i ch(\mathcal{F}_i)$.

We compute now the image of $ch(f_*E)$ in $CH^*(M)_{\mathbf{Q}}$. We use deformation from 0 to ∞. We have

$$j_{0*}(ch(f_*E)) = j_{0*}(ch(j_0^*G)) = ch(G) \cdot j_{0*}[Y] ,$$

where the first equality follows from (i) and the second by the projection formula for j_0. Since

$$[M_0] - [M_\infty] = [\mathrm{div}(\rho)] = 0 \quad \text{in} \quad CH^*(M)_{\mathbf{Q}} ,$$

the last expression can by rewritten as

$$ch(G)(a_*\mathbf{P}(N \oplus \mathbf{1}) + b_*[\tilde{Y}])$$

and using two times the projection formula, as

$$a_*(ch(a^*G)) + b_*(ch(b^*G)) = a^*(ch(\bar{f}_*E)) + 0 ,$$

where the last equality follows from (ii) and (iii).

We are now in the situation of (42) for $\bar{f}$, and thus we have

$$a_*(ch(\bar{f}_*(E))) = a_*(\bar{f}_*(td(N)^{-1}chE)) . \qquad (44)$$

Consider the composite map

$$q\colon M \to Y \times \mathbf{P}^1 \to Y,$$

where the first map is the blown-down, and the second one is the projection on the first factor. We have

$$q \circ j_0 = id_Y \quad \text{and} \quad q \circ a \circ \bar{f} = f\,. \tag{45}$$

Applying q_* to (44) and using (45), we have

$$ch(f_*E) = f_*(td(N)^{-1}chE).$$

This completes the proof of the Grothendieck–Riemann–Roch for a closed imbedding.

To prove it for an arbitrary proper map $f\colon X \to Y$ between nonsingular varieties we consider a factorization of f:

$$X \xrightarrow{g} Y \times \mathbf{P}^m \xrightarrow{p} Y\,.$$

(For a quasiprojective X, pick an imbedding $i\colon X \to \mathbf{P}^m$ as a locally closed subset; then take $g = (f, i)$). Consider

$$\tau_X\colon K(X) \to CH^*(X) \quad \text{such that} \quad \tau_X(E) = ch(E)\cdot td(X)$$

for a vector bundle E on X. We want to show that

$$f_* \circ \tau_X = \tau_Y \circ f_*\,. \tag{46}$$

It is easy to see that if (46) holds for both g and p, then it holds for f. To prove (46) for p, we note that $K(Y \times \mathbf{P}^m)$ is generated over the ring $K(Y)$ by the classes $[\mathcal{O}_Y(n)]$. Then to prove it for $E = \mathcal{O}(n)$, we can assume that Y is a point Spec(k); so that it amounts to check:

$$\int_{\mathbf{P}^m} ch(\mathcal{O}(n))\cdot td(\mathbf{P}^m) = \chi(\mathbf{P}^m, \mathcal{O}(n))\,.$$

(This is an exercise.)

The Grothendieck–Riemann–Roch theorem has been proved. □

7. Plücker formulas

In this section, the references are [8, 20] and [27].

7.1. Preliminaries and statement of the main results

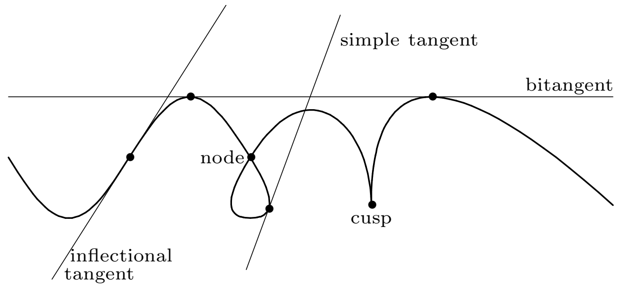
In this section, we work over the any ground field, but not of characteristic 2.

For a preliminary discussion, suppose that $C \subset \mathbf{P}^2$ is an irreducible plane curve different from a line. Suppose that C has only simple singularities, that is, it has only κ cusps and δ nodes as singularities (later, we shall drop this assumption). Recall that a cusp is a singular point whose tangent cone is a double line, and a node (ordinary double point) has two local nonsingular components intersecting transversally.

We have two basic numerical invariants of curves. By the degree of C, deg(C), we mean the degree of the polynomial defining C, or the number of points in the intersection of C

with a general line. By the genus of C, $g(C)$, we understand the genus of a desingularization of C; it follows that $g(C)$ is a birational invariant.

We shall also use different tangents: simple tangents, bitangents, and inflectional tangents. Let f denote the number of inflectional tangents, and b the number of bitangents.



There is a natural duality in this story.

Let $d^{\vee}$ = be the number of regular points of C the tangents at which pass through a fixed general point $P \in \mathbf{P}^2$.

Let $C^{\circ} = C - \mathrm{Sing}(C)$. Denoting by $\mathbf{P}^{2^{\vee}}$ the dual plane, i.e. the set of lines in $\mathbf{P}^2$, we define the map

$$\Phi \colon C^{\circ} \to \mathbf{P}^{2^{\vee}},$$

such that $P \to T_P C \subset \mathbf{P}^2$. Let $C^{\vee} = \overline{\Phi(C^{\circ})}$ be the dual curve of C. Its degree, $\deg C^{\vee}$, is the number of points of intersection with a general line in $\mathbf{P}^{2^{\vee}}$, that is, $d^{\vee}$.

Lemma 78. *If C is nonsingular, then $d^{\vee} = d(d-1)$.*

Proof. Take a general point $P \in \mathbf{P}^2$. Pick the coordinates $(x_0 : x_1 : x_2)$ such that $P = (0 : 0 : 1)$ and C is given by the vanishing of $f(x_1, x_2, x_3)$. Take a tangent to C, at point $Q \in C$, say. The condition of that this tangent passes through P is: $\frac{\partial f}{\partial x_2}(Q) = 0$. The lemma follows by multiplying the degree $d-1$ of this last equation by $d = \deg C$ (from $Q \in C$). $\square$

Poncelet remarked that the appearance of cusps and nodes in C makes $d^{\vee}$ smaller. Plücker then proved:

Theorem 79. *We have*

$$d^{\vee} = d(d-1) - 2\delta - 3\kappa\,. \tag{1}$$

We use the formula for genus of a nonsingular curve on a nonsingular surface X.

Theorem 80. (Adjunction Formula) *We have*

$$g(C) = \frac{C \cdot (K_X + C)}{2} + 1\,, \tag{47}$$

where K_X is the canonical class.

For $X = \mathbf{P}^2$ and a nonsingular curve of degree d, we get, by substituting in the last expression: $C = d[L], K = -3[L]$,

$$g(C) = \frac{d(d-3)}{2} + 1 = \frac{(d-1)(d-2)}{2}\,.$$

Again, an appearance of cusps and nodes makes the genus of a curve smaller:

Theorem 81. (Clebsch) *We have*

$$g(C) = \frac{(d-1)(d-2)}{2} - \delta - \kappa\,. \tag{2}$$

In order to give the remaining Plücker formulas, we make the notation more systematic:

$$g = g(C) = g(C^\vee)$$

$$b = \#\text{of bitangents of } C, \quad b^\vee = \#\text{of bitangents of } C^\vee$$

$$f = \#\text{of inflectional tangents of } C, \quad f^\vee = \#\text{of inflectional tangents of } C^\vee$$

$$\kappa = \#\text{of cusps of } C, \quad \kappa^\vee = \#\text{of cusps of } C^\vee$$

$$\delta = \#\text{of nodes of } C, \quad \delta^\vee = \#\text{of nodes of } C^\vee$$

We have $b = \delta^\vee$, $b^\vee = \delta$, $f = k^\vee$, $f^\vee = \kappa$.

By duality, the following formulas hold:

$$d = d^\vee(d^\vee - 1) - 2\delta^\vee - 3\kappa^\vee = d^\vee(d^\vee - 1) - 2b - 3f\,, \tag{$1^\vee$}$$

$$g = \frac{(d^\vee - 1)(d^\vee - 2)}{2} - b - f\,. \tag{$2^\vee$}$$

From [8], we record

Theorem 82. *We have*

$$\kappa^\vee = 3d(d-2) - 6\delta - 8\kappa = f\,, \tag{3}$$

$$\kappa = 3d^\vee(d^\vee - 2) - 6\delta^\vee - 8\kappa^\vee = 3d^\vee(d^\vee - 2) - 6b - 8f = f^\vee\,. \tag{$3^\vee$}$$

The formulas (1), (3), ($1^\vee$) and ($3^\vee$) are not independent. Each three imply the remaining fourth.

Any three from the numbers $d, \kappa, \delta, d^\vee, \kappa^\vee, \delta^\vee$ are determined by the remaining three. Assuming d, κ, δ, we get $d^\vee$ and $\kappa^\vee$ by (1) and (3). By substituting these expressions to ($1^\vee$), we obtain

$$\delta^\vee = \frac{1}{2}d(d-2)(d^2-9) - (2\delta + 3\kappa)(d^2 - d - 6) + 2\delta(\delta - 1) + \frac{9}{2}\kappa(\kappa - 1) + 6\delta\kappa\,.$$

From [27], we record

Theorem 83. *We have*

$$d^\vee = 2d + 2g - 2 - \kappa \tag{4}$$

$$d = 2d^\vee + 2g - 2 - \kappa^\vee = 2d^\vee + 3g - f \tag{$4^\vee$}$$

Let us look at three examples. For example, for a nonsingular cubic, the dual curve is a sextic with 9 cusps. If a cubic has 1 node, the dual curve is a quartic with 3 cusps (and one bitangent). If a cubic has 1 cusp, the dual curve is a cubic with 1 cusp.

We shall now relax our assumptions on C and pass to generalizations of these formulas with use of the multiciplities of infinitely near singular points.

Let us record the following well-known result.

Theorem 84. *Let C be a curve on a nonsingular surface X. Then there is a nonsingular surface Y and a morphism $f\colon Y \to X$ which is a composition of blow-ups at some points such, that the proper preimage $C' \subset Y$ of C is nonsingular.*

See [46].

Suppose that $P \in C$ is a singular point. Set $X_0 = X$. Fix $i \geqslant 1$. For $j = 1, \dots, i$, we perform a sequence of blows-ups

$$\pi_j \colon X_j \to X_{j-1}\,,$$

where the first blow-up π_1 is the blow-up at P. (We note that for $j \geqslant 2$, π_j can consist of several blow-ups at different points, see below.) Let

$$f_i = \pi_i \circ \cdots \circ \pi_2 \circ \pi_1\,.$$

Denote by Y_i the proper preimage of C in X_i under f_i, and set $E_i = f_i^{-1}(P)$.

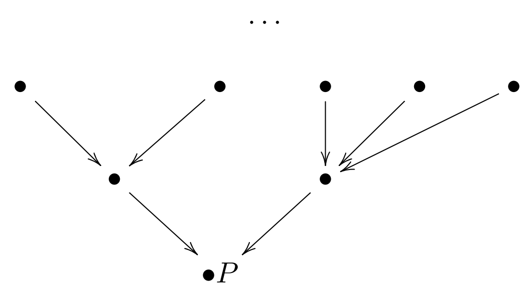
We now describe π_j by induction on j. For $j \geqslant 2$, suppose that X_j is obtained from X_{j-1} by the blow-ups of points from the set

$$E_{j-1} \cap Y_{j-1}\,.$$

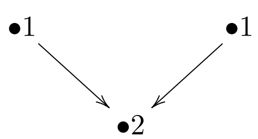
Then the points from the set $E_i \cap Y_i$ are called infinitely near singular points of (P, C) in X_i (relative to f_i).

Let us denote by $m_1, m_2, \dots$ the multiplicities of all infinitely near singular points of (C, P) (i.e., those in X_i for $i >> 0$).

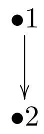
We have the tree of infinitely close points:



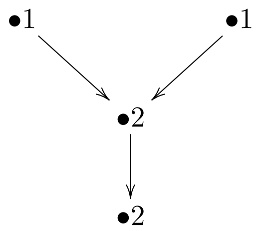
Here is the tree for a node: $y^2 = x^2$ (the attached numbers to vertices, are the multiplicities of the corresponding points),



Here is the tree for the cusp: $y^2 = x^3$:



Here is the tree for: $y^2 = x^2y^4 + x^4$,



We shall now, following Weierstrass and M. Noether, discuss generalizations of Plücker formulas with the help of the multiplicities of the infinitely near singular points.

Theorem 85. *Let C be a plane curve without multiple components and without lines as components. Suppose that* $\mathrm{Sing}(C) = \{P_1, \dots, P_k\}$*. For a fixed $i = 1, \dots, k$, let $m_{i1}, m_{i2}, \dots$ be the multiplicities of all infinitely near singular points of (C, P_i). Then we have the following two formulas:*

$$d^\vee = d(d-1) - \sum m_{ij}(m_{ij}-1) - \sum (m_r - 1)\,, \tag{48}$$

where the last sum is over all local components of C passing through singular points, and $\{m_r\}$ are the multiplicities of these local components;

$$\chi(C) = d(3-d) - \sum (c_i - 1) + \sum m_{ij}(m_{ij}-1)\,, \tag{49}$$

where c_i is the number of local components of C passing through P_i.

Before proving these formulas, we test them on examples, and discuss some consequences.

The formula (48) has as a corollary the formula (1) of Plücker: Let C be an irreducible plane curve of degree $d > 1$ having only simple singularities. Then we have

$$d^\vee = d(d-1) - 2\delta - 3\kappa\,.$$

Indeed both cusp and node contribute to $\sum m_{ij}(m_{ij}-1)$ with 2. The multiplicities of local components through a node are 1 and through cusp are 2, so the contribution of the last summand on the RHS of (48) is $-\kappa$.

Corollary 86. *Assume that C is irreducible. Then*

$$g(C) = \frac{(d-1)(d-2)}{2} - \frac{1}{2}\sum m_{ij}(m_{ij}-1)\,. \tag{50}$$

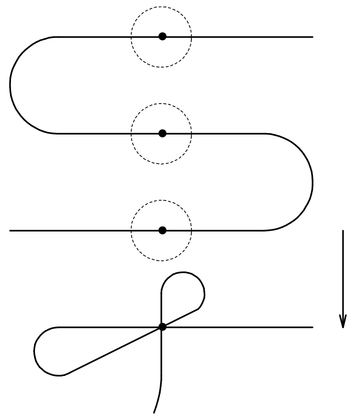
To see this, let $f\colon C' \to C$ be a desingularization of C. We have

$$\chi(C') = 2 - 2g(C)\,.$$

We also have

$$\chi(C') = \chi(C) + \sum (c_i - 1)$$

since C is gotten from C' by identifying c_i preimages of P_i corresponding to c local components of C passing through P_i.



The assertion follows. □

The formula (50) has as a corollary the formula (2) of Clebsch: Let C be an irreducible curve of degree d having only simple singularities. Then

$$g(C) = \frac{(d-1)(d-2)}{2} - \delta - \kappa\,.$$

Indeed both cusp and node contribute to $\sum m_{ij}(m_{ij}-1)$ with 2. In particular, a nonsingular cubic has genus 1, and if a cubic has either 1 cusp or 1 node, then its genus is 0.

Remark 87. It follows from the formula

$$g(C) = \frac{1}{2}(d-1)(d-2)$$

for a nonsingular curve C that only a triangular number can be its genus. What about singular curves? Let $r \in \mathbf{N}$ be arbitrary. Consider the curve defined by the equation:

$$y^2 = h(x) = x^{2r+2} + a_1 x^{2r+1} + \cdots + a_{2r+2}\,,$$

where $h(x)$ has no multiple roots (so this is the equation of a hyperelliptic curve). Its projectivization, in the coordinates $(x : y : z)$, is:

$$y^2 z^{2r} = x^{2r+2} + a_1 x^{2p+1} z + \cdots + a_{2r+1} z^{2r+2}\,.$$

This curve has only one singular point $P = (0, 1, 0)$, and the curve in its neighbourhood $(y \neq 0)$ is:

$$z^{2r} = x^{2r+2} + a_1 x^{2p+1} z + \cdots + a_{2r+1} z^{2r+2}\,.$$

The multiplicities of all infinitely near singular points of (C, P) are: $2r, 2, \ldots, 2$ (r times). By the formula (50), we have

$$g(C) = (2r+1)2r/2 - [2r(2r-1) + r \cdot 2(2-1)]/2 = r\,.$$

Remark 88. Let P be a singular point on a curve C lying on a nonsingular surface X. Denote by f the equation of C at P. Then one defines the Milnor number of P by

$$\mu = l(\mathcal{O}_{X,P}/(\partial f/\partial x, \partial f/\partial y))\,,$$

where x, y are local parameters at P. We have

$$\mu = \frac{1}{2}\sum m_i(m_i - 1)\,,$$

an expression in terms of the multiplicities of all infinitely near singular points of (C, P). The sum of such Milnor numbers evaluates the difference between the true and expected Euler characteristic of the curve. This property admits a generalization to singular hypersurfaces with isolated singularities and even non-isolated singularities.

The main tools to prove Theorem 85 are: polars and the Milnor fibre.

7.2. Polars

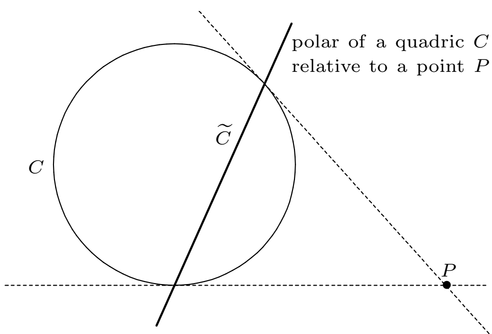
Suppose that $C \in \mathbf{P}^2$, in the coordinates $(x_0 : x_1 : x_2)$, is given by the equation $f(x_0, x_1, x_2) = 0$. The equation of the tangent line to C at a regular point (x_0, x_1, x_2) is:

$$\xi_0 \frac{\partial f}{\partial x_0} + \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} = 0\,.$$

Let us fix a point $P = (a_0, a_1, a_2) \in \mathbf{P}^2$. If P belongs to this tangent line, then

$$a_0 \frac{\partial f}{\partial x_0}(x_0, x_1, x_2) + a_1 \frac{\partial f}{\partial x_1}(x_0, x_1, x_2) + a_2 \frac{\partial f}{\partial x_2}(x_0, x_1, x_2) = 0\,.$$

Let us fix a_0, a_1, a_2; then this last condition defines a certain curve $\tilde{C}$ of degree $d - 1$. The curve $\tilde{C}$ is called the polar of the curve C relative to the point P. The intersection $C \cap \tilde{C}$ consists of those points whose tangents to C pass through P.



To be more precise: there are the points $Q_r \in C$, say, for which the line through Q_r and P intersects C with multiplicity greater than 1. Among them are the points Q_s, say, for which the line through Q_s and P is a tangent; the number of these points counted with the proper multiplicity is $d^\vee$. But singular points of C also appear. If P is sufficiently general, however, the tangents of C through singular points do not pass through P. Hence singular points of C need not be counted in computation of $d^\vee$. Thus we get the following formula:

$$d^\vee = (C \cdot \tilde{C}) - \sum_R (C \cdot \tilde{C})_R = d(d-1) - \sum_R (C \cdot \tilde{C})_R \,, \tag{51}$$

where R runs over the singular points of C. Our strategy will be to compute $(C \cdot \tilde{C})_R$ for a singular point R.

Remark 89. We invoke a (local) resolution of singularities by means of Puiseux expansion. Let $(Y,0) \subset (\mathbf{C}^2,0)$ be an irreducible germ of a curve defined by the equation $f(x,y)=0$. Then there exist power series $x(z)$ and $y(z)$ such that

$$f(x(z),y(z)) \equiv 0 \,.$$

It can be shown that we can take

$$x(z) = z^m \quad \text{and} \quad y(z) = \sum_{i \geqslant m} c_i z^i \,, \tag{52}$$

where m is the multiplicity of Y at 0.

The map

$$\pi \colon (\mathbf{C},0) \to (\mathbf{C}^2,0) \quad \text{such that} \quad z \mapsto (x(z),y(z))$$

is a resolution of the singularity $(Y,0)$.

We can rewrite (52) formally as

$$y = \sum_{i \geqslant m} c_i x^{i/m} \,. \tag{53}$$

Note that 0 is a nonsingular point on Y iff in the expansion (53) the coefficient c_i does not vanish only for those i which are multiples of m.

We shall use the following rule to compute the intersection multiplicity of two germs of curves. Suppose that $(Y,0)$ and $(Y',0)$ are two different irreducible curve germs in $(\mathbf{C}^2,0)$, defined respectively by f and g. If $\pi \colon (\mathbf{C},0) \to (Y,0)$ is the resolution of the singularity of Y by the Puiseux expansion, then $(Y \cdot Y')_0$ is equal to the degree of the map

$$g \circ \pi \colon (\mathbf{C},0) \to (\mathbf{C},0)$$

at the origin.

For details, see [8] pp. 376–395; and especially Theorem 1 on p.386.

We shall now investigate polar curves in affine coordinates. Let $C \subset \mathbf{P}^2$ be a curve, which in the coordinates $(x_0 : x_1 : x_2)$ is given by the equation $f(x_0,x_1,x_2) = 0$. Let $P = (0,0,1)$. We shall work in the affine coordinates: $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$. Let us denote by $f = f(x,y)$ the function $f(1,x,y)$, for brevity. The point P becomes now the point at infinity. The affine equation of the polar curve $\widetilde{C}$ reads

$$\frac{\partial f}{\partial y}(x,y) = 0 \,.$$

The pencil of lines passing through P is identified with the family of parallel lines $x = a$, $a \in \mathbf{C}$. Therefore the polar curve $\tilde{C}$ becomes the locus of points, where the curve

$$C_t : \ f(x, y) = t\,, \qquad t \in \mathbf{C}\,,$$

is singular, or has "vertical" tangent line $x = a$ for a fixed $a \in \mathbf{C}$.

We generalize now this presentation of the polar curve to families of curves on complex surfaces.

Let X be a nonsingular surface. Let $f \colon X \to \mathbf{C}$ be a holomorphic map, whose fibres $C_t = f^{-1}(t)$ are curves without multiple components. Suppose that $x \colon X \to \mathbf{C}$ denote a holomorphic map of rank 1 everywhere. Thus each fibre $x^{-1}(a)$, $a \in \mathbf{C}$, is a nonsingular curve.

The polar of the family $\{C_t\}$ relative to the family $\{L_a\}$ is the locus of points $Q \in X$, for which the curve C_t passing through Q is singular or tangent to the curve L_a passing through Q.

To give an analytic description of this polar, let us pick another independent holomorphic coordinate y on X. A point Q belongs to the polar, iff the curve C_t, passing through Q, is singular at Q: $\frac{\partial f}{\partial x}(Q) = \frac{\partial f}{\partial y}(Q) = 0$, or if C_t and L_a passing through Q are not transverse. These conditions are rephrased as follows: the Jacobian of f and x vanishes at Q:

$$\begin{vmatrix} \frac{\partial f}{\partial x}(Q) & \frac{\partial f}{\partial y}(Q) \\ 1 & 0 \end{vmatrix} = 0\,.$$

Thus we get the following local equation of the polar: $\frac{\partial f}{\partial y} = 0$.

Lemma 90. *Fix $Q \in X$. Let $C = C_{t_0}$ and $L = L_{a_0}$ be the curves from the families $\{C_t\}$ and $\{L_a\}$, passing through Q. Then*

$$(C \cdot \tilde{C})_Q - (C \cdot L)_Q$$

depends only on the singularity of C at Q, and not on the choice of families $\{C_t\}$ and $\{L_a\}$.

Proof. Let C_i be a component of C passing through Q. Let

$$\pi_i \colon (\mathbf{C}, 0) \to (C_i, Q)$$

be the resolution of singularities, which we shall use to compute intersection multiplicities of C_i with curves through Q according to the above rule (cf. Remark 89).

Suppose that instead of the family $\{C_t\}$ with local equation $f(x, y) = t$ and polar $\tilde{C}$, we consider the family of curves $\{D_s\}$ with local equation $g(x, y) = s$ such that the germ of the curve D_s through Q is equal to the germ (C, Q). We then have $g = u \cdot f$ in a neighborhood of Q, for some nonvanishing function u. Therefore the new polar $\tilde{D}$ has the equation

$$\frac{\partial g}{\partial y} = u \cdot \frac{\partial f}{\partial y} + f \cdot \frac{\partial u}{\partial y}\,.$$

Since $f \circ \pi_i = 0$, we thus obtain

$$\frac{\partial g}{\partial y} \circ \pi_i = u \cdot \frac{\partial f}{\partial y} \circ \pi_i \,,$$

which implies

$$(C_i \cdot \tilde{D})_Q = (C_i \cdot \tilde{C})_Q \,.$$

This shows the independence of the intersection numbers from the choice of the family $\{C_t\}$.

Suppose now that instead of the family $\{L_a\}$, we consider the family of nonsingular curves $\{M_b\}$, and denote by $\tilde{C}'$ the polar of the family $\{C_t\}$ relative to $\{M_b\}$. Since one can always compare each family with a third locally defined family whose curves cut both others transversally, we can assume without loss of generality that the curves $L = L_{a_0}$ and $M = M_{b_0}$ through Q meet transversally at Q. Then we can choose such local coordinates x, y in the neighborhood of Q that L is given by $x = a$ and M by $y = b$. If $f(x, y) = t$ is the equation of C_t, then $\frac{\partial f}{\partial y} = 0$ defines $\tilde{C}$ and $\frac{\partial f}{\partial x} = 0$ defines $\tilde{C}'$.

Suppose that $\pi_i(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are two power series. We have

$$f(x(t), y(t)) \equiv 0 \,,$$

which after differentiation w.r.t. t gives

$$\frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt} = 0 \,.$$

This last equality implies that

$$\deg(\frac{\partial f}{\partial x} \circ \pi_i) + \deg((x - a_0) \circ \pi_i) = \deg(\frac{\partial f}{\partial y} \circ \pi_i) + \deg((y - b_0) \circ \pi_i) \,.$$

If we interpret these degrees as intersection numbers, then it follows that

$$(C_i \cdot \tilde{C}')_Q + (C_i \cdot L_{a_0})_Q = (C_i \cdot \tilde{C})_Q + (C_i \cdot M_{b_0})_Q \,,$$

so the difference

$$(C \cdot \tilde{C})_Q - (C \cdot L_{a_0})_Q$$

of intersection numbers, does not depend on the choice of a family $\{L_a\}$. □

Proposition 91. *With the above notation, the following equality holds:*

$$(C_{t_0} \cdot \tilde{C})_Q - (C_{t_0} \cdot L_{a_0})_Q = \sum_{i \geqslant 0} m_i(m_i - 1) - c \,, \tag{54}$$

where $m_0, m_1, \ldots$ are the multiplicities of all infinitely near singular points of (C_{t_0}, Q), and c is the number of components of C_{t_0} passing through Q.

Proof. By previous lemma, we can assume, without loss of generality, that the tangent of any L_a does not coincide with the tangent of a component of C_{t_0}. Then, for any a, $(C_{t_0} \cdot L_a)_Q$ is the multiplicity m_0 of C_{t_0} at Q. The multiplicity of $\tilde{C}$ at Q is $m_0 - 1$.

We blow-up X at Q. Let x, y be local coordinates around Q such that $C = C_{t_0}$ has equation $f(x,y) = 0$, $L = L_{a_0}$ has equation $x = 0$ and $\tilde{C}$ has equation $\frac{\partial f}{\partial y}(x,y) = 0$. Let C' and $\tilde{C}'$ be the proper preimages of C and $\tilde{C}$ under the blow-up, and let E be the exceptional curve. The infinitely near points of (C, Q) on the blown-up surface, i.e. the points from the set $C' \cap E$, lie in coordinate neighborhood on the blown-up surface with coordinates u, v, where

$$x = u, \qquad y = uv.$$

The equation of E is: $v = 0$.

We want to interpret $\tilde{C}'$ as a polar of some family of curves relative to some family of nonsingular curves. If we expand f in a power series:

$$f(x,y) = \sum_{i+j \geqslant m_0} c_{ij} x^i y^j,$$

then C' has equation $g(u,v) = 0$, where

$$g = \sum_{i+j \geqslant m_0} c_{ij} v^{i+j-v_0} u^j \,.$$

Claim: *The curve $\tilde{C}'$ is the polar of the system of curves $g(u,v) = s$ relative to the system of nonsingular curves L'_b defined by $v = b$.*

Indeed, since the multiplicity of $\tilde{C}$ at Q is $m_0 - 1$, $\tilde{C}'$ has equation $h(u,v) = 0$, where

$$h(u,v) = v^{-(m_0-1)} \cdot \frac{\partial f}{\partial y}(v, uv) = \sum_{i+j \geqslant m_0} j c_{ij} v^{i+j-m_0} u^{j-1}.$$

Hence we get: $h = \frac{\partial g}{\partial u}$, which implies the claim.

To compute

$$(C', \tilde{C}')_R - (C', L'_{b_0})_R = (C', \tilde{C}')_R - (C', E)_R$$

for a singular point R from the set $C' \cap E$, we can therefore assume the formula (54) by induction. If we sum over all such points R, we get

$$(C' \cdot \tilde{C}') - (C' \cdot E) = \sum_{i>0} m_i(m_i - 1) - c\,,$$

and using $(C' \cdot E) = m_0$,

$$(C' \cdot \tilde{C}') = \sum_{i>0} m_i(m_i - 1) - c + m_0 \tag{55}$$

We now record the following identity:

$$(C \cdot \tilde{C})_Q = (C' \cdot \tilde{C}') + m_0(m_0 - 1)\,, \tag{56}$$

which can be obtained by the Noether method of computing the intersection numbers (see [8], p.519) - excercise.

The equalities (55) and (56) imply

$$(C \cdot \tilde{C})_Q = \sum_{i \geqslant 0} m_i(m_i - 1) + m_0 - c\,,$$

that is, the assertion of the proposition. □

Note that

$$m_0 - c = \sum (m_s - 1)\,,$$

where m_s are the multiplicities of the local components of C passing through Q.

We want now to prove the formula (48) for the degree of a dual curve. We recall the following expression for the class of C from (51):

$$d^\vee = d(d-1) - \sum (C \cdot \tilde{C})_{P_i}\,,$$

where $\mathrm{Sing}(C) = \{P_1, \ldots, P_k\}$. By the previous proposition and a the sentence after it, we have for fixed i,

$$(C \cdot \tilde{C})_{P_i} = \sum_j m_{ij}(m_{ij} - 1) + \sum (m_s - 1)\,,$$

where $m_{i1}, m_{i2}, \ldots$ are the multiplicities of all singular points infinitely near of (C, P_i), and $\{m_s\}$ are the multiplicities of the local components of C passing through P_i.

Summing up such contributions from all the singularities: $P_1, \ldots, P_k$, the formula (48) follows. □

7.3. Milnor fibre

We shall now work over $\mathbf{C}$. Consider a germ $(Y, 0)$ of a curve in $(\mathbf{C}^2, 0)$, defined by the equation $f(z_1, z_2) = 0$. Let B be a ball around 0 with sufficiently small radius.

Lemma 92. *For a sufficiently small $\delta > 0$, the restriction of $f \colon \mathbf{C}^2 \to \mathbf{C}$:*

$$f_| \colon B \cap \{(z_1, z_2) \in \mathbf{C}^2 : |f(z_1, z_2)| = \delta\} \to S_\delta$$

onto the circle $S_\delta = \{y \in \mathbf{C} : |y| = \delta\}$ is a locally trivial differentiable fibre bundle.

For a proof of this lemma, see [35]. The obvious analogue of this result is true for any hypersurface singularity (loc. cit.).

The fibre F of this fibration is called the Milnor fibre (of Y) attached to the origin.

Example 93. If $f = z_1^2 + z_2^2$, then the Milnor fibre attached to $(0,0)$ is the tangent bundle to the sphere S^1. For a node, the Milnor fibre is a cylinder. For a cusp, the Milnor fibre is a doted torus.

Remark 94. This notion allows us to study an important relation between the local complex analytic situation and global algebro-geometric situation. Let $f(z_1, z_2)$ be a polynomial without multiple factors. We consider a family of plane affine algebraic curves

$$C_t = \{(z_1, z_2) \in \mathbf{C}^2 : f(z_1, z_2) = t\}.$$

Suppose that C_0 has a singularity at the origin. If we choose B to be the ball around 0 in $\mathbf{C}^2$ of sufficiently small radius, and then we choose a sufficiently small $\delta > 0$ and t such that $|t| < \delta$, then the intersection $C_t \cap B$ is just the Milnor fibre attached to 0. It turns out that the Euler characteristic of this Milnor fibre can be used to determine the Euler characteristic of the special singular fibre from the Euler characteristic of the general nonsingular fibre in the family of curves, see Lemma 98.

This connection between the local and global situation will be used to prove the Noether formula (49).

Remark 95. The Milnor fibration admits a description with the help of a standard resolution of the singularity $(Y, 0) \subset (\mathbf{C}^2, 0)$. Let $\{E_j\}$ be the lines of the exceptional divisor of the resolution with multiplicities $\{e_j\}$. Suppose that the line E_j meets exactly r_j times other curves of the total preimage (that is, the components of the proper preimage of Y and the remaining lines $\{E_p\}_{p \neq j}$). Then we have the following expression for the Euler characteristic of the Milnor fibre:

$$\chi(F) = \sum e_j(2 - r_j). \tag{57}$$

(see [8], pp. 561–568).

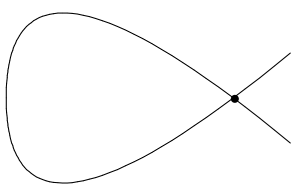
We now give a (local) formula for the computation of the Euler characteristic of the Milnor fibre using the multiplicities of infinitely near singular points.

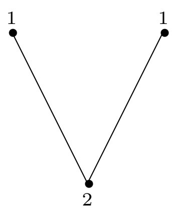
Theorem 96. *Suppose that $(Y, 0) \subset (\mathbf{C}^2, 0)$ is a germ of a curve with c irreducible components. Then we have the following formula for the Euler characteristic of the Milnor fibre attached to the origin:*

$$\chi(F) = c - \sum m_i(m_i - 1), \tag{58}$$

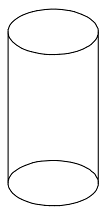
where m_i are the multiplicities of all infinitely near singular points of $(Y, 0)$.

Example 97. For a node, $\chi(F) = 0$

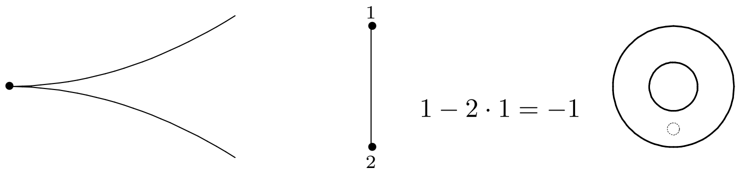




$2 - 2 \cdot 1 = 0$



and for a cusp, $\chi(F) = -1$



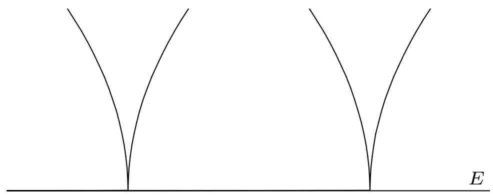
Proof. Suppose that the singularity Y is defined by the function $f = f(z_1, z_2)$.

If Y is nonsingular at 0, the formula holds true. We use induction on the number of blow-ups that are necessary to resolve the singularity of Y. We choose a small ball B around 0 and represent the Milnor fibre F attached to 0 by a nonsingular fibre $F = B \cap f^{-1}(t)$.

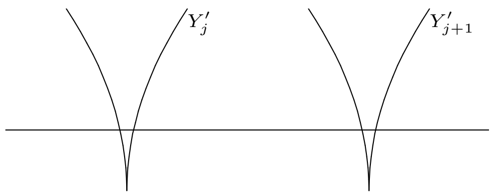
We blow up the point $0 \in \mathbf{C}^2$ once by

$$\pi \colon \tilde{X} \to \mathbf{C}^2 \,.$$

Assume that the proper preimage $\tilde{Y}$ of Y consist of components $\{Y_j\}$, and let E be the exceptional line $\pi^{-1}(0)$. Then $\tilde{Y}$ meets the exceptional line E with multiplicity $m = \mathrm{mult}_0(Y)$, that is, we have $\sum(Y_j \cdot E) = m$.



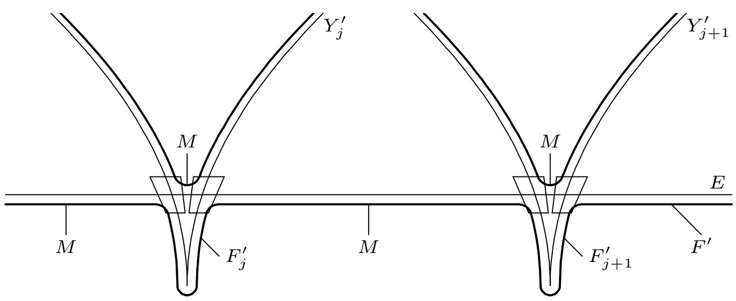
Moving the Y_j slightly, we obtain analytic sets Y_j' which meet E transversally in $(Y_j \cdot E)$ points and give analytic curves $\overline{Y_j}$ in B when blown down.



Let $\bar{f}$ be the function defining $\bar{Y} = \bigcup \bar{Y}_j$. Set

$$\bar{F} = \bar{f}^{-1}(t) \cap B \,.$$

If the deformations are chosen to be sufficiently small, then $\bar{F}$ is homeomorphic to F. The proper preimage F' of $\bar{F}$ is homeomorphic to $\bar{F}$, and hence also to F. The total preimage $E \cup \bigcup Y_j'$ looks like:



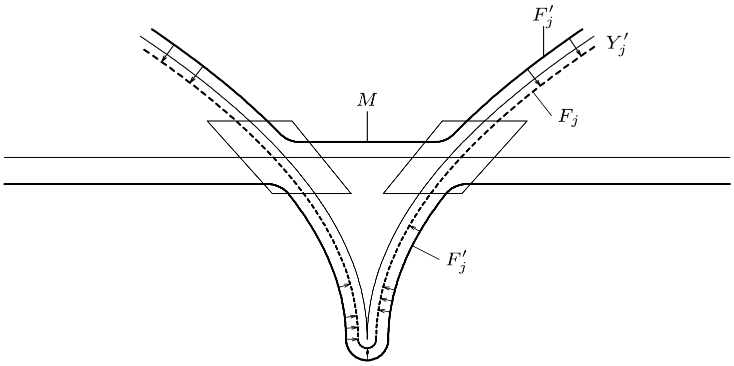
We choose a small polydisc around each of intersection points of Y_j' and E. F' is now composed of:

– a part M which covers the perforated line E;
– the intersection with the polydiscs;
– the parts of F' which lie over the Y_j'.

M is an m-tuple covering of the $[\sum(Y_j \cdot E) = m]$-tuply perforated projective line E. We thus have by Lemma 1,

$$\chi(M) = m(2-m)\,. \tag{59}$$

The intersections of F' with polydiscs are cylinders, so their Euler characteristics are equal to 0. It remains to compute the Euler characteristics of the parts F_j' of F' which lie over the Y_j'. Outside the polydiscs, F_j' is deformable into the Milnor fibre F_j of Y_j':



Hence F_j' is homeomorphic to the $(Y_j \cdot E)$-tuply perforated Milnor fibre F_j of Y_j', and also of Y_j. We thus have

$$\chi(F_j') = \chi(F_j) - (Y_j \cdot E)\,. \tag{60}$$

Summing up, we have

$$\begin{aligned}\chi(F') = \chi(M) + \sum \chi(F_j') &= -m^2 + 2m + \sum \chi(F_j) - \sum (Y_j \cdot E) \\ &= \sum \chi(F_j) - m(m-1)\,,\end{aligned}$$

where we have used the formulas (59), (60) and the identity: $\sum(Y_j \cdot E) = m$.

Since F' is homeomorphic to F, we obtain

$$\chi(F) = \sum \chi(F_j) - m(m-1)\,,$$

where F_j is the Milnor fibre of the component Y_j of the proper preimage of Y. The wanted formula (58) follows by induction. □

To complete the proof of the formula (49), one constructs a family $\{C_t\}$ of curves in the projective plane such that C_0 is isomorphic to C, and C_t is a nonsingular curve of degree d for $t \neq 0$.

More generally, let us consider a surface X equipped with a proper projection

$$f\colon X \to S = \{t \in \mathbf{C} : |t| < 1\}\,,$$

such that the special fibre $X_0 = f^{-1}(0)$ has singularities at $P_1, \ldots, P_k$, and for $t \neq 0$, the general fibre $X_t = f^{-1}(t)$ is a nonsingular. We choose a small ball B_i around each P_i and represent the Milnor fibre attached to P_i by a nonsingular fibre $F_i = B_i \cap f^{-1}(t)$.

We want to understand in which way the Euler characteristic of X_0 differs from that of X_t, and hope that this is encoded in the singularities P_i. In fact, this information is encoded in the Euler characteristics of the Milnor fibres F_i. We state

Lemma 98. *With this notation, we have*

$$\chi(X_0) = \chi(X_t) - \sum(\chi(F_i) - 1)\,.$$

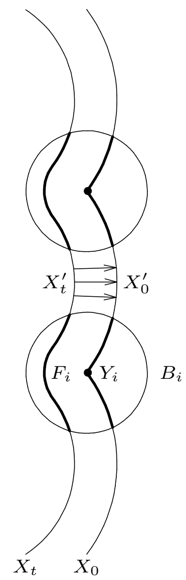
We note that $\chi(X_t)$ is known from the nonsingular case, and the $\chi(F_i)$'s are known from the previous theorem.

Proof. For a ball B, we set $B' = B - \partial B$, and define

$$X_t' = X_t - \bigcup B_i' \quad \text{and} \quad X_0' = X_0 - \bigcup B_i'\,.$$

We also set

$$Y_i = B_i \cap X_0 \quad \text{and} \quad F_i = B_i \cap X_t\,.$$



Here F_i is diffeomorphic to the Milnor fibre of the singularity of X_0 at P_i. The intersections $F_i \cap \partial B$ and $Y_i \cap \partial B$ are disjoint unions of circles, and hence have Euler characteristics equal to 0. By Lemma 1, we have

$$\chi(X_t) = \chi(X_t') + \sum \chi(F_i)$$

and

$$\chi(X_0) = \chi(X_0') + \sum \chi(Y_i)\,.$$

Since X_t' is homeomorphic to X_0', we have

$$\chi(X_t') = \chi(X_0').$$

Therefore we have

$$\chi(X_0) = \chi(X_t) - \sum(\chi(F_i) - \chi(Y_i)) = \chi(X_t) - \sum(\chi(F_i) - 1)$$

since Y_i is contractible (more precisely, Y_i is diffeomorphic to the cone over $Y_i \cap \partial B_i$ with vertex at 0 – cf. [8], Theorem 8 on p.426). □

End of the proof of the formula (49):

We have

$$\chi(C) = \chi(C_0) = \chi(C_t) - \sum(\chi(F_j) - 1)\,,$$

where F_j is the Milnor fibre of (C, P_j). Using the formula (58) and

$$\chi(C_t) = d(3-d)\,,$$

we thus obtain

$$\chi(C) = d(3-d) - \sum (c_i - 1) + \sum m_{ij}(m_{ij} - 1)\,,$$

as wanted. □

We finish this section with the following result about the genus of a singular curve on a nonsingular surface, generalizing the Adjunction formula (47) and the Noether formula (49).

Theorem 99. *Let C be a curve on a nonsingular surface X. Suppose that* $\operatorname{Sing} C = \{P_1, \dots, P_k\}$*. Let $m_{i1}, m_{i2}, \dots$ denote the multiplicities of all infinitely near singular points of (C, P_i), $i = 1, \dots, k$. Then*

$$g(C) = \frac{C \cdot (C + K_X)}{2} + 1 - \sum \frac{m_{ij}(m_{ij} - 1)}{2}\,.$$

This theorem is an example of a result "from local to global", and can be proved with the help of Chern classes.

8. Chern classes for singular varieties

In this section, we are not too systematic as for what concerns giving references. Consult a survey article by J-P. Brasselet [7] and the references therein.

8.1. Conjecture of Deligne and Grothendieck

We start with discussion of the Deligne–Grothendieck conjecture from 1969. We shall use constructible sets: when we want to compute the Euler characteristic of some topological space, we often take the partition of this space into locally closed subsets, or constructible subsets, and use the additivity property of χ. Recall that constructible sets are those obtained by finitely many unions, intersections and complements from closed subsets.

A function $\alpha\colon X \to \mathbf{Z}$ is called constructible if the set $\alpha^{-1}(n)$ is constructible for any $n \in \mathbf{Z}$. This equivalent to say that there exists a partition $X = \bigcup X_i$, X_i constructible, such that $\alpha_{|X_i} =$ constant. Constructible functions are a tool and not a subject of the theory of characteristic asses for singular varieties. The abelian group of constructible functions on X is denoted by $F(X)$.

For a subvariety $V \subset X$, we define a function $\mathbf{1}_V \in F(X)$ by the condition: $\mathbf{1}_V(x) = 1$ iff $x \in V$.

Note that the set $\{\mathbf{1}_V\}$, where V runs over subvarieties V of X, is a basis of $F(X)$ (exercise.)

For a proper map $f\colon X \to Y$, a subvariety $V \subset X$ and $y \in Y$, we set

$$f_*(\mathbf{1}_V)(y) = \chi(f^{-1}(y) \cap V)\,,$$

and extend this definition to arbitrary constructible function α by linearity.

Proposition 100. *There is a unique covariant functor F from compact complex algebraic varieties to abelian groups whose value on a variety X is $F(X)$ and whose value f_* on a map f is as defined above.*

Proof. For any constructible function α on a variety X, $f_*(\alpha)(y)$ is uniquely determined since the functions of the form $\mathbf{1}_V$ form a basis over the integers for $F(X)$. If $\{S_i\}$ is a stratification of X subordinate to both α and f, then

$$f_*(\alpha)(y) = \sum_i \alpha(S_i)\chi(S_i \cap f^{-1}(y))\,,$$

Note that stratification can be chosen in such a way that the closures of the strata are complex algebraic. Then we use the fact that χ of a stratified object is the sum of χ of the strata (see Proposition 2). Using this characterization of $f_*(\alpha)$ we also get:

– If α is constructible, then $f_*(\alpha)$ is constructible.
– If $g\colon W \to X$ is proper, then $(g \circ f)_*\alpha = g_*(f_*\alpha)$ for $\alpha \in F(W)$.

Use stratification theory and the multiplicativity of χ for fibre bundles (see Lemma 1). □

We shall work with complex analytic varieties. The characteristic classes considered here will be located in the singular homology groups for compact varieties, and in the Borel–Moore homology groups for any varieties. For algebraic varieties, we also may use the Chow groups. For these homology theories, a proper morphism $f\colon X \to Y$ induces a map $f_*\colon H(X) \to H(Y)$.

Conjecture 101. (Deligne–Grothendieck) *There exists a natural transformation from the functor F to homology which, on a nonsingular variety X, assigns to the constant function $\mathbf{1}$ the Poincaré dual of the total Chern class of X.*
This is equivalent to say that for any variety X there is a map

$$c_*\colon F(X) \to H(X)$$

such that

(1) $f_*c_*(\alpha) = c_*f_*(\alpha)$ *for* $\alpha \in F(X)$ *and proper map* $f\colon X \to Y$*;*
(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$*;*
(3) $c_*(\mathbf{1}_X) = c(TX) \cap [X]$ *for a nonsingular variety* X*.*

We set for a possibly singular variety X

$$c(X) := c_*(\mathbf{1}_X)\,.$$

Thus for a nonsingular variety X, $c(X) = c(TX) \cap [X] \in H_*(X, \mathbf{Z})$. (Warning: In the literature, we often have $c(X) = c(TX) \in H^*(X, \mathbf{Z})$.)

Proposition 102. *If such a c_* exists, then it must be unique.*

Proof. For any constructible function α on X, $c_*(\alpha)$ can be determined using a resolution of singularities. Indeed, we find maps f_j of nonsingular varieties X_j to X and integers m_j such that

$$\alpha = \sum m_j (f_j)_*(\mathbf{1}_{X_j})\,.$$

More precisely, assuming that we already have f_j for $j = 1, \ldots, i-1$, each $f_i\colon X_i \to X$ may be chosen as a resolution of an irreducible subvariety of the support of

$$\alpha - \sum_{j=1}^{i-1} m_j (f_j)_*(\mathbf{1}_{X_j})$$

of maximal dimension. Then $c_*(\alpha)$ must be $\sum m_i (f_i)_* c(X_i)$. □

Note that we even obtain the following nontrivial result about the Chern classes of nonsingular varieties.

Corollary 103. *For nonsingular X and X_i, if m_i and $f_i\colon X_i \to X$ are chosen so that for any $x \in X$,*

$$\sum m_i (f_i)_* c(X_i) = 1\,,$$

then

$$c(X) = \sum_i m_i (f_i)_* c(X_i)\,.$$

Finally, note that for any possibly singular variety X

$$\int_X c(X) = \chi(X)\,.$$

8.2. Nash blow-up

So the problem is how to define $c_i(X)$ without having the tangent vector bundle. The first idea is to try to use all limits of tangent spaces at regular points approaching a singular point. Let $\dim X = n$ and suppose that $X \subset M$, where M is a manifold of dimension m. Let $X^\circ \subset X$ be the set of regular points of X. Consider the map

$$\Phi\colon X^\circ \to G_n(TM) \quad \text{such that} \quad x \mapsto T_x X \subset T_x M\,.$$

Define $\tilde{X} = \overline{\Phi(X^\circ)}$, and denote by $\nu\colon \tilde{X} \to X$ the projection (The map ν is often called the "Nash blow-up".) Denote by $\tilde{T}$ the restriction to $\tilde{X}$ of the tautological rank n subbundle on $G_n(TM)$. (It can happen that $\tilde{X}$ is nonsingular but $\tilde{T} \neq T\tilde{X}$.)

Example 104. Assume that $X \subset M$ is a hypersurface, given around a point $x \in X \subset M$ by the vanishing of f. Let $z_1, \ldots, z_m$ be local coordinates around $x \in M$. Then

$G_{m-1}(TM) = \mathbf{P}(T^*M)$, the projectivization of the cotangent bundle of M, and we have

$$\Phi(x) = (x, \frac{\partial f}{\partial z_1}(x), \dots, \frac{\partial f}{\partial z_m}(x)) .$$

This suggests a connection with blow-up along the subscheme of M defined by the Jacobian ideal $(f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_m})$, i.e. along $\mathrm{Sing}(X)$.

Indeed, we have the following result that we state without proof (cf. [37]).

Lemma 105. *For a hypersurface $X \subset M$ the Nash blow-up $\tilde{X}$ is the proper image of X under the blow-up of M along* $\mathrm{Sing}(X)$.

Remark 106. In [37], the author discusses the following issues.
(i) The Nash blow-up ν is an isomorphism if and only if X is nonsingular.
(ii) Is it possible to get a resolution of singularities using a sequence of Nash blow-ups? As an example of Whitney's umbrella shows, we do not get, in general, a minimal desingularization (loc.cit.).

8.3. Mather classes

We set

$$c_M(X) := \nu_*(c(\tilde{T}) \cap [\tilde{X}]) \in H(X) ,$$

and call it the Mather class of X.

Remark 107. Suppose that there exists $\bar{X}$ together with a map $\mu\colon \bar{X} \to \tilde{X}$ such that $\bar{\nu} = \nu \circ \mu$ is birational and proper and there exists $\bar{\Omega}$ a locally free sheaf of rank n on X together with a surjection $\bar{\nu}^*(\Omega^1_X) \twoheadrightarrow \bar{\Omega}$. Then we have

$$c_M(X) = \overline{\nu}_*(c(\overline{\Omega}^*) \cap [\overline{X}]) .$$

(Use the projection formula.)

Example 108. Let X be an irreducible curve, and $f\colon \bar{X} \to X$ its desingularization. We get an exact sequence

$$f^*\Omega^1_X \to \Omega^1_{\bar{X}} \to \Omega^1_{\bar{X}/X} \to 0 .$$

We set

$$\bar{\Omega} = \mathrm{Im}(f^*\Omega^1_X \to \Omega^1_{\bar{X}}) = f^*\Omega^1_X/(\text{torsion}) ,$$

a line bundle on $\bar{X}$ since a torsion free rank 1 sheaf on a nonsingular curve is invertible. We have an exact sequence

$$0 \to \bar{\Omega} \to \Omega^1_{\bar{X}} \to \Omega^1_{\bar{X}/X} \to 0 , \qquad (61)$$

and the equality

$$c_M(X) = f_*(c(\bar{\Omega}^*) \cap [\bar{X}]) .$$

Denoting by $R = \sum \alpha_y[y]$ the ramification cycle of f, we obtain from the sequence (61)

$$c_M(X) = f_* c(\bar{X}) - \sum_y \alpha_y [f(y)] .$$

A natural question emerges: Does $c_* = c_M$ satisfy the Deligne–Grothendieck conjecture?

The answer to this question is negative. Consider the curve $X \subset \mathbf{P}^2$ defined by the equation $y^2 z = x^3$ with a cusp at $P = (0, 0, 1)$.

As topology doesn't see this singularity, we have $X \sim S^2$. The proper preimage $\tilde{X}$ of X under the blow-up of $\mathbf{P}^2$ at P is isomorphic to $\mathbf{P}^1$ and the canonical projection $\nu \colon \bar{X} \to X$ is a homeomorphism. Hence

$$\nu_* \mathbf{1}_{\tilde{X}} = \mathbf{1}_X .$$

Here $R = 1 \cdot [\nu^{-1}(P)]$. From the exact sequence

$$0 \to \bar{\Omega} \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}/X} \to 0 ,$$

we get $\deg(\tilde{\Omega}) = -3$. (This implies, in particular, that $\tilde{T} \neq T\tilde{X}$.) Since

$$\nu_* c_M(\mathbf{1}_{\tilde{X}}) = \nu_* c(\tilde{X}) = \nu_* c(\mathbf{P}^1) ,$$

we get $\int \nu_* c_M(\mathbf{1}_{\tilde{X}}) = 2$. By virtue of

$$c_M \nu_*(\mathbf{1}_{\tilde{X}}) = c_M(X) = \nu_*(c(\tilde{T}) \cap [\tilde{X}]) ,$$

we get $\int c_M \nu_*(\mathbf{1}_{\tilde{X}}) = 3$. Therefore

$$\nu_* c_M(\mathbf{1}_{\tilde{X}}) \neq c_M \nu_*(\mathbf{1}_{\tilde{X}}) .$$

So c_M needs to be modified to give the wanted transformation c_*.

Remark 109. Let $X \subset \mathbf{P}^m$ be an algebraic variety of dimension n. Suppose first that X is nonsingular. We define the ith polar variety by

$$M_i = \{x \in X : \dim(T_x X \cap V^{n-i+2}) \geqslant i - 1\} ,$$

where V^{n-i+2} is a general linear subspace of $\mathbf{P}^m$ of codimension $n - i + 2$. Note that $\operatorname{codim}_X M_i = i$.

Let $X \subset \mathbf{P}^3$ be a surface. Fix a point $P \in \mathbf{P}^3$ and line $l \in \mathbf{P}^3$. We have

$$M_1 = \{x \in X : P \in T_x X\} \quad \text{and} \quad M_2 = \{x \in X : l \subset T_x X\} .$$

Set $L = \mathcal{O}_{\mathbf{P}^m}(1)|_X$. There is the following formula of Eger and Todd. The component of $c(X)$ in $H_{n-j}(X, \mathbf{Z})$ is equal to

$$\sum_{i=0}^{j} (-1)^i \binom{n-j+1}{n-i+1} c_1(L)^{j-i} \cap [M_i] .$$

Reversing it, we obtain a formula for $[M_i]$ in terms of the Chern classes of X and $c_1(L)$. This formula may be proved using the Thom–Porteous formula, after introducing the bundle of principal parts, which is an algebraic counterpart of jets (see [39] and [40]) (exercise).

For a possibly singular projective variety X, we define the ith polar by

$$M_i = \overline{\{x \in X^\circ : \dim(T_x X \cap V^{n-i+2}) \geqslant i-1\}}\,,$$

where V^{n-i+2} is a general linear subvariety of $\mathbf{P}^m$ of codimension $n-i+2$. Piene [39] shows that the component of the Mather class $c_M(X)$ in $H_{n-j}(X,\mathbf{Z})$ is equal to

$$\sum_{i=0}^{j}(-1)^i\binom{n-j+1}{n-i+1}c_1(L)^{j-i}\cap[M_i]\,.$$

The MacPherson class will be a combination of the Mather classes of some subvarieties of X, that is the Mather class of some cycle on X. To find the coefficients in this cycle, we shall need some ingenious constructible function invented by MacPherson who associated it with name of Euler.

8.4. Local Euler obstruction

In this section we shall define, following R. MacPherson, a certain constructible function which together with the Mather classes will yield a solution of the Deligne–Grothendieck conjecture. This will be the obstruction to extend a certain section defined and nonzero outside of $\nu^{-1}(x)$ of the Nash bundle to a section on all $\tilde{X}$ vanishing nowhere.

Suppose that $x \in X \subset M$, where $\dim X = n$ and M is a complex manifold of dimension m. Let $z_1, \dots, z_m$ be holomorphic coordinates in M around x. We invoke the Nash blow-up $\nu\colon \tilde{X} \to X$ and the rank n bundle $\tilde{T}$ on $\tilde{X}$.

Consider $||z||^2 = \sum z_i \bar{z}_i$. Then $d||z||^2$ is a local section of $(TM)^*$ (real dual), and it induces a local section s of $\tilde{T}^*$.

Let B_ε (resp. S_ε) be a closed ball (resp. sphere) with radius ε.

Claim: For a sufficiently small ε, the section s does not vanish in any point of $\nu^{-1}(B_\varepsilon - 0)$.

This follows from the following result of Whitney, formulated using the local coordinates. If a regular point y tends to 0 along $\overline{0y}$, then the n-plane $\nu^{-1}y = T_y X$ is not orthogonal to $\overline{0y}$ for y sufficiently close to 0.

The obstruction $Eu(\tilde{T}^*, s)$ to extend s to an everywhere nonzero section on $\nu^{-1}(B_\varepsilon)$ lies in $H^{2n}(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon;\mathbf{Z})$. Indeed, for any $y \in \nu^{-1}B_\varepsilon$, $\tilde{T}^*_y$ is an oriented vector space of dimension $2n$ and $\pi_i(\tilde{T}^*_y - \{0\}) = 0$ for $i < 2n-1$, $\pi_{2n-1}(\tilde{T}^*_y - \{0\}) = \mathbf{Z}$. (A good reference to obstruction theory is [48].)

Let $\mathbf{O}$ be the fundamental class of orientation in $H_{2n}(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon;\mathbf{Z})$. Consider the composite map

$$H_{2n}(\tilde{X},\mathbf{Z}) \to H_{2n}(\nu^{-1}B_\varepsilon - \nu^{-1}S_\varepsilon, \mathbf{Z}) \simeq H_{2n}(\nu^{-1}B_\varepsilon, \nu^{-1}S_\varepsilon;\mathbf{Z})\,.$$

The class $[\tilde{X}]$ goes to $\mathbf{O}$ under this map.

We define the local Euler obstruction $X \to \mathbf{Z}$ by

$$Eu_X(x) =< Eu(\tilde{T}^*, s), \mathbf{O} > , \qquad (62)$$

where $<,>: H^{2n} \times H_{2n} \to \mathbb{Z}$ is the $\cap$-map.

Note that if $x \in X$ is regular, then $Eu_X(x) = 1$.

Now, MacPherson's definition of c_* goes as follows. Let $T: Z(X) \to F(X)$ be defined for a subvariety $V \subset X$ by

$$T([V]) = Eu_V .$$

To show that T is well defined, that is Eu_V is constructible one works with a stratification of the Nash blow-up ν and obstruction theory.

Lemma 110. *The transformation T is an isomorphism from the group of algebraic cycles to the group of constructible functions.*

Proof. To show the assertion, one works modulo subvarieties of dimension d and proceeds by descending induction on d. □

The transformation $c_*: F(X) \to H(X)$ is defined to be

$$c_* = c_M \circ T^{-1} ,$$

so it combines Mather classes and the local Euler obstruction. We also set $c(X) = c_*(\mathbf{1}_X)$.

Let X be a plane curve having a cusp P as the unique singular point (we continue previous considerations). We claim that

$$\nu_* c_M T^{-1}(\mathbf{1}_{\tilde{X}}) = c_M T^{-1} \nu_*(\mathbf{1}_{\tilde{X}}) .$$

Let us compute the degrees of both sides. Since $\tilde{X} = \mathbf{P}^1$ is nonsingular, $T^{-1}(\mathbf{1}_{\tilde{X}}) = \tilde{X}$, and so

$$\nu_* c_M T^{-1}(\mathbf{1}_{\tilde{X}}) = \nu_* c_M(\tilde{X}) .$$

Hence $\int LHS = 2$.

Since $Eu_X(P) = 2$ (see Proposition 112 below) and $Eu_X(Q) = 1$ for $Q \neq P$, we have

$$T^{-1}(\mathbf{1}_X) = X - P ,$$

the equality of cycles. Therefore $\int RHS$ is equal to

$$\int c_M T^{-1} \nu_*(\mathbf{1}_{\tilde{X}}) = \int c_M(X - P) = \int c_M(X) - 1 = \int \nu_*(c(\tilde{T}) \cap [\tilde{X}]) - 1 = 3 - 1 = 2 .$$

Theorem 111. (MacPherson 1974) *The map $c_*: F(X) \to H(X)$ satisfies the Deligne–Grothendieck conjecture. It is the unique such a transformation.*

Before proving the theorem, we shall discuss its various applications.

There are, in general, two methods of computing $c(X)$ for singular X. They follow from the proof of the theorem of MacPherson.

(i) Suppose that there exist proper maps $g_r\colon W_r \to X$ with nonsingular W_r, and $m_r \in \mathbf{Z}$ such that

$$\sum m_r (g_r)_*(\mathbf{1}_{W_r}) = \mathbf{1}_X\,.$$

Then we have

$$c(X) = \sum m_r (g_r)_* c(W_r)\,.$$

(ii) Suppose that

$$T(\sum n_r V_r) = \mathbf{1}_X$$

(that is, $\sum n_r Eu_{V_r}(x) = 1$ for any $x \in X$). Then we have

$$c(X) = \sum n_r (i_r)_* c_M(V_r)\,,$$

where $i_r\colon V_r \hookrightarrow X$ is the inclusion.

8.5. Computation of Eu for a curve

Let X be an irreducible curve, and $f\colon \bar{X} \to X$ its desingularization. Let $R = \sum \alpha_y[y]$ be the ramification cycle of f. In $H_0(X)$, we have

$$c_M = f_* c(\bar{X}) + \sum \alpha_y [f(y)]\,.$$

We shall now give relations between constructible functions on X. Let $x \in X$ be a singular point. Denote by c_x the number of local components of X through x. We have $c_x = \chi(f^{-1}(x)$ (see our study of the Noether formula of Plücker). We have

$$f_* \mathbf{1}_{\bar{X}} - \sum (c_x - 1)(i_x)_* \mathbf{1}_{\{x\}} = \mathbf{1}_X\,,$$

where $i_x\colon \{x\} \to X$ is the inclusion. By (i), we thus have in $H_0(X)$

$$c(X) = f_* c(\bar{X}) - \sum (c_x - 1)(i_x)_*[x]\,,$$

which together with the expression for the Mather class gives

$$c_M(X) - c(X) = \sum_x \Big(\sum_{y: f(y)=x} \alpha_y + c_x - 1\Big)(i_x)_*[x]\,. \tag{63}$$

Note that

$$c_x + \sum_{y: f(y)=x} \alpha_y$$

is the multiplicity of X at x.

We have the following tautological relation:

$$\mathbf{1}_X = Eu_X - (Eu_X - \mathbf{1}) = Eu_X - \sum_x (i_x)_*(Eu_X(x) - 1)[x]\,,$$

which by (ii) implies in $H_0(X)$

$$c(X) = c_M(X) - \sum_x (Eu_X(x) - 1)(i_x)_*[x]\,. \tag{64}$$

Comparing the expressions (63) and (64), we get

Proposition 112. *For a curve X and $x \in X$, we have*

$$Eu_X(x) = \mathrm{mult}_x X\,.$$

8.6. Properties of the local Euler obstruction

– If $x \in X$ and $y \in Y$, then

$$E_{X\times Y}(x,y) = Eu_X(x)\cdot Eu_Y(y)\,.$$

– If $X = \cup X_i$ at x, then

$$Eu_X(x) = \sum Eu_{X_i}(x)\,.$$

– There is an algebraic formula for Eu with the help of characteristic classes due to Gonzalez-Sprinberg and Verdier [19]:

$$Eu_X(x) = \int_{\nu^{-1}(x)} c(\tilde{T}) \cap s(\nu^{-1}(x), \tilde{X})\,.$$

Here $s(\nu^{-1}(P), \tilde{X})$ is the Segre class. More generally, for an imbedding $X \hookrightarrow Y$, consider the blow-up of Y along X with the exceptional divisor E, and the projection $\eta\colon E \to X$. Then we have the Segre class

$$s(X,Y) = \sum_{i\geqslant 1} (-1)^{i-1}\eta_*(E^i)$$

in $CH_*(X)$. It generalizes the Segre class for a vector bundle. This is one of the most important characteristic classes in algebraic geometry [13, 14].

– There is a formula using multiplicities of polar classes. Let X by an n-dimensional subvariety of $\mathbf{P}^m$. Then

$$Eu_X(x) = \sum_{i=0}^{n-1} \mathrm{mult}_x(M_i)\,.$$

– There is a formula for a hypersurface $X \subset \mathbf{P}^n$ having only isolated singularities:

$$Eu_X(x) = 1 + (-1)^n \mu(x, X\cap H)\,,$$

where H is a general hyperplane passing through P and we regard $X \cap H$ as a hypersurface in H.

Note that in this situation, there exist formulas for $\int c_M X$ and $\int c(X)$ in terms of the Euler characteristic of a nonsingular hypersurface (we already know a formula for it) and the so called Teissier numbers.

More precisely, we have

$$\int c_M(X) = \sum_{i=0}^{n-1}(-1)^{n-1-i}(i+1)(d-1)^{n-1-i}d - (-1)^{n-1}\sum(\mu_x^{(n)} + \mu_x^{(n-1)})$$

and

$$\int c(X) = \sum_{i=0}^{n-1}(-1)^{n-1-i}(i+1)(d-1)^{n-1-i}d - (-1)^{n-1}\sum \mu_x^{(n)}\,.$$

– Let X be a cone over a nonsingular curve in $\mathbf{P}^2$ of degree d, and x its vertex. Then

$$Eu_X(x) = 2d - d^2\,.$$

– There is a topological formula for Eu due to Dubson. Suppose that $X \subset \mathbf{C}^m$ is stratified by a Whitney stratification $\{S_1, \dots, S_r\}$. Then

$$Eu_X(x) = \sum_{i=0}^{r} \chi(S_i \cap B_\varepsilon \cap H) \cdot E_i\,,$$

where B_ε is a sufficiently small closed ball centered at x, H is a hyperplane passing sufficiently close to x (but not through x), and $E_i = Eu_X(y)$ for $y \in S_i$.

8.7. Proof of the theorem of MacPherson

It is sufficient to show that

$$c_M T^{-1} f_* = f_* c_M T^{-1}\,.$$

Claim *It suffices to prove this identity for maps $f\colon X \to Y$, where X is nonsingular variety and for the function identically one on X.*

We already know that any α is of the form $\sum m_i (f_i)_*(\mathbf{1}_{X_i})$, where the sources X_i of f_i are nonsingular. We have

$$f_* c_M T^{-1}(\alpha) = \sum m_i f_* c_M T^{-1}(f_i)_* \mathbf{1} = \sum m_i c_M T^{-1} f_* (f_i)_* \mathbf{1} = c_M T^{-1} f_*(\alpha)\,,$$

where the middle equality follows from the application of the result for f_i and from the functoriality of $(-)_*$.

Since for nonsingular X, $c_M(T^{-1}(\mathbf{1}) = c(X)$, it suffices to show

$$f_* c(X) = c_M(T^{-1} f_* \mathbf{1})\,.$$

This is equivalent to prove that for a cycle on Y, $\sum n_i V_i$, such that $T(\sum n_i V_i) = f_*(\alpha)$, we have

$$f_* c(X) = c_M(T^{-1} f_* \mathbf{1})\,.$$

By virtue of the definition of $f_*(\alpha)$, this assertion translates into the following statement.

It is sufficient to prove that for a proper map $f\colon X \to Y$ with X nonsingular, there exists an algebraic cycle $\sum m_j V_j$ on Y such that the following two properties hold:

(i) $f_*c(X) = \sum_j m_j (i_j)_* c_M(V_j)$, *where* $i_j \colon V_j \hookrightarrow Y$ *is the inclusion*;
(ii) $\chi(f^{-1}(y)) = \sum_j m_j Eu_{V_j}(y)$ *for any* $y \in Y$.

We invoke now the following MacPherson's graph construction. (This is an incarnation of the deformation to the normal cone). Suppose that X is a nonsingular variety. Let

$$f \colon X \to Y \subset M \,,$$

where M is a manifold. We have the differential $df \colon TX \to f^*TM$, a morphism of vector bundles on X. We shall write TM instead of f^*TM for short. We have the following imbedding of bundles on X

$$\mathrm{Hom}(TX, TM) \hookrightarrow G_n(TX \oplus TM) \,. \tag{65}$$

Denote by $R \subset TX \oplus TM$ the tautological rank n bundle on $G_n(TX \oplus TM)$.

For any $\lambda \in \mathbf{C}$, we consider the section $\lambda \cdot df$ of $\mathrm{Hom}(TX, TM)$. It induces a map

$$s_\lambda \colon X \to G_n(TX \oplus TM)$$

by (65). We set

$$W := \overline{\mathrm{Im}((s_\lambda \times 1) \colon X \times \mathbf{C} \to G_n(TX \oplus TM) \times \mathbf{P}^1)}$$

and

$$W_\infty := W \cap (G_n(TX \oplus TM) \times \{\infty\}) \,.$$

The cycle W_∞ is crucial for our reasoning. Suppose that

$$W_\infty = \sum n_j W_j \quad \text{on } G_n(TX \oplus TM) \times \{\infty\}.$$

Let $\pi \colon \mathrm{Supp}(W_\infty) \to X$ be the restriction of the projection $G_n(TX \oplus TM) \to X$. We set

$$V_j = f \circ \pi(W_j) \subset Y \,.$$

These are the subvarieties in the wanted cycle $\sum m_j V_j$. The coefficients m_j are, however, more subtle to define.

Let $\tilde{V}_j$ be the Nash blow-up (taken w.r.t. the imbedding $V_j \hookrightarrow M$), and let $\tilde{T}_j$ be the corresponding Nash bundle. Denote by Z_j the join of $\tilde{V}_j$ and W_j, that is, the closure in $\tilde{V}_j \times_{V_j} W_j$ of the inverse image of the open set of $\tilde{V}_j$ projecting isomorphically to V_j. Let $\rho_j \colon Z_j \to \tilde{V}_j$ denote the projection. Analyzing the geometry of Z_j, one can show the following result (cf. [33]).

Lemma 113. *We have the inclusion of vector bundles on* Z_j: $\tilde{T}_j \subset R$.

Granting this lemma, we define the integers r_j by the following equation:

$$(\rho_j)_*(c_{\mathrm{top}}(R/\tilde{T}_j) \cap [Z_j]) = r_j[\tilde{V}_j] \,.$$

The wanted cycle is now defined by $\sum m_j V_j$, where $m_j = n_j \cdot r_j$.

It is proved in [33] that this cycle satisfies the needed two properties (i) and (ii). □

8.8. Riemann–Roch for singular varieties

Let X be a (possibly singular) projective variety. Let $K^0(X)$ be the Grothendieck group of vector bundles on X, and $K_0(X)$ that of coherent sheaves on X. For nonsingular varieties X, the map

$$K^0(X) \to H_*(X, \mathbf{Q}) \quad \text{such that} \quad [\mathcal{F}] \mapsto \tau_X([\mathcal{F}])\,,$$

is a natural transformation of covariant functors, the fundamental ingredient being the Todd genus $td(X)$.

Theorem 114. (Baum–Fulton–MacPherson) *For a possible singular variety X, there exists a natural transformation of contravariant functors*

$$\tau \colon K_0 \to H_*(-, \mathbf{Q})$$

such that
(i) *the diagram*

$$\begin{array}{ccc} K^0(X) \otimes K_0 X & \xrightarrow{\otimes} & K_0 X \\ \downarrow{\scriptstyle ch\otimes\tau} & & \downarrow{\scriptstyle \tau} \\ H^*(X, \mathbf{Q}) \otimes H_*(X, \mathbf{Q}) & \xrightarrow{\cap} & H_*(X, \mathbf{Q}) \end{array}$$

is commutative;

(ii) *for a nonsingular X,*

$$\tau(\mathcal{O}_X) = td(X) \cap [X]\,.$$

For a proof, see [1] and [13].

9. Some possible continuations

9.1. Characteristic classes for real vector bundles

Let us now pass to topology. There is a natural formation of characteristic classes for real vector bundles, the Stiefel–Whitney classes, such that the ith class is in $H^i(M, \mathbf{Z}_2)$ for a vector bundle E on a differentiable manifold M. These classes were historically one of the first characteristic classes, and have important applications, e.g., to the questions of imbeddings of manifolds. If we want to have a larger ring of coefficients, we can still make a use of Chern classes and define the Pontrjagin classes. Let E be a real vector bundle on a differentiable manifold M. In order to use Chern classes, we take the complexification $E \otimes_{\mathbf{R}} \mathbf{C}$, and set

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbf{C}) \in H^{4i}(M, \mathbf{Z})\,.$$

(Note that $E \otimes_{\mathbf{R}} \mathbf{C} \simeq \overline{E \otimes_{\mathbf{R}} \mathbf{C}} = (E \otimes_{\mathbf{R}} \mathbf{C})^*$, which implies $2c_{2j+1}(E \otimes_{\mathbf{R}} \mathbf{C}) = 0$; we thus neglect the classes of odd degree.) Write

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots$$

Since $E \otimes_{\mathbf{R}} \mathbf{C} = E \oplus \overline{E}$, we have

$$p(E) = c(E)c(\overline{E})\,.$$

We deduce from the corresponding properties of Chern classes: the naturality and the additivity formula of Pontrjagin classes.

We want now to discuss the following question: Decide if $\mathbf{P}^4_{\mathbf{C}}$ can be differentiably imbedded in $\mathbf{R}^9$. We wish to approach this question using the Pontrjagin classes. We have

$$p(\mathbf{P}^4_{\mathbf{C}}) = c(T\mathbf{P}^4_{\mathbf{C}})c(\overline{T\mathbf{P}^4_{\mathbf{C}}}) = (1+h)^5(1-h)^5 = (1-h^2)^5\,.$$

If $\mathbf{P}^4_{\mathbf{C}}$ can be differentiably imbedded in $\mathbf{R}^9$, then there is an exact sequence

$$0 \to (T\mathbf{P}^4_{\mathbf{C}})_{\mathbf{R}} \to T\mathbf{R}^9|_{\mathbf{P}^4_{\mathbf{C}}} \to N \to 0\,,$$

where $(T\mathbf{P}^4_{\mathbf{C}})_{\mathbf{R}}$ is the realization of the holomorphic tangent bundle $(T\mathbf{P}^4_{\mathbf{C}})$ and N is the normal bundle of $(T\mathbf{P}^4_{\mathbf{C}})$ in $\mathbf{R}^9$.

We have, by additivity,

$$p(T\mathbf{R}^9|_{\mathbf{P}^4_{\mathbf{C}}}) = p\big((T\mathbf{P}^4_{\mathbf{C}})_{\mathbf{R}}\big) \cdot p(N)\,.$$

But $p(T\mathbf{R}^9|_{\mathbf{P}^4_{\mathbf{C}}}) = 1$ by naturality. Thus

$$p(N) = p(T\mathbf{P}^4_{\mathbf{C}})^{-1} = \frac{1}{(1-h^2)^5} = 1 + 5h^2 + 15h^4\,.$$

Since N is a line bundle over $\mathbf{R}$, the top component of $p(N)$ should be in $H^2(\mathbf{P}^4_{\mathbf{C}}, \mathbf{Z})$ – a contradiction.

If $\mathbf{P}^4_{\mathbf{C}}$ can be imbedded in $\mathbf{R}^n$ then the normal bundle has rank at least 4. It follows that $\mathbf{P}^4_{\mathbf{C}}$ cannot be imbedded in $\mathbf{R}^n$ where $n \leqslant 11$.

Here is another application of Pontrjagin classes.

In mathematics, cobordism is a fundamental equivalence relation on the class of compact manifolds of the same dimension, set up using the concept of the boundary of a manifold. Two manifolds are cobordant if their disjoint union is the boundary of a manifold one dimension higher.

Suppose that M is a compact oriented manifold of dimension $4n$. Consider the monomials labelled by partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n:

$$p^{\lambda}(M) = p^{\lambda_1}(M) \cdots p^{\lambda_n}(M)\,.$$

The following result is due to Thom.

Theorem 115. *Two compact, oriented manifolds M and N are cobordant if and only if*

$$p^\lambda(M) = p^\lambda(N)$$

for any λ.

We see again that we need to work with polynomials in characteristic classes and not only with single classes. A reference for this subsection is [36].

9.2. Arithmetic characteristic classes

This is a mixture of algebraic geometry and complex analysis. We work with arithmetic varieties and arithmetic Chow groups.

Arakelov geometry studies a scheme X over $\mathbf{Z}$, by putting Hermitian metrics on holomorphic vector bundles over $X(\mathbf{C})$, the complex points of X. This extra Hermitian structure is applied as a substitute, for the failure of the scheme Spec($\mathbf{Z}$) to be a complete variety.

An arithmetic cycle of codimension p is a pair (Z, g), where $Z \in Z^p(X)$ and g is a Green current for Z, a higher dimensional generalization of a Green function on a Riemann surface (used by Arakelov). Currents are operators on differential forms, induced by integration. The arithmetic Chow group $\widehat{CH}^p(X)$ is the abelian group of arithmetic cycles of codimension p, modulo some "trivial cycles", mimicking rational equivalence and suitable differentiation of currents.

Let $\overline{E}$ be a Hermitian vector bundle on X. One attaches to $\overline{E}$ characteristic classes with values in arithmetic Chow groups. Among them are arithmetic Chern classes $\widehat{c}_i(\overline{E})$. They satisfy naturality, but they fail to have the Whitney sum property (for short exact sequences). One needs the secondary Chern class, defined by Bott and Chern to control this failure.

This is an enormous machinery. Some explicit applications concern the height of projective varieties. Consider a system of diophantine equations with integral coefficients, which defines an arithmetic variety X in projective space $\mathbf{P}^n_{\mathbf{Z}}$. The Faltings height $h(X)$ of X is a measure of the arithmetic complexity of the system of equations. It is an arithmetic analog of the geometric notion of the degree of a projective variety. The height $h(X)$ generalizes the classical height of a rational point of projective space, used by Siegel, Northcott and Weil to study the questions of diophantine approximation. If $\overline{\mathcal{O}(1)}$ denotes the canonical Hermitian line bundle on $\mathbf{P}^n$, then we define

$$h(X) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}}(1))^{\dim(X)}|_X)\,.$$

We have

$$h(\mathbf{P}^n) = \frac{1}{2}\sum_{i=1}^{n} H_i\,,$$

where

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

is a harmonic number. Recently, the heights of some homogeneous spaces have been calculated using Schubert calculus. A reference for this subsection is [47].

9.3. Thom polynomials

Let $f\colon M \to N$ be a holomorphic map of manifolds. A point $x \in M$ is a singularity of f if df drops rank at x. We want to introduce a characteristic class which will measure the complexity of singularities. Thom introduced a polynomial for that purpose. For a singularity η, he proposed to look, for a general f, at the geometry of the cycle

$$\overline{\{x \in M : f \text{ has singularity } \eta \text{ at } x\}}\,.$$

The Poincaré dual of its class is the Thom polynomial of η. It depends of $\dim M + \dim N$ variables.

The case $\dim M = \dim N = 1$ and $\eta = A_1\colon z \mapsto z^2$ was already known to Riemann and Hurwitz in the 19th century! Let $f\colon M \to N$ be a holomorphic, surjective map of compact Riemann surfaces. For $x \in M$, we set

$$e_x := \text{number of branches of } f \text{ at } x.$$

Then the ramification divisor of f is equal to

$$\sum (e_x - 1)x\,.$$

The *Riemann–Hurwitz formula* asserts that

$$\sum_{x \in M} (e_x - 1) = 2g(M) - 2 - \deg(f)\big(2g(N) - 2\big)\,. \tag{66}$$

The right-hand side of Eq.(66) can be rewritten as

$$f^* c_1(N) - c_1(M)\,.$$

This is the simplest Thom polynomial. References for this subsection are [25] and [38].

9.4. Warning

The methods developed in this course do not solve all problems of enumerative geometry. Consider the following question: How many rational plane curves of degree d pass through $3d-1$ general points in $\mathbf{P}^2$? A solution of this problem needs ideas of quantum cohomology, see [17].

9.5. An interesting open problem

There are many open problems of different character. Here is one. Let F be the flag variety of complete flags in $\mathbf{C}^n$. This is an algebraic variety of dimension $n(n-1)/2$ equipped in a cellular decomposition consisting of $n!$ algebraic Schubert cells. The closures of these cells are Schubert varieties X_w indexed by permutations $w \in S_n$, and their classes form an additive basis of the cohomology ring $H^*(F, \mathbf{Z})$.

Problem: Give a combinatorial formula for the constants c^u_{wv}, appearing in the decomposition of the product

$$[X^w] \cdot [X^v] = \sum_u c^u_{wv} [X^u]$$

of Schubert classes. Note that for a Grassmannian, a similar question is solved thanks to the Littlewood–Richardson rule that was first discovered in representation theory.

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Piotr Pragacz

Piotr Pragacz jest matematykiem pracującym od 1981 roku w IM PAN w Warszawie. W swych badaniach naukowych zajmuje się geometrią algebraiczną. Od 2000 roku kieruje Zakładem Algebry i Geometrii Algebraicznej IM PAN. Przedtem w latach 1977–1981 pracował na UMK w Toruniu.

Piotr Pragacz studiował algebrę w Toruniu, geometrię algebraiczną w Warszawie, oraz kombinatorykę i geometrię enumeratywną w Paryżu u Alain Lascoux, którego uważa za swojego głównego nauczyciela. Habilitacja Piotra Pragacza dotyczyła geometrycznych i algebraicznych aspektów rozmaitości Schuberta oraz miejsc degeneracji morfizmów wiązek. Głównym jej wkładem było wprowadzenie i badanie P-ideałów miejsc degeneracji. Tej tematyce poświęcona jest książka [Book] napisana wspólnie z Williamem Fultonem. Spektrum zainteresowań matematycznych Piotra Pragacza obejmuje: geometrię algebraiczną ze szczególnym uwzględnieniem teorii przecięć: [15, 21, 32, [Book], 50]; algebraiczną kombinatoryką ze szczególnym uwzględnieniem funkcji symetrycznych: [13, 18, 34] oraz globalną teorię osobliwości ze szczególnym uwzględnieniem klas charakterystycznych rozmaitości osobliwych i wielomianów Thoma: [28, 38, 48, 49, 56, 57]. Terminy: “The Lascoux–Pragacz ribbon identity” (za W. Chen, 2004) oraz “The Sergeev–Pragacz formula” (za I. G. Macdonaldem w słynnej monografii *Symmetric Functions and Hall Polynomials*, 1995), weszły na stałe do terminologii algebraicznej. W 2000 roku Piotr Pragacz powołał do życia Seminarium Impanga, które stało się ogólnopolskim i międzynarodowym forum geometrii algebraicznej [O5]. Cytowana literatura: http://www.impan.pl/~pragacz/publications.htm