Averaging for dissipative PDEs Geometric methods for systems with fast oscillation of the vector field

Piotr Zgliczyński

Institute of Computer Science and Computional Mathematics,
Jagiellonian University, Kraków

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Plan of the talk

1. Results
2. About the approach

The problem

Let $z \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, for $k = 1, \ldots, m$ $u_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ $g_k : \mathbb{R} \to \mathbb{R}$, smooth. We consider

$$z' = f(z) + \sum_{k=1}^{m} g_k(\omega_k t) u_k(t, z) \tag{1}$$

where there exist bounded functions G_k, such that $G_k'(t) = g_k(t)$ for $k = 1, \ldots, m$.

$$z' = f(z). \tag{2}$$

Question: Assume that we know a dynamics of (2) then what can be set about the dynamics of (1).

Partial answer for ODEs: Processes induced by time shift by (2) and (1) are C^k-close for any k on bounded sets and bounded times intervals

What about the dissipative PDEs?

Results - ODEs

Theorem

Consider

$$z' = f(z) + \sum_{k=1}^{m} g_k(\omega_k t) u_k(t, z) \qquad (3)$$

where $f(0) = 0$ *and* $df(0)$ *is hyperbolic and there exist bounded functions* G_k*, such that* $G_k'(t) = g_k(t)$ *for* $k = 1, \ldots, m$.
Assume uniform in time (may depend on z*) bounds for* $\|u_k(t, z\|$, $\left\|\frac{\partial u_k}{\partial z}(t, z)\right\|$, $\left\|\frac{\partial u_k}{\partial t}(t, z)\right\|$, $\left\|\frac{\partial^2 u_k}{\partial t \partial z}(t, z)\right\|$, $\left\|\frac{\partial^2 u_k}{\partial z^2}(t, z)\right\|$.
Then there exists $\bar{\omega}$*, such that if* $\min |\omega_k| > \bar{\omega}$*, then there exists* $r = O(1/\min |\omega_k|)$ *such that in the set* $\mathbb{R} \times B(0, r)$ *there exists unique full trajectory of (3), which is normally hyperbolic.*

No smallness assumption on $g_k u_k$, just rapid oscillations of the perturbation
If (3) is T-periodic, then this orbits is also T-periodic.

Results - PDEs, Burgers equation

Consider the viscous Burgers equation

$$u_t + u \cdot u_x - \nu u_{xx} = f(t, x), \quad \nu > 0 \tag{4}$$

with periodic boundary conditions $x \in \mathbb{R}/2\Pi$ and constraint

$$\int_0^{2\pi} u(t, x) dx = 2\pi c. \tag{5}$$

Assume that f is a finite trygonometric polynomial is bounded with bounded time derivatives of Fourier coefficients.
Then there exists a full orbit of size $O(1/c)$ (we fix f and ν), which attracts all orbits.
Known result: Jauslin, Kreiss, Moser.

Results - PDEs, Burgers equation what this have to do with rapid oscillation

In the coordinate frame moving with the velocity c the Burgers equation have the same form, but the forcing becomes rapidly oscillating.
After transformation:

$$v(t,x) = u(t, x+ct) - c \tag{6}$$

our viscous Burgers equation becomes

$$v_t + v \cdot v_x - \nu v_{xx} = f(t, x+ct), \qquad \int_0^{2\pi} v(t,x)dx = 0. \tag{7}$$

$$f(t,x) = \sum_{k\in\mathbb{Z}} a_k(t)e^{ikx}, \quad f(t,x+ct) = \sum_{k\in\mathbb{Z}} a_k(t)e^{ikct}e^{ikx}$$

Results - PDEs - Navier-Stokes

Perodic boundary conditions $x \in \mathbb{T}^d = (\mathbb{R}/2\Pi)^d$ with $d = 2, 3$ and constraint

$$\int_{\mathbb{T}^d} u(t,x)dx = c. \tag{8}$$

With forcing term $f(t,x)$ with zero mean.
Assume that f is a finite trygonometric polynomial, which is bounded with bounded time derivatives of Fourier coefficients and $k \cdot c$ for all modes in $k \cdot c \neq 0$ (generic condition on the velocity)
Then there exists a full orbit of size $O(1/\min(|k \cdot c|))$ (we fix f and ν). If $d = 2$, then it attracts all orbits.
Idea: we move to moving frame and we have equation with fast oscillation, just as in the previous case.

Results - PDEs increasing the existence domain

Consider

$$\frac{du}{dt} = F(u) + \sum_j P_j(u, \nabla u) \sin(\omega_j t) \tag{9}$$

where F is for example Burgers or Navier-Stokes, Kuramoto-Sivashinsky vector field and P_j are arbitrary polynomials plus periodic boundary conditions.
Observe that we allow for expressions like $P_j(\nabla u) = (\nabla u | \nabla u)^j$ for $j \in \mathbb{N}$, so j can be as large as we want.

Theorem

*When $|\omega_j| \to \infty$, then the region U_ω in which the forward solution is defined for all $t \geq 0$ is increasing, so that $\bigcup_\omega U_\omega =$ (**full phase space**) and all these solutions are attracted to zero (or (periodic) orbit there is some (periodic) forcing)*

Results - increasing domain of existence an ODE version

$$x' = -x + a\cos(\omega t)x^3, x \in \mathbb{R}, a \in \{0, 1\} \tag{10}$$

- for $a = 0$ $x_0 = 0$ is globally attracting solution, when $a \neq 0$ then x_0 is only locally attracting.
- for $a \neq 0$ some solutions go to infinity in finite time.
- But for $\omega \to \infty$ the basin of attraction of x_0 increases to enclose the whole

Some details on Burgers equation -

The Burgers equation for Fourier modes $u(t,x) = \sum_{k\in\mathbb{Z}} z_k \exp(ikz)$ is

$$z_k' = -\lambda_k z_k - ikcz_k + N_k(z) + g_k(t) \tag{11}$$

where $c = z_0 \in \mathbb{R}$, $\lambda_k = \nu k^2$

$$N_k(z) = \frac{-ik}{2} \sum_{k_1\in\mathbb{Z}\setminus\{0,k\}} z_{k_1} z_{k-k_1}. \tag{12}$$

Therefore

$$\omega_k = -kc. \tag{13}$$

Observe that the transformation

$$x_k = z_k \exp(-i\omega_k t) \tag{14}$$

preserves the reality condition, i.e. if $\overline{z_{-k}} = z_k$, then $\overline{x_{-k}} = x_k$ and

$$N_k(z(x)) = N_k(x). \tag{15}$$

Therefore we obtain equation

$$x_k' = -\lambda_k x_k + N_k(x) + g_k(t)\exp(ikc). \tag{16}$$