Homotopy homomorphisms and the classifying space functor

Abstract

Let $\Omega_M : \text{Top}^* \to \text{Mon}$ be the Moore loop space functor from the category of well-pointed pathconnected spaces (here spaces always will mean compactly generated spaces without any separation conditions) to the category of topological monoids and continuous homomorphisms. The classifying space functor $B : \text{Mon} \to \text{Top}^*$ almost behaves like a left adjoint of $\Omega_M$. We will show that $B$ and $\Omega_M$ are an adjoint pair in a derived sense: Let $\text{HoMon}$ be obtained from $\text{Mon}$ by formally inverting all homomorphisms whose underlying maps are homotopy equivalences, and let $\text{HoTop}^*$ be the usual homotopy category. We show

**Theorem:** The category $\text{HoMon}$ exists, and the functors $B$ and $\Omega_M$ induce an adjoint pair

$$B : \text{HoMon} \leftrightarrow \text{HoTop}^* : \Omega_M$$

In fact we prove more: We construct $\text{HoMon}$ as the homotopy category of the category of monoids and homotopy homomorphisms. For that purpose we improve the classical definition of a homotopy homomorphism. Let $HMor(A, B)$ denote the space of homotopy homomorphisms from $A$ to $B$. We show that there is a natural homotopy equivalence

$$HMor(A, \Omega_M(X)) \to \text{Top}^*(B(A), X),$$

which implies the theorem.

The theorem has a number of interesting consequences:

- Let $J : \text{Top}^* \to \text{Mon}$ be the free based monoid functor of James. As an immediate corollary of the theorem we obtain that

$$BJ(X) \simeq \Sigma(X).$$

This in turn gives another short proof of Puppe’s version of James’s theorem:

If $X$ is a pathconnected numerably contractible well-pointed space, then there is a natural homotopy equivalence

$$J(X) \simeq \Omega_M \Sigma(X)$$

which is a homotopy homomorphism.

- The classifying space functor preserves homotopy colimits up to homotopy equivalences. (I am not aware of a proof of this result in its full generality. I know a proof of the analogue statement up to weak homotopy equivalences, which is quite involved.)
• Recall that a monoid $A$ is called grouplike if the monoid structure has homotopy inverses. There is a natural homotopy homomorphism

$$\mu(A) : A \to \Omega_M B(A),$$

which is a homotopy equivalence iff $A$ is grouplike. The map $\mu(A)$ is a group completion in the following sense: Given a solid arrow diagram

in $\text{HoMon}$ with a grouplike $B$, then there is a unique morphism $g' : \Omega_M B(A) \to B$ in $\text{HoMon}$ making the diagram commute.