Spectral measures, dispersive estimates, and the low Mach number limits of compressible viscous fluids

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Scaled compressible Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}$$

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right), \ \mu > 0$$

$$\varrho = \varrho(t,x), \ \mathbf{u} = \mathbf{u}(t,x), \ t \in (0,T), \ x \in \Omega$$

Boundary conditions

$\Omega \subset R^3$ an (unbounded) domain with smooth boundary

COMPLETE SLIP BOUNDARY CONDITIONS:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

NAVIER'S SLIP BOUNDARY CONDITIONS WITH FRICTION:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \beta[\mathbf{u}]_{\rm tan} + [\mathbb{S}\mathbf{n}]_{\rm tan}|_{\partial\Omega} = 0, \ \beta \approx \varepsilon^{-\alpha}$$

Ill-prepared initial data

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon r_{0,\varepsilon},\ \{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty(\Omega)$$

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty(\Omega; R^3)$$

Asymptotic limit

Suppose

$$p(\varrho) \approx \varrho^\gamma,\ \gamma > 3/2$$

$$r_{0,\varepsilon} \to r_0 \text{ weakly in } L^2(\Omega)$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega; R^3)$$

Then

$$r_\varepsilon \equiv \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r \text{ weakly-(*) in } L^\infty(0,T; L^2 \oplus L^\gamma(\Omega))$$

$$\mathbf{u}_\varepsilon \to \ \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3))$$

$$\text{and strongly in } L^2((0,T) \times K; R^3) \text{ for any compact } K \subset \Omega$$

Incompressible Navier-Stokes system

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})\Big) + \nabla_x \Pi = \mu \Delta \mathbf{U}$$

Energy inequality

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2} E(\varrho, \overline{\varrho}) \right)(\tau, \cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} E(\varrho_{0,\varepsilon}, \overline{\varrho}) \right)\ \mathrm{d}x$$

$$E(\varrho, \overline{\varrho}) = H(\varrho) - H'(\overline{\varrho})(\varrho - \overline{\varrho}) - H(\overline{\varrho})$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

Helmholtz decomposition

$$\mathbf{u}_\varepsilon = \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\mathbf{u}_\varepsilon], \ \mathbf{H}^\perp[\mathbf{u}_\varepsilon] = \nabla_x \Phi_\varepsilon$$

$$\Delta \Phi_\varepsilon = \mathrm{div}_x \mathbf{u}_\varepsilon \text{ in } \Omega, \ \nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Compactness of the convective term

Ebin [1977], Klainerman and Majda [1981], Schochet [1984]...

$$\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \mathbf{U} \otimes \mathbf{U} \text{ (in some sense)}$$

- Local method (Lions and Masmoudi [1998])

$$\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon = \langle \text{compact terms} \rangle + \langle \text{gradient} \rangle$$

- No-slip + special geometry properties (Desjardins, Grenier, Lions, Masmoudi [1999])

$$\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \mathbf{U} \times \mathbf{U} \text{ a.a. in } \Omega$$

Ω a bounded regular domain

- Strichartz estimates (Desjardins and Grenier [1999])

$$\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \longrightarrow \mathbf{U} \times \mathbf{U} \text{ a.a.}$$

$\Omega = R^3$

Lighthill's equation

$$p'(\overline{\varrho}) = 1$$

$$\varepsilon \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \nabla_x r_\varepsilon = \varepsilon \mathbf{f}_\varepsilon$$

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon},\ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$\mathbf{f}_\varepsilon = \mathrm{div}_x \mathbb{S}_\varepsilon - \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \frac{1}{\varepsilon^2} \nabla_x \Big(p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}) \Big)$$

Wave equation - for acoustic potential

$$\varepsilon \partial_t r_\varepsilon + \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \Phi_\varepsilon + r_\varepsilon = \varepsilon G_\varepsilon$$

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\nabla_x \Phi_\varepsilon = \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \approx \mathbf{H}^\perp[\mathbf{u}_\varepsilon]$$

Abstract formulation

$$A[v] = -\Delta v \text{ in } \Omega,\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0$$

A a non-negative self-adjoint operator on $L^2(\Omega)$

Abstract acoustic equation

$$\varepsilon \partial_t r_\varepsilon - A[\Phi_\varepsilon] = 0$$

$$\varepsilon \partial_t \Phi_\varepsilon + r_\varepsilon = \varepsilon F(A)[g_\varepsilon]$$

$F(A)$ may become singular for $A \to 0+$, $A \to \infty$

Variation-of-constants formula

$$\Phi_\varepsilon(t) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon}\right)\left[\Phi_{0,\varepsilon} + \mathrm{i}\frac{1}{\sqrt{A}}[r_{0,\varepsilon}]\right]$$
$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{A}\frac{t}{\varepsilon}\right)\left[\Phi_{0,\varepsilon} - \mathrm{i}\frac{1}{\sqrt{A}}[r_{0,\varepsilon}]\right]$$
$$\frac{1}{2}\int_0^t \left(\exp\left(\mathrm{i}\sqrt{A}\frac{t-s}{\varepsilon}\right) + \exp\left(-\mathrm{i}\sqrt{A}\frac{t-s}{\varepsilon}\right)\right) F(A)[g_\varepsilon(s)]\ \mathrm{d}s,$$

$$\{\Phi_{0,\varepsilon}\}_{\varepsilon>0},\ \{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega)$$

$$\{g_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega))$$

GOAL

$$\left\{ t \mapsto \int_\Omega \varphi G(A)[\Phi_\varepsilon(t,\cdot)] \ \mathrm{d}x \right\} \to 0 \ \text{ in } L^2(0,T) \text{ as } \varepsilon \to 0$$

for any $\varphi \in C_0^\infty(\Omega)$, $G \in C_0^\infty(0,\infty)$

We need:

$$\left(\int_0^T \left| \left\langle \exp\left({\rm i}\sqrt{A}\frac{t}{\varepsilon} \right) [\Psi], G(A)[\varphi] \right\rangle \right|^2 \ {\rm d}t \right)^{1/2} \leq \omega(\varepsilon, G, \varphi) \|\Psi\|_{L^2(\Omega)}$$

$$\omega(\varepsilon, G, \varphi) \to 0 \text{ as } \varepsilon \to 0$$

for any $\varphi \in C_0^\infty(\Omega)$, $G \in C_0^\infty(0, \infty)$

Uniform decay of "non-homogeneous" components :

$$\int_0^T \int_0^t \left| \left\langle \exp \left(-\mathrm{i}\sqrt{A}\frac{t-s}{\varepsilon} \right) F(A)[g_\varepsilon(s)], G(A)[\varphi] \right\rangle \right|^2 \ \mathrm{d}s \ \mathrm{d}t$$

$$\leq \int_0^T \int_0^T \left| \left\langle \exp \left(-\mathrm{i}\sqrt{A}\frac{t-s}{\varepsilon} \right) [g_\varepsilon(s)], G(A)F(A)[\varphi] \right\rangle \right|^2 \ \mathrm{d}t \ \mathrm{d}s$$

$$\leq \omega^2(\varepsilon, G \cdot F, \varphi) \int_0^T \left\| \exp \left(\mathrm{i}\sqrt{A}\frac{s}{\varepsilon} \right) [g_\varepsilon(s)] \right\|^2_{L^2(\Omega)} \ \mathrm{d}s$$

$$= \omega^2(\varepsilon, G \cdot F, \varphi) \int_0^T \| G_\varepsilon(s) \|^2_{L^2(\Omega)} \ \mathrm{d}s.$$

Reformulation via spectral measures:

$$\left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon} \right)[\Psi], G(A)\varphi \right\rangle = \int_0^\infty \exp\left(\mathrm{i}\sqrt{\lambda}\frac{t}{\varepsilon} \right) G(\lambda)\tilde{\Psi}(\lambda)\ \mathrm{d}\mu_\varphi(\lambda)$$

where μ_φ is the spectral measure associated to the function φ

$$\tilde{\Psi} \in L^2(\Omega; \mathrm{d}\mu_\varphi),\ \|\tilde{\Psi}\|_{L^2_{\mu_\varphi}(\Omega)} \leq \|\Psi\|_{L^2(\Omega)}.$$

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon} \right) [\Psi], G(A)\varphi \right\rangle \right|^2 \ \mathrm{d}t$$

$$= \int_0^T \int_0^\infty \int_0^\infty \exp\left(\mathrm{i} \left(\sqrt{x} - \sqrt{y} \right) \frac{t}{\varepsilon} \right) \times$$

$$\times G(x) G(y) \tilde{\Psi}(x)\ \overline{\tilde{\Psi}}(y)\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y)\ \mathrm{d}t$$

$$\leq e \int_0^\infty \int_0^\infty \left(\int_{-\infty}^\infty \exp\left(-(t/T)^2 \right) \exp\left(\mathrm{i} \left(\sqrt{x} - \sqrt{y} \right) \frac{t}{\varepsilon} \right)\ \mathrm{d}t \right) \times$$

$$\times G(x) G(y) \Psi(x) \overline{\tilde{\Psi}(y)}\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y)$$

$$\leq eT\sqrt{\pi}\int_0^\infty\int_0^\infty \exp\left(-\frac{T^2|\sqrt{x}-\sqrt{y}|^2}{4\varepsilon^2}\right)\times$$

$$\times|\tilde{\Psi}(x)|\ |\tilde{\Psi}(y)|\ |G(x)|\mathrm{d}\mu_\varphi(x)\ |G(y)|\mathrm{d}\mu_\varphi(y)$$

by Cauchy-Schwartz inequality

$$\leq \omega^2(\varepsilon, G, \varphi)\|\Psi\|^2_{L^2(\Omega)}$$

$$\omega^4(\varepsilon, G, \varphi) \leq$$

$$c\int_0^\infty\int_0^\infty \exp\left(-\frac{T^2|\sqrt{x}-\sqrt{y}|^2}{2\varepsilon^2}\right)\ |G(x)||G(y)|\mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y)$$

Conclusion via RAGE theorem

$$\omega(\varepsilon, G, \varphi) \to 0 \text{ as } \varepsilon \to 0$$

$$\iff$$

μ_φ does not charge points

$$\iff$$

the point spectrum of A is empty

Following Y.Last [1996]:

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon} \right) [\Psi], G(A)[\varphi] \right\rangle \right|^2 \, \mathrm{d}t$$

$$\leq eT\sqrt{\pi} \int_0^\infty |\Psi(x)|^2 \left(\int_0^\infty \exp\left(-\frac{|\sqrt{x} - \sqrt{y}|^2}{\varepsilon^2} \frac{T^2}{4} \right) \mathrm{d}\mu_\varphi(y) \right) \times$$

$$\times G^2(x) \, \mathrm{d}\mu_\varphi(x)$$

$$\int_0^\infty \exp\left(-\frac{|\sqrt{x}-\sqrt{y}|^2}{\varepsilon^2}\frac{T^2}{4}\right) \mathrm{d}\mu_\varphi(y)$$

$$= \sum_{n=0}^\infty \int_{\varepsilon n \leq |\sqrt{y}-\sqrt{x}| < \varepsilon(n+1)} \exp\left(-\frac{|\sqrt{x}-\sqrt{y}|^2}{\varepsilon^2}\frac{T^2}{4}\right) \mathrm{d}\mu_\varphi(y)$$

$$\leq \sup_{n \geq 0} \int_{\varepsilon n \leq |\sqrt{y}-\sqrt{x}| < \varepsilon(n+1)} 1 \mathrm{d}\mu_\varphi(y) \sum_{n=0}^\infty \exp\left(-\frac{n^2 T^2}{4}\right)$$

for $x \in \mathrm{supp}[G]$

Stone's formula:

$$\mu_{\varphi}(a,b)$$

$$= \lim_{\delta \to 0+} \lim_{\eta \to 0+} \int_{a+\delta}^{b-\delta} \left\langle \left(\frac{1}{A - \lambda - \mathrm{i}\eta} - \frac{1}{A - \lambda + \mathrm{i}\eta} \right) \varphi, \varphi \right\rangle \ \mathrm{d}\lambda$$

Limiting absorption principle:

$$\left[\begin{array}{c} \textit{Operators} \\ \\ \mathcal{V} \circ (A - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V} : L^2(\Omega) \to L^2(\Omega), \\ \\ \mathcal{V}[v] = (1 + |x|^2)^{-s/2},\ s > 1 \\ \\ \textit{are bounded uniformly for } \lambda \in [a, b],\ 0 < a < b,\ \eta > 0, \end{array} \right]$$

Conclusion via Kato's result

Operator A satisfies Limiting Absorption Principle

$$\Longrightarrow$$

$$\mu_\varphi[I] \leq c_\delta |I| \text{ for any interval } I \subset [\delta, 1/\delta],\ \delta > 0$$

$$\Longrightarrow$$

$$\omega(\varepsilon, G, \varphi) \leq \sqrt{\varepsilon} c(G, \varphi)$$

Uniform decay for varying domains

- $$\Omega_\varepsilon = R^3 \setminus K_\varepsilon, \ K_\varepsilon \subset \{|x| \leq r\}$$

- $$\Omega \subset \Omega_{\varepsilon_2} \subset \Omega_{\varepsilon_1} \text{ for } \varepsilon_2 > \varepsilon_1$$

- $$|\Omega_\varepsilon \setminus \Omega| \to 0 \text{ as } \varepsilon \to 0$$

- $$\Omega_\varepsilon \text{ satisfy the uniform cone condition}$$

$$\Longrightarrow \ \omega(\varepsilon, G, \varphi) \approx \sqrt{\varepsilon}$$

Stone's formula $\Rightarrow$

$$\mu_{\varepsilon,\varphi}(a,b) = \int_a^b \left\langle \left(w^-_{\varepsilon,\lambda} - w^+_{\varepsilon,\lambda} \right), \varphi \right\rangle_{L^2(\Omega_\varepsilon)} \ \mathrm{d}\lambda,\ 0 < a < b,$$

where $w^\pm_{\varepsilon,\lambda}$ solve

$$\Delta w^\pm_{\varepsilon,\lambda} + \lambda w^\pm_{\varepsilon,\lambda} = \varphi \text{ in } \Omega_\varepsilon,\ \nabla_x w^\pm_{\varepsilon,\lambda} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

Sommerfeld radiation condition

$$\lim_{r\to\infty} r \left(\partial_r \pm \mathrm{i}\sqrt{\lambda} \right) w^\pm_{\varepsilon,\lambda} = 0,\ r = |x|$$

Reduction to bounded domain case:

$$\mathrm{supp}[\varphi] \subset \{|x| \leq R\}$$

$$w^{\pm}_{\varepsilon,\lambda_\varepsilon}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta,\phi) \text{ for } |x| = 2R$$

(r, θ, ϕ) polar coordinates

$$w^{\pm}_{\varepsilon,\lambda_\varepsilon}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta,\phi) \frac{h_l^{(1)}(\pm\sqrt{\lambda} r)}{h_l^{(1)}(\pm\sqrt{\lambda} 2R)} \text{ for all } x \in R^3 \setminus \overline{B}_{2R},$$

Y_l^m spherical harmonics of order l

,

$h_l^{(1)}$ spherical Bessel functions

“Perforated” domains

$$\Omega_\varepsilon = \Omega \setminus \sum_{i=1}^{M} B_{\delta(\varepsilon)}[x_i]$$

$$B_\delta[x_i] = \{ |x - x_i| \leq \delta \}$$

$$\delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

Stratified fluids

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho} \nabla_x F$$

Anelastic constraint

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0$$

$$A = \frac{1}{\tilde{\varrho}} \mathrm{div}_x(\tilde{\varrho} \nabla_x \mathbf{U})$$