Averaging for dissipative PDEs
Geometric methods for systems with fast oscillation of the vector field

Piotr Zgliczyński

Institute of Computer Science and Computational Mathematics,
Jagiellonian University, Kraków

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The problem

Let \( z \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n \), for \( k = 1, \ldots, m \) \( u_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \)
\( g_k : \mathbb{R} \to \mathbb{R} \), smooth. We consider

\[
\dot{z} = f(z) + \sum_{k=1}^{m} g_k(\omega_k t) u_k(t, z)
\]  

where there exist bounded functions \( G_k \), such that \( G_k'(t) = g_k(t) \) for \( k = 1, \ldots, m \).

\[
\dot{z} = f(z).
\]

**Question:** Assume that we know a dynamics of (2) then what can be set about the dynamics of (1).

**Partial answer for ODEs:** Processes induced by time shift by (2) and (1) are \( C^k \)-close for any \( k \) on bounded sets and bounded times intervals

**What about the dissipative PDEs?**
Consider

\[ z' = f(z) + \sum_{k=1}^{m} g_k(\omega_k t)u_k(t, z) \tag{3} \]

where \( f(0) = 0 \) and \( df(0) \) is hyperbolic and there exist bounded functions \( G_k \), such that \( G'_k(t) = g_k(t) \) for \( k = 1, \ldots, m \).

Assume uniform in time (may depend on \( z \)) bounds for \( \|u_k(t, z)\|, \|\partial u_k / \partial z(t, z)\|, \|\partial u_k / \partial t(t, z)\|, \|\partial^2 u_k / \partial t \partial z(t, z)\|, \|\partial^2 u_k / \partial z^2(t, z)\|. \)

Then there exists \( \bar{\omega} \), such that if \( \min |\omega_k| > \bar{\omega} \), then there exists \( r = O(1/\min |\omega_k|) \) such that in the set \( \mathbb{R} \times B(0, r) \) there exists unique full trajectory of (3), which is normally hyperbolic.

No smallness assumption on \( g_k u_k \), just rapid oscillations of the perturbation.

If (3) is \( T \)-periodic, then this orbits is also \( T \)-periodic.
Consider the viscous Burgers equation

\[ u_t + u \cdot u_x - \nu u_{xx} = f(t, x), \quad \nu > 0 \]  

(4)

with periodic boundary conditions \( x \in \mathbb{R}/2\pi \) and constraint

\[ \int_0^{2\pi} u(t, x) \, dx = 2\pi c. \]  

(5)

Assume that \( f \) is a finite trigonometric polynomial is bounded with bounded time derivatives of Fourier coefficients. Then there exists a full orbit of size \( O(1/c) \) (we fix \( f \) and \( \nu \)), which attracts all orbits.

Known result: Jauslin, Kreiss, Moser.
In the coordinate frame moving with the velocity $c$ the Burgers equation have the same form, but the forcing becomes rapidly oscillating. After transformation:

$$v(t, x) = u(t, x + ct) - c$$ \hspace{1cm} (6)

our viscous Burgers equation becomes

$$v_t + v \cdot v_x - \nu v_{xx} = f(t, x + ct), \quad \int_0^{2\pi} v(t, x) \, dx = 0. \hspace{1cm} (7)$$

$$f(t, x) = \sum_{k \in \mathbb{Z}} a_k(t) e^{ikx}, \quad f(t, x + ct) = \sum_{k \in \mathbb{Z}} a_k(t) e^{ikct} e^{ikx}$$
Perodic boundary conditions $x \in \mathbb{T}^d = (\mathbb{R}/2\pi)^d$ with $d = 2, 3$ and constraint

$$\int_{\mathbb{T}^d} u(t, x) dx = c. \quad (8)$$

With forcing term $f(t, x)$ with zero mean.
Assume that $f$ is a finite trygonometric polynomial, which is bounded with bounded time derivatives of Fourier coefficients and $k \cdot c$ for all modes in $k \cdot c \neq 0$ (generic condition on the velocity)
Then there exists a full orbit of size $O(1/\min(|k \cdot c|))$ (we fix $f$ and $\nu$). If $d = 2$, then it attracts all orbits.
Idea: we move to moving frame and we have equation with fast oscillation, just as in the previous case.
Consider
\[
\frac{du}{dt} = F(u) + \sum_j P_j(u, \nabla u) \sin(\omega_j t)
\] (9)

where \(F\) is for example Burgers or Navier-Stokes, Kuramoto-Sivashinsky vector field and \(P_j\) are arbitrary polynomials plus periodic boundary conditions.

Observe that we allow for expressions like \(P_j(\nabla u) = (\nabla u|\nabla u)^j\) for \(j \in \mathbb{N}\), so \(j\) can be as large as we want.

**Theorem**

*When \(|\omega_j| \to \infty\), then the region \(U_\omega\) in which the forward solution is defined for all \(t \geq 0\) is increasing, so that \(\bigcup_\omega U_\omega = (\text{full phase space})\) and all these solutions are attracted to zero (or (periodic) orbit there is some (periodic) forcing)*
\[
x' = -x + a \cos(\omega t)x^3, \quad x \in \mathbb{R}, \quad a \in \{0, 1\}
\]

for \(a = 0\) \(x_0 = 0\) is globally attracting solution, when \(a \neq 0\) then \(x_0\) is only locally attracting.

for \(a \neq 0\) some solutions go to infinity in finite time.

But for \(\omega \to \infty\) the basin of attraction of \(x_0\) increases to enclose the whole
Some details on Burgers equation -

The Burgers equation for Fourier modes $u(t, x) = \sum_{k \in \mathbb{Z}} z_k \exp(ikz)$ is

$$z_k' = -\lambda_k z_k - ikcz_k + N_k(z) + g_k(t)$$  \hspace{1cm} (11)

where $c = z_0 \in \mathbb{R}$, $\lambda_k = \nu k^2$

$$N_k(z) = \frac{-ik}{2} \sum_{k_1 \in \mathbb{Z} \setminus \{0,k\}} z_{k_1} z_{k-k_1}.  \hspace{1cm} (12)$$

Therefore

$$\omega_k = -kc.  \hspace{1cm} (13)$$

Observe that the transformation

$$x_k = z_k \exp(-i\omega_k t)  \hspace{1cm} (14)$$

preserves the reality condition, i.e. if $\overline{z_{-k}} = z_k$, then $\overline{x_{-k}} = x_k$ and

$$N_k(\overline{z(x)}) = N_k(x).  \hspace{1cm} (15)$$

Therefore we obtain equation

$$x_k' = -\lambda_k x_k + N_k(x) + g_k(t) \exp(ikc).  \hspace{1cm} (16)$$