Homotopy homomorphisms and the classifying space functor

Abstract

Let $\Omega_M : \mathcal{T}op^* \to \mathcal{M}on$ be the Moore loop space functor from the category of well-pointed pathconnected spaces (here spaces always will mean compactly generated spaces without any separation conditions) to the category of topological monoids and continuous homomorphisms. The classifying space functor $B : \mathcal{M}on \to \mathcal{T}op^*$ almost behaves like a left adjoint of Ω_M. We will show that B and Ω_M are an adjoint pair in a derived sense: Let $Ho\mathcal{M}on$ be obtained from $\mathcal{M}on$ by formally inverting all homomorphisms whose underlying maps are homotopy equivalences, and let $Ho\mathcal{T}op^*$ be the usual homotopy category. We show

Theorem: The category $Ho\mathcal{M}on$ exists, and the functors B and Ω_M induce an adjoint pair

$$B : Ho\mathcal{M}on \longleftrightarrow Ho\mathcal{T}op^* : \Omega_M$$

In fact we prove more: We construct $Ho\mathcal{M}on$ as the homotopy category of the category of monoids and homotopy homomorphisms. For that purpose we improve the classical definition of a homotopy homomorphisms. Let $HMor(A, B)$ denote the space of homotopy homomorphisms from A to B. We show that there is a natural homotopy equivalence

$$HMor(A, \Omega_M(X)) \to \mathcal{T}op^*(B(A), X),$$

which implies the theorem.

The theorem has a number of interesting consequences:

- Let $J : \mathcal{T}op^* \to \mathcal{M}on$ be the free based monoid functor of James. As an immediate corollary of the theorem we obtain that

  $$BJ(X) \simeq \Sigma(X).$$

  This in turn gives another short proof of Puppe's version of James's theorem:
  If X is a pathconnected numerably contractible well-pointed space, then there is a natural homotopy equivalence

  $$J(X) \simeq \Omega_M \Sigma(X)$$

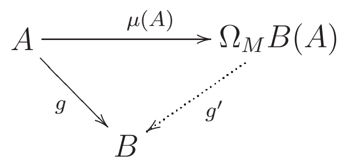
  which is a homotopy homomorphism.

- The classifying space functor preserves homotopy colimits up to homotopy equivalences. (I am not aware of a proof of this result in its full generality. I know a proof of the analogue statement up to weak homotopy equivalences, which is quite involved.)

- Recall that a monoid A is called grouplike if the monoid structure has homotopy inverses. There is a natural homotopy homomorphism

$$\mu(A) : A \to \Omega_M B(A),$$

which is a homotopy equivalence iff A is grouplike. The map $\mu(A)$ is a group completion in the following sense: Given a solid arrow diagram



in $Ho\mathcal{M}on$ with a grouplike B, then there is a unique morphism $g' : \Omega_M B(A) \to B$ in $Ho\mathcal{M}on$ making the diagram commute.