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Selected Topics in Intersection Theory

(an expanded abstract)

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Contents

We start with some preliminaries about homology and cohomology classes determined by subvarieties of complex varieties. Unless otherwise is stated, we shall work with (co)homology groups with integer coefficients.

1. Cycles and their classes

The cohomology groups form a graded ring, with the cup product $\cup$, and the homology groups form a module over the cohomology ring, by means of the cap product $\cap$. When X is a nonsingular complex projective variety of dimension n, it is an oriented real $2n$-manifold, and the group $H_{2n}(X)$ has a canonical generator $[X]$; the canonical map $H^i(X) \to H_{2n-i}(X)$, taking α to $\alpha \cap [X]$, often called the Poincaré duality map, is an isomorphism.

We have pushforward maps

$$f_* : H_i(X) \to H_i(Y)$$

and pullback maps

$$f^* : H^i(Y) \to H^i(X)$$

for a continuous map $f : X \to Y$. These are related by the projection formula

$$f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta)$$

for $\alpha \in H^i(Y)$ and $\beta \in H_j(X)$. If X and Y are nonsingular and projective, and one identifies homology with cohomology, one gets pushforward maps

$$f_* \, H^i(X) \to H^{i+2d}(Y)\,,$$

where $d = \dim(Y) - \dim(X)$.

A (closed) subvariety V of a projective variety X determines a class denoted $[V]$ in $H_{2r}(X)$, where r is the dimension of V. This can be extended to arbitrary subschemes V of pure dimension r, by setting

$$[V] = \sum m_i[V_i]$$

if V_i are the irreducible components of V and m_i is the multiplicity of V_i (i.e., the length of the local ring of V at the generic point of V_i). If X is nonsingular, by Poincaré duality, we have the class $[V]$ in $H^{2c}(X) = H_{2r}(X)$, where c is the codimension of V in X.

If $f\ X \to Y$ is a morphism of projective varieties, and V is a subvariety of X, with $W = f(V)$, then $f_*[V] = 0$ if $\dim(W) < \dim(V)$, and $f_*[V] = d{\cdot}[W]$ if the map from V to W is generically d to 1; in particular, $f_*[V] = [W]$ if f maps V birationally onto W. If f is a smooth morphism from X to Y (such as the projection of a projective bundle), and V is a subvariety of Y, then $f^*[V] = [f^{-1}(V)]$.

Subvarieties V and W of a smooth projective variety X are said to meet transversally if their intersection is a union of subvarieties $Z_1, \dots, Z_p$, with $\mathrm{codim}(Z_i, X) = \mathrm{codim}(V, X) + \mathrm{codim}(W, X)$ for each i, and the tangent space to Z_i at a general point is the transversal intersection of the tangent spaces to V and to W at that point. In this case

$$[V]{\cup}[W] = [Z_1] + \cdots + [Z_p]\,.$$

If c is any cohomology class on a projective variety X, the notation $\int_X c$ is often used to denote the image of $c{\cap}[X]$ by the degree mapping (push-forward) $H_*(X) \to H_*(pt) = \mathbf{Z}$. The intersection number $\int_X [V] \cup [W]$ of two subvarieties V and W of complementary dimension is often denoted $\langle V, W\rangle$.

If a projective X has a filtration

$$X = X_s \supset X_{s-1} \supset \cdots \supset X_0 = \emptyset$$

by closed algebraic subsets, and each $X_i \setminus X_{i-1}$ is a disjoint union of varieties $U_{i,j}$ each isomorphic to an affine space $\mathbf{C}^{n(i,j)}$, then the classes $[\overline{U}_{i,j}]$ of their closures give a basis for $H_*(X) = \oplus H_i(X)$ over $\mathbf{Z}$. (In particular, all odd groups $H_{2i+1}(X)$ must vanish.) These conditions apply to Schubert varieties in flag manifolds, which are the closures of Schubert cells.

It is also a general fact that if a connected algebraic group G acts on a variety X, the corresponding action on the cohomology is trivial, so $[g{\cdot}V] = [V]$ for a subvariety V and an element g in G. If G acts transitively on X, one can use this to make $g{\cdot}V$ meet a given subvariety W transversally.

A vector bundle E of rank d on a variety X has Chern classes $c_i(E)$ in $H^{2i}(X)$, with $c_0(E) = 1$ and $c_i(E) = 0$ if $i > d$. If $f\ Y \to X$ is a morphism, then

$$c_i(f^*E) = f^*(c_i(E))\,.$$

These classes satisfy the Whitney formula: if E' is a subbundle of E, with quotient bundle E'', then

$$c_i(E) = \sum_{j+h=i} c_j(E') \cup c_h(E'') .$$

If X is nonsingular, and s is a section of E that is transversal to the zero section, then $c_d(E) = [Z(s)]$, where $Z(s)$ is the zero locus of s.

In topology, we use also Stiefel–Whitney classes and Pontrjagin classes, associated with real vector bundles.

We end this section with a definition of the *Schur determinant.*

There is one for any sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers, and any sequence $c_0, c_1, \ldots$ of commuting elements in a ring. This is often written as a formal series

$$c = c_0 + c_1 + \ldots .$$

Usually λ is a *partition*, i.e.,

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0 ,$$

but occasionally the general case is needed. Usually one also has $c_0 = 1$. In any case, one sets $c_h = 0$ for $h < 0$. Define

$$\Delta_\lambda(c) := \det \left(c_{\lambda_i+j-i}\right)_{1\leqslant i,j\leqslant n} ,$$

i.e.,

$$\Delta_\lambda(c) = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+n-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{\lambda_n-n+1} & \cdots & c_{\lambda_n-1} & c_{\lambda_n} \end{vmatrix}$$

For example, $\Delta_{(n)}(c) = c_n$, and $\Delta_{(1,1)}(c) = {c_1}^2 - c_0c_2$, $\Delta_{(2,1)}(c) = c_1c_2 - c_0c_3$, and $\Delta_{(1,1,1)}(c) = {c_1}^3 - 2c_1c_2 + c_0c_3$. When $c_0 = 1$, the determinant is unchanged if zeros are added to the sequence λ. When c_r is the r^{th} complete symmetric polynomial in some variables $x_1, \ldots, x_l$, this is the *Schur polynomial*:

$$\Delta_\lambda(c) = s_\lambda(x_1, \ldots, x_l) .$$

More generally, when c_r is defined by

$$\sum_{r\geqslant 0} c_r = \frac{\prod_{j=1}^m (1+q_j)}{\prod_{i=1}^l (1-p_i)} ,$$

then $\Delta_\lambda(c)$ is a symmetric polynomial known variously as a *hook Schur polynomial*, *supersymmetric Schur polynomial* or *bisymmetric polynomial.*

2. Real vector bundles

Let us now pass to topology. There is a natural formation of characteristic classes for real vector bundles, the Stiefel–Whitney classes, such that the ith class is in $H^i(M, \mathbf{Z}_2)$ for a vector bundle E on a differentiable manifold M. These classes were historically one of the first characteristic classes, and have important applications, e.g., to the questions of embeddings of manifolds. If we want to have a larger ring of coefficients, we can still make a use of Chern classes and define the Pontrjagin classes. Let E be a real vector bundle on a differentiable manifold M. In order to use Chern classes, we take the complexification $E \otimes_{\mathbf{R}} \mathbf{C}$, and set

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbf{C}) \in H^{4i}(M, \mathbf{Z}) .$$

(Note that $E \otimes_{\mathbf{R}} \mathbf{C} \simeq \overline{E \otimes_{\mathbf{R}} \mathbf{C}} = (E \otimes_{\mathbf{R}} \mathbf{C})^*$, which implies $2c_{2j+1}(E \otimes_{\mathbf{R}} \mathbf{C}) = 0$; we thus neglect the latter classes.) Write

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots$$

Since $E \otimes_{\mathbf{R}} \mathbf{C} = E \oplus \overline{E}$, we have

$$p(E) = c(E)c(\overline{E}) .$$

We deduce from the corresponding properties of Chern classes: the naturality and the Whitney sum formula of Pontrjagin classes.

We want now to discuss the following question: Decide if $\mathbf{P}^4_{\mathbf{C}}$ can be differentiably imbedded in $\mathbf{R}^9$. We wish to approach this question using the Pontrjagin classes. We have

$$p(\mathbf{P}^4_{\mathbf{C}}) = c(T\mathbf{P}^4_{\mathbf{C}})c(\overline{T\mathbf{P}^4_{\mathbf{C}}}) = (1+h)^5(1-h)^5 = (1-h^2)^5 .$$

If $\mathbf{P}^4_{\mathbf{C}}$ can be differentiably imbedded in $\mathbf{R}^9$, then there is an exact sequence

$$0 \to (T\mathbf{P}^4_{\mathbf{C}})_{\mathbf{R}} \to T\mathbf{R}^9|_{\mathbf{P}^4_{\mathbf{C}}} \to N \to 0 ,$$

where $(T\mathbf{P}^4_{\mathbf{C}})_{\mathbf{R}}$ is the realization of the holomorphic tangent bundle $(T\mathbf{P}^4_{\mathbf{C}})$ and N is the normal bundle of $(T\mathbf{P}^4_{\mathbf{C}})$ in $\mathbf{R}^9$.

We have, by additivity,

$$p(T\mathbf{R}^9|_{\mathbf{P}^4_{\mathbf{C}}}) = p((T\mathbf{P}^4_{\mathbf{C}})_{\mathbf{R}}) \cdot p(N) .$$

But $p(T\mathbf{R}^9|_{\mathbf{P}^4_{\mathbf{C}}}) = 1$ by naturality. Thus

$$p(N) = p(T\mathbf{P}^4_{\mathbf{C}})^{-1} = \frac{1}{(1-h^2)^5} = 1 + 5h^2 + 15h^4 .$$

Since N is a line bundle over $\mathbf{R}$, the top component of $p(N)$ should be in $H^2(\mathbf{P}^4_{\mathbf{C}}, \mathbf{Z})$ – a contradiction.

If $\mathbf{P}^4_{\mathbf{C}}$ can be imbedded in $\mathbf{R}^n$ then the normal bundle has rank at least 4. It follows that $\mathbf{P}^4_{\mathbf{C}}$ cannot be imbedded in $\mathbf{R}^n$ where $n \leqslant 11$.

Here is another application of Pontrjagin classes.

In mathematics, cobordism is a fundamental equivalence relation on the class of compact manifolds of the same dimension, set up using the concept of the boundary of a manifold. Two manifolds are cobordant if their disjoint union is the boundary of a manifold one dimension higher.

Suppose that M is a compact oriented manifold of dimension $4n$. Consider the monomials labelled by partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n:

$$p^\lambda(M) = p^{\lambda_1}(M) \cdots p^{\lambda_n}(M)\,.$$

The following result is due to Thom.

Theorem 1. *Two compact, oriented manifolds M and N are cobordant if and only if*

$$p^\lambda(M) = p^\lambda(N)$$

for any λ.

3. Degeneracy loci

The classical period, which was investigation of matrices of forms with rank conditions, ended with Giambelli's work in the early part of this century.

The modern period started in the 1950's, when R. Thom considered a map $\varphi\, F \to E$ of complex vector bundles, of ranks m and l, on a variety (then a differentiable manifold, here a complex manifold) X. On open sets where the bundles are trivial, the map is given by a matrix of complex-valued functions on the base. Set

$$D_r(\varphi) = \{\, x \in X : \operatorname{rank}(\varphi(x)) \leqslant r \,\},$$

a subvariety (usually singular) of X. If the map φ is sufficiently generic, this should have codimension $c = (l-r)(m-r)$, and so should define a cohomology class $[D_r(\varphi)]$ in $H^{2c}(X)$.

The case considered classically, in our first lecture, is that when X is projective space $\mathbf{P}^N$, and $E = \oplus_{i=1}^{l} \mathcal{O}(p_i)$, $F = \oplus_{j=1}^{m} \mathcal{O}(-q_j)$, where $\mathcal{O}(t) = \mathcal{O}(1)^{\otimes t}$, with $\mathcal{O}(1)$ the standard line bundle on $\mathbf{P}^N$. An $l \times m$ matrix A determines a map φ (with A acting on the left on columns), and the locus we called V_r is just $D_r(\varphi)$. Since $H^{2d}(\mathbf{P}^N) = \mathbf{Z} \cdot [L]$, where L is a linear subspace of codimension d, knowing the cohomology class of $D_r(\varphi)$ is the same as knowing its degree.

Thom's basic observation was:

"There must be a polynomial in the Chern classes of E and the Chern classes of F that is equal to this class."

The reason is simple. Consider a "universal" situation. Let G_d be the Grassmannian of rank d subspaces of a large vector space. On $G_m \times G_l$ there are universal bundles E and F of ranks l and m, and a bundle

$$H = \operatorname{Hom}(F, E) \to G_m \times G_l\,,$$

with a universal map φ from F to E on H. (As is common, we omit notation for pullbacks of bundles, here for the maps from H to G_m and H to G_l.) On H, *all* cohomology classes are polynomials in these Chern classes, so the assertion is clear in this case. The given situation on X is the pullback of the universal map, for some differentiable map from X to H. And the classes and Chern classes pull back as needed, at least if the map is suitably generic.

Thom stated the problem: *Find such a polynomial.*

Porteous gave the answer in 1962 (although it was published only in 1971): Define c by the formula

$$c = c(E - F) = c(E)/c(F),$$

where $c(E) = 1 + c_1(E) + c_2(E) + \dots$ denotes the total Chern class, and the division is carried out formally. Then

$$[D_r(\varphi)] = \Delta_\lambda(c), \quad \text{where} \quad \lambda = ((l-r)^{m-r}).$$

Equivalently, by looking at the dual map $\varphi^*\ E^* \to F^*$, one can write

$$[D_r(\varphi)] = \Delta_\mu(c(F^*)/c(E^*)),$$

with $\mu = ((m-r)^{l-r})$.

For a generalization, we need double Schubert polynomials. These *double Schubert polynomials* are characterized (and can be calculated) by the following two properties:

(1) If $w = w_0 = n\, n-1 \dots 2\, 1$, i.e., $w(i) = n+1-i$ for $1 \leqslant i \leqslant n$, then

$$\mathfrak{S}_w = \mathfrak{S}_w(x_1, \dots, x_n, y_1, \dots, y_n) = \prod_{i+j \leqslant n} (x_i - y_j).$$

(2) If, for some i, $w(i) > w(i+1)$, and $v = w \cdot s_i$ interchanges the values of w in positions i and $i+1$, then

$$\mathfrak{S}_v = \partial_i\, \mathfrak{S}_w,$$

where ∂_i is the *divided difference operator* defined on polynomials by the formula

$$\partial_i\, P = \frac{P\ -\ s_i(P)}{x_i\ -\ x_{i+1}}$$

and $s_i(P)$ is obtained from P by interchanging x_i and x_{i+1}.

Note that the variables y_j act as scalars in this operation. Since any permutation can be obtained from w_0 by a sequence of such interchanges, (1) and (2) determine all double Schubert polynomials. It is a first fundamental fact about them that they are independent of choices. It is also a fact that one can use any n for which w is in S_n to start this procedure. (We work in $S_\infty = \bigcup S_n$.)

Suppose we are given a map $h : F \to E$ of vector bundles on a nonsingular variety X, and suppose each bundle comes with a partial flag of subbundles. Noting that giving a subbundle is equivalent to giving a quotient bundle, we may write this

$$F_1 \subset F_2 \subset \cdots \subset F_t \subset F \to E \twoheadrightarrow E_s \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1.$$

Let w be a permutation such that $w(i) < w(i+1)$ if i is not the rank of any E_a, and $w^{-1}(j) < w^{-1}(j+1)$ if j is not the rank of any F_b. Set

$$\Omega_w(h) = \{x \in X : \operatorname{rank}(F_b(x) \to E_a(x)) \leqslant \#\{i \leqslant \operatorname{rank}(E_a) : w(i) \leqslant \operatorname{rk}(F_b) \ \forall\ a, b\}.$$

Theorem. (*Fulton 1992*) *If the map h is sufficiently generic, then Ω_w has pure codimension $l(w)$, and $[\Omega_w] = \mathfrak{S}_w(x, y)$.*

Here the x's and y's are the Chern roots of E and F, taken in increasing order by ranks: $x_1, \ldots, x_a$ are the Chern roots for E_1, if $a = rk(E_1)$, $x_{a+1}, \ldots, x_{a'}$ are the Chern roots for $\operatorname{Ker}(E_2 \to E_1)$, etc. And $y_1, \ldots, y_b$ are the Chern roots for F_1, $y_{b+1}, \ldots, y_{b'}$ are the Chern roots for F_2/F_1, etc. The fact that $\mathfrak{S}_w$ is symmetric in the Chern roots of each of these bundles means that it can be expressed in terms of their Chern classes, and it is in this sense that the formula is to be interpreted.

The classical case of matrices of forms is recovered by taking

$$E_a = \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_a) \quad \text{and} \quad F_b = \mathcal{O}(-q_1) \oplus \cdots \oplus \mathcal{O}(-q_b),$$

so the upper left $a \times b$ submatrix of A gives the bundle map from F_b to E_a.

Now let us give a sketch of the proof of the theorem. First, it suffices to do the case where the bundles are completely filtered, and the bundle map is the identity:

$$F_1 \subset \cdots \subset F_n = F = E = E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_1.$$

To see this, given $F \to E$, look at the map $F \subset F \oplus E \twoheadrightarrow E$, where the first map is the graph of the map, and the second is the projection. Then pass to appropriate projective bundles over X to fill in all the steps. (Here one uses the fact that the pullback maps for projective bundles are injective on cohomology or Chow groups.)

Moreover, there is a universal case. Suppose we are given a vector bundle V of rank n and a filtration $F_1 \subset \cdots \subset F_{n-1} \subset V$ on X. Let $\mathcal{F} = Fl(V) \to X$ be the flag bundle, on which one has the tautological flag

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = V$$

of subbundles of (the pullback of) V. Let $Q_i = V/U_{n-i}$, so we have the situation

$$F_1 \subset \cdots \subset F_{n-1} \subset V \twoheadrightarrow Q_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow Q_1$$

on $\mathcal{F}$. It suffices to prove the formula in this case, because the previous case is obtained from this by a section $s : X \to \mathcal{F}$. On $\mathcal{F}$ we have, for any w in S_n, the universal subvariety

$$\Omega_w = \{\, x \in \mathcal{F} : \operatorname{rank}(F_q(x) \to Q_p(x)) \leqslant \#\{i \leqslant p : w(i) \leqslant q\}\}.$$

The given locus in X is $s^{-1}(\Omega_w)$. In fact, the question of whether a given situation is "generic" enough is transferred to the question of whether this section s is in sufficiently general position with respect to the universal subvariety Ω_w of $\mathcal{F}$.

The formula for the cohomology of a flag manifold generalizes to families. Let

$$x_i = c_1(\mathrm{Ker}(Q_i \to Q_{i-1})) = c_1(U_{n-i+1}/U_{n-i})\,.$$

Then

$$H^*(\mathcal{F}) = H^*(X)[x_1, \ldots, x_n]/(e_1(x) - c_1(V), \ldots, e_n(x) - c_n(V)).$$

This is easily proved from the construction of $\mathcal{F}$ from X by a sequence of projective bundles, using the standard presentation of the cohomology of a projective bundle over the cohomology of the base. Note that the operators ∂_i determine operators on this ring, with elements in $H^*(X)$ acting as scalars. Let $y_i = c_1(F_i/F_{i-1})$. Our goal is to prove that $[\Omega_w] = \mathfrak{S}_w(x, y)$ for all w. We need two lemmas:

Lemma. *We have* $[\Omega_{w_0}] = \prod_{i+j\leqslant n}(x_i - y_j)$.

This locus is the analogue of a point in the case of the flag manifold. It is the image of the tautological section from X to $\mathcal{F}$, that takes a point x to the flag $L_\bullet = F_\bullet(x)$. To compute this, one realizes Ω_{w_0} as the zero section of a vector bundle K of rank $n(n-1)/2$ on $\mathcal{F}$:

$$K = \mathrm{Ker}\Big(\bigoplus_{i=1}^{n-1} \mathrm{Hom}(F_i, Q_{n-i}) \to \bigoplus_{i=1}^{n-2} \mathrm{Hom}(F_i, Q_{n-i-1})\Big),$$

where the map takes a collection of homomorphisms from F_i to Q_{n-i} to the differences of the two canonical maps

$$F_i \subset F_{i+1} \to Q_{n-i-1} \quad \text{and} \quad F_i \to Q_{n-i} \twoheadrightarrow Q_{n-i-1}.$$

It is not hard to verify that the canonical section of K, given by the maps

$$F_i \subset V \twoheadrightarrow Q_{n-i}\,,$$

vanishes precisely (scheme-theoretically) on the locus Ω_{w_0}, and that its top Chern class is the indicated product.

Next we have to get from the class of this smallest locus to the classes of larger loci. One cannot hope to do this by a *morphism*, but one can look for a *correspondence*. Let $\mathcal{F}(n-i)$ be the partial flag bundle without the $(n-i)^{\text{th}}$ term, i.e., consisting of flags of subspaces of all dimensions except $n-i$. On $\mathcal{F}(n-i)$ there is a universal flag

$$U_1 \subset \cdots \subset U_{n-i-1} \subset U_{n-i+1} \subset \cdots \subset U_n = V,$$

where we have used the same notation as for these bundles as for their pullbacks to $\mathcal{F}$. There is a canonical projection $\mathcal{F} \to \mathcal{F}(n-i)$, which is the $\mathbf{P}^1$-bundle $\mathbf{P}(U_{n-i+1}/U_{n-i-1})$. We have a commutative Cartesian diagram, where all maps are projections of $\mathbf{P}^1$-bundles:

$$\begin{array}{ccc} \mathcal{F} \times_{\mathcal{F}(n-i)} \mathcal{F} & \xrightarrow{p_2} & \mathcal{F} \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{F}(n-i) \end{array}$$

Lemma. (1) *The map*

$$p_{1*} \circ p_2{}^* : H^{2d}(\mathcal{F}) \to H^{2d+2}(\mathcal{F})$$

is equal to ∂_i.

(2) *We have*

$$p_{1*} \circ p_2{}^*[\Omega_w] = [\Omega_{w \cdot s_i}]$$

if $w(i) > w(i+1)$.

Proof. Both parts of this lemma are rather easy. For the first, one just needs to know about the structure of $\mathbf{P}^1$-bundles. If U is a bundle of rank 2 on a variety Y, $\mathcal{O}(1)$ is the tautological quotient line bundle of U on $\mathbf{P}(U)$, $p : \mathbf{P}(U) \to Y$ is the projection, and $x = c_1(\mathcal{O}(1))$, then any cohomology class in $H^*(\mathbf{P}(U))$ has a unique expression of the form $\alpha x + \beta$, with α and β in $H^*(Y)$. And $p_*(\alpha x + \beta) = \alpha$. Part (1) follows easily from this. (Note that when U corresponds to U_{n-i+1}/U_{n-i-1}, then $\mathcal{O}(1)$ corresponds to U_{n-i+1}/U_{n-i}, so x corresponds to x_i, and $\partial_i(x_i) = 1$.)

Part (2) follows from the fact that, when $w(i) > w(i+1)$, p_1 maps $p_2{}^{-1}(\Omega_w)$ birationally onto $\Omega_{w \cdot s_i}$. This is a local question, so one is reduced to the case of the flag manifold itself. This is a simple calculation, using the "row echelon" descriptions of Schubert cells. It is also true that if $w(i) < w(i+1)$ then p_1 maps $p_2{}^{-1}(\Omega_w)$ into Ω_w, from which it follows that $p_{1*} \circ p_2{}^*[\Omega_w] = 0$. □

The theorem follows immediately from the two lemmas, together with the two properties that characterize Schubert polynomials.

4. Schubert calculus

Schubert calculus is a branch of algebraic geometry introduced in the nineteenth century by Hermann Schubert, in order to solve various counting problems of of enumerative geometry. It was a precursor of several more modern theories, for example characteristic classes, and, in particular, its algorithmic aspects are still of current interest. The objects introduced by Schubert are the Schubert cells, which are locally closed sets in a Grassmannian defined by conditions of incidence of a linear subspace in projective space with a given flag. Schubert classes are the classes of the closures of Schubert cells. The intersection theory of these classes, which can be seen as the product structure in the cohomology ring of the Grassmannian, in principle allows the prediction of the cases where intersections of cells results in a finite set of points; which are potentially concrete answers to enumerative questions. A supporting theoretical result is that the Schubert classes span the whole cohomology ring of Grassmannian. In detailed calculations the combinatorial aspects enter as soon as the cells have to be indexed. Lifted from the Grassmanian, which is a homogeneous space, to the general linear group that acts on it, similar questions are involved in the Bruhat decomposition and classification of parabolic subgroups (by block matrix). Putting Schubert's system on a rigorous footing is Hilbert's 15th problem.

We shall study 3 fundamental theorems: basis theorem, Pieri formula and determinantal formula in the case the classical Grassmannian and the Lagrangian Grassmannian as well.

Here is a very interesting open problem in Schubert calculus. Let F be the flag variety of complete flags in $\mathbf{C}^n$. This is an algebraic variety of dimension $n(n-1)/2$ equipped in a cellular decomposition consisting of $n!$ algebraic Schubert cells. The closures of these cells are Schubert varieties X_w indexed by permutations $w \in S_n$, and their classes form an additive basis of the cohomology ring $H^*(F, \mathbf{Z})$.

Problem: Give a combinatorial formula for the constants c^u_{wv}, appearing in the decomposition of the product

$$[X^w] \cdot [X^v] = \sum_u c^u_{wv}[X^u]$$

of Schubert classes. Note that for a Grassmannian, a similar question is solved thanks to the Littlewood–Richardson rule that was first discovered in representation theory.

Symmetric function play an important role in Schubert calculus. They also play a basic role for degeneracy loci. Among the symmetric functions, these are the supersymmetric functions which are especially important.

5. Supersymmetric functions

Let $\mathbb{A} = \mathbb{A}_n = (a_1, \ldots, a_n)$ and $\mathbb{B} = \mathbb{B}_m = (b_1, \ldots, b_m)$ be two sequences of independent variables. We say that $F \in \mathbf{Z}[\mathbb{A}, \mathbb{B}]$ is *supersymmetric* if F is symmetric in $\mathbb{A}$ and $\mathbb{B}$ separately and $F(a_1 = t, b_1 = -t)$ is independent on t (this variant of the independence condition will be used just temporarily to make the life easier by avoiding problems with signs). Here is a family of suppersymmetric polynomials which immediately comes to mind. Define $s_r(\mathbb{A}/\mathbb{B})$ as the coefficients of the series

$$\prod_{i=1}^{n}(1-a_i)^{-1}\prod_{j=1}^{m}(1+b_j) = \sum s_r(\mathbb{A}/\mathbb{B}).$$

We see that $s_r(\mathbb{A}/\mathbb{B})$ interpolate between $s_r(\mathbb{A})$ (complete homogeneous polynomials in $\mathbb{A}$) and $e_r(\mathbb{B})$ (elementary symmetric polynomials in $\mathbb{B}$), and they are supersymmetric. Also supersymmetric is therefore any Schur determinant

$$s_\lambda(\mathbb{A}/\mathbb{B}) := \det\big(s_{\lambda_i+j-i}(\mathbb{A}/\mathbb{B})\big)_{1\leqslant i,j\leqslant l(\lambda)}\,.$$

For example, $s_1(\mathbb{A}/\mathbb{B}) = s_1(\mathbb{A}) + e_1(\mathbb{B})$, $s_2(\mathbb{A}/\mathbb{B}) = s_2(\mathbb{A}) + s_1(\mathbb{A})e_1(\mathbb{B}) + e_2(\mathbb{B})$, $s_3(\mathbb{A}/\mathbb{B}) = s_3(\mathbb{A}) + s_2(\mathbb{A})e_1(\mathbb{B}) + s_1(\mathbb{A})e_2(\mathbb{B}) + e_3(\mathbb{B})$ etc , and $s_{21}(\mathbb{A}/\mathbb{B}) = s_2(\mathbb{A}/\mathbb{B})s_1(\mathbb{A}/\mathbb{B}) - s_3(\mathbb{A}/\mathbb{B})$.

Observe that, for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, $s_\lambda(\mathbb{A}/0)$ is the classical Schur S-polynomial denoted $s_\lambda(\mathbb{A})$ and originally defined by Jacobi in the form

$$\frac{1}{\Delta(\mathbb{A})}\begin{vmatrix} a_1^{\lambda_1+n-1} & a_1^{\lambda_2+n-2} & \dots & a_1^{\lambda_n} \\ a_2^{\lambda_1+n-1} & a_2^{\lambda_2+n-2} & \dots & a_2^{\lambda_n} \\ \vdots & \vdots & & \vdots \\ a_n^{\lambda_1+n-1} & a_n^{\lambda_2+n-2} & \dots & a_n^{\lambda_n} \end{vmatrix} = \sum_{w\in S_n} w\Big[\frac{a_1^{\lambda_1+n-1}a_2^{\lambda_2+n-2}\dots a_n^{\lambda_n}}{\Delta(\mathbb{A})}\Big],$$

where

$$\Delta(\mathbb{A}) = \prod_{i<j}(a_i - a_j)$$

and S_n acts on $a_1,\dots,a_n$ via permutations. This is perhaps a good moment to introduce some notation which will make many formulas more concise. Given elements $a_1,\dots,a_n$ of a commutative ring and $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbf{N}^n$, we set

$$a^\alpha := a_1^{\alpha_1} \cdot \dots \cdot a_n^{\alpha_n}\,.$$

For two sequences of integers $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$, we define

$$\alpha + \beta := (\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)\,.$$

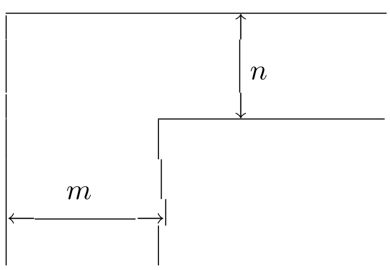
In particular, using this notation, the above formula becomes

$$s_\lambda(\mathbb{A}) = \sum_w w\Big[\frac{a^{\lambda+\rho(n-1)}}{\Delta(\mathbb{A})}\Big].$$

(Recall that one can always allow one or more zeros to occur at the end of a partition, and identify sequences that differ only by such zeros.)

It turns out that the **Z**-module of supersymmetric polynomials is freely generated by $\{s_\lambda(\mathbb{A}/\mathbb{B})\}$, where λ runs over the set of partitions whose diagrams D_λ are contained in the (n,m)-hook:

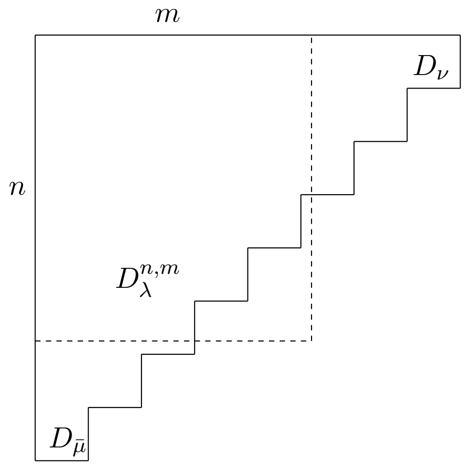
$D^{n,m} =$



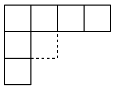
Perhaps the best way to see this, is via the following expression which, we will soon see, is equal to $s_\lambda(\mathbb{A}/\mathbb{B})$. Supposing that $D_\lambda \subset D^{n,m}$, set

$$F_\lambda(\mathbb{A}/\mathbb{B}) := \sum_{w\in S_n\times S_m} w\left[\frac{a^{\nu+\rho(n-1)}b^{\mu+\rho(m-1)}\prod\limits_{(i,j)\in D_\lambda^{n,m}}(a_i+b_j)}{\Delta(\mathbb{A})\ \Delta(\mathbb{B})}\right],$$

where ν, μ and $D_\lambda^{n,m}$ are defined by the following picture displaying D_λ:



For example, for $\lambda = (4,1,1)$ and $m = n = 2$, we have the picture:



and the above expression becomes

$$\frac{a_1^3 b_1^2 (a_1+b_1)(a_1+b_2)(a_2+b_1)}{(a_1-a_2)(b_1-b_2)} + \frac{a_2^3 b_1^2 (a_2+b_1)(a_2+b_2)(a_1+b_1)}{(a_2-a_1)(b_1-b_2)}$$

$$+\frac{a_1^3 b_2^2 (a_1+b_2)(a_1+b_1)(a_2+b_2)}{(a_1-a_2)(b_2-b_1)} + \frac{a_2^3 b_2^2 (a_2+b_2)(a_2+b_1)(a_1+b_2)}{(a_2-a_1)(b_2-b_1)}.$$

What one can say about this expression? Clearly F_λ is a polynomial which is homogeneous of degree $|\lambda|$ and symmetric in $\mathbb{A}$ and $\mathbb{B}$.

Exercise. Verify that the polynomial in the above example is supersymmetric.

Here is a general argument. Under the substitution $a_1 = -b_1 = t$, F_λ becomes a polynomial in t and the degree of t in $\Delta(\mathbb{A})\Delta(\mathbb{B})$ is $n+m-2$. Hence, if we show that the degree of t in the numerators of all $n!m!$ summands in F_λ cannot exceed $n+m-2$, then we know that $F_\lambda(a_1 = -b_1 = t)$ doesn't depend on t. Or, equivalently, it suffices to show that for

$$N = a^{\nu+\rho(n-1)} b^{\mu+\rho(m-1)} \prod_{(i,j)\in D_\lambda^{n,m}} (a_i + b_j),$$

the degree of t in $N(a_i = -b_j = t)$ does not exceed $n+m-2$ for $i = 1,\dots,n$, $j = 1,\dots,m$. Of course, if $(i,j) \in D_\lambda^{n,m}$ (matrix coordinates), then after the substitution we get zero. So suppose $(i,j) \notin D_\lambda^{n,m}$. This clearly implies $j > \lambda_i$ and $i > \widetilde{\lambda}_j$ (in particular, $\nu_i = 0$ and $\mu_j = 0$). The factors in N containing a_i or b_j are

$$(a_i + b_1), \dots, (a_i + b_{\lambda_i}), (a_1 + b_j), \dots, (a_{\widetilde{\lambda}_j} + b_j), a_i^{n-i}, b_j^{m-j}$$

and the degree of t is therefore

$$\lambda_i + \widetilde{\lambda}_j + (n-i) + (m-j) \leqslant n + m - 2.$$

Note that if $D_\lambda^{n,m}$ is the entire $n \times m$ rectangle, we have the following factorization

$$F_\lambda(\mathbb{A}/\mathbb{B}) = s_\nu(\mathbb{A}) s_\mu(\mathbb{B}) \prod_{i,j}(a_i + b_j).$$

Using an induction argument, one shows the following result.

Proposition. *$\{F_\lambda(\mathbb{A}/\mathbb{B})\}$, where λ runs over the set of partitions with $D_\lambda \subset D^{n,m}$, is a basis of the* **Z***-module of supersymmetric polynomials.*

We give a sketch of the proof. First one observes that for the cases $n = 0$ or $m = 0$ starting the induction, the assertion is a simple consequence of the Jacobi presentation of a Schur polynomial and the fact that multiplication by $\Delta(\mathbb{A})$ establishes an isomorphism of the **Z**-modules of alternating and symmetric polynomials in $\mathbb{A}$. Then one shows, using the above factorization formula, that those F_λ, where $D_\lambda^{n,m}$ is the full $n \times m$-rectangle, form a basis of the **Z**-module of supersymmetric polynomials which vanish for $a_n = b_m = 0$. To perform the induction step from $(n-1, m-1)$ to (n, m), one takes an arbitrary supersymmetric polynomial $P = P(\mathbb{A}_n, \mathbb{B}_m)$ and substitutes $a_n = b_m = 0$ in P, thus obtaining a polynomial $P' = P'(\mathbb{A}_{n-1}, \mathbb{B}_{m-1})$ which is supersymmetric in $\mathbb{A}_{n-1}$ and $\mathbb{B}_{m-1}$. By noticing that the polynomial $P - P'$ is supersymmetric (in $\mathbb{A}_n$ and $\mathbb{B}_m$) and vanishes under the substitution $a_n = b_m = 0$, one proves the induction assertion by using the induction assumption and the above characterization of supersymmetric polynomials which vanish for $a_n = b_m = 0$.

Finally, we show that

$$F_\lambda(\mathbb{A}/\mathbb{B}) = s_\lambda(\mathbb{A}/\mathbb{B})$$

which is, of course, our main goal. To this end, we can assume that $n >> 0$ (in fact, $n \geqslant |\lambda|$ will do the job). Indeed, letting in either expression some last variables be zero, we get the analogous polynomials associated with λ but depending only on the preceding variables. By the proposition, there exist integers a_σ such that

$$s_\lambda(\mathbb{A}/\mathbb{B}) = \sum_\sigma a_\sigma F_\sigma(\mathbb{A}/\mathbb{B}),$$

where all partitions σ have n parts at most. Therefore, setting now all the y's to be zero, we do not lose any summand on the right-hand side. But then, invoking the Jacobi presentation of $s_\lambda(\mathbb{A})$ again, we get $a_\lambda = 1$ and $a_\sigma = 0$ for all $\sigma \neq \lambda$.

We have paid so much attention to this formula because it is a key result in the theory of supersymmetric polynomials. It implies immediately

$$s_{(m^n)}(\mathbb{A}/\mathbb{B}) = \prod_{i,j}(a_i + b_j),$$

and, as we have already noticed, the following factorization formula: for any partition λ with $D_\lambda \subset D^{n,m}$, if $D_\lambda^{m,n}$ is the entire $n \times m$-rectangle,

$$s_\lambda(\mathbb{A}/\mathbb{B}) = s_\nu(\mathbb{A}) s_\mu(\mathbb{B}) s_{(m^n)}(\mathbb{A}/\mathbb{B}).$$

Exercise. *Show that if $D_\lambda \not\subset D^{n,m}$, then $s_\lambda(\mathbb{A}_n/\mathbb{B}_m) = 0$.*

Observe that the formula implies also, for any partition λ, the following duality formula

$$s_\lambda(\mathbb{A}/\mathbb{B}) = s_{\widetilde{\lambda}}(\mathbb{B}/\mathbb{A}).$$

Moreover, it "incorporates" one of the central results on Schur S-functions, namely the Littlewood–Richardson rule for multiplying two Schur functions.

Let us now come back to our initial condition of independence under the substitution $a_1 = t = b_1$ imposed by geometry. This is obtained by a change of sign in the basic generating function. Define $s_r(\mathbb{A} - \mathbb{B})$ to be the coefficients of the power series

$$\prod_{i=1}^{n}(1-a_i)^{-1}\prod_{j=1}^{m}(1-b_j) =: \sum_r s_r(X - \mathbb{B}),$$

and $s_\lambda(\mathbb{A} - \mathbb{B})$ as the Schur determinant

$$s_\lambda(\mathbb{A}-\mathbb{B}) := \Delta_\lambda(1 + s_1(\mathbb{A}-\mathbb{B}) + s_2(\mathbb{A}-\mathbb{B}) + \ldots) = \det\big(s_{\lambda_i+j-i}(\mathbb{A}-\mathbb{B})\big).$$

Of course the family $\{s_\lambda(\mathbb{A}-\mathbb{B})\}$, where λ runs over partitions with $D_\lambda \subset D^{n,m}$, form a basis of the **Z**-module of polynomials P symmetric separately in $\mathbb{A}$ and $\mathbb{B}$ and such that the specialization $P(a_1 = t = b_1)$ yields a polynomial independent of t. The polynomials $s_\lambda(\mathbb{A}-\mathbb{B})$ are often called *Schur polynomials in a difference of alphabets.*

Given two vector bundles E and F on a variety X, we define

$$s_\lambda(E-F) := \Delta_\lambda(s(E-F)) = \Delta_\lambda(s(E)/s(F))$$

$$\text{and}\quad \Delta_\lambda(E-F) := \Delta_\lambda(c(E-F)) = \Delta_\lambda(c(E)/c(F)).$$

In other words, letting $\mathbb{A}$, $\mathbb{B}$ be the sequences of Chern roots of E and F, we have $s_\lambda(E - F) = s_\lambda(\mathbb{A}-\mathbb{B})$. We also set

$$s_\lambda(E) := \Delta_\lambda\big(s(E)\big).$$

For given two partitions $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_r)$ we denote by λ, μ their juxtaposition, which is the sequence $(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_r)$. From the above we get, for vector bundles E and F of respective ranks n and m, the following formulas:

$$c_{mn}(E \otimes F^\vee) = s_{(m^n)}(E-F) = \Delta_{(n^m)}(E-F);$$

the *factorization formula*: for a partition $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_1 \leqslant m$ and a partition $\nu = (\nu_1, \ldots, \nu_n)$,

$$\begin{aligned} s_{((m^n)+\nu),\mu}(E-F) &= s_\nu(E)s_\mu(-F)s_{(m^n)}(E-F) \\ &= (-1)^{|\mu|}s_\nu(E)s_{\widetilde{\mu}}(F)s_{(m^n)}(E-F); \end{aligned}$$

and the *duality formula*: for any partition λ,

$$s_\lambda(E-F) = (-1)^{|\lambda|} s_{\widetilde{\lambda}}(F-E) = \Delta_{\widetilde{\lambda}}(E-F).$$

The symmetric functions studied in this section are useful tools to study degeneracy loci and Thom polynomials.

6. Arithmetic intersection theory

This is a mixture of algebraic geometry and complex analysis. We work with arithmetic varieties and arithmetic Chow groups.

Arakelov geometry studies a scheme X over $\mathbf{Z}$, by putting Hermitian metrics on holomorphic vector bundles over $X(\mathbf{C})$, the complex points of X. This extra Hermitian structure is applied as a substitute, for the failure of the scheme $\mathrm{Spec}(\mathbf{Z})$ to be a complete variety.

An arithmetic cycle of codimension p is a pair (Z, g), where $Z \in Z^p(X)$ and g is a Green current for Z, a higher dimensional generalization of a Green function on a Riemann surface (used by Arakelov). Currents are operators on differential forms, induced by integration. The arithmetic Chow group $\widehat{CH}^p(X)$ is the abelian group of arithmetic cycles of codimension p, modulo some "trivial cycles", mimicking rational equivalence and suitable differentiation of currents.

Let $\overline{E}$ be a Hermitian vector bundle on X. One attaches to $\overline{E}$ characteristic classes with values in arithmetic Chow groups. Among them are arithmetic Chern classes $\widehat{c}_i(\overline{E})$. They satisfy naturality, but they fail to have the Whitney sum property (for short exact sequences). One needs the secondary Chern class, defined by Bott and Chern to control this failure.

This is an enormous machinery. Some explicit applications concern the height of projective varieties. Consider a system of diophantine equations with integral coefficients, which defines an arithmetic variety X in projective space $\mathbf{P}^n_{\mathbf{Z}}$. The Faltings height $h(X)$ of X is a measure of the arithmetic complexity of the system of equations. It is an arithmetic analog of the geometric notion of the degree of a projective variety. The height $h(X)$ generalizes the classical height of a rational point of projective space, used by Siegel, Northcott and Weil to study the questions of diophantine approximation. If $\overline{\mathcal{O}(1)}$ denotes the canonical Hermitian line bundle on $\mathbf{P}^n$, then we define

$$h(X) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}}(1))^{\dim(X)}|_X)\,.$$

We have

$$h(\mathbf{P}^n) = \frac{1}{2}\sum_{i=1}^{n} H_i\,,$$

where

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

is a harmonic number. Recently, the heights of some homogeneous spaces have been calculated using Schubert calculus. We shall present these calculations.

7. Thom polynomials

A prototype of the formulas which we shall consider is the following classical result. Let $f : M \to N$ be a holomorphic, surjective map of compact Riemann surfaces. For $x \in M$, we set

$$e_x := \text{number of branches of } f \text{ at } x.$$

Then the ramification divisor of f is equal to

$$\sum (e_x - 1)x\,.$$

The *Riemann–Hurwitz formula* asserts that

$$\sum_{x\in M} (e_x - 1) = 2g(M) - 2 - \deg(f)(2g(N) - 2)\,.$$

The right-hand side of the equation can be rewritten as

$$f^*c_1(N) - c_1(M)\,,$$

and gives us the Thom polynomial of the singularity A_1 of maps between curves.

Fix $m, n, p \in \mathbf{N}$. Consider the space $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ of p-jets of analytic functions from $\mathbf{C}^m$ to $\mathbf{C}^n$ which map 0 to 0. Consider the natural right-left action of the group $\mathrm{Aut}^p_m \times \mathrm{Aut}^p_n$ on $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$, where Aut^p_n denotes the group of p-jets of automorphisms of $(\mathbf{C}^n, 0)$. By a *singularity class* we shall mean a closed algebraic right-left invariant subset of $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$. Given complex analytic manifolds M^m and N^n, a singularity class $\Sigma \subset \mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ defines the subset $\Sigma(M, N) \subset \mathcal{J}^p(M, N)$, where $\mathcal{J}^p(M, N)$ is the space of p-jets from M to N.

Theorem. *Let $\Sigma \subset \mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ be a singularity class. There exists a universal polynomial $\mathcal{T}^\Sigma$ over $\mathbf{Z}$ in $m+n$ variables $c_1, \dots, c_m, c'_1, \dots, c'_n$ which depends only on Σ, m and n such that for any complex analytic manifolds M^m, N^n and for almost any map $f : M \to N$, the class of*

$$\Sigma(f) := f_p^{-1}(\Sigma(M, N))$$

is equal to

$$\mathcal{T}^\Sigma(c_1(M), \dots, c_m(M), f^*c_1(N), \dots, f^*c_n(N)),$$

where $f_p : M \to \mathcal{J}^p(M, N)$ is the p-jet extension of f.

This is a theorem due to Thom.

If a singularity class Σ is *stable* (e.g. closed under the contact equivalence, then $\mathcal{T}^\Sigma$ depends on the virtual vector bundle $TM - f^*TN$.

Let $f : M \to N$ be a map of complex analytic manifolds. We shall work with the tangent map

$$f_* : TM \to f^*TN$$

and also with cotangent one

$$f^* : f^*T^*N \to T^*M\,.$$

Given a partition λ, we define

$$s_\lambda(f^*TN - TM)$$

to be the effect of the following specialization of $S_\lambda(\mathbb{A}-\mathbb{B})$: the indeterminates of $\mathbb{A}$ are set equal to the Chern roots of f^*TN, and the indeterminates of $\mathbb{B}$ to the Chern roots of TM.

Given a singularity class Σ, the Poincaré dual of $\Sigma(f)$, for almost any map $f : M \to N$, will be written in the form

$$\sum_\lambda \alpha_\lambda s_\lambda(f^*TN - TM)$$

with integer coefficients α_λ.

Accordingly, we shall write

$$\mathcal{T}^\Sigma = \sum_\lambda \alpha_\lambda S_\lambda\,,$$

where S_λ is identified with $S_\lambda(\mathbb{A}-\mathbb{B})$ for the universal Chern roots $\mathbb{A}$ and $\mathbb{B}$.

For example, consider the singularity class $\Sigma = \Sigma^i$. So, $m - i \leqslant n$, and we have

$$\mathcal{T}^{\Sigma^i} = S_{(i^{n-m+i})}\,,$$

the Giambelli–Thom–Porteous formula.

A basic result on Schur function expansions of Thom polynomials of singularity classes is

Theorem. *Let Σ be a nontrivial stable singularity class. Then for any partition λ, the coefficient α_λ in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^\Sigma = \sum \alpha_\lambda S_\lambda$$

is nonnegative and $\sum_\lambda \alpha_\lambda > 0$.

The original proof used the classification space of singularities and the Fulton–Lazarsfeld theorem.

Sketch of proof of the theorem First, using some Veronese map, we "materialize" all singularity classes in sufficiently large Grassmannians.

We fix a singularity class Σ and take the Schur function expansion of $\mathcal{T}^\Sigma$. We take sufficiently large Grassmannian containing Σ and such that specializing $\mathcal{T}^\Sigma$ in the Chern classes of the tautological (quotient) bundle Q, we do not lose any Schur summand.

We identify by the Giambelli formula a Schur polynomial of Q with the corresponding Schubert cycle.

To test a coefficient in the Schur function expansion of $\mathcal{T}^{\Sigma}$, we intersect $[\Sigma]$ with the corresponding *dual* Schubert cycle. Using the Bertini–Kleiman theorem we put the cycles in a general position, so that we can reduce to set-theoretic intersection, which is nonnegative. □

Note. If $\alpha_\lambda \neq 0$, then we shall say that λ belongs to the indexing set of the Schur function expansion of $\mathcal{T}^{\Sigma}$, or that the partition λ appears in the Schur function expansion of $\mathcal{T}^{\Sigma}$, or just λ appears in $\mathcal{T}^{\Sigma}$.

We record now a variant valid for *not necessary* stable singularity classes.

Theorem. *Let Σ be a nontrivial singularity class. Then for any partitions α, μ, the coefficient $\alpha_{\lambda,\mu}$ in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^{\Sigma} = \sum \alpha_{\lambda,\mu} s_\lambda(T^*M) s_\mu(f^*TN)$$

is nonnegative, and $\sum_{\lambda,\mu} \alpha_{\lambda,\mu} > 0$.

(It is important that we use the cotangent bundle to the source M and the tangent bundle to the target N.) The latter result implies the former.

More generally, it is natural to consider the $\mathcal{P}$-ideal of a singularity class Σ, denoted by $\mathcal{P}^{\Sigma}$. This is the subset in the polynomial ring $\mathbf{Z}[c_1, \dots, c_m, c'_1 \dots, c'_n]$, consisting of all polynomials P which satisfy the following universality property. For any complex analytic manifolds M^m, N^n and almost any map $f : M \to N$,

$$P(c_1(M), \dots, c_m(M), f^*c_1(N), \dots, f^*c_n(N))$$

is supported on $\Sigma(f)$ that the class of a cycle on M in $H(M)$ is in the image of $H(\Sigma(f)) \to H(M)$.)

For $\Sigma = \Sigma^i$, $\mathcal{P}^{\Sigma}$ is simply the ideal of polynomials which – after specialization to the Chern classes of M and N – support cycles in the locus D, where

$$\dim \operatorname{Ker}(f_* : TM \to f^*TN) \geqslant i\,.$$

for almost any map $f : M \to N$. (This means that the class of a cycle on M in $H(M)$ is in the image of $H(D) \to H(M)$.)

Note that in terms of the cotangent map, D is the locus where

$$\operatorname{rank}(f^* : f^*T^*N \to T^*M) \leqslant m - i\,,$$

for almost any map $f : M \to N$.

Of course, the component of minimal degree of $\mathcal{P}^{\Sigma}$ is generated over $\mathbf{Z}$ by $\mathcal{T}^{\Sigma}$. Usefulness of $\mathcal{P}$-ideals come from the following observation. Suppose that $\Sigma \subset \Sigma'$, where Σ' is another singularity class. Then $\mathcal{T}^{\Sigma}$ belongs to $\mathcal{P}^{\Sigma'}$. Thus if one knows the algebraic structure of $\mathcal{P}^{\Sigma'}$, one can use it to compute $\mathcal{T}^{\Sigma}$. In this way, the degeneracy loci of the cotangent map appear to be useful objects to study Thom polynomials.

Set $\mathcal{P}^i := \mathcal{P}^{\Sigma^i}$. We know the algebraic structure of $\mathcal{P}^i$, i.e., a certain *finite* set of its algebraic generators and its $\mathbf{Z}$-basis The arguments combine geometry of Grassmann bundles with algebra of Schur functions.

Before proceeding further, let us state the following result which is rather useful to compute the Schur function expansions of Thom polynomials.

Theorem. *Suppose that a stable singularity class Σ is contained in Σ^i. Then all summands in the Schur function expansion of $\mathcal{T}^\Sigma$ are indexed by partitions containing (i^{n-m+i}).*

Thus the partitions not containing this rectangle cannot appear in the Schur function expansion of $\mathcal{T}^\Sigma$.

Let $\mathbb{A}$ and $\mathbb{B}$ be two alphabets such that

$$\sum c_i = \prod_{a\in\mathbb{A}}(1+a) \quad \text{and} \quad \sum c'_j = \prod_{b\in\mathbb{B}}(1+b).$$

We have

Proposition. *No nonzero $\mathbf{Z}[c_1,\dots,c_m]$-linear combination of the Schur functions $S_\lambda(\mathbb{A}-\mathbb{B})$'s, where all λ's do not contain (i^{n-m+i})), belongs to $\mathcal{P}^i$.*

The idea of the proof is to interpret $\mathcal{P}^i$ as a "generalized resultant", and use some specialization trick.

Thus, in particular, no nonzero $\mathbf{Z}$-linear combination of the $s_\lambda(\mathbb{A}-\mathbb{B})$'s, where all λ's do not contain (i^{n-m+i}), belongs to $\mathcal{P}^i$.

Also, we have

Proposition. *Any $S_\lambda(\mathbb{A}-\mathbb{B})$, where λ contains (i^{n-m+i}) belongs to $\mathcal{P}^i$.*

The idea of the proof is to use a desingularization of D in the product of two Grassmann bundles, and apply appropriate pushforward formulas.

We are now ready to justify the theorem. Since Σ is contained in Σ^i, the Thom polynomial $\mathcal{T}^\Sigma$ belongs to $\mathcal{P}^i$. By the stability assumption, the Thom polynomial $\mathcal{T}^\Sigma$ is a (unique) $\mathbf{Z}$-linear combination of the $s_\lambda(\mathbb{A}-\mathbb{B})$'s. The two propositions imply that only Schur functions indexed by partitions containing the rectangle (i^{n-m+i}) appear in this sum. □

In the computations of Thom polynomials, it is convenient to "split" them into pieces supported on the consecutive degeneracy loci of the cotangent map. Let $\mathcal{T}$ be the Thom polynomial of a singularity class. By the *h-part* of $\mathcal{T}$ we mean the sum of all Schur functions appearing in $\mathcal{T}$ (multiplied by their coefficients) such that the corresponding partitions satisfy the following condition: λ contains the rectangle partition (h^{n-m+h}), but it does not contain the larger diagram $((h+1)^{n-m+h+1})$. The polynomial $\mathcal{T}$ is a sum of its h-parts, $h=1,2,\dots$.

We shall mostly study Thom polynomials of singularities.

Let $l \geqslant 0$ be a fixed integer and $\bullet \in \mathbf{N}$. Two stable germs $\kappa_1, \kappa_2 : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+l}, 0)$ are said to be right-left equivalent if there exist germs of biholomorphisms φ of $(\mathbf{C}^\bullet, 0)$ and ψ of $(\mathbf{C}^{\bullet+l}, 0)$ such that $\psi \circ \kappa_1 \circ \varphi^{-1} = \kappa_2$. A suspension of a germ is its trivial unfolding:

$(x, v) \mapsto (\kappa(x), v)$. Consider the equivalence relation (on stable germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+l}, 0)$) generated by right-left equivalence and suspension. A *singularity* η is an equivalence class of this relation.[1]

According to Mather's classification, singularities are in one-to-one correspondence with finite dimensional (local) $\mathbf{C}$-algebras. We shall use the following notation of Mather:

– A_i will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geqslant 0$;

– $I_{a,b}$ (of Thom–Boardman type $\Sigma^{2,0}$) for stable germs with local algebra

$$\mathbf{C}[[x, y]]/(xy, x^a + y^b), \quad b \geqslant a \geqslant 2;$$

– $III_{a,b}$ (of Thom–Boardman type $\Sigma^{2,0}$) for stable germs with local algebra

$$\mathbf{C}[[x, y]]/(xy, x^a, y^b), \quad b \geqslant a \geqslant 2 \quad \text{(here } l \geqslant 1\text{)}.$$

With a singularity η, there is associated Thom polynomial $\mathcal{T}^\eta$ in the formal variables $c_1, c_2, \ldots$ which after the substitution of c_i to

$$c_i(f^*TN - TM) = [c(f^*TN)/c(TM)]_i\,,$$

for a general map $f : M \to N$ between complex analytic manifolds, evaluates the Poincaré dual of $[\eta(f)]$, where $\eta(f)$ is the cycle carried by the closure of the set

$$\{x \in M : \text{the singularity of } f \text{ at } x \text{ is } \eta\}\,.$$

By codim(η), we mean the codimension of $\eta(f)$ in X.

Codimensions of above singularities are as follows:

– A_i associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+l}, 0))$, where $i \geqslant 0$ and $l \geqslant 0$ has codimension $(l + 1)i$.

– $I_{a,b}$ associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+l}, 0))$, where $b \geqslant a \geqslant 2$ and $l \geqslant 0$ has codimension $(l + 1)(a + b - 1) + 1$.

– $III_{a,b}$ associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+l}, 0))$, where $b \geqslant a \geqslant 2$ and $l \geqslant 1$ has codimension $(l + 1)(a + b - 2) + 2$.

Let $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+l}, 0)$ be a prototype of a singularity η. It is possible to choose a maximal compact subgroup G_η of the *right-left symmetry group*

$$\operatorname{Aut}\kappa = \{(\varphi, \psi) \in \operatorname{Aut}_n \times \operatorname{Aut}_{n+l} : \psi \circ \kappa \circ \varphi^{-1} = \kappa\}\,,$$

such that images of its projections to the factors Aut_n and Aut_{n+l} are linear [2]. That is, projecting on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+l}$, we obtain representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$. Let E'_η and E_η denote the vector bundles associated with the universal principal

[1] A singularity corresponds to a single R–L orbit.
[2] By Aut_n we mean here the space of automorphisms of $(\mathbf{C}^n, 0)$.

G_η-bundles $EG_\eta \to BG_\eta$ that correspond to $\lambda_1(\eta)$ and $\lambda_2(\eta)$, respectively. The *total Chern class*, $c(\eta) \in H^*(BG_\eta)$, and the *Euler class*, $e(\eta) \in H^{2\,\mathrm{codim}(\eta)}(BG_\eta)$, of η are defined by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)} \quad \text{and} \quad e(\eta) := e(E'_\eta)\,.$$

We end this section by recalling the *method of restriction equations* due to Rimányi et al.

Theorem. *Let η be a singularity. Suppose that the number of singularities of codimension less than or equal to* $\mathrm{codim}(\eta)$ *is finite. Moreover, assume that the Euler classes of all singularities of codimension smaller than* $\mathrm{codim}(\eta)$ *are not zero-divisors. Then we have*

1. *if* $\xi \neq \eta$ *and* $\mathrm{codim}(\xi) \leqslant \mathrm{codim}(\eta)$, *then* $\mathcal{T}^\eta(c(\xi)) = 0$;
2. $\mathcal{T}^\eta(c(\eta)) = e(\eta)$.

This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

Solving of these equations is rather difficult. This method is well suited for compter experiments, though the bounds of such computations are quite sharp.

For $\eta = A_i$, a suitable maximal compact subgroup can be chosen as $G_{A_i} = U(1)\times U(l)$. The Chern class is

$$c(A_i) = \frac{1+(i+1)x}{1+x}\prod_{j=1}^{l}(1+y_j)\,,$$

where x and $y_1,\ldots,y_l$ are the Chern roots of the universal bundles on $BU(1)$ and $BU(l)$. The Euler class is

$$e(A_i) = i!\; x^i \prod_{j=1}^{l}(y_j - ix)\cdots(y_j-2x)(y_j-x)\,.$$

In case of $\eta = I_{2,2}$, we consider the extension of $U(1)\times U(1)$ by $\mathbf{Z}/2\mathbf{Z}$. Denoting this group by H, a maximal compact subgroup is $G_\eta = H \times U(l)$ for all $l \geqslant 0$. But to make computations easier, we use the subgroup $U(1)\times U(1)\times U(l)$ as G_η We have

$$c(I_{2,2}) = \frac{(1+2x_1)(1+2x_2)}{(1+x_1)(1+x_2)}\prod_{j=1}^{l}(1+y_j)\,.$$

Here x_1, x_2 and $y_1,\ldots,y_l$ are the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(l)$. The Euler class is

$$e(I_{2,2}) = x_1x_2(2x_1-x_2)(2x_2-x_1)\prod_{j=1}^{l}(y_j-a_1)(y_j-a_2)(y_j-a_1-a_2)\,.$$

Next, we consider $\eta = III_{2,2}$. This time we use the maximal compact group $G_\eta = U(2) \times U(l{-}1)$ for $l \geqslant 1$. We have

$$c(III_{2,2}) = \frac{(1{+}2x_1)(1{+}2x_2)(1{+}x_1{+}x_2)}{(1{+}x_1)(1{+}x_2)} \prod_{j=1}^{l-1}(1 + y_j)\,,$$

where x_1, x_2 and $y_1, \ldots, y_{l-1}$ denote the Chern roots of the universal bundles on $BU(2)$ and $BU(l{-}1)$. The Euler class is

$$e(III_{2,2}) = (x_1x_2)^2(x_1{-}2x_2)(x_2{-}2x_1) \prod_{j=1}^{l-1}(x_1{-}y_j) \prod_{j=1}^{l-1}(x_2{-}y_j)\,.$$

For the singularity $III_{2,3}$, we can use the action of the $U(1) \times U(1) \times U(l-1)$. We have

$$c(III_{2,3}) = \frac{(1{+}2x_1)(1{+}3x_2)(1{+}x_1{+}x_2)}{(1{+}x_1)(1{+}x_2)} \prod_{j=1}^{l-1}(1 + y_j)\,.$$

This time x_1, x_2 and $y_1, \ldots, y_l$ are the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(l-1)$. The Euler class is

$$\begin{aligned} e(III_{2,3}) =& 4x_1^2x_2^3(x_1 - x_2)(x_1 - 3x_2)(x_2 - 2x_1) \\ &\times \prod_{j=1}^{l-1}(x_1 - y_j)(x_2 - y_j)(2x_2 - y_j)\,. \end{aligned}$$

For the singularity $III_{3,3}$, the maximal compact group is $U(2) \times U(l-1)$. The Chern class is

$$c(III_{3,3}) = \frac{(1 + 3x_1)(1 + 3x_2)(1 + x_1 + x_2)}{(1 + x_1)(1 + x_2)} \prod_{j=1}^{l-1}(1 + y_j)\,,$$

where x_1, x_2 and $y_1, \ldots, y_{l-1}$ are the Chern roots of the universal bundles $BU(2)$ and $BU(l-1)$. The Euler class is

$$\begin{aligned} e(III_{3,3}) =& 4x_1^3x_2^3(3x_1 - x_2)(3x_1 - 2x_2)(3x_2 - x_1)(3x_2 - 2x_1) \\ &\times \prod_{j=1}^{l-1}(x_1 - y_j)(2x_1 - y_j)(x_2 - y_j)(2x_2 - y_j)\,. \end{aligned}$$

We display now the Chern or/and Euler classes of some other singularities (we omit to interpret the variables x_i and y_j). We have

$$\begin{aligned} c(I_{a,b}) &= \frac{(1 + \frac{a+b}{\mathrm{nwd}(a,b)}x_1)(1 + \frac{ab}{\mathrm{nwd}(a,b)}x_2)}{(1 + \frac{a}{\mathrm{nwd}(a,b)}x_1)(1 + \frac{b}{\mathrm{nwd}(a,b)}x_2)} \prod_{j=1}^{l-1}(1 + y_j)\,; \\ e(I_{a,b}) &= \frac{a!b!a^{b-1}b^{a-1}x^{a+b}}{\mathrm{nwd}(a,b)^{a+b}} \prod_{j=1}^{l} \left(\prod_{i=1}^{a}(i\frac{\mathrm{nwd}(a,b)}{b}x - y_j) \prod_{i=1}^{b-1}(i\frac{\mathrm{nwd}(a,b)}{a}x - y_j) \right)\,; \\ c(III_{a,b}) &= \frac{(1 + ax_1)(1 + bx_2)(1 + x_1 + x_2)}{(1 + x_1)(1 + x_2)} \prod_{j=1}^{l-1}(1 + y_j)\,; \end{aligned}$$

$$e(III_{a,b}) = (a-1)!(b-1)!\prod_{i=1}^{b-1}(ax_1 - ix_2)\prod_{i=1}^{a-1}(bx_2 - ix_1)$$
$$\times \prod_{j=1}^{l-1}\left(\prod_{i=1}^{a-1}(y_j - ix_1)\prod_{i=1}^{b-1}(y_j - ix_2)\right).$$

We show now how to compute the Euler class of $III_{2,3}$. Assume that $l = 1$ and consider the germ $g(x, y) = (x^2, y^3, xy)$. A prototype of $III_{2,3}$ can be written as the unfolding

$$g + \sum_{i=1}^{8} u_i h_i\,,$$

where h_i form a basis of the space

$$\frac{\mathfrak{m}_{x,y}^3}{\mathfrak{m}_{x,y}\cdot\{\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\} + \mathbf{C}^3 \cdot I(g)}\,,$$

and where $I(g)$ is the subspace generated by the component functions of g. We shall work with the basis consisting of the following germs:

$$\begin{aligned} h_1(x,y) &= (x,0,0), & h_5(x,y) &= (0,y,0),\\ h_2(x,y) &= (y,0,0), & h_6(x,y) &= (0,y^2,0),\\ h_3(x,y) &= (y^2,0,0), & h_7(x,y) &= (0,0,x),\\ h_4(x,y) &= (0,x,0), & h_8(x,y) &= (0,0,y). \end{aligned}$$

Let ρ_{h_i} denote the representation of the action of the group $U(1) \times U(1)$ on the space generated by h_i. Then, denoting the one-dimensional representations of the first and the second copies of $U(1)$ by λ and μ, we have

$$\begin{aligned} \rho_{h_1} &= \lambda, & \rho_{h_5} &= \mu^2,\\ \rho_{h_2} &= \lambda^2 \otimes \mu^{-1}, & \rho_{h_6} &= \mu,\\ \rho_{h_3} &= \lambda^2 \otimes \mu^{-2}, & \rho_{h_7} &= \mu,\\ \rho_{h_4} &= \lambda^{-1} \otimes \mu^3, & \rho_{h_8} &= \lambda. \end{aligned}$$

Therefore for $l = 1$, using the representation $\bigoplus \rho_{h_i}$, we can write the Euler class as

$$e(III_{2,3}) = 4x_1^2x_2^3(x_1 - x_2)(x_1 - 3x_2)(x_2 - 2x_1)\,,$$

where x_1 and x_2 denote the Chern roots of the universal bundles on the two copies of $BU(1)$.

For $l = 2$, in addition to h_i above, we need to consider the representations of the action of the group $U(l-1) = U(1)$ on the spaces generated by $(x, y) \mapsto (0, 0, 0, x)$, $(x, y) \mapsto (0, 0, 0, y)$ and $(x, y) \mapsto (0, 0, 0, y^2)$. These can be written as $\nu \otimes \lambda^{-1}$, $\nu \otimes \mu^{-1}$ and

$\nu \otimes \mu^{-2}$, where ν denotes the one-dimensional representation of this copy of $U(1)$. Hence, in this case, the Euler class can be written as

$$e(III_{2,3}) = 4x_1^2x_2^3(x_1 - x_2)(x_1 - 3x_2)(x_2 - 2x_1)(x_1 - y_1)(x_2 - y_1)(2x_2 - y_1)\,,$$

where x_i are as above and y_1 denotes the Chern root of the universal bundle on $BU(1)$.

For $l \geqslant 1$, we need to consider $U(l-1)$ instead of $U(1)$, giving rise to $y_1, \dots, y_{l-1}$ (and respectively to the product $\prod_{j=1}^{l-1}(x_1-y_j)(x_2-y_j)(2x_2-y_j))$ instead of y_1 (and respectively of $(x_1 - y_1)(x_2 - y_1)(2x_2 - y_1))$.

We shall need the following alphabets:

Definition. *We set*

$$\begin{aligned}
\mathbb{D} &= \boxed{x_1} + \boxed{x_2} + \boxed{x_1 + x_2}\,,\\
\mathbb{E} &= \boxed{2x_1} + \boxed{2x_2}\,,\\
\mathbb{F} &= \boxed{2x_1} + \boxed{3x_2} + \boxed{x_1 + x_2}\,,\\
\mathbb{G} &= \boxed{3x_1} + \boxed{3x_2} + \boxed{x_1 + x_2}\,,\\
\mathbb{H} &= \boxed{2x_1} + \boxed{4x_2} + \boxed{x_1 + x_2}\,.
\end{aligned}$$

Notation. In the rest of the paper we shall use the shifted parameter

$$r := l + 1\,.$$

When we need to emphasize the dependence on r we shall write $\eta(r)$ for the singularity $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+r-1}, 0)$, and denote the Thom polynomial of $\eta(r)$ by $\mathcal{T}_r^\eta$, or $\mathcal{T}_r$ for short. (In this notation, the result of Thom, $\mathcal{T}_r^{A_1} = S_r$, has a transparent form.)

We now specify, with the help of these alphabets, some equations characterizing Thom polynomials $\mathcal{T}_r$ imposed by different singularities.

Note. The variables below will be specialized to the Chern roots of the *cotangent* bundles.

First, we give the vanishing equations coming from the Chern classes of singularities. Let $\mathbb{B}_j$ denote an alphabet of cardinality j. We have the following equations:

$$\begin{aligned}
A_i(r) :&\quad \mathcal{T}_r(x - \mathbb{B}_{r-1} - \boxed{(i+1)x}) = 0 \qquad \text{for } \ i = 0, 1, 2, \dots\,;\\
I_{2,2}(r) :&\quad \mathcal{T}_r\,(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = 0\,;\\
I_{2,3}(r) :&\quad \mathcal{T}_r\left(\boxed{2x} + \boxed{3x} - \boxed{5x} - \boxed{6x} - \mathbb{B}_{r-1}\right) = 0\,;\\
III_{2,2}(r) :&\quad \mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,;\\
III_{2,3}(r) :&\quad \mathcal{T}_r\,(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}) = 0\,;\\
III_{2,4}(r) :&\quad \mathcal{T}_r\,(\mathbb{X}_2 - \mathbb{H} - \mathbb{B}_{r-2}) = 0\,.
\end{aligned}$$

Using the Chern classes displayed above, one can write down other vanishing equations.

We give now some normalizing equations coming from the Euler classes of singularities. We have

$$
\begin{aligned}
A_i(r): \quad & \mathcal{T}_r(x-\mathbb{B}_{r-1}-\boxed{(i+1)x}) = R(x+\boxed{2x}+\boxed{3x}+\cdots+\boxed{ix}\,, \mathbb{B}_{r-1}+\boxed{(i+1)x}\,)\,; \\
I_{2,2}(r): \quad & \mathcal{T}_r(\mathbb{X}_2-\mathbb{E}-\mathbb{B}_{r-1}) = x_1x_2(x_1-2x_2)(x_2-2x_1)\ R(\mathbb{X}_2+\boxed{x_1+x_2}, \mathbb{B}_{r-1})\,; \\
I_{2,3}(r): \quad & \mathcal{T}_r\left(\boxed{2x}+\boxed{3x}-\boxed{5x}-\boxed{6x}-\mathbb{B}_{r-1}\right) \\
& = 2xR(\boxed{2x}+\boxed{3x}, \boxed{5x}+\boxed{6x}+\mathbb{B}_{r-1}) \\
& \times \prod_{j=1}^{r-1}(4x-b_j)(6x-b_j)\,; \\
III_{2,2}(r): \quad & \mathcal{T}_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2}) = R(\mathbb{X}_2, \mathbb{D}+\mathbb{B}_{r-2})\,; \\
III_{2,3}(r): \quad & \mathcal{T}_r\left(\mathbb{X}_2-\mathbb{F}-\mathbb{B}_{r-2}\right) = 2x_2(x_1-x_2)R(\mathbb{X}_2, \mathbb{F}+\mathbb{B}_{r-2})\prod_{j=1}^{r-2}(2x_2-b_j)\,; \\
III_{3,3}(r): \quad & \mathcal{T}_r\left(\mathbb{X}_2-\mathbb{G}-\mathbb{B}_{r-2}\right) = x_1x_2(3x_1-2x_2)(3x_2-2x_1) \\
& \times R(\mathbb{X}_2, \mathbb{G}+\mathbb{B}_{r-2})\prod_{j=1}^{r-2}(2x_1-b_j)(2x_2-b_j)\,.
\end{aligned}
$$

Using the Euler classes displayed above, one can write down other normalizing equations.

The above data will be used in the lectures to compute the Thom polynomials of the singularities: A_i, I_{22}, etc.

The suitable parts of the following books and papers can serve as introductory texts to the lectures.

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Piotr Pragacz

Piotr Pragacz jest matematykiem pracującym od 1981 roku w IM PAN w Warszawie. W swych badaniach naukowych zajmuje się geometrią algebraiczną. Od 2000 roku kieruje Zakładem Algebry i Geometrii Algebraicznej IM PAN. Przedtem w latach 1977–1981 pracował na UMK w Toruniu. Piotr Pragacz studiował algebrę w Toruniu, geometrię algebraiczną w Warszawie, oraz kombinatorykę i geometrie enumeratywną w Paryżu u Alain Lascoux, którego uważa za swojego głównego nauczyciela. Habilitacja Piotra Pragacza dotyczyła geometrycznych i algebraicznych aspektów rozmaitości Schuberta oraz miejsc degeneracji morfizmów wiązek. Głównym jej wkładem było wprowadzenie i badanie P-ideałów miejsc degeneracji. Tej tematyce poświęcona jest książka [Book] napisana wspólnie z Williamem Fultonem. Spektrum zainteresowań matematycznych Piotra Pragacza obejmuje: geometrię algebraiczna ze szczególnym uwzględnieniem teorii przeciec: [15], [21], [32], [Book], [50]; algebraiczną kombinatoryką ze szczególnym uwzględnieniem funkcji symetrycznych: [13], [18], [34] oraz globalną teorię osobliwości ze szczególnym uwzględnieniem klas charakterystycznych rozmaitości osobliwych i wielomianów Thoma: [28], [38], [48], [49], [56], [57]. Terminy: "The Lascoux–Pragacz ribbon identity" (za W. Chen, 2004) oraz "The Sergeev–Pragacz formula" (za I. G. Macdonaldem w słynnej monografii *Symmetric Functions and Hall Polynomials*, 1995), weszły na stale do terminologii algebraicznej. W 2000 roku Piotr Pragacz powołał do życia Seminarium Impanga, które stało się ogólnopolskim i międzynarodowym forum geometrii algebraicznej [O5]. Cytowana literatura: http://www.impan.pl/~pragacz/publications.htm