BRAIDS, CONFIGURATION SPACES AND KNOTS

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Lecture 1

Basic concepts: Braids and configuration spaces

1. Systems of n curves in three-dimensional space and braid groups

First of all braids naturally arise as objects in 3-space. Let us consider two parallel planes P_0 and P_1 in $\mathbb{R}^3$, which contain two ordered sets of points A_1, ..., $A_n \in P_0$ and B_1, ..., $B_n \in P_1$. These points are lying on parallel lines L_A and L_B respectively. The space between the planes P_0 and P_1 we denote by Π. Suppose that the point B_i is lying under the point A_i, as a result of the orthogonal projection of the plane P_0 onto the plane P_1. Let us connect the set of points A_1, ..., A_n with the set of points B_1, ..., B_n by simple non-intersecting curves C_1, ..., C_n lying in the space Π and such that each curve meets only once each parallel plane P_t lying in the space Π (see Figure 1).

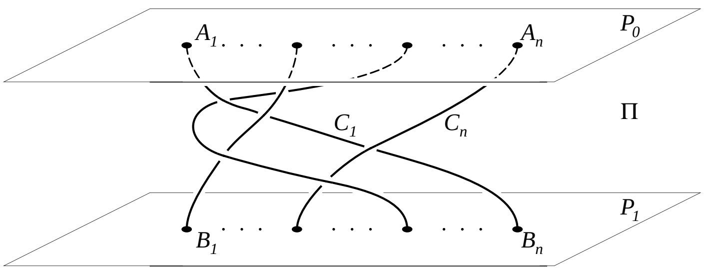

Figure 1.

This object is called a *braid* and the curves are called the *strings* of a braid. Usually braids are depicted by projections on the plane passing through the lines L_A and L_B. This projection is supposed to be in general position so that there is only finite number of double points of intersection which are lying on pairwise different levels and intersections are transversal. The simplest braid σ_i (Fig. 2) corresponds to the transposition $(i, i+1)$.

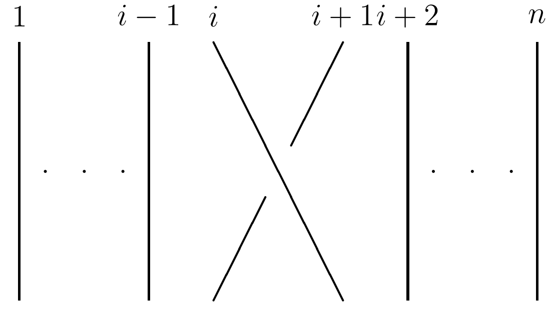

Figure 2.

Let us introduce the following equivalence relation on the set of all braids with n strings and with fixed P_0, P_1, A_i and B_i. It is defined by homeomorphisms $h : \Pi \to \Pi$, identical on $P_0 \cup P_1$ and such that $h(P_t) = P_t$. Braids β and β' are equivalent if there exists such a homeomorphism h that $h(\beta) = \beta'$.

On the set Br_n of equivalence classes under the considered relation the structure of a group introduces as follows. We put a copy Π' of the domain Π under the Π in such a way that P_0' coincides with P_1 and each A_i coincides with B_i and it is possible to glue braids β and β'. This gluing gives a composition of braids $\beta\beta'$ (Fig. 3).

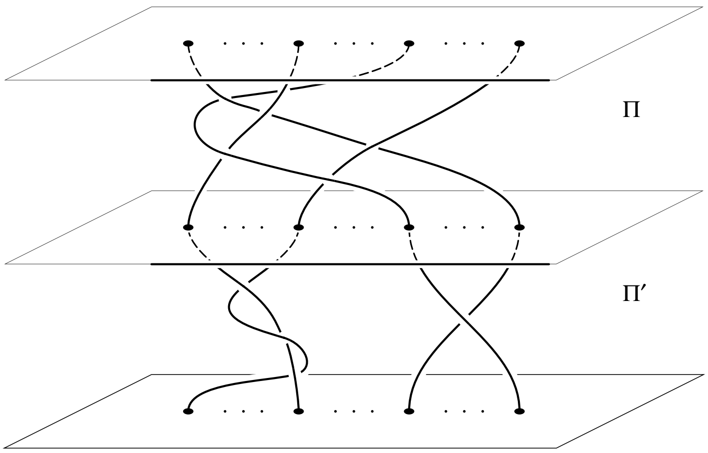

FIGURE 3.

Unit element is the equivalence class containing a braid of n parallel intervals, the braid β^{-1} inverse to β is defined by reflection of β with respect to the plane $P_{1/2}$. A string C_i of a braid β connects the point A_i with the pont B_{k_i} defining the permutation S^β. If this permutation is identical then the braid β is called *pure*. The map $\beta \to S^\beta$ defines an epimorphism τ_n of the braid group Br_n on the permutation group $\Sigma_n$with the kernel consisting of all pure braids:

$$1 \to P_n \to Br_n \xrightarrow{\tau_n} \Sigma_n \to 1. \tag{1}$$

The following presentation of the braid group Br_n with generators σ_i, $i = 1, ..., n-1$ and two types of relations:

$$\begin{cases} \sigma_i\sigma_j & = \sigma_j\,\sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{cases} \tag{2}$$

is the algebraic expression of the fact that any isotopy of braids can be broken down into “elemntary moves” of two types that correspond to two types of relations.

If we add a vertical interval to the system of curves on Figure 1 we can get a canonical inclusion j_n of the group Br_n into the group Br_{n+1}

$$j_n : Br_n \to Br_{n+1}.$$

If the symmetric group Σ_n is given by its canonical presentation with generators s_i, $i = 1, ..., n-1$ and relations:

$$\begin{cases} s_is_j & = s_j\,s_i, \quad \text{if} \quad |i-j| > 1, \\ s_is_{i+1}s_i & = s_{i+1}s_is_{i+1} \\ s_i^2 = 1, \end{cases} \tag{3}$$

then the homomorphism τ_n is given by the formula

$$\tau_n(\sigma_i) = s_i, \quad i = 1, \ldots, n-1.$$

2. Configuration spaces and braid groups

If we look at the Figure 1, then this picture can be interpreted as a graph of a loop in the *configuration space* of n points on a plane, that is the space of unordered sets of n points on a plane, see Figure 4.

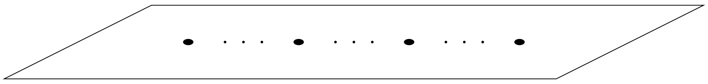

Figure 4.

So, it is possible to interpret the braid group as the fundamental group of the configuration space. Formally it is done as follows. The symmetric group Σ_m acts on the Cartesian power $(\mathbb{R}^2)^m$ of the space $\mathbb{R}^2$:

$$w(y_1, ..., y_m) = (y_{w(1)}, ..., y_{w(m)}), \quad w \in \Sigma_m. \tag{4}$$

Denote by $F(\mathbb{R}^2, m)$ the space of m-tuples of pairwise different points in $\mathbb{R}^2$:

$$F(\mathbb{R}^2, m) = \{(p_1, ..., p_m) \in (\mathbb{R}^2)^m : p_i \neq p_j \text{ for } i \neq j\}.$$

This is the space of regular points of this action. We call the orbit space of this action $B(\mathbb{R}^2, m) = F(\mathbb{R}^2, m)/\Sigma_m$ the *configuration space of n points on a plane.* The braid group Br_m is the fundamental group of configuration space

$$Br_m = \pi_1(B(\mathbb{R}^2, m)).$$

The pure braid group P_m is the is the fundamental group of the space $F(\mathbb{R}^2, m)$. The covering

$$p : F(\mathbb{R}^2, m) \to B(\mathbb{R}^2, m)$$

defines the exact sequence:

$$1 \to \pi_1(F(\mathbb{R}^2, m)) \xrightarrow{p_*} \pi_1(B(\mathbb{R}^2, m)) \to \Sigma_n \to 1. \tag{5}$$

which is equivalent to sequence (1).

It can be used for proving the canonical presentation of the braid group (2).

3. Braid groups as automorphism groups of free groups and the word problem

Another important approach to the braid group bases on the fact that this group may be considered as a subgroup of the automorphism group of a free group.

Let F_n be the free group of rank n with the set of generators $\{x_1, \ldots, x_n\}$. Assume further that $\operatorname{Aut} F_n$ is the automorphism group of F_n.

The inclusion of the braid group Br_n into $\operatorname{Aut} F_n$ may be described as follows. Let $\overline{\sigma}_i \in \operatorname{Aut} F_n, i = 1, 2, \ldots, n-1$, be given by the formula which describes its action on generators:

$$\begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases} \tag{6}$$

Let us define a map ν of the generators σ_i, $i = 1, \dots, n-1$ of the braid group Br_n to these automorphisms:

$$\nu(\sigma_i) = \overline{\sigma}_i. \tag{7}$$

Theorem 1. *Formula (7) define correctly a homomorphism*

$$\nu : Br_n \to \operatorname{Aut} F_n.$$

which is a monomorphism.

Theorem 1 gives a solution of the word problem for the braid groups. It was done first by E. Artin.

The free group F_n is a fundamental group of a disc D_n without n points and the generator x_i corresponds to a loop going around the i-th point. The braid group Br_n is the mapping class group of a disc D_n with its boundary fixed and so it acts on the fundamental group of D_n. This action is described by the formulas (6) where x_i correspond to the canonical loops on D_n which form the generators of the fundamental group. Geometrically this action is depicted in the Figure 5.

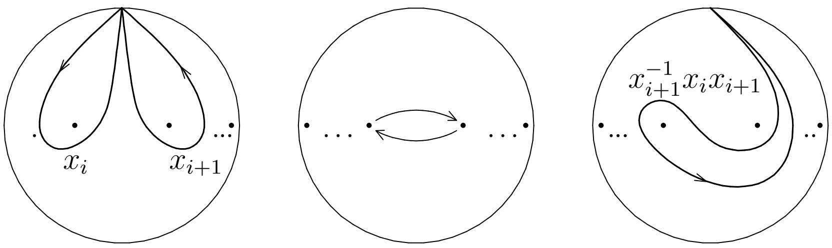

FIGURE 5.

4. PRESENTAION OF THE PURE BRAID GROUP AND MARKOV NORMAL FORM

Let $f(y_1, ..., y_m)$ be a word with (possibly empty) entries of y_i^ϵ, where y_i are some letters and ϵ may be ± 1. If y_i are elements of a group G then $f(y_1, ..., y_m)$ will be considered as the corresponding element of G.

Let us define the elements $s_{i,j}$, $1 \le i < j \le m$, of the braid group Br_m by the formula:

$$s_{i,j} = \sigma_{j-1} ... \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} ... \sigma_{j-1}^{-1}.$$

These elements satisfy the following Burau relations ([1], [3]):

$$\begin{aligned} &s_{i,j} s_{k,l} = s_{k,l} s_{i,j} \text{ for } i < j < k < l \text{ and } i < k < l < j, \\ &s_{i,j} s_{i,k} s_{j,k} = s_{i,k} s_{j,k} s_{i,j} \text{ for } i < j < k, \\ &s_{i,k} s_{j,k} s_{i,j} = s_{j,k} s_{i,j} s_{i,k} \text{ for } i < j < k, \\ &s_{i,k} s_{j,k} s_{j,l} s_{j,k}^{-1} = s_{j,k} s_{j,l} s_{j,k}^{-1} s_{i,k} \text{ for } i < j < k < l. \end{aligned} \tag{8}$$

W. Burau and later A. A. Markov proved that the elements $s_{i,j}$ with the relations (8) give a presentation of the pure braid group P_m [3]. The following formula is a consequence of the Burau relations and also belongs to A. A. Markov:

$$[s_{i,l}, s_{j,k}^\epsilon] = f(s_{1,l}, ..., s_{l-1,l}), \quad \epsilon = \pm 1, \quad k < l. \tag{9}$$

Let us define the elements $\sigma_{k,l}$, $1 \le k \le l \le m$ by the formulas

$$\sigma_{k,k} = e,$$
$$\sigma_{k,l} = \sigma_k^{-1}...\sigma_{l-1}^{-1}.$$

Let P_m^k be the subgroup of P_m generated by the elements $s_{i,j}$ with $j \le k$.

Theorem 1. (A. A. Markov) *(i) Every element of the group Br_m can be uniquely written in the form*

$$f_m(s_{1,m}, ..., s_{m-1,m})...f_j(s_{1,j}, ..., s_{j-1,j})...f_2(s_{1,2})\sigma_{i_m,m}...\sigma_{i_j,j}...\sigma_{i_2,2}. \tag{10}$$

(ii) $1 = P_m^1 \subset P_m^2 \subset \cdots \subset P_m^m = P_m$.
(iii) The factor group P_m^k / P_m^{k-1} is the free group on free generators $s_{i,k}$, $1 \le i \le k$.

The form (10) is called the *Markov normal form*, it also gives the solution of the word problem for the braid groups.

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Lecture 2

Basic concepts about knots

Definition 1. *Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an embedding if $f : X \to f(X)$ is a homeomorphism.*

Let I denote the closed interval $I = [0, 1]$.

Definition 2. *Two embeddings $f_0, f_1 : X \to Y$ are called isotopic if there is an embedding*

$$F : X \times I \to Y \times I$$

such that $F(x, t) = (f(x, t), t),\ x \in X,\ t \in I$ with $f(x, 0) = f_0(x),\ f(x, 1) = f_1(x)$.

JORDAN CURVE THEOREM *If J is a embedding of a circle S^1 in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus J(S^1)$ has two components, and $J(S^1)$ is the boundary of each.*

Definition 3. *A knot is a continuous embedding of a circle S^1 into $\mathbb{R}^3$ or into $\mathbb{S}^3$. If a direction of passing through on a knot (one of two possible) is given we are speaking about oriented knots.*

Definition 4. *An n-component link is a continuous embedding of a disjoint union of n circles $\amalg_{i=1}^n S_i^1$ into $\mathbb{R}^3$ or into $\mathbb{S}^3$.*

Definition 5. *Two embeddings $f_0, f_1 : X \to Y$ are called ambient isotopic if there is a level preserving isotopy*

$$H : Y \times I \to Y \times I$$

such that $H(y, t) = (h(y, t), t),\ y \in Y,\ t \in I$ with $h_0 = id_Y$ and $f_1 = h_1 f_0$.

Topological embeddings are too general.

Definition 6. *A knot $\mathcal{K}$ is called tame if it is ambient isotopic to a simple closed polygon in $\mathbb{R}^3$ or in $\mathbb{S}^3$. A knot which is not tame is called wild.*

Proposition 1. *Wild knots exist.* $\square$

Now let X and Y be parts of $\mathbb{R}^n$. We will consider maps (and isotopies, in particular) $f : X \to Y$ which are *piecewise linear* (p.l).

Definition 7. *Two p.l. knots are equivalent if they are ambient isotopic.*

Digression: polyhedra and piecewise linear maps.

Let $x_0, x_1, \dots, x_r$ be points in $\mathbb{R}^n$. They are *linearly dependent* (in the affine sense) if there exist real numbers $\lambda_0, \lambda_1, \dots, \lambda_r$, not all zero, such that

$$\sum_{i=0}^{r} \lambda_i x_i = 0 \quad \text{and} \quad \sum_{i=0}^{r} \lambda_i = 0$$

Let A be a part of $\mathbb{R}^n$. The *convex hull* or *span* of A is the set of all linear combinations of elements of A with positive coefficients whose sum is one, that is points x of $\mathbb{R}^n$:

$$x = \sum_{i=1}^{p} \lambda_i a_i,$$

$a_i \in A$, coefficients λ_i are positive and $\sum_{i=1}^{p} \lambda_i = 1$.

A *convex cell* in $\mathbb{R}^n$ is a convex hull of a finite number (m) of points, $m \geq 1$.

A convex cell can be also defined as a compact, non-empty subset in $\mathbb{R}^n$ which is the solution of a finite number of linear equations $f_i(x) = 0$ and linear inequalities $g_j(x) \geq 0$.

A *face* of a cell is obtained by changing some of inequalities $g_i(x) \geq 0$ to equalities. It is written $B \geq A$ if A is a face of B.

A *vertex* is a face consisting of one point.

An *n-simplex* is a convex hull of a set of (n+1) points which are linearly independent.

A *polyhedron* is a union of finite number of convex cells.

Let P and Q be two polyhedra, $f : P \to Q$ is a map. The map f is *piecewise linear* (p.l.) if

(i) f is continuous,

(ii) the graph of f :

$$\Gamma_f = \{(x, f(x)) : x \in P\}$$

is a polyhedron.

Proposition 1. *The composition of two p.l. maps is a p.l. map.* □

A *convex linear cell complex* in $\mathbb{R}^n$ is a finite set K of convex cells in $\mathbb{R}^n$ such that

1) if $A \in K$, then every face of A is in K,

2) if A and B are in K, then $A \cap B = \emptyset$ or $A \cap B$ is a common face of A and B.

A *simplicial complex* is a cell complex whose cells are all simplices.

Let K be a simplicial complex, $A \in K$, then we define

$$\operatorname{star}(A; K) = \{B \in K : B \geq A\},$$

$$\overline{\operatorname{star}}(A; K) = \{B \in K : B \text{ is a face of an element of } \operatorname{star}(A; K),$$

Let X be a topological space. A *coordinate map* is an embedding (homeomorphism on its image)

$$f : P \to X,$$

such that P is polyhedron. Usually we write (f, P) to denote such a map. Two coordinate maps (f, P) and (g, Q) are *compatible* if either

$$f(P) \cap g(Q) = \emptyset$$

ot there exists a coordinate map (h, R) such that

$$(1) \quad h(R) = f(P) \cap g(Q)$$

$$(2) \quad f^{-1}h \text{ and } g^{-1}h \text{ are p.l.}$$

A p.l. m-ball is a polyhedron p.l. homeomorphic to m-simplex. A p.l. m-sphere is a polyhedron p.l. homeomorphic to a boundary an $(m+1)$-simplex.

A *p.l. manifold* M^m of dimension m is a polyhedron such that its every point has a (closed) neighbourhood p.l. homeomorphic to m-ball.

Regular neighbourhoods

Let P be a polyhedron and P_0 is its sub-polyhedron $P_0 \subset P$. We suppose that $B = \mathrm{cl}_P(P - P_0)$ is a p.l. ball of dimension m, such that $B \cap P_0$ is a union of cells of P_0 p.l. homeomorphic to a ball of dimension $m-1$. In this case we say that P *collapses* to P_0 by an *elementary collapse* and we write $P \searrow^e P_0$. We say that P *collapses* to P_0 and we write $P \searrow P_0$ if there exists a finite sequence

$$P = P_r \searrow^e P_{r-1} \searrow^e \cdots \searrow^e P_0.$$

We say that P is *collapsible* if P collapses to a point, we write $P \searrow 0$.

Let X be a polyhedron in a p.l. manifold M of dimension m. Sub-polyhedron N of M is called *regular neighbourhood* of X in M if we have

1) N is a closed neighbourhood of X in M.
2) N is a sub-manifold with a boundary of M of dimension m.
3) $N \searrow X$.

Theorem 1. *(i) For each polyhedron X in a manifold M there exists a regular neighbourhood N of X.*

(ii) If N_1 and N_2 are two regular neighbourhoods of X in M, then there exists a p.l. homeomorphism

$$h : N_1 \to N_2$$

which is identical on X.

(iii) If X is collapsible ($X \searrow 0$), then every its regular neibourhood is a p.l. m-ball.

Theorem 2. *If N_1 and N_2 are regular neighbourhoods of X in M, then there exists an ambient isotopy h_t of M such that $h_t(x) = x$ for all $x \in X$ and for all $t \in [0,1]$ and such that $h_1(N_1) = N_2$.*

Knot equivalences

Definition 8. *Let u be a straight segment of a polygonal knot $\mathcal{K}$ in $\mathbb{R}^3$ and D is a triangle n $\mathbb{R}^3$, $\partial D = u \cup v \cup w$, where u, v, and w are 1-faces of D. If $D \cap \mathcal{K} = u$, then $\mathcal{K}' = (\mathcal{K} \setminus u) \cup v \cup w$ defines another polygonal knot. We say $\mathcal{K}'$ results from $\mathcal{K}$ by a Δ-move. If $\mathcal{K}$ is oriented, $\mathcal{K}'$ is to carry the orientation induced by that of $\mathcal{K} \setminus u$. The inverse process is denoted by Δ^{-1}.*

Definition 9. *Two p.l. knots are called combinatorially equivalent or isotopic by Δ-moves if there is a finite sequence of Δ-moves which transforms one knot to another.*

Theorem 1. (of Alexander-Schönflies) *Let $i : S^2 \to S^3$ be a p.l. embedding. Then*

$$S^3 = B_1 \cup B_2, \ i(S^2) = B_1 \cap B_2 = \partial B_1 = \partial B_2,$$

where B_i, $i = 1, 2$, is a 3-ball, i.e. p.l. homeomorphic to a 3-simplex. □

Theorem 2. (of Alexander-Tietze) *Any p.l. homeomorphism of n-ball B keeping the boundary fixed is isotopic to the identity by a p.l. ambient isotopy keeping the boundary fixed.* □

Theorem 3. (on equivalence of equivalences) *Let $\mathcal{K}_0$ and $\mathcal{K}_1$ be p.l. knots in $\mathbb{S}^3$. The following assertions are equivalent.*

(1) There is an orientation preserving homeomorphism

$$f : \mathbb{S}^3 \to \mathbb{S}^3$$

which carries $\mathcal{K}_0$ onto $\mathcal{K}_1$: $f(\mathcal{K}_0) = \mathcal{K}_1$.

(2) The knots $\mathcal{K}_0$ and $\mathcal{K}_1$ are ambient isotopic.

(3) The knots $\mathcal{K}_0$ and $\mathcal{K}_1$ are combinatorially equivalent.

Proof. (1) $\Longrightarrow$ (2): We begin by showing that there is an ambient isotopy $H(x,t) = (h_t(x), t)$ of S^3 such that $h_1 f$ leaves fixed a 3-simplex $[P_0, P_1, P_2, P_3]$. If

$$f : S^3 \to S^3$$

has a fixed point, choose it as P_0; if not, let P_0 be any interior point of a 3-simplex $[s^3]$ of S^3. There is an ambient isotopy of S^3 which leaves $S^3 \setminus [s^3]$ fixed and carries P_0 over to any other interior point of $[s^3]$. If $[s^3]$ and $[\tilde{s}^3]$ have a common 2-face, one can easily construct an ambient isotopy moving an interior point P_0 of $[s^3]$ to an interior point P_0' of $[\tilde{s}^3]$ which is the identity outside $[s^3] \cup [\tilde{s}^3]$.

So there is an ambient isotopy H^0 with

$$h_1^0 f(P_0) = P_0,$$

since any two 3-simplices can be connected by a chain of adjoining ones. Next we choose a point $P_1 \neq P_0$ in the simplex star of P_0, and by similar arguments we construct an ambient isotopy H^1 with $h_1^1 h_1^0 f$ leaving fixed the 1-simplex $[P_0, P_1]$. A further step leads to $h_1^2 h_1^1 h_1^0 f$ with a fixed 2-simplex $[P_0, P_1, P_2]$. At this juncture the assumption comes in that f is required to preserve the orientation. A point $P_3 \notin [P_0, P_1, P_2]$, but in the star of $[P_0, P_1, P_2]$, will be mapped by $h_1^2 h_1^1 h_1^0 f$ onto a point P_3' in the same halfspace with regard to the plane spanned by P_0, P_1, P_2. This ensures the existence of the final ambient isotopy H^3 such that $h_1^3 h_1^2 h_1^1 h_1^0 f$ leaves fixed $[P_0, P_1, P_2, P_3]$. The assertion follows from the fact that $H = H^3 H^2 H^1 H^0$ is an ambient isotopy, $H(x,t) = (h_t(x), t)$.

By Theorem of Alexander-Schönflies the complement of $[P_0, P_1, P_2, P_3]$ is a combinatorial 3-ball and by Theorem of Alexander-Tietze there is an ambient isotopy which connects $h^1 f$ with the identity of S^3.

(2) $\Longrightarrow$ (1) follows from the definition of an ambient isotopy.

Next we prove (1) $\Longrightarrow$ (3): Let

$$h : S^3 \to S^3$$

be an orientation preserving homeomorphism and $\mathcal{K}_1 = h(\mathcal{K}_0)$. The preceding argument shows that there is another orientation preserving homeomorphism

$$g : S^3 \to S^3, \quad g(\mathcal{K}_0) = \mathcal{K}_1$$

such that g leaves fixed some 3-simplex $[s^3]$ which will have to be chosen outside a regular neighbourhood of $\mathcal{K}_0$ and $\mathcal{K}_1$. For an interior point P of $[s^3]$ consider $S^3 \setminus \{P\}$ as Euclidean 3-space $\mathbb{R}^3$. There is a translation τ of $\mathbb{R}^3$, which moves $\mathcal{K}_0$ into $[s^3] \setminus P$. It is easy to prove that $\mathcal{K}_0$ and $\tau(\mathcal{K}_0)$ are isotopic by moves. We claim that $\mathcal{K}_1 = g(\mathcal{K}_0)$ and $g\tau(\mathcal{K}_0) = \tau(\mathcal{K}_0)$ are isotopic by moves also, which would complete the proof. Choose a subdivision of the triangulation of S^3 such that the triangles used in the isotopy by moves between $\mathcal{K}_0$ and $\tau(\mathcal{K}_0)$ form a subcomplex of S^3. There is an isotopy by moves

$$\mathcal{K}_0 \to \tau(\mathcal{K}_0)$$

which is defined on the triangles of the subdivision. Homeomorphism $g : S^3 \to S^3$ maps the subcomplex onto another one carrying over the isotopy by moves.

(3) $\Longrightarrow$ (1). It is not difficult to construct a homeomorphism of S^3 onto itself which realizes a $\Delta^{\pm 1}$-move and leaves fixed the rest of the knot. Choose a regular neighbourhood U of the 2-simplex which defines the $\Delta^{\pm 1}$-move whose boundary meets the knot in two points. By linear extension one can obtain a homeomorphism producing the $\Delta^{\pm 1}$-move in U and leaving $S^3 \setminus U$ fixed. $\square$

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Lecture 3

Properties of braids

1. Abelianization and commutator subgroup

Let us define a homomorphism from braid group to integers by taking the sum of exponents of the entries of the generators σ_i in the expression of any element of the group through these canonical generators:

$$\deg : Br_n \to \mathbb{Z},\ \deg(b) = \sum_j m_j,\ \text{ where } b = (\sigma_{i_1})^{m_1} \dots (\sigma_{i_k})^{m_k}.$$

Proposition 1. *The homomorphism*

$$\deg : Br_n \to \mathbb{Z}$$

gives the abelianization of the braid group and the commutator subgroup Br'_n is characterized by the condition

$$b \in Br'_n \ \text{ if and only if } \ \deg(b) = 0.$$

Proof. Let $a : Br_n \to A$ be a homomorphism to any other abelian group A, then from the canonical relations we have:

$$a(\sigma_i)a(\sigma_{i+1})a(\sigma_i) = a(\sigma_{i+1})a(\sigma_i)a(\sigma_{i+1}).$$

the commutativity of A gives that $a(\sigma_{i+1}) = a(\sigma_i)$. This means that the homomorphism deg is universal. □

2. Garside Normal Form, Center and Conjugacy Problem

Essential role in Garside work [1] plays the monoid of *positive* braids Br_n^+, that is the monoid which has a presentation the same as the canonical presentation of braid group with generators σ_i, (but without σ_i^{-1}) $i = 1, ..., n$ and the same relations. So each element of this monoid can be represented as a word on the elements σ_i, $i = 1, ..., n$ with no entries of σ_i^{-1}. Two positive words A and B in the alphabet $\{\sigma_i, (i = 1, \dots, n-1)\}$ will be said to be *positively equal* if they are equal as elements of Br_n^+. In this case we shall write $A \doteq B$.

First of all Garside proves the following statement.

Proposition 2. *In Br_n^+ for $i, k = 1, ..., n-1$, given $\sigma_i A \doteq \sigma_k B$, it follows that*

$$\text{if } \ k = i, \ \text{ then } \ A \doteq B,$$

$$\text{if } \ |k - i| = 1, \ \text{ then } \ A \doteq \sigma_k \sigma_i Z, \ B \doteq \sigma_i \sigma_k Z \ \text{ for some } \ Z,$$

$$\text{if } \ |k - i| \geq 2, \ \text{ then } \ A \doteq \sigma_k Z, \ B \doteq \sigma_i Z \ \text{ for some } \ Z.$$

The same is true for the right multiples of σ_i.

Corollary 1. *If $A \doteq P$, $B \doteq Q$, $AXB \doteq PYQ$, ($L(A) \geq 0$, $L(B) \geq 0$), then $X \doteq Y$. That is, monoid Br_n^+ is left and right cancellative.*

Garside's *fundamental word* Δ in the braid group Br_{n+1} is defined by the formula:

$$\Delta = \sigma_1 \dots \sigma_n \sigma_1 \dots \sigma_{n-1} \dots \sigma_1 \sigma_2 \sigma_1.$$

If we use Garside's notation $\Pi_t \equiv \sigma_1 \dots \sigma_t$, then $\Delta \equiv \Pi_n \dots \Pi_1$.

For a positive word W in σ_i, $i = 1, ..., n$ we say that Δ is a *factor* of W or simply W contains Δ, if $W \doteq A\Delta B$ with A and B being arbitrary positive words, probably empty. If W does not contain Δ we shall say W is *prime to* Δ.

Garside's transformation of words $\mathcal{R}$ is defined by the formula

$$\mathcal{R}(\sigma_i) \equiv \sigma_{n-i}.$$

This gives the automorphism of Br_n and of the positive braid monoid Br_n^+.

Proposition 3. *In* Br_n

$$\sigma_i \Delta \doteq \Delta \mathcal{R}(\sigma_i).$$

Proposition 4. *If* W *is an arbitrary positive word in* Br_n^+ *such that either*

$$W \doteq \sigma_1 A_1 \doteq \sigma_2 A_2 \doteq \dots \doteq \sigma_{n-1} A_{n-1},$$

or

$$W \doteq B_1 \sigma_1 \doteq B_2 \sigma_2 \doteq \dots \doteq B_{n-1} \sigma_{n-1},$$

then $W \doteq \Delta Z$ *for some* Z*.*

Proposition 5. *The canonical homomorphism*

$$Br_n^+ \to Br_n$$

is a monomorphism.

Among positive words on the alphabet $\{\sigma_1 \dots \sigma_n\}$ let us introduce a lexicographical ordering with the condition that $\sigma_1 < \sigma_2 < \dots < \sigma_n$. For a positive word W the *base* of W is the smallest positive word which is positively equal to W. The base is uniquely determined. If a positive word A is prime to Δ, then for the base of A the notation $\overline{A}$ will be used.

Theorem 1. (F. A. Garside) *Every word* W *in* Br_{n+1} *can be uniquely written in the form* $\Delta^m \overline{A}$*, where* m *is an integer.*

The form of a word W established in this theorem we call the *Garside left normal form* and the index m we call the *power* of W. The same way the *Garside right normal form* is defined and the corresponding variant of Theorem 1 is true. The Garside normal form also gives a solution to the word problem in the braid group.

Theorem 2. (F. A. Garside) *The necessary and sufficient condition that two words in* Br_{n+1} *are equal is that their Garside normal forms are identical.*

Garside normal form for the braid groups was refined in the subsequent works of S. I. Adyan W. Thurston, E. El-Rifai and H. R. Morton. Namely, there was introduced the *left-greedy form* (in the terminology of W. Thurston)

$$\Delta^t A_1 \dots A_k,$$

where A_i are the succesive possible longest *fragments of the word* Δ (in the terminology of S. I. Adyan) or *positive permutation braids* (in the terminology of E. El-Rifai and H. R. Morton). Certainly, the same way the *right-greedy form* is defined.

The center of the braid group was fist found by W.-L. Chow.

Garside normal form gives an elegant proof of the following theorem.

Theorem 3. *(i) When $n = 1$ the center of the group Br_{n+1} is generated by Δ.*
(ii) When $n > 1$ the center of the group Br_{n+1} is generated by Δ^2.

Proof. Let W be any element in the center. Then by definition for any element $X \in Br_{n+1}$ we have

$$WX = XW. \tag{1}$$

There are three possible forms of W:

a) $W = \Delta^m \overline{A}$, where $L(\overline{A} > 0$;
b) $W = \Delta^{2k+1}$;
c) $W = \Delta^{2k}$.

We consider each.

a) let $\overline{A} = a_i A_i$. Let $|s - i| = 1$. Cosider first case m even, put $X \equiv a_s a_i$. Then commutation equality (1) gives

$$\Delta^m a_i A_i a_s a_i = a_s a_i \Delta^m a_i A_i = \Delta^m a_s a_i a_i A_i.$$

Hence

$$a_i A_i a_s a_i = a_s a_i a_i A_i,$$

and so

$$a_i A_i a_s a_i \doteq a_s a_i a_i A_i.$$

Then by Proposition 2 $a_i a_i A_i \doteq a_i a_s A_s$ for some A_s so that

$$a_i A_i \doteq a_s A_s \tag{2}$$

Repeated application of (2) now gives

$$a_1 A_1 \doteq a_2 A_2 \doteq \ldots \doteq a_n A_n \doteq \overline{A}.$$

Hence $\overline{A}$ is divisible by Δ what is impossible. □

Let α be a positive word such that $\Delta \doteq \alpha X$, where X is an arbitrary positive word, probably empty. For any word W in Br_{n+1}, the word $\alpha^{-1} W \alpha$, reduced to Garside normal form is called an *α-transformation of W*.

For any word W in Br_{n+1} with the Garside normal form $\Delta^m \overline{A} \equiv W_1$ consider the following chains of α-transformations: take all the α-transformations of W_1 and let those which are of power $\geq m$ and which are distinct from each other be W_2, W_3, $\ldots$, W_t. Now repeat the process for each of the words W_2, W_3, $\ldots$, W_t in turn, denoting successively by W_{t+1}, W_{t+2}, $\ldots$, any new words occurring, the condition being always that each new word must be of power $\geq m$. Continue to repeat the process for every new distinct word arrising, as the sequence W_1, W_2, W_3, $\ldots$, expands.

Proposition 6. *The set W_1, W_2, W_3, $\ldots$, is finite.*

Suppose that in the set W_1, W_2, W_3, $\ldots$, the highest power reached is s and that the words of power s form the subset V_1, V_2, $\ldots$. Then this set V_1, V_2, $\ldots$ is called the *summit set* of W.

Theorem 4. (F. A. Garside) *Two elements A and B of the group Br_{n+1} are conjugate if and only if their summit sets are identical.*

3. Ordering of Braids

A group G is said *totally* (or *linearly*) *left* (correspondingly *right*) *ordered* if it has a total order $<$ invariant by left (right) multiplication, i.e. if $a < b$, then $ca < cb$ for any $a, b, c \in G$. If this order is also invariant by right (left) multiplication, then the group G is called *ordered*.

For any left ordered group G denote by P the set of positive elements $\{x \in G : x > 1\}$, then the set of negative elements is defined by the formula: $P^{-1} = \{x \in G : x^{-1} \in P\}$. The total character of an order on G is expressed by the partition

$$G = P \coprod \{1\} \coprod P^{-1}.$$

The invariance of multiplication is expressed by the inclusion $P^2 \subset P$, where P^2 is formed by products of couples of elements of P. Conversely, if there exist a subset P of a group G with the properties:

$$G = P \coprod \{1\} \coprod P^{-1}, \quad P^2 \subset P,$$

then G is left ordered by the order defined by: $x < y$ if and only if $x^{-1}y \in P$. A group G then is ordered if and only if $xPx^{-1} \subset P$ for all $x \in G$.

Let $i \in \{1, \dots, n\}$ and a word w on the alphabet $\{\sigma_1^{\pm 1}, \dots, \sigma_n^{\pm 1}\}$ is expressed in the form

$$w_0 \sigma_i w_1 \sigma_i \dots \sigma_i w_r,$$

where the subwords $w_0, \dots, w_r$ are the words on the letters $\sigma_j^{\pm 1}$ with $j > i$. Then such a word is called *σ_i-positive*. This means that all entries of $\sigma_i^{\pm 1}$ in the word w with i minimal must be positive. If all such entries are negative then a word w is called *σ_i-negative*. A braid of Br_{n+1} is called *σ_i-positive* (*σ_i-negative*) if there exists its expression as a word on the standard generators which is *σ_i-positive* (*σ_i-negative*). A braid is called *σ-positive* (*σ-negative*) if it exists a number i, such that it is *σ_i-positive* (*σ_i-negative*).

Theorem 5. (P. Dehornoy) *Every braid in Br_{n+1} different from 1 is either σ-positive or σ-negative.*

Corollary 2. *For all n the braid group Br_{n+1} is left ordered.*

Corollary 3. *For all n the braid group Br_{n+1} does not have elements of finite order.*

4. Representations

4.1. **Burau representation.** We denote the ring $\mathbb{Z}[t, t^{-1}]$ by Λ. Let us map the generators of the braid group Br_n to the following elements of the group $GL_n\Lambda$

$$\sigma_i \mapsto U_i = \begin{pmatrix} E_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & E_{n-i-1} \end{pmatrix}, \tag{3}$$

where E_i is the unit $i \times i$ matrix. The formula (3) define correctly the representation of the braid group in $GL_n\Lambda$:

$$\psi_n : Br_n \to GL_n\Lambda,$$

which is called *Burau representation.*

Theorem 6. *Let $n \geq 3$ and $V_1, V_2, ..., V_{n-1}$ be the $(n-1) \times (n-1)$ matrices over Λ given by*

$$V_1 = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & E_{n-3} \end{pmatrix}, \tag{4}$$

$$V_{n-1} = \begin{pmatrix} E_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}, \tag{5}$$

and for $1 < i < n-1$,

$$V_i = \begin{pmatrix} E_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & E_{n-i-2} \end{pmatrix}, \tag{6}$$

Then for all $i = 1, ..., n-1$,

$$C^1 U_i C = \begin{pmatrix} V_i & 0 \\ R_i & 1 \end{pmatrix}, \tag{7}$$

where C *is the* $n \times n$ *matrix*

$$C = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \tag{8}$$

and R_i *is the row of length* $n-1$ *equal to* 0 *if* $i < n-1$ *and to* $(0, ..., 0, 1)$ *if* $i = n-1$.

The formula

$$\psi_n^r(\sigma_i) = V_i$$

defines a group homomorphism

$$\psi_n^r : Br_n \to GL_{n-1}\Lambda$$

for all $n \geq 3$. It is called the *reduced Burau representation.*

Theorem 1. *Burau representation is faithful for* $n = 3$.

Proof. Consider the group homomorphism

$$\phi : GL_2\Lambda \to SL_2\mathbb{Z}$$

obtained by the substitution $t \mapsto -1$. It transforms the reduced Burau matrices

$$V_1 = \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix},$$

into the integral matrices

$$a_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

The group $SL_2\mathbb{Z}$ is generated by a_1, a_2 with defining relations

$$a_1 a_2 a_1 = a_2 a_1 a_2$$

and

$$(a_1 a_2 a_1)^4 = 1.$$

The homomorphism

$$\phi \circ \psi_n^r : Br_3 \to SL_2\mathbb{Z}$$

sends the standard braid generators σ_1, σ_2 to a_1, a_2, respectively. This homomorphism is surjective and its kernel is the normal subgroup generated by the braid $(\sigma_1\sigma_2\sigma_1)^4$. Since this braid is central in Br_3, the kernel in question is the cyclic group $((\sigma_1\sigma_2\sigma_1)^4) \subset Br_3$. Consequently,

$$\operatorname{Ker}\psi_3 \subset \operatorname{Ker}(\phi \circ \psi_3^r = ((\sigma_1\sigma_2\sigma_1)^4) \subset Br_3.$$

Observe that

$$V_1V_2V_1 = \begin{pmatrix} 0 & -t^2 \\ -t & 0 \end{pmatrix} \quad \text{and} \quad (V_1V_2V_1)^2 = \begin{pmatrix} t^3 & 0 \\ 0 & t^3 \end{pmatrix},$$

Therefore, for any nonzero $k \in \mathbb{Z}$,

$$\psi_3^r((\sigma_1\sigma_2\sigma_1)^{4k}) = (V_1V_2V_1)^{4k} = \begin{pmatrix} t^{6k} & 0 \\ 0 & t^{6k} \end{pmatrix} \neq E_2.$$

Hence

$$\operatorname{Ker}\psi_3 = \operatorname{Ker}(\phi \circ \psi_3^r) = \{1\}.$$

□

Theorem 2. (J. A. Moody; D. D. Long and M. Paton; S. Bigelow) Burau representation is unfaithful for $n \geq 5$.

The case $n = 4$ remains open.

4.2. **Lawrence-Krammer representation.** Consider the ring $K = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ of Laurent polynomials in two variables q, t, and the free K-module

$$V = \bigoplus_{1 \leq i < j \leq n} K\, x_{i,j}.$$

For $k \in \{1, 2, \ldots, n-1\}$, define the action of the braid generators σ_k on the basis of V by the formula:

$$\sigma_k(x_{i,j}) = \begin{cases} x_{i,j}, & k < i-1 \text{ or } j < k; \\ x_{i-1,j} + (1-q)x_{i,j}, & k = i-1; \\ tq(q-1)x_{i,i+1} + qx_{i+1,j}, & k = i < j-1; \\ tq^2 x_{i,j}; & k = i = j-1 \\ x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1}, & i < k < j-1; \\ x_{i,j-1} + tq^{j-i}(q-1)x_{j-1,j}, & k = j-1; \\ (1-q)x_{i,j} + qx_{i,j+1}, & k = j. \end{cases} \tag{9}$$

Direct comutation shows that this defines a representation

$$\rho_n : Br_n \to GL(V),$$

which was firstly defined by R. Lawrence in topological terms and in the explicit form (9) by D. Krammer.

Theorem 3. (S. Bigelow, D. Krammer) *The representation*

$$\rho_n : Br_n \to GL\,(V)$$

is faithful for all $n \geq 1$.

Actually, S Bigelow proved this theorem for the representation ρ_n characterized in homological terms and D. Krammer proved the following. Let $K = \mathbb{R}[t^{\pm 1}]$, $q \in \mathbb{R}$, and $0 < q < 1$. Then the representation ρ_n defined by (9) is faithful for all $n \geq 1$. This result implies Theorem 3: if a representation over $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ becomes faithful after assigning a real value to q, then it is faithful itself.

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BRAIDS, CONFIGURATION SPACES AND KNOTS

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November 2012

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Lecture 4

Knot projections

Let E be a plane in $\mathbb{R}^3$. We consider an orthogonal projection of $\mathbb{R}^3$ onto a plane E

$$p : \mathbb{R}^3 \to E$$

and the image of a knot $\mathcal{K}$ by p to E which is called a *projection of a knot.* A point $P \in p(\mathcal{K}) \subset E$ whose preimage $p^{-1}(P)$ under the projection p contains more that one point is called a *multiple* point.

Definition 1. *A double point P of a knot projection, that is $p^{-1}(P)$ contains two points, is also called a crossing.*

Definition 2. *A projection of a knot $\mathcal{K}$ is called regular if*
(1) there are only finitely many multiple points $\{P_1, \dots, P_n\}$, and all multiple points are double points;
(2) no vertex of $\mathcal{K}$ is mapped onto a double point.

Definition 3. *The minimal number of crossings in a regular projection of a knot is called the order of a knot.*

Proposition 1. *The set of regular projections is open and dense in the space of all projections.*
□

The projection of a knot does not determine the knot, but if at every double point in a regular projection the overcrossing line is marked the knot can be reconstructed from the projection.

If a knot is oriented the projection inherits the orientation. The regular projection of a knot with this additional information is called a *knot projection* or *knot diagram.* Two knot diagrams will be regarded as *equal* if they are isotopic in E as graphs, where the isotopy is required to respect overcrossing and undercrossing.

Definition 4. *Two knots diagrams are called equivalent if they are connected by a finite sequence of Reidemeister moves R1, R2, R3, which are described in Figure 1.*

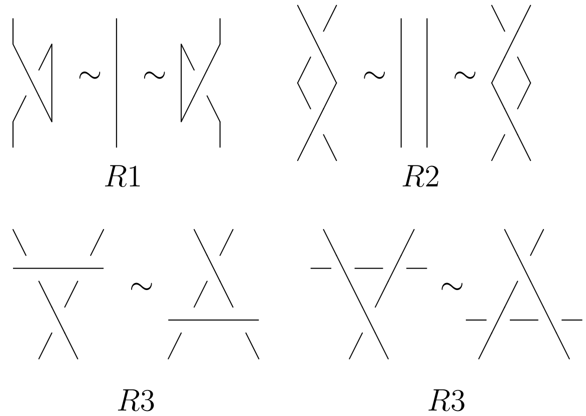

Fig. 1: Reidemeister moves for knots

Each Reidemeister move effect local changes in the diagram. Evidently all these operations can be realized by an ambient isotopy of the knot; equivalent diagrams therefore define equivalent knots. The converse is also true.

Proposition 2. *Two knots are equivalent if and only if all their diagrams are equivalent.*

Proof. The first step in the proof will be to verify that any two regular projections p_1, p_2 of the same simple closed polygon K are connected by $R_i^{\pm 1}$-moves. Let p_1, p_2 be represented by points on S^2, and choose on S^2 a polygonal path s from p_1 to p_2 in general position with respect to the lines of singular projections on S^2. When such a line is crossed the diagram will be changed by an operation $R_i^{\pm 1}$, the actual type depending on the type of singularity, corresponding to the line that is crossed.

It remains to show that for a fixed projection equivalent knots possess equivalent diagrams. According to Theorem on equivalence of equivalences it suffices to show that a $\Delta^{\pm 1}$-move induces $R_i^{\pm 1}$-operations on the projection. This again is easily verified. □

Knot group

Definition 5. *The knot group of a knot K is defined as the fundamental group of the knot complement of K in $\mathbb{R}^3$,*

$$\pi_1(\mathbb{R}^3 \backslash K).$$

THE WIRTINGER PRESENTATION

On a diagram of a knot K the is finite number of arcs $a_1, ..., a_n$. Each a_i is assumed connected to a_{i-1} and $a_{i+1}(\text{mod } n)$ by undercrossing arcs. The union of these arcs is the knot K. Let the letter x_i, $i = 1, \ldots, n$ corresponds to the loop which goes around the arc a_i,. Now at each crossing, where meet arcs a_i, a_{i+1} and some a_k there is a certain relation r_i among the loops x_i, x_{i+1} and x_k obviously must hold:

$$x_i x_k = x_k x_{i+1} \quad or \quad x_k x_i = x_{i+1} x_k$$

according to the type of the crossing.

Theorem 1. *The group $\pi_1(\mathbb{R}^3 \setminus K)$ is generated by the (homotopy classes of the) x_i and has presentation*

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, ..., x_n \, ; \, r_1, ..., r_n \rangle$$

Moreover, any one of the r_i may be omitted and the above remains true.

Seifert surface, genus of a knot and linking number

Theorem 2. *A knot $\mathcal{K}$ in $\mathbb{R}^3$ (a simple closed curve $\mathcal{K} \subset \mathbb{R}^3$) is the boundary of an orientable surface, embedded in $\mathbb{R}^3$. It is called a Seifert surface.*

Proof. Let $D(\mathcal{K})$ be a regular projection of $\mathcal{K}$ equipped with an orientation. By altering $D(\mathcal{K})$ in the neighbourhood of double points according to the orientation of $\mathcal{K}$ diagram $D(\mathcal{K})$ dissolves into a number of disjoint oriented simple closed curves which are called *Seifert cycles*. Choose an oriented 2-cell for each Seifert cycle, and embed the 2-cells in $\mathbb{R}^3$ as a disjoint union such that their boundaries are projected onto the Seifert cycles. The orientation of a Seifert cycle is to coincide with the orientation induced by the oriented 2-cell. We may place the 2-cells into planes $z =$ const parallel to the projection plane ($z = 0$), and choose planes $z = a_1$, $z = a_2$ for corresponding Seifert cycles c_1, c_2 with $a_1 < a_2$ if c_1 contains c_2. Now we can undo the cut-and-paste-process by joining the 2-cells at each double point by twisted bands such as to obtain a connected surface S with $\partial S = \mathcal{K}$. □

Definition 6. *A minimal genus g of a Seifert surface spanning a knot $\mathcal{K}$ is called the genus of the knot $\mathcal{K}$.*

Note that Theorem on equivalence of equivalences is true for links. Notions of knot diagram can be evidently generalized to link diagrams, Reidemeister moves are the same and equivalence of diagrams corresponds to equivalence of links.

Let $\mathcal{L}$ be an oriented two-component link and α and β are its components. Consider the crossing points of these two components $P_1, \ldots, P_n$. Put $\epsilon(P_i) = 1$ if the arrow of the over-crossing branch meets the arrow of undercrossing branch going anticlockwise, and $\epsilon(P_i) = -1$ in the opposite case. Define the linking number $\mathrm{lk}(\mathcal{L})$ by the formula

$$\mathrm{lk}(\mathcal{L}) = \mathrm{lk}(\alpha, \beta) = \frac{1}{2} \sum_{i=1}^{n} \epsilon(P_i).$$

Note that the number of crossing points of two components is even ($n = 2k$), so $\mathrm{lk}(\mathcal{L})$ is always an integer.

Theorem 3. *The integer* $\mathrm{lk}(\alpha, \beta)$ *is a correctly defined invariant of 2-component links.*

Proof. The integer $\mathrm{lk}(\alpha, \beta)$ in invariant under the Reidemeister moves R2 and R3. $\square$

Examples. H. Hopf link, J.H.C.Whitehead link, Borromean rings.

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BRAIDS, CONFIGURATION SPACES AND KNOTS

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Lecture 5

Alexander-Conway polynomial

Alexander-Conway polynomial invariant is described by three axioms:
AXIOM 1. To each oriented knot or link K there is associated a polynomial $\nabla_K(z) \in \mathbb{Z}[z]$. Equivalent knots and links receive identical polynomials:

$$K \sim K' \implies \nabla_K(z) = \nabla_{K'}(z).$$

AXIOM 2. If $K \sim 0$ (the unknot) then $\nabla_K(z) = 1$.
AXIOM 3. Suppose that three knots or links L_+, L_- and L_0 differ at the site of one crossing in the standard way.

Then

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z)$$

We call this the *skein relation*.

Definition 1. *A link is split if it is equivalent to a link with diagram containing two nonempty parts that live in disjoint neighborhoods.*

Theorem 1. *If L is a split link then $\nabla_L(z) = 0$.*

Corollary 1. *If L is an n-component unlink ($n > 1$) then $\nabla_L(z) = 0$.*

Jones polynomial

Jones polynomial invariant is described by three axioms:
AXIOM 1. To each oriented knot or link K there is associated a Laurent polynomial $V_K \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$. Equivalent knots and links receive identical polynomials:

$$K \sim K' \implies V_K = V_{K'}.$$

AXIOM 2. If $K \sim 0$ (the unknot) then $V_K = 1$.
AXIOM 3. Suppose that three knots or links L_+, L_- and L_0 differ at the site of one crossing in the standard way.

Then

$$t^{-1}V_{L_+} - tV_{L_-} = (t^{1/2} - t^{-1/2})V_{L_0}$$

Jones Polynomial f in Kauffman variable $A = t^{-1/4}$:

AXIOM 3.

$$A^4 f_{L_+} - A^{-4} f_{L_-} = (A^{-2} - A^2) f_{L_0}$$

Jones Polynomial K in Khovanov setting; variable $q = -A^{-2} = -t^{1/2}$:

AXIOM 2. If $L \sim 0$ (the unknot) then $K_L = q + q^{-1}$.

AXIOM 3.

$$q^{-2}K_{L_+} - q^2 K_{L_-} = (q^{-1} - q)K_{L_0}$$

Suppose a braid is placed in a cube. On the boundary of the cube join the point A_i (Figure 1) to the point B_i by a mutually disjoint simple arc C_i. Since our initial braid does not intersect the boundary of the cube except at the points $A_1, \dots, A_n$ and $B_1, \dots, B_n$ we obtain a link (or, in particular, a knot) i.e. a system of simple closed curves in $\mathbb{R}^3$. A link obtained in such a manner is called the *closure* of the braid.

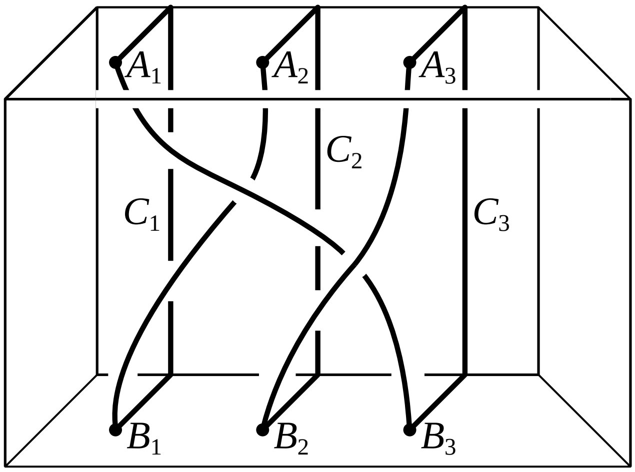

FIGURE 1.

Theorem 2. (J. W. Alexander) *Any link can be represented by a closed braid.*

The next step is to understand equivalence classes of braids which correspond to links. Markov Theorem gives an answer to this question. At first we define two types of Markov moves for braids.

Type 1 Markov move replaces a braid β on n strings by its conjugate $\gamma\beta\gamma^{-1}$.

Type 2 Markov move replaces a braid β on n strings by the braid $j_n(\beta)\sigma_n$ on $n+1$ strings or by $j_n(\beta)\sigma_n^{-1}$ where j_n is the canonical inclusion of the group Br_n into the group Br_{n+1}

$$j_n : Br_n \to Br_{n+1}.$$

Theorem 3. (A. A. Markov) *Suppose that β and β' are two braids (not necessary with the same number of strings). Then, the closures of β and β' represent the same link if and only if β can be transformed into β' by means of a finite number of type 1 and type 2 Markov moves. Namely there existes the following sequence,*

$$\beta = \beta_0 \to \beta_1 \to \cdots \to \beta_m = \beta',$$

such that, for $i = 0, 1 \dots, m-1$, β_{i+1} is obtained from β_i by the application of a type 1 or 2 Markov moves or their inverses.

In other words, if we consider the disjoint union of all braid groups

$$\amalg_{n=1}^{n=\infty} Br_n,$$

then the Markov moves of types 1 and 2 define the equivalence relation on this set $\sim$ such that the quotient set

$$\amalg_{n=1}^{n=\infty} Br_n / \sim$$

is in one-to-one correspondence with isotopy classes of links.

There exists a lot of proofs of Markov Theorem, see for example the work of P. Traczyk [2].

References

[1] *G. Burde; H. Zieschang*, Knots. Second edition. de Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 2003. xii+559 pp.

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[3] *P. Vogel*, Representation of links by braids: a new algorithm. Comment. Math. Helv. 65 (1990), no. 1, 104113.