Homotopy homomorphisms and the classifying space functor

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1 Introduction

Throughout this course **space** will mean a well-pointed k-space and **monoid** a topological monoid with well-pointed unit. Here *well-pointed* means that the inclusion of the basepoint is a closed cofibration, and a space X is a *k-space* if $A \subset X$ is closed iff $p^{-1}(A)$ is closed in C for each map $C \to X$ where C is a compact Hausdorff space.

Let $\mathcal{T}op^*$ denote the category of spaces, $\mathcal{M}on$ the category of monoids and $\mathcal{CM}on \subset \mathcal{M}on$ its full subcategory of commutative monoids. There is a important functor

$$B : \mathcal{M}on \longrightarrow \mathcal{T}op^*,$$

the classifying space functor which preserves colimits and finite limits (an explicit definition will be given later). If M is a commutative monoid, its multiplication

$$\mu : M \times M \longrightarrow M$$

is a homomorphism and induces a multiplication

$$BM \times BM \cong B(M \times M) \xrightarrow{B\mu} BM$$

which makes BM a commutative monoid. The following is easy to prove.

1.1 Theorem: $B : \mathcal{CM}on \to \mathcal{CM}on$ is left adjoint to the loop space functor $\Omega : \mathcal{CM}on \to \mathcal{CM}on$, where ΩM is given the pointwise multiplication.

In this course we will try to obtain a similar result for

$$B : \mathcal{M}on \longrightarrow \mathcal{T}op^*.$$

In view of Theorem 1.1 the *Moore loop space functor*

$$\Omega' : \mathcal{T}op^* \longrightarrow \mathcal{M}on$$

is a candidate for the right adjoint.

1.2 The Moore path space: Let X be a (not necessarily based) space. The *Moore path space* of X is the subspace $\mathrm{Path}(X) \subset X^{\mathbb{R}_+} \times \mathbb{R}_+$ consisting of all pairs (w, r) such that $w(t) = w(r)$ for all $t \geq r$. We call r the *length* of w and denote it by $r = l(w)$.
There are maps $\delta_+, \delta_- : \mathrm{Path}(X) \to X$ defined by $\delta_+(w, r) = w(r)$ and $\delta_-(w, r) = w(0)$.
For two paths (w_1, r_1) and (w_2, r_2) with $\delta_+(w_1, r_1) = \delta_-(w_2, r_2)$ we define *path addition* by

$$(w_1, r_1) + (w_2, r_2) = (w, r_1 + r_2)$$

with

$$w(t) = \begin{cases} w_1(t), & 0 \leq t \leq r_1, \\ w_2(t - r_1) & r_1 \leq t. \end{cases}$$

If $(X, *)$ is a based space, the *Moore loop space* $\Omega'(X) \subset \mathrm{Path}(X)$ is the subspace of all pairs (w, r) with $\delta_+(w, r) = \delta_-(w, r) = *$. Path addition defines a monoid structure on $\Omega' X$ with $(c, 0)$ as unit, where $c(0) = *$.

1.3 Exercise: (1) Show that Ω' defines a functor $\mathcal{T}op^* \to \mathcal{M}on$. (You have to show that $\Omega' X$ is a well-pointed topological monoid!)

(2) Let ΩX be the usual loop space. It is embedded in $\Omega' X$ as the loops of lengths 1. Show that ΩX is a deformation retract of $\Omega_M X$.

(Hint: Investigate the homotopy

$$\varphi : \Omega' X \times I \longrightarrow \Omega' X, \quad \varphi : ((w, r); t) = (w_t, r_t)$$

with $r_t = (1 - t) \cdot r + t$

$$w_t(s) = w\left(\frac{r \cdot s}{r_t}\right) \text{ if } r > 0$$

and $\varphi((c, 0), t) = (c, 0)$.)

Now any right adjoint preserves products, but we have

1.4 Exercise: There is no product preserving functor $L : \mathcal{T}op^* \to \mathcal{M}on$ such that $LX \simeq \Omega X$ (homotopy equivalent) for each $X \in \mathcal{T}op^*$.

(Hint: Take $X = \Omega' Y$ and show that $L(X)$ is a commutative monoid. If $LX \simeq \Omega X$, this would imply that $\Omega^2(Y)$ is homotopy equivalent to a commutative monoid for each $Y \in \mathcal{T}op^*$ which is known to be false)

So Ω' has no chance to be right adjoint to B. Never-the-less, there is evidence that Ω' is right adjoint to B **up to homotopy**. This evidence comes from results about homotopy homomorphisms.

2 Homotopy homomorphisms

In 1960 Sugawara introduced the notion of a strongly homotopy multiplicative map between monoids, which we will call a homotopy homomorphisms or h-morphism, for short [6].

2.1 Definition: A *homotopy homomorphism* or *h-morphism* $f : M \to N$ between two monoids is a sequence of maps

$$f_n : M^{n+1} \times I^n \longrightarrow N \quad n \in \mathbb{N}$$

such that $(x_i \in M, t_j \in I)$

$$f_n(x_0, t_1, x_1, t_2, \dots, t_n, x_n) = \begin{cases} f_{n-1}(x_0, t_1, \dots, x_{i-1} \cdot x_i, \dots, t_n, x_n) & \text{if } t_i = 0 \\ f_{i-1}(x_0, t_1, \dots, x_{i-1}) \cdot f_{n-i}(x_i, t_{i+1}, \dots, x_n) & \text{if } t_i = 1 \end{cases}$$

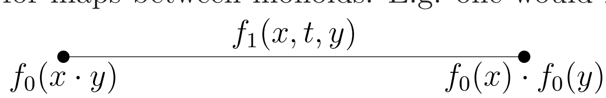
We call $f_0 : M \to N$ the *underlying map* of f.

If in addition

$$f_n(x_0, t_1, x_1, t_2, \dots, t_n, x_n) = \begin{cases} f_{n-1}(x_1, t_2, \dots, x_n) & \text{if } x_0 = e \\ f_{n-1}(x_0, \dots, x_{i-1}, \max(t_i, t_{i+1}), x_{i+1}, \dots, x_n) & \text{if } x_i = e \\ f_{n-1}(x_0, t_1, \dots, x_{n-1}) & \text{if } x_n = e \end{cases}$$

where $e \in M$ is the unit, we call f a *unitary homotopy homomorphism* or *uh-morphism*, for short.

Since an h-morphism does not pay tribute to the unit it does not seem to be the right notion for maps between monoids. E.g. one would like the path



to be the constant one, if x or y is the unit. Unitary h-morphisms have this property. Never-the-less, in the past one usually considered h-morphisms because the additional conditions for uh-morphisms make it harder to work with them.

The most extensive study of h-morphisms and their induced maps on classifying spaces was done by Fuchs [1], who constructed composites of h-morphisms, proved that composition is homotopy associative and stated that an h-morphism $f : M \to N$ whose underlying map is a homotopy equivalence has a homotopy inverse h-morphisms $g : N \to M$. In fact, he constructed g_0, g_1 and the homotopies $g \circ f \simeq \mathrm{id}$ and $f \circ g \simeq \mathrm{id}$ in dimensions 0 and 1 in [1, p.205-p.208], but left the rest to the reader.

We handle these problems by interpreting uh-morphisms as genuine homomorphisms of a "cofibrant" replacement of M.

By a *semigroup* we will mean a k-space with an associative multiplication. Let $\mathcal{S}\mathrm{gp}$ denote the category of semigroups and continuous homomorphisms.

2.2 Constructions: We will construct continuous funtctors

$$\overline{W} : \mathcal{S}\mathrm{gp} \longrightarrow \mathcal{S}\mathrm{gp} \quad \text{and} \quad W : \mathcal{M}on \longrightarrow \mathcal{M}on$$

and natural transformations

$$\overline{\varepsilon} : \overline{W} \longrightarrow \mathrm{Id} \quad \text{and} \quad \varepsilon : W \longrightarrow \mathrm{Id}$$

as follows:

$$\overline{W}M = \left(\coprod_{n=0}^{\infty} M^{n+1} \times I^n \right) / \sim$$

with the relation

(1) $(x_0, t_1, \ldots, t_n, x_n) \sim (x_0, t_1, \ldots, x_{i-1} \cdot x_i, \ldots, t_n, x_n) \qquad \text{if } t_i = 0$

and WM is the quotient of $\overline{W}M$ by imposing the additional relations

(2)
$$\begin{aligned} (x_0, t_1, \ldots, t_n, x_n) &\sim (x_1, t_2, \ldots, t_n, x_n) && \text{if } x_0 = e \\ &\sim (x_0, \ldots, x_{i-1}, \max(t_i, t_{i+1}), x_{i+1}, \ldots, x_n) && \text{if } x_i = e \\ &\sim (x_0, t_1, \ldots, t_{n-1}, x_{n-1}) && \text{if } x_n = e \end{aligned}$$

The multiplications of $\overline{W}M$ and WM are given on representatives by

$$(x_0, t_1, \ldots, x_k) \cdot (y_0, u_1, \ldots, y_l) = (x_0, t_1, \ldots x_k, 1, y_0, u_1, \ldots, y_l).$$

The natural transformations $\overline{\varepsilon}$ and ε are defined by

$$\overline{\varepsilon}(M),\ \varepsilon(M) : (x_0, t_1, \ldots, x_k) \longmapsto x_0 \cdot x_1 \cdot \ldots \cdot x_k.$$

Their underlying maps have natural sections

$$\overline{\iota}(M),\ \iota(M) : x \longmapsto (x)$$

which are not homomorphisms, and there is a homotopy over M

$$h_s : (x_0, t_1, x_1, \ldots, x_n) \longmapsto (x_0, s \cdot t_1, \ldots, s \cdot t_n, x_n)$$

from $\overline{\iota}(M) \circ \overline{\varepsilon}(M)$ respectively $\iota(M) \circ \varepsilon(M)$ to the identity. In particular, $\overline{\varepsilon}(M)$ and $\varepsilon(M)$ are shrinkable as maps.

By inspection we see

2.3 Observation: (1) h-morphisms $(f_n) : M \to N$ correspond bijectively to homomorphisms $\overline{f} : \overline{W}M \to N$ of semigroups, and $f_0 = \overline{f} \circ \overline{\iota}(M)$

(2) uh-morphisms $(f_n) : M \to N$ correspond bijectively to homomorphisms $f : WM \to N$ of monoids, and $f_0 = f \circ \iota(M)$

2.4 Definition: We call a homomorphism $f : M \to N$ in $\mathcal{M}on$ or $\mathcal{S}\text{gp}$ a *weak equivalence* if its underlying map of spaces is a homotopy equivalence.

Both constructions have a universal property, which is a consequence of the following result. We give $\mathcal{T}op^*(X, Y)$ the k-function space topology, obtained by turning the space $\text{Maps}(X, Y)$ of all maps from X to Y with the compact-open topology into a k-space. We give $\mathcal{M}on(M, N)$ and $\mathcal{S}\text{gp}(M, N)$ the subspace topologies of the corresponding function spaces in $\mathcal{T}op^*$.

2.5 Proposition: (1) Let M be a monoid and $p : X \to Y$ a homomorphism of monoids. Let

$$p_* : \mathcal{M}on(WM, X) \longrightarrow \mathcal{M}on(WM, Y)$$

be the induced map. If p is a fibration of underlying spaces, so is p_*. If p is a weak equivalence, p_* is a homotopy equivalence.

(2) The same holds for $\overline{W}$ in the category $\mathcal{S}\text{gp}$.

The proof is by induction on the filtration of WM by the submonoids $W^{(n)}M$ generated by elements z having a representative $(x_0, t_1, \ldots, t_k, x_k)$ with $k \leq n$. For the fibration part the inductive step uses Strøm's results about fibrations and cofibrations [5] and for the homotopy equivalence part it uses the following lemma which is of separate interest.

2.6 HELP-Lemma: In the category $\mathcal{T}op$ of non-based k-spaces the following statements are equivalent:

(1) $p : X \to Y$ is a homotopy equivalence.

(2) For any homotopy commutative square with i a cofibration

$$\begin{array}{ccc} A & \xrightarrow{f_A} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

and any homotopy $H_A : A \times I \to Y$ from $p \circ f_A$ to $g \circ i$ there is a map $f : B \to X$ and a homotopy $H : B \times I \to Y$ from $p \circ f$ to g such that

$$f \circ i = f_A \quad \text{and} \quad H \circ (i \times \mathrm{id}) = H_A.$$

2.7 Exercise: Prove the lemma.

(Hint: Replace p by the associated path space fibration and use Strøm's results about fibrations and cofibrations [5].)

As an immediate consequence of Proposition 2.5 we obtain the

2.8 Lifting Theorem: Given homomorphisms of monoids

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle p} \\ WM & \xrightarrow{f} & Y \end{array}$$

such that p is a weak equivalence, then there exists a homomorphism $g : WM \to X$, unique up to homotopy in $\mathcal{M}on$ (i.e. a homotopy through homomorphisms) such that $f \simeq p \circ g$ in $\mathcal{M}on$.
If in addition the underlying map of p is a fibration there is a homomorphism $g : WM \to X$, unique up to homotopy in $\mathcal{M}on$, such that $f = p \circ g$.

(2) For $\overline{W}$ the analogous results hold in the category $\mathcal{S}$gp.

2.9 Exercise: (1) Deduce 2.8 from 2.5 (easy)

(2) Prove 2.8 directly by construction g inductively on $W^{(n)}M$.
(Use the HELP-Lemma.)

The maps

$$\begin{aligned} \varepsilon(N)_* : \mathcal{M}on(WM, WN) &\to \mathcal{M}on(WM, N) \\ \varepsilon(N)_* : \mathcal{S}\mathrm{gp}(\overline{W}M, \overline{W}N) &\to \mathcal{S}\mathrm{gp}(\overline{W}M, N) \end{aligned}$$

are homotopy equivalences. So from a homotopy theoretic point of view it makes sense to change our notations of homotopy homomorphisms:

2.10 Definition: A *unitary homotopy homomorphism*, *uh*-morphism (in the new sense) from M to N is a homomorphism $f : WM \to WN$. Its *underlying map* is $\varepsilon_N \circ f \circ \eta_M$.
Analogously, an *h-morphism* (in the new sense) from the semigroup M to the semigroup N is a homomorphism $\overline{W}M \to \overline{W}N$.

This solves the problem of composition, and from 2.5 we obtain

2.11 Proposition: If $f : WM \to WN$ is a weak equivalence, i.e. a *uh*-morphism from M to N, whose underlying map is a homotopy equivalence, then f is a homotopy equivalence in the category $\mathcal{M}on$.
The analogous statement in $\mathcal{S}$gp holds for homomorphisms $\overline{W}M \to \overline{W}N$.

3 The classifying space and the Moore loop space functors as homotopically adjoint pair

3.1 The 2-sided bar construction: Let $\mathcal{C}$ be a small topologically enriched category, X a $\mathcal{C}^{op}$-diagram and Y a $\mathcal{C}$-diagram in $\mathcal{T}op$. We define a simplicial space $B_\bullet(X, \mathcal{C}, Y)$ by

$$B_n(X, M, Y) = \coprod_{A,B\in\mathcal{C}} X(B) \times \mathcal{C}_n(A, B) \times Y(A),$$

where $\mathcal{C}_n(A, B)$ is the space of all composable n-tuples of morphisms $(f_1, \ldots, f_n)$ such that source$(f_n) = A$ and target$(f_1) = B$

with boundary and degeneracy maps given by

$$\begin{array}{ll} d^i(x, f_1, \ldots, f_n, y) = (X(f_1)(x), f_2, \cdots, f_n, y) & i = 0 \\ d^i(x, f_1, \ldots, f_n, y) = (x, f_1, \ldots, f_i \cdot f_{i+1}, \ldots, f_n, y) & 0 < i < n \\ d^i(x, f_1, \ldots, f_n, y) = (x, f_1, \ldots, f_{n-1}, Y(f_n)(y)) & i = n \\ s^i(x, f_1, \ldots, f_n, y) = (x, f_1, \ldots, f_i, \mathrm{id}, f_{i+1}, \ldots, f_n, y) & 0 \le i \le n \end{array}$$

We consider a topological monoid as a topologically enriches category with one object and define the *classifying space functor*

$$B : \mathcal{M}on \longrightarrow \mathcal{T}op^*$$

by $BM = B(*, M, *)$. We will also make use of the variant

$$\widetilde{B} : \mathcal{M}on \longrightarrow \mathcal{T}op^*$$

where the topological realizations of $B_\bullet(*, M, *)$ is replaced by the fat realization which disregards degeneracies.

Let

$$\Omega' : \mathcal{T}op^* \longrightarrow \mathcal{M}on$$

be the Moore loop space functor defined in the introduction.

Let $\mathcal{HM}on$ denote the category of monoids and uh-morphisms, so that $\mathcal{HM}on(M, N) = \mathcal{M}on(WM, WN)$.

In this section we outline a proof of

3.2 Theorem: The functors

$$B : \mathcal{HM}on \rightleftarrows \mathcal{T}op^* : W\Omega'$$

are a homotopically adjoint pair. More precisely, there is a continuous natural map

$$\lambda : \mathcal{HM}on(M, \Omega' X) = \mathcal{M}on(WM, W\Omega' X) \longrightarrow \mathcal{T}op^*(BWM, X)$$

which is a homotopy equivalence.

The proof uses methods from enriched category theory. All our categories are topologically enriched: their morphism sets are topologized and composition is continuous.

3.3 Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be topologically enriched categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *continuous* if

$$F : \mathcal{A}(A, B) \longrightarrow \mathcal{B}(FA, FB)$$

is continuous for all A and B in $\mathcal{A}$.
A natural transformation $\alpha : F \Rightarrow G$ of continuous functors $F, G : \mathcal{A} \to \mathcal{B}$ is called *continuous* if

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{F} & \mathcal{B}(FA,FB) \\ \downarrow G & & \downarrow \alpha(B)_* \\ \mathcal{B}(GA,GB) & \xrightarrow{\alpha(A)^*} & \mathcal{B}(FA,GB) \end{array}$$

is a commutative diagram of continuous maps for all A and B in $\mathcal{A}$.
A **collection** of morphisms $\{\beta(A) : FA \to GA;\ A \in \mathrm{ob}\,\mathcal{A}\}$ is called a *continuous natural transformation up to homotopy* if the above diagram is homotopy commutative.

3.4 Exercise: Show that B, Ω and Ω' are continuous functors.

Two continuous functors

$$F : \mathcal{A} \rightleftarrows \mathcal{B} : G$$

form a continuous adjoint pair such that

$$\mathcal{B}(FA, X) \cong \mathcal{A}(A, GX)$$

is a homeomorphism iff there are continuous natural transformations

$$\eta : FG \longrightarrow \mathrm{Id}_{\mathcal{B}} \quad \text{and} \quad \mu : \mathrm{Id}_{\mathcal{A}} \longrightarrow GF,$$

the *counit* and *unit*, so that

$$G\eta \circ \mu G = \mathrm{id}_G \quad \text{and} \quad \eta F \circ F\mu = \mathrm{id}_F$$

(e.g. see [3, p. 49]).
Adapting the proof of this to our situation, we have to construct a homotopy unit and homotopy counit

$$\mu(WM) : WM \longrightarrow W\Omega' BWM \quad \text{and} \quad \eta(X) : BW\Omega' X \longrightarrow X,$$

i.e. continuous natural transformations up to homotopy such that

3.5 $W\Omega'\eta(X) \circ \mu(W\Omega' X) \simeq \mathrm{id}_{W\Omega' X}$ and $\eta(BWM) \circ B\mu(WM) \simeq \mathrm{id}_{BWM}$.

Then the map λ of Theorem 3.2 is defined by

$$\lambda : \mathcal{M}on(WM, W\Omega' X) \xrightarrow{B} \mathcal{M}on(BWM, BW\Omega' X) \xrightarrow{\eta(X)_*} \mathcal{T}op^*(BWM, X)$$

and a homotopy inverse κ by

$$\kappa : \mathcal{T}op^*(BWM, X) \xrightarrow{W\Omega'} \mathcal{M}on(W\Omega' BWM, W\Omega' X) \xrightarrow{\mu(WM)^*} \mathcal{M}on(WM, W\Omega' X)$$

3.6 The homotopy counit: Let X be a based space and let

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}; \quad \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \text{ for all } i\}$$

denote the standard n-simplex. The *evaluation map*

$$\mathrm{ev}(X) : B\Omega' X = \left(\coprod_{n \geq 0} (\Omega' X)^n \times \Delta^n \right) / \sim \longrightarrow X$$

is defined by

$$\mathrm{ev}(X)((w_1, \ldots, w_n)(t_0, \ldots, t_n)) = (w_1 + \ldots + w_n)(\sum_{i=1}^{n} t_i \cdot \sum_{j=1}^{i} l(w_j))$$

where $l(w_j)$ is the length of w_j.

The homotopy counit η is the composite

$$\eta(X) : BW\Omega' X \xrightarrow{B\varepsilon(\Omega' X)} B\Omega' X \xrightarrow{\mathrm{ev}(X)} X$$

It is natural in X, so that λ is natural in X.

3.7 The homotopy unit: For a monoid M let EM denote the 2-sided bar construction $B(M, M, *)$. Then

$$z \cdot (x_0, x_1, \ldots, x_n) = (z \cdot x_0, x_1, \ldots, x_n)$$

defines a left M-action on the simplicial space $B_\bullet(M, M, *)$ and hence on EM.

Let $P(EM)$ denote the space of Moore paths in EM starting at the basepoint (e) in the 0-skeleton of EM. The endpoint projection

$$P(EM) \longrightarrow EM$$

is known to be a fibration. Moreover, it is a homotopy equivalence because $P(EM)$ and EM are contractible. Let $P(EM, M)$ be the pullback

$$\begin{array}{ccc} P(EM, M) & \longrightarrow & P(EM) \\ \downarrow{\scriptstyle \pi(M)} & & \downarrow \\ M & \xrightarrow{i} & EM \end{array}$$

where i is the inclusion of the 0-skeleton, i.e. $P(EM, M)$ is the space of Moore paths in EM starting at (e) and ending in M. Then $\pi(M)$ is a fibration and a homotopy equivalence. We define a monoid structure $\oplus$ in $P(EM, M)$ by

$$w_1 \oplus w_2 = w_1 + x \cdot w_2$$

where $+$ is the usual path addition, $x \in M$ is the endpoint of w_1 and $x \cdot w_2$ is the path $t \mapsto x \cdot w_2(t)$. Then $\pi(M) : P(EM, M) \to M$ is a homomorphism and hence a weak equivalence of monoids.

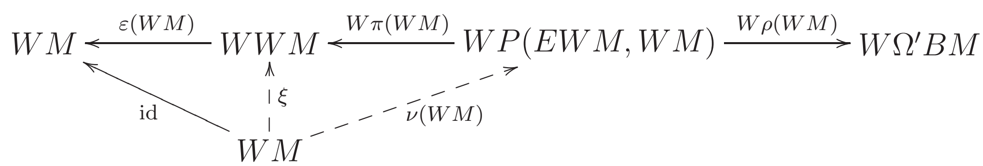
Factoring out the operation of M on EM we obtain a projection

$$EM \to BM$$

inducing a homomorphism

$$\rho(M) : (P(EM, M), \oplus) \longrightarrow (\Omega' BM, +).$$

We apply this construction to WM rather than M and obtain a sequence of natural maps between continuous functors.



By the Lifting Theorem there is a homomorphism ξ, unique up to homotopy, such that

$$\varepsilon(WM) \circ \xi \simeq \mathrm{id}_{WM} \quad \text{in } \mathcal{M}on.$$

Since $W\pi(WM)$ is a weak equivalence, there is a homomorphism

$$\nu(WM) : WM \longrightarrow WP(EWM, WM),$$

unique up to homotopy such that

$$W\pi(WM) \circ \nu(WM) \simeq \xi \quad \text{in } \mathcal{M}on.$$

We define our homotopy unit by

$$\mu(WM) : WM \xrightarrow{\nu(WM)} WP(EWM, WM) \xrightarrow{W\rho(WM)} W\Omega' BWM$$

ν and hence μ is a continuous natural transformation up to homotopy.

3.8 Exercise: Let $\alpha : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ be a continuous natural transformation of continuous functors such that each $\alpha(A)$ is a homotopy equivalence. Choose a homotopy inverse $\beta(A)$ of $\alpha(A)$ for each A in $\mathcal{A}$. Then the $\beta(A)$ form a continuous natural transformation $\beta : G \Rightarrow F$ up homotopy.

The verification of the conditions 3.5 depends on an explicit description of an h-morphism $M \to \Omega' BM$ defined by a homomorphism

$$\zeta' : \overline{W}(M) \longrightarrow \Omega' BM$$

and the interplay of $\overline{W}(M)$ and WM.

We define ζ' as a composite of homomorphisms

$$\overline{W}(M) \xrightarrow{\ \zeta\ } P(EM, M) \xrightarrow{\ \rho(M)\ } \Omega' BM$$

The homomorphism ζ maps the element represented by $(x_0, t_1, \ldots, x_n)$ to the path

$$v_0 + v_1 + \ldots + v_n$$

of length $t_1 + \ldots + t_n + 1$ in the simplex $(e, x_0, x_1, \ldots, x_n) \times \Delta^{n+1} \subset EM$, where

$$v_k(s) = (e, x_0, \ldots, x_n) \times (u_0, \ldots, u_{n+1}) \quad l(v_k) = t_{k+1}$$

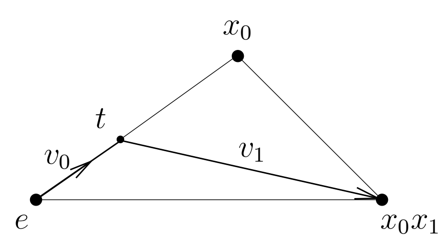
with

$$u_r = \begin{cases} (1-s) \cdot t_r \cdot \prod_{j=r+1}^{k} (1 - t_j) & r \leq k \\ s & r = k+1 \\ 0 & r \geq k+2 \end{cases}$$

and the convention that $t_{n+1} = 1$.

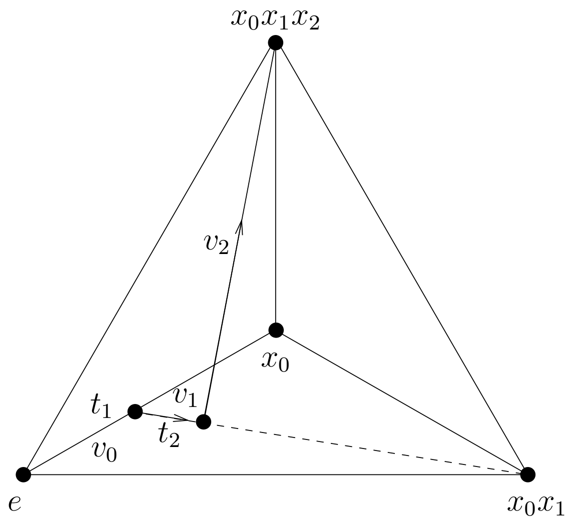
Observe that $+$ is the usual path addition of Moore paths in EM and not the monoid structure of $P(EM, M)$.

Examples: (1) (x_0, t, x_1) is mapped to the path $v_0 + v_1$ of length $t + 1$ given by

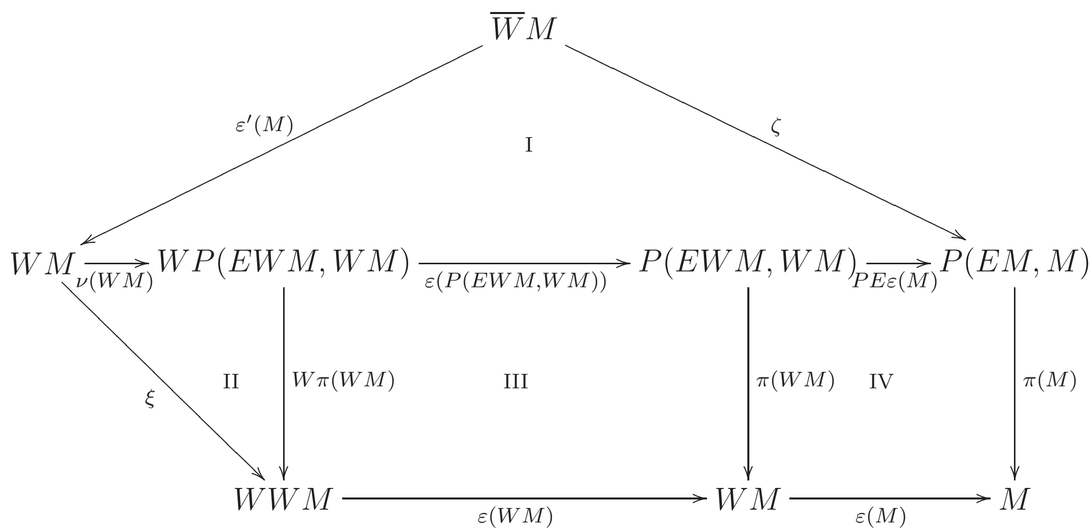


where $l(v_0) = t$ and $l(v_1) = 1$.

(2) $(x_0, t_1, x_1, t_2, x_2)$ is mapped to the path $v_0 + v_1 + v_2$



We now consider the following diagram



The squares III and IV commute by naturality of ε and π, triangle II commutes up to homotopy, and the outmost hexagon commutes by construction of ζ. Since $\pi(M)$ is a weak equivalence I commutes up to homotopy by the uniqueness part of the Lifting Theorem. As a consequence we have

3.9 $\zeta : \overline{W}M \to P(EM, M)$ is a weak equivalence of semigroups.

We are now in the position to prove

3.10 Proposition: $\eta(BWM) \circ B\mu(WM) \simeq \mathrm{id}_{BWM}$ in $\mathcal{T}op^*$.

3.11 Proposition: $W\Omega'\eta(X)\circ\mu(W\Omega'X)\simeq \mathrm{id}_{W\Omega'X}$ in $\mathcal{M}on$.

Proposition 3.10 is a fairly easy consequence of

3.12 Lemma:

$$\begin{array}{ccccccc}
\widetilde{B}\overline{W}M & & \xrightarrow{\quad\widetilde{B}\varepsilon'(M)\quad} & \widetilde{B}WM & \xrightarrow{\quad\widetilde{B}\varepsilon(M)\quad} & & \widetilde{B}M \\
\Big\downarrow{\scriptstyle \widetilde{B}\zeta} & & & & & & \Big\downarrow{\scriptstyle \mathrm{proj}} \\
\widetilde{B}P(EM,M) & \xrightarrow{\widetilde{B}\rho(M)} & \widetilde{B}\Omega'BM & \xrightarrow{\quad\mathrm{proj}\quad} & B\Omega'BM & \xrightarrow{\mathrm{ev}(BM)} & BM
\end{array}$$

commutes up to homotopy.

We have to take $\widetilde{B}$ because $\overline{W}M$ does not have a unit. The lemma is proven by introduction on the filtration of $\widetilde{B}\overline{W}M$ using the fact that both maps map a simplex $(z_1,\ldots,z_n)\times\Delta^n$ in $\widetilde{B}\overline{W}M$ into some higher dimensional simplex in BM. Hence the restrictions of both maps to $(z_1,\ldots,z_n)\times\Delta^n$ are homotopic by a linear homotopy.

It is easy to show that the two maps of Proposition 3.11 are homotopic in $\mathcal{T}op^*$ but considerably harder that they are homotopic in $\mathcal{M}on$.

3.13 Definition: A monoid M is called *grouplike* if it has homotopy inverses.

3.14 Proposition: (1) If M is grouplike, then $\rho(M)$ and hence $\mu(WM)$ are weak equivalences.

(2) $\Omega'\,\mathrm{ev}(Y):\Omega'B\Omega'Y\to\Omega Y$ is a weak equivalence. Hence so is $\Omega'\eta(Y)$.

(3) $B\mu(WM):BWM\to BW\Omega'BWM$ is a homotopy equivalence.

The proof makes use of the following adjunction.

Let $\mathcal{ST}op^*$ denote the category of reduced simplicial spaces $X_\bullet$, i.e. X_0 is a single point. Then the topological realization

$$\mathcal{ST}op^*\longrightarrow\mathcal{T}op^*,\quad X_0\longmapsto|X_0|$$

is canonically based and has a right adjoint

$$\mathcal{S}\mathrm{ing}^*_\bullet:\mathcal{T}op^*\to\mathcal{ST}op^*,\quad \mathcal{S}\mathrm{ing}^*_\bullet(X)=\mathcal{T}op^*((\Delta^n,\Delta^n_0),(X,*))$$

where $\Delta^n_0\subset\Delta^n$ is the 0-skeleton.

There is a simplical map

$$\alpha_\bullet(X) : B_\bullet(*, \Omega' X, *) \longrightarrow \mathcal{S}\mathrm{ing}^*_\bullet(X)$$

which is dimensionwise a homotopy equivalence and allows us to compare $\rho(M)$ with the unit and $\mathrm{ev}(X)$ with the counit of this adjunction.

4 Immediate consequences

Localizations:

4.1 Definition: Let $\mathcal{C}$ be a category and $\mathscr{W}$ a class of morphisms in $\mathcal{C}$. The *localization* of $\mathcal{C}$ with respect to $\mathscr{W}$ is a category $\mathcal{C}[\mathscr{W}^{-1}]$ with $\mathrm{ob}\,\mathcal{C}[\mathscr{W}^{-1}] = \mathrm{ob}\,\mathcal{C}$ and a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathscr{W}^{-1}]$ such that

(1) γ is the identity on objects

(2) $\gamma(f)$ is an isomorphism for all $f \in \mathscr{W}$

(3) if $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(f)$ is an isomorphism for all $f \in \mathscr{W}$ then there exists a unique functor $\overline{F} : \mathcal{C}[\mathscr{W}^{-1}] \to \mathcal{D}$ such that $F = \overline{F} \circ g$.

4.2 Exercise: Let $\mathscr{W} \subset \mathcal{T}op^*$ be the class of homotopy equivalences. Show that $\mathcal{T}op^*[\mathscr{W}^{-1}]$ is the homotopy category $\mathcal{T}op^*/\simeq$.

4.3 Proposition: Let $\mathscr{W} \subset \mathcal{M}on$ be the class of weak equivalences. Then $\mathcal{M}on[\mathscr{W}^{-1}]$ exists and is defined as follows

$$\mathcal{M}on[\mathscr{W}^{-1}](M, N) = \mathcal{M}on(WM, WN)/\simeq$$

($\simeq$ is the homotopy relation in $\mathcal{M}on$) and

$$\gamma : \mathcal{M}on \to \mathcal{M}on[\mathscr{W}^{-1}] \quad f \mapsto [Wf]$$

where $[Wf]$ is the homotopy class of Wf.

The proof is left as an exercise.

4.4 Notation: As is common usage we write $\mathrm{Ho}\,\mathcal{M}on$ for $\mathcal{M}on[\mathscr{W}^{-1}]$ and $\mathrm{Ho}\,\mathcal{T}op^*$ for $\mathcal{T}op^*/\simeq$.

Theorem 3.2 implies

4.5 Theorem: The functors B and Ω' induce an adjoint pair

$$\operatorname{Ho} B : \operatorname{Ho} \mathcal{M}on \rightleftarrows \operatorname{Ho} \mathcal{T}op^* : \operatorname{Ho} \Omega$$

James's theorem:

The underlying space functor $U : \mathcal{M}on \to \mathcal{T}op^*$ has a left adjoint

$$J : \mathcal{T}op^* \to \mathcal{M}on$$

commonly called the *James construction*, which associates with each based space X the free based monoid on X.

4.6 Proposition: (James [2]) For each path-connected based space there is a weak homotopy equivalence of spaces

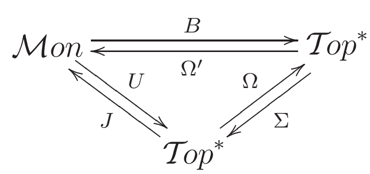
$$JX \simeq \Omega\Sigma X.$$

D. Puppe investigated the conditions which would imply for this weak homotopy equivalence to be a genuine homotopy equivalence.

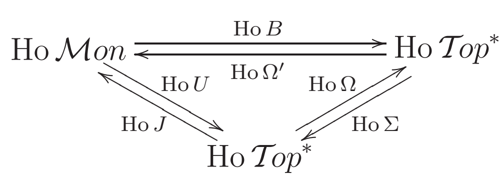
4.7 Proposition: (Puppe [4]): If X is a well-pointed path-connected numerably contractible space then there is a homotopy equivalence

$$JX \simeq \Omega\Sigma X.$$

Consider the diagram of functors



All functors preserve weak equivalences where the weak equivalences in $\mathcal{T}op^*$ are the genuine homotopy equivalences. Hence they induce a diagram of localizations



consisting of adjoint pairs. By Exercise 1.3 there is a natural transformation

$$\tau(X) : U \circ \Omega'(X) \to \Omega(X)$$

which is a homotopy equivalence. Hence $\mathrm{Ho}\,\Omega = \mathrm{Ho}\,U \circ \mathrm{Ho}\,\Omega'$. Since their left adjoints are unique up to natural isomorphisms this implies that $\mathrm{Ho}\,B \circ \mathrm{Ho}\,J$ and $\mathrm{Ho}\,\Sigma$ are naturally isomorphic. We obtain

4.8 Proposition: For each $X \in \mathcal{T}op^*$ there is a homotopy equivalence

$$BJ(X) \simeq \Sigma(X)$$

natural up to homotopy. □

We obtain Puppe's result by combining 4.8 with another result:

4.9 Proposition: If M is a numerably contractible monoid such that $\pi_0(M)$ is a group, then M is grouplike [4].

Proof of 4.7: If X is path-connected and numerably contractible, so is JX. Hence JX is grouplike and the uh-morphism $\mu(WJX) : JX \to \Omega' BJX$ is a weak equivalence in $\mathcal{HM}on$ by 3.14. In particular, its underlying map is a homotopy equivalence. So we have a sequence of homotopy equivalences

$$JX \simeq \Omega' BJX \simeq \Omega BJX \simeq \Omega\Sigma X$$

Homotopical group completion: Homotopical group completion is the replacement of a monoid by a grouplike one having a universal property.

4.10 Proposition: The uh-morphism $\mu(WM) : M \to \Omega' BM$ is a group completion in the following sense: Given a diagram

$$\begin{array}{ccc} M & \xrightarrow{\mu(WM)} & \Omega' BM \\ & {\scriptstyle g}\searrow \qquad \swarrow{\scriptstyle \overline{g}} & \\ & N & \end{array}$$

in $\mathrm{Ho}\,\mathcal{M}on$ with N grouplike, there exists a unique morphism $\overline{g} : \Omega' BM \to N$ making the diagram commute.

Proof Consider the diagram

$$\begin{array}{ccccc} WM & \xrightarrow{\mu(WM)} & W\Omega' BM & \xleftarrow{W\Omega' B\varepsilon(M)} & W\Omega' BWM \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle W\Omega' Bg} \\ WN & \xrightarrow{\mu(WN)} & W\Omega' BN & \xleftarrow{W\Omega' B\varepsilon(N)} & W\Omega' BWN \end{array}$$

All maps apart from g and $\mu(WM)$ are homotopy invertible in $\mathcal{M}on$. So we obtain a map

$$\overline{g} : W\Omega' BM \longrightarrow WN$$

such that $\overline{g} \circ \mu(WM) \simeq g$ in $\mathcal{M}on$. The uniqueness of $\overline{g}$ follows from the fact that $B\mu(WM)$ is a homotopy equivalence (see 3.14). $\square$

5 Diagrams of monoids

We want to show that the classifying space functor preserves homotopy colimits of diagrams of monoids. We start with defining homotopy colimits in the category $\mathcal{M}on$.

$\mathcal{M}on$ is topologically enriched tensored and cotensored over $\mathcal{T}op^*$. The latter means that there are functors

$$\begin{array}{rcll} \mathcal{T}op^* \times \mathcal{M}on & \to & \mathcal{M}on, & (X, M) \mapsto X \otimes M \\ (\mathcal{T}op^*)^{\text{op}} \times \mathcal{M}on & \to & \mathcal{M}on, & (X, M) \mapsto M^X \end{array}$$

and natural homeomorphism

$$\mathcal{M}on(X \otimes M, N) \cong \mathcal{T}op^*(X, \mathcal{M}on(M, N)) \cong \mathcal{M}on(M, N^X)$$

M^X is the k-function space with pointwise multiplication, $X \otimes M$ is more complicated: as a set, it is a free product of copies M, one copy for each $x \in X$.

Let $\mathcal{C}$ be a small indexing category and $\mathcal{M}on^{\mathcal{C}}$ the category of $\mathcal{C}$-diagrams in $\mathcal{M}on$.

5.1 Definition: A map $f : D_1 \to D_2$ of $\mathcal{C}$-diagrams in $\mathcal{M}on$ is called a *weak equivalence* if it is objectwise a weak equivalence in $\mathcal{M}on$. We denote the class of weak equivalence in $\mathcal{M}on^{\mathcal{C}}$ by $\mathscr{W}^{\mathcal{C}}$.

Our first aim is to show that the localization $\mathcal{M}on^{\mathcal{C}}[(\mathscr{W}^{\mathcal{C}})^{-1}] = \operatorname{Ho} \mathcal{M}on^{\mathcal{C}}$ exists. We proceed as in Section 2.

We define a $\mathcal{C} \times \mathcal{C}^{\text{op}}$-diagram $B(\mathcal{C}, \mathcal{C}, \mathcal{C})$ in $\mathcal{T}op$ as follows:

$$B(\mathcal{C}, \mathcal{C}, \mathcal{C})(b, a) = B(\mathcal{C}(-, b), \mathcal{C}, \mathcal{C}(a, -))$$

where the right side is the 2-sided bar construction of 3.1.

The $\mathcal{C} \times \mathcal{C}^{\text{op}}$ structure on $B_n(\mathcal{C}, \mathcal{C}, \mathcal{C})$ is given by

$$(g, h) \cdot (f_0, \ldots, f_{n+1}) = (g \circ f_0, f_1, \ldots, f_n, f_{n+1} \circ h)$$

Analogously we define a $\mathcal{C}^{\mathrm{op}}$-diagram $B(*, \mathcal{C}, \mathcal{C})$ in $\mathcal{T}op$, where $*$ denotes the constant $\mathcal{C}^{\mathrm{op}}$-diagram on a single point.

Let D be a $\mathcal{C}$-diagram in $\mathcal{M}on$ and X a $\mathcal{C}^{\mathrm{op}}$-diagram in $\mathcal{T}op^*$. We define $X \otimes_{\mathcal{C}} D$ to be the coequalizer in $\mathcal{M}on$ of

$$\coprod_{f \in \text{ mor } \mathcal{C}} X(\text{target}(f)) \otimes D(\text{source}(f)) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{a \in \text{ob}\, \mathcal{C}} X(a) \otimes D(a)$$

where for $f : a \to b$ in $\mathcal{C}$ the f-summand $X(b) \otimes D(a)$ is mapped as follows

$$\begin{array}{rcrcl} \alpha & = & X(f) \otimes \mathrm{id}: & X(b) \otimes D(a) & \longrightarrow & X(a) \otimes D(a) \\ \beta & = & \mathrm{id} \otimes D(f): & X(b) \otimes D(a) & \longrightarrow & X(b) \otimes D(b) \end{array}$$

By $B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} D$ we denote the $\mathcal{C}$-diagram in $\mathcal{M}on$ defined by

$$a \mapsto B(\mathcal{C}(-, a), \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} D$$

where $X_+ = X \sqcup \{+\} \in \mathcal{T}op^*$ with basepoint $+$ for $X \in \mathcal{T}op$.

5.2 Proposition: Let D_0, D_1, D_2 be $\mathcal{C}$ diagrams in $\mathcal{M}on$ and let $p : D_1 \to D_2$ be a weak equivalence. Then p induces a homotopy equivalence

$$p_* : \mathcal{M}on^{\mathcal{C}}(B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} WD_0, D_1) \to \mathcal{M}on^{\mathcal{C}}(B(\mathcal{C}, \mathcal{C}, \mathcal{C}) \otimes_{\mathcal{C}} WD_0, D_2)$$

in $\mathcal{T}op^*$.

The proof is similar to the proof of Proposition 2.5 and proceeds by induction on the skeleton of $B(\mathcal{C}, \mathcal{C}, \mathcal{C})$.

If we now did continue in the spirit of Section 2 we would define a homotopy homomorphism between $\mathcal{C}$-diagrams D_1 and D_2 of monoids as a strict map of $\mathcal{C}$-diagrams

$$B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} WD_1 \to B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} WD_2$$

We will not pursue this approach but only list

5.3 Corollary: If $p : D_1 \to D_2$ is a weak equivalence of $\mathcal{C}$-diagrams in $\mathcal{M}on$, then

$$\mathrm{id} \otimes_{\mathcal{C}} p : B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} WD_1 \to B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} D_2$$

is a homotopy equivalence in $\mathcal{M}on^{\mathcal{C}}$.

The following is a well-known fact about the 2-sided bar construction: Let $\mathcal{C}_\bullet$ denote the $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-diagram of simplical sets sending (a, b) to the constant simplicial set $\mathcal{C}(a, b)$. The maps

$$\begin{array}{rrcl} \varepsilon_n : & B_n(\mathcal{C}, \mathcal{C}, \mathcal{C})(a, b) & \longrightarrow & \mathcal{C}(a, b) \\ & (f_0, \ldots, f_{n+1}) & \longmapsto & f_0 \circ \ldots \circ f_{n+1} \end{array}$$

define a simplicial map $B_\bullet(\mathcal{C}, \mathcal{C}, \mathcal{C}) \to \mathcal{C}_\bullet$. Let $\varepsilon : B(\mathcal{C}, \mathcal{C}, \mathcal{C}) \to \mathcal{C}$ be its realization.

5.4 Proposition: $\varepsilon \otimes_{\mathcal{C}} \mathrm{id} : B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} D \to \mathcal{C}_+ \otimes_{\mathcal{C}} D \cong D$ is a weak equivalence in $\mathcal{M}on^{\mathcal{C}}$.

5.5 Proposition: $\mathrm{Ho}\,\mathcal{M}on^{\mathcal{C}}$ exists and $B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} WD$ and D are isomorphic in $\mathrm{Ho}\,\mathcal{M}on^{\mathcal{C}}$.

Proof The existence of $\mathrm{Ho}\,\mathcal{M}on^{\mathcal{C}}$ follows from Proposition 5.2 as in the localization part of Section 4. The second statement follows from Proposition 5.4. □

We have a continuous adjoint pair

$$\mathrm{colim} : \mathcal{M}on^{\mathcal{C}} \rightleftarrows \mathcal{M}on : \ \mathrm{const}$$

where const is the constant diagram functor. It induces on adjoint pair.

$$\mathrm{hocolim} : \mathrm{Ho}\,\mathcal{M}on^{\mathcal{C}} \rightleftarrows \mathrm{Ho}\,\mathcal{M}on : \ \mathrm{hoconst}$$

where hoconst in the constant diagram functor and

$$\mathrm{hocolim}\, D = \mathrm{colim}(B(\mathcal{C}, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} WD) = B(*, \mathcal{C}, \mathcal{C})_+ \otimes_{\mathcal{C}} WD$$

There is a corresponding pair of adjoint functors

$$\mathrm{hocolim} : \mathrm{Ho}((\mathcal{T}op^*)^{\mathcal{C}}) \rightleftarrows \mathrm{Ho}\,\mathcal{T}op^* : \ \mathrm{hoconst}$$

where hoconst is the constant diagram functor and

$$\mathrm{hocolim}\, D = B(*, \mathcal{C}, D)/B(*, \mathcal{C}, *)$$

where $* \to D$ is the inclusion of base points.

5.6 Proposition: The diagram

$$\begin{array}{ccc} \mathrm{Ho}\,\mathcal{M}on^{\mathcal{C}} & \xrightarrow{\mathrm{hocolim}} & \mathrm{Ho}\,\mathcal{M}on \\ {\scriptstyle \mathrm{Ho}\,B^{\mathcal{C}}} \downarrow & & \downarrow {\scriptstyle \mathrm{Ho}\,B} \\ \mathrm{Ho}(\mathcal{T}op^*)^{\mathcal{C}} & \xrightarrow{\mathrm{hocolim}} & \mathrm{Ho}\,\mathcal{T}op^* \end{array}$$

commutes up to natural equivalence.

Proof The corresponding diagram of right adjoints commute. □

5.7 Corollary: The classifying space functor preserves homotopy colimits up to homotopy equivalences.

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